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Well-Posedness of Stochastic Continuity Equations on Riemannian Manifolds*

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Abstract The authors analyze continuity equations with Stratonovich stochasticity,

$$\partial\rho + \mathrm{div}_h \Big[\rho \circ \Big(u(t,x) + \sum_{i=1}^{N} a_i(x)\dot{W}_i(t)\Big)\Big] = 0$$

defined on a smooth closed Riemannian manifold M with metric h. The velocity field u is perturbed by Gaussian noise terms $\dot{W}_1(t), \cdots, \dot{W}_N(t)$ driven by smooth spatially dependent vector fields $a_1(x), \cdots, a_N(x)$ on M. The velocity u belongs to $L^1_t W^{1,2}_x$ with $\mathrm{div}_h\, u$ bounded in $L^p_{t,x}$ for $p > d+2$, where d is the dimension of M (they do not assume $\mathrm{div}_h\, u \in L^\infty_{t,x}$). For carefully chosen noise vector fields a_i (and the number N of them), they show that the initial-value problem is well-posed in the class of weak L^2 solutions, although the problem can be ill-posed in the deterministic case because of concentration effects. The proof of this "regularization by noise" result is based on a L^2 estimate, which is obtained by a duality method, and a weak compactness argument.

Keywords Stochastic continuity equation, Riemannian manifold, Hyperbolic equation, Non-smooth velocity field, Weak solution, Existence, Uniqueness
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1 Introduction and Main Results

One of the basic equations in fluid dynamics is the continuity equation

$$\partial_t\rho + \mathrm{div}(u\rho) = 0 \quad \text{in } [0,T]\times\mathbb{R}^d,$$

where $u = u(x,t)$ is the velocity field describing the flow and ρ is the fluid density. It encodes the familiar law of conservation of mass. Mathematically speaking, if the velocity field u is Lipschitz continuous, then the continuity equation (and the related transport equation) can be solved explicitly by means of the method of characteristics. Unfortunately, in realistic applications, the velocity is much rougher than Lipschitz, typically u belongs to some spatial Sobolev space and one must seek well-posedness of the continuity equation in suitable classes of weak solutions. Well-posedness of weak solutions follows from the theory of renormalized solutions (see [1, 13,

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33–34]), assuming that $u \in L^1_t W^{1,1}_x$ (or even $L^1_t BV_x$) with $\operatorname{div} u \in L^\infty_{t,x}$. A key step in this theory is to show that a weak solution ρ is also a renormalized solution, that is, $S(\rho)$ is a weak solution for all "reasonable" nonlinear functions $S : \mathbb{R} \to \mathbb{R}$. It is the validity of this chain rule property that asks for $W^{1,1}_x$ (or BV_x) regularity of the velocity u. The assumption that $\operatorname{div} u$ is bounded cannot be relaxed (unbounded divergence leads to concentration effects).

Recently there has been significant interest in studying fluid dynamics equations supplemented with stochastic terms. This (renewed) interest is partly motivated by the problem of turbulence. Although the basic (Navier-Stokes) equations are deterministic, some of their solutions exhibit wild random-like behavior, with the basic problem of existence and uniqueness of smooth solutions being completely open. There is a vague hope that "stochastic perturbations" can render some of the models "well-posed" or "better behaved", thereby providing some insight into the onset of turbulence. We refer to [18] for a general discussion of "regularization by noise" phenomena, which has been a recurring theme in many recent works on stochastic transport and continuity equations of the form

$$\partial \rho + \nabla \rho \circ (u + a\dot{W}) = 0, \quad \partial \rho + \operatorname{div}[\rho \circ (u + a\dot{W})] = 0, \tag{1.1}$$

posed on $\mathbb{R}^d$ with a given initial condition $\rho|_{t=0} = \rho_0$. Here $W = W(t)$ is a Wiener process with noise coefficient a and the symbol $\circ$ refers to the Stratonovich stochastic differential. It is not our purpose here to review the (by now vast) literature on regularization by noise (i.e., improvements in regularity, existence, uniqueness, stability, etc., induced by noise). Instead we emphasize some of the papers that develop an analytical (PDE) approach [3, 6, 23–24, 39], related to the one taken in the present paper. There is another flexible approach that study the stochastic flow associated with the SPDE (1.1), relying on regularizing properties of the corresponding SDE to supply a flow that is more regular than its coefficient u, see e.g. [19] for the stochastic transport equation and [35–36] for the stochastic continuity equation. A good part of the recent literature is motivated by the article [19] of Flandoli, Gubinelli, and Priola, which in turn built upon an earlier work by Davies [11]. One of the main results in [19] is that if u is x-Hölder continuous, then the initial-value problem for the transport equation in (1.1) is well-posed under the weak assumption that $\operatorname{div} u \in L^2$. Most of the works just cited assume that the noise coefficient a is constant. Well-posedness results for continuity equations with x-dependent noise coefficients can be found in [39] (see also [40] and [29]). Subtle regularization by noise results for some nonlinear SPDEs can be found in [23–24]. Let us also recall that first order stochastic partial differential equations (SPDEs for short) with "Lipschitz coefficients" have been deeply analyzed in Kunita's works (see [9, 31]).

In recent years there has been a growing interest in analyzing the basic equations of fluid dynamics on Riemannian manifolds instead of flat domains, with the nonlinearity (curvature) of the domains altering the underlying dynamics in nontrivial ways (see e.g. [2, 7, 43]). A Riemannian manifold provides a more general framework in which to study fluid dynamics than a "physical surface", with the relevant quantities becoming independent of coordinates and a distance function. Partial differential equations (PDEs for short) on manifolds arise in many applications, including geophysical flows (atmospheric models of fluids constrained to flow on the surface of a planet) and general relativity in which the Einstein-Euler equations are posed on a manifold with the metric being one of the unknowns. Transport equations on manifolds have been analyzed in [14, 17], where the DiPerna-Lions theory of weak solutions is extended to (some classes of) Riemannian manifolds.

The mathematical literature on SPDEs on manifolds is at the moment scanty, see [15, 21, 27–28] for equations in which the noise enters the equation as an Itô source term. In [22] we established the renormalization property for weak solutions of stochastic continuity equations on manifolds, under the assumption that the irregular velocity field u belongs to $L^1_t W^{1,2}_x$. Corollaries of this result included L^2 estimates and uniqueness (provided $\operatorname{div}_h u \in L^\infty$). The purpose of the present paper is to establish the existence and uniqueness of weak L^2 solutions without the assumption $\operatorname{div}_h u \in L^\infty$.

To be more precise, we are given a d-dimensional ($d \geq 1$) smooth, closed and compact manifold M, endowed with a (smooth) Riemannian metric h. We are interested in the initial-value problem for the stochastic continuity equation

$$\mathrm{d}\rho + \operatorname{div}_h(\rho\, u)\mathrm{d}t + \sum_{i=1}^{N} \operatorname{div}_h(\rho\, a_i) \circ \mathrm{d}W^i(t) = 0 \quad \text{in } [0,T] \times M, \tag{1.2}$$

where $T > 0$ denotes a fixed final time, $u : [0,T] \times M \to TM$ is a given time-dependent irregular vector field on M, $a_1, \cdots, a_N : M \to TM$ are suitable smooth vector fields on M (to be fixed later), $W^1, \cdots, W^N$ are independent real-valued Brownian motions, and the symbol $\circ$ means that the equation is understood in the Stratonovich sense. We recall that for a vector field X (locally of the form $X^j \partial_j$), the divergence of X is given by $\operatorname{div}_h X = \partial_j X^j + \Gamma^j_{ij} X^i$, where Γ^k_{ij} are the Christoffel symbols associated with the Levi-Civita connection ∇ of the metric h (Einstein's summation convention is used throughout the paper).

Roughly speaking, the proof of well-posedness for (1.2) consists of two main steps. In the first step we construct an appropriate noise term that has the potential to suppress concentration effects. Indeed, to remove the assumption $\operatorname{div}_h u \in L^\infty_{t,x}$, we are led to consider a specific noise term linked to the geometry of the underlying curved domain M, implying a structural effect of noise and nonlinear domains on improving the well-posedness of weak solutions (more on this below). Related results on Euclidean domains (with x-independent noise coefficients) can be found in [3, 6] (see also [35–36, 39]). In the second step, with help of the noise term, we establish a crucial $L^\infty_t L^2_{\omega,x}$ estimate for weak solutions that do not depend on $\|\operatorname{div}_h u\|_{L^\infty}$. To this end, we make use of a duality method, inspired by Beck, Flandoli, Gubinelli and Maurelli [6], Gess and Maurelli [23–24], and Gess and Smith [24] (more on this below).

We use the following concept of weak solution for (1.2) (for unexplained notation and background material, see Section 2).

Definition 1.1 (weak L^2 solution, Stratonovich formulation) *Given $\rho_0 \in L^2(M)$, a weak L^2 solution of* (1.2) *with initial datum $\rho|_{t=0} = \rho_0$ is a function ρ that belongs to $L^\infty([0,T]; L^2(\Omega \times$*

M)) such that $\forall \psi \in C^\infty(M)$ the stochastic process $(\omega, t) \mapsto \int_M \rho(t)\psi \, \mathrm{d}V_h$ has a continuous modification which is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-semimartingale and for any $t \in [0,T]$ the following equation holds $\mathbb{P}$-a.s.:

$$\begin{aligned}\int_M \rho(t)\psi \, \mathrm{d}V_h &= \int_M \rho_0 \psi \, \mathrm{d}V_h + \int_0^t \int_M \rho(s)\, u(\psi) \, \mathrm{d}V_h \, \mathrm{d}s \\ &\quad + \sum_{i=1}^N \int_0^t \int_M \rho(s)\, a_i(\psi) \, \mathrm{d}V_h \circ \mathrm{d}W^i(s).\end{aligned}$$

We have an equivalent concept of solution using the Itô stochastic integral and the corresponding SPDE

$$\mathrm{d}\rho + \mathrm{div}_h(\rho\, u)\, \mathrm{d}t + \sum_{i=1}^N \mathrm{div}_h(\rho\, a_i)\, \mathrm{d}W^i(t) - \frac{1}{2}\sum_{i=1}^N \Lambda_i(\rho)\, \mathrm{d}t = 0, \tag{1.3}$$

where Λ_i is a second order differential operator linked to the vector field a_i, defined by $\Lambda_i(\rho) := \mathrm{div}_h(\mathrm{div}_h(\rho a_i) a_i)$ for $i = 1, \cdots, N$. Recall that, for a smooth function $f : M \to \mathbb{R}$ and a vector field X, we have $X(f) = (X, \mathrm{grad}_h f)_h$ (which locally becomes $X^j \partial_j f$). Moreover, $X(X(f)) = (\nabla^2 f)(X, X) + (\nabla_X X)(f)$, where $\nabla^2 f$ is the covariant Hessian of f and $\nabla_X X$ is the covariant derivative of X in the direction X. In the Itô SPDE (1.3) the operator $\Lambda_i(\cdot)$ is the formal adjoint of $a_i(a_i(\cdot))$.

According to [22], the next definition is equivalent to Definition 1.1.

Definition 1.2 (weak L^2 solution, Itô formulation) *Given $\rho_0 \in L^2(M)$, a weak L^2 solution of* (1.2) *with initial datum $\rho|_{t=0} = \rho_0$ is a function ρ that belongs to $L^\infty([0,T]; L^2(\Omega \times M))$ such that $\forall \psi \in C^\infty(M)$ the process $(\omega, t) \mapsto \int_M \rho(t)\psi \, \mathrm{d}V_h$ has a continuous modification which is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process and for any $t \in [0,T]$ the following equation holds $\mathbb{P}$-a.s.:*

$$\begin{aligned}\int_M \rho(t)\psi \, \mathrm{d}V_h &= \int_M \rho_0 \psi \, \mathrm{d}V_h + \int_0^t \int_M \rho(s)\, u(\psi) \, \mathrm{d}V_h \, \mathrm{d}s \\ &\quad + \sum_{i=1}^N \int_0^t \int_M \rho(s)\, a_i(\psi) \, \mathrm{d}V_h \, \mathrm{d}W^i(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M \rho(s)\, a_i(a_i(\psi)) \, \mathrm{d}V_h \, \mathrm{d}s.\end{aligned} \tag{1.4}$$

To guarantee that these definitions make sense, we need the vector field u to fulfill some basic conditions. First, we require spatial Sobolev regularity:

$$u \in L^1([0,T]; \overrightarrow{W^{1,2}(M)}), \tag{1.5}$$

see Section 2 for unexplained notation. This means that $u \in L^1([0,T]; \overrightarrow{L^2(M)})$, which is sufficient to ensure that the mapping $t \mapsto \int_0^t \int_M \rho(s) u(s)(\psi) \, \mathrm{d}V_h \, \mathrm{d}s$ is absolutely continuous, $\mathbb{P}$-a.s., for any $\rho \in L^\infty_t L^2_{\omega,x}$ and $\psi \in C^\infty(M)$, and hence it is not contributing to cross-variations against W^i. These cross-variations appear when passing from Stratonovich to Itô integrals in the SPDE (1.2).

In addition, we will assume that

$$u \in L^\infty([0,T]; \overrightarrow{L^\infty(M)}), \tag{1.6}$$

and, more importantly, that the distributional divergence of u satisfies

$$\operatorname{div}_h u \in L^p([0,T] \times M) \quad \text{for some } p > d+2. \tag{1.7}$$

To derive a priori estimates, we need the following concept of renormalization (see [22] for details and comments).

Definition 1.3 (renormalization, Itô formulation) *Let ρ be a weak L^2 solution of (1.2) with initial datum $\rho|_{t=0} = \rho_0 \in L^2(M)$. We say that ρ is renormalizable if, for any $F \in C^2(\mathbb{R})$ with F, F', F'' bounded on $\mathbb{R}$, and for any $\psi \in C^\infty(M)$, the stochastic process $(\omega, t) \mapsto \int_M F(\rho(t))\psi \,\mathrm{d}V_h$ has a continuous modification which is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process, and, setting $G_F(\xi) := \xi F'(\xi) - F(\xi)$, for $\xi \in \mathbb{R}$, the function $F(\rho)$ satisfy the SPDE*

$$\begin{aligned}
&\mathrm{d}F(\rho) + \operatorname{div}_h(F(\rho)u)\,\mathrm{d}t + G_F(\rho)\operatorname{div}_h u\,\mathrm{d}t \\
&\quad + \sum_{i=1}^N \operatorname{div}_h(F(\rho)a_i)\,\mathrm{d}W^i(t) + \sum_{i=1}^N G_F(\rho)\operatorname{div}_h a_i\,\mathrm{d}W^i(t) \\
&= \frac{1}{2}\sum_{i=1}^N \Lambda_i(F(\rho))\,\mathrm{d}t - \frac{1}{2}\sum_{i=1}^N \Lambda_i(1)G_F(\rho)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i=1}^N F''(\rho)\,(\rho\operatorname{div}_h a_i)^2\,\mathrm{d}t + \sum_{i=1}^N \operatorname{div}_h(G_F(\rho)\overline{a}_i)\,\mathrm{d}t,
\end{aligned} \tag{1.8}$$

weakly (in x), $\mathbb{P}$-a.s., where the first order differential operator $\overline{a}_i$ is defined by $\overline{a}_i := (\operatorname{div}_h a_i)a_i$ and $\Lambda_i(1) = \operatorname{div}_h \overline{a_i}$ for $i = 1, \cdots, N$; that is, for all $\psi \in C^\infty(M)$ and for any $t \in [0,T]$, the following equation holds $\mathbb{P}$-a.s.:

$$\begin{aligned}
\int_M F(\rho(t))\psi\,\mathrm{d}V_h &= \int_M F(\rho_0)\psi\,\mathrm{d}V_h + \int_0^t\int_M F(\rho(s))\,u(\psi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \sum_{i=1}^N \int_0^t\int_M F(\rho(s))\,a_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}W^i(s) \\
&\quad + \frac{1}{2}\sum_{i=1}^N \int_0^t\int_M F(\rho(s))\,a_i(a_i(\psi))\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \int_0^t\int_M G_F(\rho(s))\operatorname{div}_h u\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \sum_{i=1}^N \int_0^t\int_M G_F(\rho(s))\operatorname{div}_h a_i\,\psi\,\mathrm{d}V_h\,\mathrm{d}W^i(s) \\
&\quad - \frac{1}{2}\sum_{i=1}^N \int_0^t\int_M \Lambda_i(1)\,G_F(\rho(s))\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^N \int_0^t\int_M F''(\rho(s))(\rho(s)\operatorname{div}_h a_i)^2\,\psi\,\mathrm{d}V_h\,\mathrm{d}s
\end{aligned}$$

$$-\sum_{i=1}^{N}\int_0^t\int_M G_F(\rho(s))\overline{a}_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}s. \tag{1.9}$$

Theorem 1.1 (renormalization property see [22]) *Assume* (1.5) *and consider a weak L^2 solution ρ of* (1.2) *with initial datum $\rho_0 \in L^2(M)$, according to Definition* 1.2. *Then ρ is renormalizable in the sense of Definition* 1.3.

To prove the L^2 estimate mentioned earlier, we actually need a version of the weak formulation (1.9) that uses time-dependent test functions. Moreover, we are required to insert into that weak formulation "non-smooth" test functions. These technical aspects of the theory are developed in Sections 4–5.

One can only expect the noise term to improve the well-posedness situation for (1.2) if the resulting second order differential operator $\sum_i a_i(a_i)$ appearing in (1.4) is non-degenerate (uniformly elliptic). The required non-degeneracy is not guaranteed. The reason is a geometric one that is tied to the nonlinearity of the domain. Indeed, given an arbitrary d-dimensional smooth manifold M, it is not possible to find a global frame for it, that is, d smooth vector fields $E_1, \cdots, E_d$ that constitute a basis for T_xM for all $x \in M$. Manifolds that exhibit this property are called parallelizable. Examples of parallelizable manifolds are Lie groups (e.g. $\mathbb{R}^d$, $\mathbb{T}^d$) and the sphere $\mathbb{S}^d$ with $d \in \{1,3,7\}$. We refer to Section 6 for further details, and a proof of the following simple but useful fact.

Lemma 1.1 (non-degenerate second order operator) *There exist $N = N(M)$ smooth vector fields $a_1, \cdots, a_N$ on M such that the following identity holds*

$$\frac{1}{2}\sum_{i=1}^{N} a_i(a_i(\psi)) = \Delta_h\psi - \frac{1}{2}\sum_{i=1}^{N}\overline{a}_i(\psi), \quad \forall\psi \in C^2(M),$$

where Δ_h is the Laplace-Beltrami operator of (M,h) and $\overline{a}_1, \cdots, \overline{a}_N$ are first order differential operators: $\overline{a}_i := (\operatorname{div}_h a_i)\, a_i$ *for $i = 1, \cdots, N$.*

It is now clear that with the specific vector fields $a_1, \cdots, a_N$ constructed in Lemma 1.1, the resulting second order operator $\frac{1}{2}\sum_{i=1}^{N} a_i(a_i(\cdot))$ in (1.4) becomes uniformly elliptic. The main result of this paper, which shows how the use of noise can avoid concentration in the density ρ, is the following theorem.

Theorem 1.2 (well-posedness) *Suppose conditions* (1.5)–(1.7) *hold. Let the vector fields $a_1, \cdots, a_N$ be given by Lemma* 1.1. *Then there exists a unique weak solution of* (1.2) *with initial datum $\rho|_{t=0} = \rho_0 \in L^2(M)$.*

As far as we know, this theorem provides the first result on regularization by noise on a manifold (we are not aware of any such result even for ODEs). The proof consists of several steps. The first one establishes the well-posedness of strong solutions to (1.2) with smooth data (u, ρ_0), which is the topic of Section 3. Here the basic strategy is, with the help of a smooth partition of unity subordinate to a finite atlas, to solve localized versions of (1.2) "pulled back" to $\mathbb{R}^d$, relying on Kunita's existence and uniqueness theory for SPDEs on Euclidean domains (see [30–31]). We "glue" the localized solutions together on M, obtaining in this way a global solution. The gluing procedure is well-defined, because if two coordinate patches intersect, then their corresponding solutions must agree on the intersection, in view of the uniqueness result that is available on $\mathbb{R}^d$ (with u, ρ_0 smooth). In Section 8, we derive an L^2 estimate for general weak solutions ρ (with non-smooth u, ρ_0):

$$\mathbb{E} \int_M |\rho(t,x)|^2 \, \mathrm{d}x \leq C \int_M |\rho_0(x)|^2 \, \mathrm{d}x, \quad t \in [0,T], \tag{1.10}$$

where the constant C depends on $\| \operatorname{div}_h u\|_{L^p_{t,x}}$, see (1.7), but not $\| \operatorname{div}_h u\|_{L^\infty}$. The derivation of this estimate is based on (1.8), and a duality argument in which we construct a specific (deterministic) test function $\phi(t,x)$ that can "absorb" the bad $\operatorname{div}_h u$ term in (1.8). This function solves the terminal-value problem

$$\partial_t \phi(t) + \Delta_h \phi(t) - b(t,x)\phi = 0 \quad \text{on } [0,t_0] \times M, \quad \phi(t_0,x) = 1 \quad \text{on } M,$$

where $t_0 \in [0,T]$, Δ_h is the Laplace-Beltrami operator, and $b = b(t,x) \leq 0$ is an appropriately chosen irregular function ($b \in L^p$ with $p > d+2$). Using Fredholm theory and embedding theorems in anisotropic Sobolev spaces $W^{1,2,p}_{t,x}$ (see [8]), we prove that this problem admits a unique solution $\phi \in W^{1,2,p}_{t,x}$ that satisfies

$$\|(\phi, \nabla\phi)\|_{L^\infty_{t,x}} \leq C(p,d,T,M)\|b\|_{L^p_{t,x}}, \tag{1.11}$$

where ∇ denotes the covariant derivative. Using this ϕ as test function in the time-space weak formulation of (1.8), along with the estimates (1.11), we arrive at the L^2 estimate (1.10) via Grönwall's inequality. In the final step (Section 9), we replace the irregular vector field u and the initial function $\rho_0 \in L^2$ by appropriate smooth approximations u_τ and $\rho_{0,\tau}$, respectively, where $\tau > 0$ is the approximation parameter, and solve for each $\tau > 0$ the corresponding SPDE with smooth data $(u_\tau, \rho_{0,\tau})$, giving raise to a sequence $\{\rho_\tau\}_{\tau>0}$ of approximate solutions. In view of (1.10), we have an L^2 bound on ρ_τ that is independent of τ, which is enough to arrive at the existence of a weak solution to (1.2) by way of a compactness argument. Uniqueness is an immediate consequence of (1.10).

Before ending this (long) section, let us briefly discuss the nontrivial matter of regularizing functions and vector fields on manifolds. In the Euclidean case one uses mollification. Mollification possesses many fitting properties (e.g. it commutes with differential operators) that are not easy to engineer if the function in question is defined on a manifold. Indeed, on a Riemannian manifold, there are a number of smoothing devices currently being used, including partition of unity combined with Euclidean convolution in local charts (see e.g. [14, 21–22]), Riemannian convolution smoothing (see [25]), and the heat semigroup method (see e.g. [17, 21]), where the last two are better at preserving geometric properties. In this paper, for smoothing of the data ρ_0 (function) and u (vector field), we employ standard mollifcation in time and convolution with the heat semigroup in the spatial variables, where the heat semigroup approach is applied to functions as well as vector fields (the latter via 1-forms and the de Rham-Hodge semigroup), see Section 10 for details.

2 Background Material

In an attempt to make the paper more self-contained and fix relevant notation, we briefly review some basic aspects of differential geometry and stochastic analysis. For unexplained terminology and rudimentary results concerning the target equation (1.2), we refer to [22].

2.1 Geometric framework

We refer to [4, 12, 32] for background material on differential geometry and analysis on manifolds. Fix a closed, compact, connected and oriented d-dimensional smooth Riemannian manifold (M,h). The metric h is a smooth positive-definite 2-covariant tensor field, which determines for every $x \in M$ an inner product h_x on T_xM. Here T_xM denotes the tangent space at x, and by $TM = \coprod_{x\in M} T_xM$ we denote the tangent bundle. For two arbitrary vectors $X_1, X_2 \in T_xM$, we will henceforth write $h_x(X_1,X_2) =: (X_1,X_2)_{h_x}$ or even $(X_1,X_2)_h$ if the context is clear. We set $|X|_h := (X,X)_h^{\frac{1}{2}}$. Recall that, in local coordinates $x = (x^i)$, the partial derivatives $\partial_i := \frac{\partial}{\partial x^i}$ form a basis for T_xM, while the differential forms $\mathrm{d}x^i$ determine a basis for the cotangent space T_x^*M. Therefore, in local coordinates, h reads

$$h = h_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \quad h_{ij} = (\partial_i, \partial_j)_h.$$

We will denote by (h^{ij}) the inverse of the matrix (h_{ij}).

We denote by $\mathrm{d}V_h$ the Riemannian density associated to h, which in local coordinates takes the form

$$\mathrm{d}V_h = |h|^{\frac{1}{2}}\,\mathrm{d}x^1\cdots\mathrm{d}x^d,$$

where $|h|$ is the determinant of h. Throughout the paper, we will assume for convenience that

$$\mathrm{Vol}(M,h) := \int_M \mathrm{d}V_h = 1.$$

For $p \in [1,\infty]$, we denote by $L^p(M)$ the usual Lebesgue spaces on (M,h). In local coordinates, the gradient of a function $f : M \to \mathbb{R}$ is the vector field given by the following expression

$$\mathrm{grad}_h\, f := h^{ij}\partial_i f\,\partial_j.$$

The symbol ∇ refers to the Levi-Civita connection of h, namely the unique linear connection on M that is compatible with h and is symmetric. The Christoffel symbols associated to ∇ are given by

$$\Gamma_{ij}^k = \frac{1}{2}h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}).$$

In particular, the covariant derivative of a vector field $X = X^\alpha\partial_\alpha$ is the $(1,1)$-tensor field which in local coordinates reads

$$(\nabla X)_j^\alpha := \partial_j X^\alpha + \Gamma_{kj}^\alpha X^k.$$

The divergence of a vector field $X = X^j\partial_j$ is the function defined by

$$\operatorname{div}_h X := \partial_j X^j + \Gamma^j_{kj} X^k.$$

For any vector field X and $f \in C^1(M)$, we have $X(f) = (X, \operatorname{grad}_h f)_h$, which locally takes the form $X^j\partial_j f$. We recall that for a (smooth) vector field X, the following integration by parts formula holds:

$$\int_M X(f)\,\mathrm{d}V_h = \int_M (\operatorname{grad}_h f, X)_h\,\mathrm{d}V_h = -\int_M f\ \operatorname{div}_h X\,\mathrm{d}V_h,$$

recalling that M is closed (so all functions are compactly supported).

Given a smooth vector field X on M, we consider the norm

$$\|X\|^p_{\overrightarrow{L^p(M)}} := \begin{cases} \displaystyle\int_M |X|^p_h\,\mathrm{d}V_h, & p \in [1,\infty), \\ \||X|_h\|_{L^\infty(M)}, & p = \infty. \end{cases}$$

The closure of the space of smooth vector fields on M with respect to the norm $\|\cdot\|_{\overrightarrow{L^p(M)}}$ is denoted by $\overrightarrow{L^p(M)}$. We define the Sobolev space $\overrightarrow{W^{1,p}(M)}$ in a similar fashion. Indeed, consider the norm

$$\|X\|^p_{\overrightarrow{W^{1,p}(M)}} := \begin{cases} \displaystyle\int_M (|X|^p_h + |\nabla X|^p_h)\,\mathrm{d}V_h, & \text{if } p \in [1,\infty), \\ \|\,|X|_h + |\nabla X|_h\,\|_{L^\infty(M)}, & \text{if } p = \infty, \end{cases}$$

where, locally, $|\nabla X|^2_h = (\nabla X)^i_j\, h_{ik} h^{jm}\,(\nabla X)^k_m$. The closure of the space of smooth vector fields with respect to this norm is $\overrightarrow{W^{1,p}(M)}$. For more operative definitions, $\overrightarrow{L^p(M)}$ and $\overrightarrow{W^{1,p}(M)}$ can be seen as the spaces of vector fields whose components in any arbitrary chart belong to the corresponding Euclidean space.

We will make essential use of the anisotropic Sobolev space $W^{1,2,p}([0,T]\times M)$, with $p \in [1,\infty)$ and $T > 0$ finite. This space is defined as the completion of $C^\infty([0,T]\times M)$ under the norm

$$\|w\|_{W^{1,2,p}([0,T]\times M)} := \sum_{\substack{j,k\ge 0\\ 2j+k\le 2}} \Big[\iint_{[0,T]\times M} |\partial_t^j \nabla^k w|^p_h\,\mathrm{d}t\,\mathrm{d}V_h\Big]^{\frac{1}{p}}, \tag{2.1}$$

where $\nabla^k w$ denotes the kth covariant derivative of the function w. We have the following important embedding result (see Section 10 for a proof).

Proposition 2.1 *Suppose $p > d+2$. Then*

$$W^{1,2,p}([0,T]\times M) \subset\subset C^{0,1-\frac{1+d}{p}}([0,T]\times M);$$

the first-order x-derivatives of a function $w = w(t,x) \in W^{1,2,p}([0,T]\times M)$ are Hölder continuous with exponent $1-\frac{1+d}{p}$, such that

$$\|w\|_{C^0([0,T]\times M)} + \|\nabla w\|_{C^0([0,T]\times M)} \le C\|w\|_{W^{1,2,p}([0,T]\times M)}$$

for some constant $C = C(p,d,M)$.

Finally, we introduce the following second order differential operators associated with the vector fields $a_1, \cdots, a_N$:

$$\Lambda_i(\psi) := \mathrm{div}_h(\mathrm{div}_h(\psi a_i)a_i), \quad \psi \in C^2(M), \quad i = 1, \cdots, N. \tag{2.2}$$

It is not difficult to see that the adjoint of $\Lambda_i(\cdot)$ is $a_i(a_i(\cdot))$:

$$\begin{aligned}\int_M \Lambda_i(\psi)\phi \, \mathrm{d}V_h &= \int_M \psi \, a_i(a_i(\phi)) \, \mathrm{d}V_h \\ &= \int_M \psi \left((\nabla^2\phi)(a_i, a_i) + (\nabla_{a_i} a_i)(\psi)\right) \mathrm{d}V_h, \quad \forall \psi, \phi \in C^2(M),\end{aligned}$$

see [22] for further details.

2.2 Stochastic framework

We use the books [38, 42] as general references on the topic of stochastic analysis. From beginning to end, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a complete right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. Without loss of generality, we assume that the σ-algebra $\mathcal{F}$ is countably generated. Let $W = \{W_i\}_{i=1}^N$ be a finite sequence of independent one-dimensional Brownian motions adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We refer to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W)$ as a (Brownian) stochastic basis.

Consider two real-valued stochastic processes $Y, \widetilde{Y}$. We call $\widetilde{Y}$ a modification of Y if, for each $t \in [0, T]$, $\mathbb{P}(\{\omega \in \Omega : Y(\omega, t) = \widetilde{Y}(\omega, t)\}) = 1$. It is important to pick good modifications of stochastic processes. Right (or left) continuous modifications are often used (they are known to exist for rather general processes), since any two such modifications of the same process are indistinguishable (with probability one they have the same sample paths). Besides, they necessarily have left-limits everywhere. Right-continuous processes with left-limits are referred to as càdlàg.

An $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted, càdlàg process Y is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-semimartingale if there exist processes F, M with $F_0 = M_0 = 0$ such that

$$Y_t = Y_0 + F_t + M_t,$$

where F is a finite variation process and M is a local martingale. In this paper we will only be concerned with continuous semimartingales. The quantifier "local" refers to the existence of a sequence $\{\tau_n\}_{n\geq 1}$ of stopping times increasing to infinity such that the stopped processes $\mathbf{1}_{\{\tau_n>0\}} M_{t\wedge\tau_n}$ are martingales.

Given two continuous semimartingales Y and Z, we can define the Fisk-Stratonovich integral of Y with respect to Z by

$$\int_0^t Y(s) \circ \mathrm{d}Z(s) = \int_0^t Y(s) \, \mathrm{d}Z(s) + \frac{1}{2} \langle Y, Z \rangle_t \, ,$$

where $\int_0^t Y(s)\mathrm{d}Z(s)$ is the Itô integral of Y with respect to Z and $\langle Y, Z\rangle$ denotes the quadratic cross-variation process of Y and Z. Let us recall Itô's formula for a continuous semimartingale

Y. Let $F \in C^2(\mathbb{R})$. Then $F(Y)$ is again a continuous semimartingale and the following chain rule formula holds:

$$F(Y(t)) - F(Y(0)) = \int_0^t F'(Y(s))\mathrm{d}Y(s) + \frac{1}{2}\int_0^t F''(Y(s))\,\mathrm{d}\left\langle Y,Y\right\rangle_s.$$

Martingale inequalities are generally important for several reasons. For us they will be used to bound Itô stochastic integrals in terms of their quadratic variation (which is easy to compute). One of the most important martingale inequalities is the Burkholder-Davis-Gundy inequality. Let $Y = \{Y_t\}_{t\in[0,T]}$ be a continuous local martingale with $Y_0 = 0$. Then, for any stopping time $\tau \le T$,

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]}|Y_t|\Big)^p \le C_p\,\mathbb{E}\sqrt[p]{\langle Y,Y\rangle_\tau}, \quad p\in(0,\infty),$$

where C_p is a universal constant.

3 Smooth Data and Strong Solutions

3.1 Strong solution

We are going to construct strong solutions to (1.2) when the data ρ_0, u are smooth. More precisely, throughout this section, we will assume $\rho_0 \in C^\infty(M)$ and that $u : [0,\infty)\times M \to TM$ is a vector field on M that is smooth in both variables. The strategy we employ is the following one: Firstly we solve a local version of (1.2) "pulled back" on $\mathbb{R}^d$, applying the "Euclidean" existence and uniqueness theory developed in [30]. In a second step we glue these solutions all together on M, obtaining a global solution. The gluing procedure is well-posed because there is a uniqueness result on $\mathbb{R}^d$ for smooth data (ρ_0, u).

Fixing a point $p\in M$, we may find an open neighborhood $\mathcal{U}(p)\subset M$ of p and coordinates $\gamma_p : \mathcal{U}(p)\to\mathbb{R}^d$ such that $\gamma_p(\mathcal{U}(p)) = \mathbb{R}^d$. By compactness of M, there is a finite atlas $\mathcal{A}$ with these properties, namely, there exist $p_1,\cdots,p_K$ such that $M = \bigcup_{l=1}^K \mathcal{U}(p_l)$ and $\gamma_{p_l}(\mathcal{U}(p_l)) = \mathbb{R}^d$.

Remark 3.1 To construct these coordinates, one is usually led to use that the ball $B_1(0)\subset\mathbb{R}^d$ is diffeomorphic to the whole space, for instance via the map

$$\Phi : \mathbb{R}^d \to B_1(0), \quad z\mapsto \frac{z}{\sqrt{1+|z|^2}}.$$

If we now have a C^1 function $f : B_1(0)\to\mathbb{R}$ with bounded derivatives, then it is straightforward to check that $f\circ\Phi$ has bounded derivatives as well.

Fix a point p_l. In the coordinates given by γ_{p_l}, (1.2) looks like

$$\begin{aligned}
&\mathrm{d}\rho^l + [\rho^l \operatorname{div}_h u + \partial_j\rho^l u^j]\,\mathrm{d}t + \sum_{i=1}^N [\partial_j\rho^l a_i^j + \rho^l \operatorname{div}_h a_i]\circ \mathrm{d}W_t^i = 0 \quad \text{on } [0,T]\times\mathbb{R}^d,\\
&\rho^l(0) = \rho_0\circ\gamma_{p_l}^{-1} \quad \text{on } \mathbb{R}^d.
\end{aligned}$$

Observe that the coefficients satisfy the hypotheses on [30, pages 264 and 267]. In particular, the z-derivatives are bounded, in view of Remark 3.1. (In Kunita's notation we have

$$\begin{aligned}
&Q_0(t,x,v) = v \operatorname{div}_h u, \quad Q_j(t,x,v) = v \operatorname{div}_h a_j, \quad j = 1,\cdots,N,\\
&P_0^r = u^r, \quad P_j^r = a_j^r, \quad r = 1,\cdots,d,\ j = 1,\cdots,N,\\
&Q_0^{(1)} = \operatorname{div}_h u, \quad Q_0^{(0)} = 0,\\
&Q_j^{(1)} = \operatorname{div}_h a_j, \quad Q_j^{(0)} = 0, \quad j = 1,\cdots,N.)
\end{aligned}$$

Therefore, we may apply [30, Theorem 4.2] to obtain a unique strong solution which we call ρ^l (for the definition of strong solution, see [30, p. 255]). Let us "lift" ρ^l on M, via γ_{p_l}, namely, for $t \in [0,T]$ define

$$\widehat{\rho}^l(t,x) := \begin{cases} \rho^l(t,\gamma_{p_l}(x)), & x \in \mathcal{U}(p_l),\\ 0, & x \notin \mathcal{U}(p_l). \end{cases}$$

We repeat this procedure for all p_l, thereby obtaining $\widehat{\rho}^l$ for $l \in \{1,\cdots,K\}$.

Suppose that $\mathcal{U}(p_l) \cap \mathcal{U}(p_{l'}) \neq \emptyset$ for some $l \neq l'$. Fix $q \in \mathcal{U}(p_l) \cap \mathcal{U}(p_{l'})$. Arguing as above, we may find coordinates $\eta_q : \mathcal{V}(q) \to \mathbb{R}^d$ such that $\mathcal{V}(q)$ is an open neighborhood of q with $\mathcal{V}(q) \subset \mathcal{U}(p_l) \cap \mathcal{U}(p_{l'})$ and $\eta_q(\mathcal{V}(q)) = \mathbb{R}^d$. Once again, we can find a unique strong solution ρ_q, which we lift on M: For $t \in [0,T]$ define

$$\widehat{\rho}_q(t,x) := \begin{cases} \rho_q(t,\eta_q(x)), & x \in \mathcal{V}(q),\\ 0, & x \notin \mathcal{V}(q). \end{cases}$$

We now restrict $\widehat{\rho}^l(t,\cdot)$ on $\mathcal{V}(q)$. Trivially, the restriction satisfies (1.2) on $\mathcal{V}(q)$. This is a geometric equation (and thus coordinate-independent), which implies that the restriction of $\widehat{\rho}^l(t,\cdot)$ to $\mathcal{V}(q)$ must satisfy (1.2) when written in the coordinates given by η_q. By uniqueness in $\mathbb{R}^d$ (of strong solutions), we must have $\widehat{\rho}^l(\cdot,\eta_q^{-1}(\cdot)) = \rho_q(\cdot,\cdot)$ on $[0,\infty) \times \mathbb{R}^d$, and thus $\widehat{\rho}^l(t,x) = \widehat{\rho}_q(t,x)$ for all $t \in [0,T]$ and $x \in \mathcal{V}(q)$. By symmetry, we infer

$$\widehat{\rho}^l(t,x) = \widehat{\rho}^{l'}(t,x) \quad \text{for } (t,x) \in [0,T] \times \mathcal{V}(q).$$

Repeating the whole procedure for all $q \in \mathcal{U}(p_l) \cap \mathcal{U}(p_{l'})$, we conclude that

$$\widehat{\rho}^l(t,x) = \widehat{\rho}^{l'}(t,x) \quad \text{for } (t,x) \in [0,T] \times (\mathcal{U}(p_l) \cap \mathcal{U}(p_{l'})).$$

In view of these compatibility conditions, we may unambiguously define

$$\rho(t,x) := \widehat{\rho}^l(t,x), \quad (t,x) \in [0,T] \times M, \tag{3.1}$$

where l is an index in $\{1,\cdots,K\}$ such that $x \in \mathcal{U}(p_l)$.

We have thus arrived at the following lemma.

Lemma 3.1 (strong solution, smooth data) *The function ρ given by* (3.1) *is the unique strong solution of* (1.2) *with initial datum $\rho_0 \in C^\infty(M)$ and smooth vector field $u : [0,\infty) \times M \to TM$. Moreover, ρ is a C^∞ semimartingale.*

3.2 Elementary L^p bound

Let ρ be the solution constructed above. We observe that, in view of the results in [30], locally in the coordinates induced by γ_{p_l} on $\mathcal{U}(p_l)$, we have the following explicit expression for ρ:

$$\begin{aligned}
&\rho(t,\gamma_{p_l}^{-1}(z)) = \rho^l(t,z)\\
&= \exp\Big(\int_0^t \mathrm{div}_h\, u(s,\xi_s(y))\,\mathrm{d}s + \sum_{i=1}^N \int_0^t \mathrm{div}_h\, a_i(y)\circ \mathrm{d}W_s^i\Big)\Big|_{y=\xi_t^{-1}(z)}\\
&\quad \times \rho_0(\gamma_{p_l}^{-1}\circ \xi_t^{-1}(z))\\
&= \exp\Big(\int_0^t \mathrm{div}_h\, u(s,\xi_s(y))\mathrm{d}s\Big)\Big|_{y=\xi_t^{-1}(z)} \exp\Big(\sum_{i=1}^N \mathrm{div}_h\, a_i(\xi_t^{-1}(z))W_t^i\Big)\\
&\quad \times \rho_0(\gamma_{p_l}^{-1}\circ \xi_t^{-1}(z))
\end{aligned} \tag{3.2}$$

for $(t,z)\in[0,T]\times\mathbb{R}^d$, where ξ is a stochastic flow of diffeomorphisms, satisfying

$$\mathrm{d}\xi_t(z) = -u(t,\xi_t(z))\,\mathrm{d}t - \sum_{i=1}^N a_i(\xi_t(z))\circ \mathrm{d}W_t^i, \quad \xi_0(z)=z.$$

Here the vector fields u, a_i are seen as vectors in $\mathbb{R}^n$ through our coordinate system.

Let us derive an L^p bound. Fix $p\in[1,\infty)$ and let $(\chi_l)_l$ be a smooth partition of unity subordinated to our atlas $\mathcal{A}$. We have

$$\begin{aligned}
\int_M \chi_l(x)|\rho(t,x)|^p\,\mathrm{d}V_h(x) &= \int_{\operatorname{supp}\chi_l} \chi_l(x)|\rho(t,x)|^p\,\mathrm{d}V_h(x)\\
&= \int_{\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}(z))|\rho(t,\gamma_{p_l}^{-1}(z))|^p|h_{\gamma_{p_l}}(z)|^{\frac12}\,\mathrm{d}z,
\end{aligned}$$

where $|h_{\gamma_{p_l}}|^{\frac12}$ denotes the determinant of the metric h written in the coordinates induced by γ_{p_l}. Using (3.2) and the change of variable $z=\xi_t(w)$, we obtain

$$\begin{aligned}
&\int_M \chi_l(x)|\rho(t,x)|^p\,\mathrm{d}V_h(x)\\
&= \int_{\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}(z))\exp\Big(p\int_0^t \mathrm{div}_h\, u(s,\xi_s(y))\,\mathrm{d}s\Big)\Big|_{y=\xi_t^{-1}(z)}\\
&\quad \times \exp\Big(p\sum_{i=1}^N \mathrm{div}_h\, a_i(\xi_t^{-1}(z))W_t^i\Big)\,|\rho_0(\gamma_{p_l}^{-1}\circ\xi_t^{-1}(z))|^p|h_{\gamma_{p_l}}(z)|^{\frac12}\,\mathrm{d}z\\
&= \int_{\xi_t^{-1}\circ\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}\circ\xi_t(w))\exp\Big(p\int_0^t \mathrm{div}_h\, u(s,\xi_s(w))\,\mathrm{d}s\Big)\\
&\quad \times \exp\Big(p\sum_{i=1}^N \mathrm{div}_h\, a_i(w)W_t^i\Big)\,|\rho_0(\gamma_{p_l}^{-1}(w))|^p|\partial\xi_t(w)||h_{\gamma_{p_l}}(\xi_t(w)|^{\frac12}\,\mathrm{d}w\\
&= \int_{\xi_t^{-1}\circ\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}\circ\xi_t(w))\exp\Big(p\int_0^t \mathrm{div}_h\, u(s,\xi_s(w))\,\mathrm{d}s\Big)
\end{aligned}$$

$$\times \exp\Big(p\sum_{i=1}^N \operatorname{div}_h a_i(w)W_t^i\Big)\, |\rho_0(\gamma_{p_l}^{-1}(w))|^p |h_{\xi_t^{-1}\circ\gamma_{p_l}}(w)|^{\frac12}\,\mathrm{d}w.$$

In passing, note that $\xi_t^{-1}\circ\gamma_{p_l}$ is a bona fide smooth chart. In the following, C denotes a constant that depends only on T, p, $\|\operatorname{div}_h u\|_{L^\infty_{t,x}}$, $\|\operatorname{div}_h a_i\|_{L^\infty}$ and is allowed to vary from line to line. For convenience, set $A_i := \|\operatorname{div}_h a_i\|_{L^\infty(M)}$. We proceed as follows:

$$\begin{aligned}
&\int_M \chi_l(x)|\rho(t,x)|^p\,\mathrm{d}V_h(x)\\
&\le C\int_{\xi_t^{-1}\circ\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}\circ\xi_t(w))\\
&\quad\times \exp\Big(p\sum_{i=1}^N A_i|W_t^i|\Big)\, |\rho_0(\gamma_{p_l}^{-1}(w))|^p |h_{\xi_t^{-1}\circ\gamma_{p_l}}(w)|^{\frac12}\,\mathrm{d}w\\
&= C\exp\Big(p\sum_{i=1}^N A_i|W_t^i|\Big)\int_{\xi_t^{-1}\circ\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}\circ\xi_t(w))\\
&\quad\times\ |\rho_0(\gamma_{p_l}^{-1}(w))|^p |h_{\xi_t^{-1}\circ\gamma_{p_l}}(w)|^{\frac12}\,\mathrm{d}w\\
&\le C\exp\Big(p\sum_{i=1}^N A_i|W_t^i|\Big)\|\rho_0\|^p_{L^\infty(M)}\\
&\quad\times\int_{\xi_t^{-1}\circ\gamma_{p_l}(\operatorname{supp}\chi_l)} \chi_l(\gamma_{p_l}^{-1}\circ\xi_t(w))|h_{\xi_t^{-1}\circ\gamma_{p_l}}(w)|^{\frac12}\,\mathrm{d}w\\
&= C\exp\Big(p\sum_{i=1}^N A_i|W_t^i|\Big)\|\rho_0\|^p_{L^\infty(M)}\int_{\operatorname{supp}\chi_l}\chi_l(x)\,\mathrm{d}V_h(x).
\end{aligned}$$

Taking expectation leads to

$$\begin{aligned}
&\mathbb{E}\int_M \chi_l(x)|\rho(t,x)|^p\,\mathrm{d}V_h(x)\\
&\le C\|\rho_0\|^p_{L^\infty(M)}\int_M\chi_l(x)\,\mathrm{d}V_h(x)\,\mathbb{E}\exp\Big(p\sum_{i=1}^N A_i|W_t^i|\Big)\\
&= C\|\rho_0\|^p_{L^\infty(M)}\int_M\chi_l(x)\,\mathrm{d}V_h(x)\prod_{i=1}^N\mathbb{E}\exp(pA_i|W_t^i|)\\
&\le C\|\rho_0\|^p_{L^\infty(M)}\int_M\chi_l(x)\,\mathrm{d}V_h(x),
\end{aligned}$$

where we have used that the Brownian motions are independent and satisfy the standard estimate (see [20, page 54]),

$$\mathbb{E}\exp(\alpha|W_t^i|)\le\beta,\quad t\in[0,T],\ \alpha>0,$$

where the constant β depends on α and T. Therefore, summing over l, we obtain

$$\mathbb{E}\|\rho(t)\|^p_{L^p(M)}\le C\|\rho_0\|^p_{L^\infty(M)}\int_M\mathrm{d}V_h(x)=C\|\rho_0\|^p_{L^\infty(M)},$$

where the constant C depends on the L^∞ norms of $\operatorname{div}_h u$, $\operatorname{div} a_1, \cdots, \operatorname{div}_h a_N$. Since we are assuming that $\rho_0, u \in C^\infty$, the right-hand side of the last expression is finite, and thus $\rho \in L^\infty_t L^p_{\omega,x}$. Moreover, using [30, Theorem 1.1], we infer that the stochastic process $(\omega, t) \mapsto \int_M \rho(t)\psi \,\mathrm{d}V_h$ is a continuous $\mathcal{F}_t$-semimartingale for any $\psi \in C^\infty(M)$.

Let us summarize all these results in the following lemma.

Lemma 3.2 (L^p estimates, smooth data) *Suppose $\rho_0, u \in C^\infty$. Let ρ be the unique strong solution of* (1.2) *given by Lemma* 3.1. *Then, for any $p \in [1, \infty]$,*

$$\rho \in L^\infty([0,T]; L^p(\Omega \times M)), \qquad \sup_{t\in[0,T]} \mathbb{E}\|\rho(t)\|^p_{L^p(M)} \le C\|\rho_0\|^p_{L^\infty(M)},$$

where $C = C(p, T, \|\operatorname{div}_h u\|_{L^\infty([0,T]\times M)}, \max_i \|\operatorname{div}_h a_i\|_{L^\infty(M)})$. Besides, for any $\psi \in C^\infty(M)$, the process $(\omega, t) \mapsto \int_M \rho(t)\psi \,\mathrm{d}V_h$ is a continuous $\mathcal{F}_t$-semimartingale.

Let us bring (1.2) into its Itô form, still assuming that $\rho_0, u \in C^\infty$. We are not going to spell out all the details, referring instead to [30] for the missing pieces. The solution ρ we have constructed in Lemma 3.1 is a smooth semimartingale, and it satisfies $\mathbb{P}$-a.s. the following equation:

$$\begin{aligned}
\rho(t,x) &= \rho_0(x) - \int_0^t \operatorname{div}_h(\rho(s,x)\,u)\,\mathrm{d}s - \sum_{i=1}^N \int_0^t \operatorname{div}_h(\rho(s,x)\,a_i)\,\mathrm{d}W^i(s) \\
&\quad - \frac{1}{2}\sum_{i=1}^N \langle \operatorname{div}_h(\rho(\cdot,x)a_i), W^i_\cdot\rangle_t \\
&= \rho_0(x) - \int_0^t \operatorname{div}_h(\rho(s,x)\,u)\,\mathrm{d}s - \sum_{i=1}^N \int_0^t \operatorname{div}_h(\rho(s,x)\,a_i)\,\mathrm{d}W^i(s) \\
&\quad - \frac{1}{2}\sum_{i=1}^N \langle a_i(\rho(\cdot,x)), W^i_\cdot\rangle_t - \frac{1}{2}\sum_{i=1}^N \langle \rho(\cdot,x), W^i_\cdot\rangle_t \operatorname{div}_h a_i
\end{aligned} \tag{3.3}$$

for all $t \in [0,T]$ and $x \in M$, by definition of the Stratonovich integral. By Theorem 1.1 and [30, Lemma 1.3], we obtain

$$\begin{aligned}
a_i(\rho(t,x)) &= a_i(\rho_0(x)) - \int_0^t a_i(\operatorname{div}_h(\rho(s,x)\,u))\,\mathrm{d}s \\
&\quad - \sum_{j=1}^N \int_0^t a_i(\operatorname{div}_h(\rho(s,x)\,a_j))\,\mathrm{d}W^j(s) - \frac{1}{2}\sum_{j=1}^N \langle a_i(\operatorname{div}_h(\rho(\cdot,x)a_j)), W^j_\cdot\rangle_t
\end{aligned}$$

and

$$\begin{aligned}
\langle a_i(\rho(\cdot,x)), W^i_\cdot\rangle_t &= -\sum_{j=1}^N \Big\langle \int_0^\cdot a_i(\operatorname{div}_h(\rho(s,x)\,a_j))\,\mathrm{d}W^j(s), W^i_\cdot \Big\rangle_t \\
&= -\sum_{j=1}^N \int_0^t a_i(\operatorname{div}_h(\rho(s,x)\,a_j))\,\mathrm{d}\langle W^j, W^i\rangle_s \\
&= -\int_0^t a_i(\operatorname{div}_h(\rho(s,x)\,a_i))\,\mathrm{d}s,
\end{aligned} \tag{3.4}$$

because the Brownian motions are independent, and the time-integral involving u is absolutely continuous and thus not contributing to the quadratic variation.

Moreover, it is clear that

$$\begin{aligned}\langle \rho(\cdot,x), W^i_\cdot \rangle_t &= -\sum_{j=1}^N \Big\langle \int_0^\cdot \mathrm{div}_h(\rho(s,x)\,a_j)\,\mathrm{d}W^j(s), W^i_\cdot \Big\rangle_t \\ &= -\sum_{j=1}^N \int_0^t \mathrm{div}_h(\rho(s,x)\,a_j)\,\mathrm{d}\langle W^j, W^i\rangle_s \\ &= -\int_0^t \mathrm{div}_h(\rho(s,x)\,a_i)\,\mathrm{d}s. \end{aligned} \tag{3.5}$$

Re-starting from (3.3), using (3.4) and (3.5), we finally arrive at

$$\begin{aligned}\rho(t,x) = \rho_0(x) &- \int_0^t \mathrm{div}_h(\rho(s,x)\,u)\,\mathrm{d}s - \sum_{i=1}^N \int_0^t \mathrm{div}_h(\rho(s,x)\,a_i)\,\mathrm{d}W^i(s) \\ &+ \frac{1}{2}\sum_{i=1}^N \int_0^t a_i(\mathrm{div}_h(\rho(s,x)\,a_i))\,\mathrm{d}s \\ &+ \frac{1}{2}\sum_{i=1}^N \int_0^t \mathrm{div}_h\, a_i\ \mathrm{div}_h(\rho(s,x)\,a_i)\,\mathrm{d}s \\ = \rho_0(x) &- \int_0^t \mathrm{div}_h(\rho(s,x)\,u)\,\mathrm{d}s - \sum_{i=1}^N \int_0^t \mathrm{div}_h(\rho(s,x)\,a_i)\,\mathrm{d}W^i(s) \\ &+ \frac{1}{2}\sum_{i=1}^N \int_0^t \Lambda_i(\rho(s,x))\,\mathrm{d}s, \end{aligned}$$

where the second order differential equation Λ_i is defined in (2.2). This is the strong Itô form of (1.2), derived under the assumption that $\rho_0, u \in C^\infty$. If we now integrate this against $\psi \in C^\infty(M)$ (say), since the Itô integral admits a Fubini-type theorem, we arrive at the weak form given in Definition 1.2.

In view of this, combining Lemmas 3.1 and 3.2, we eventually arrive at the following proposition.

Proposition 3.1 (weak solution, smooth data) *Let ρ given by* (3.1) *be the unique strong solution of* (1.2) *with initial datum $\rho_0 \in C^\infty(M)$ and smooth vector field $u : [0,\infty)\times M \to TM$. Then ρ is a weak L^2 solution of* (1.2) *in the sense of Definition* 1.1.

4 Time-Dependent Test Functions

During an upcoming proof (of the L^2 estimate), we will need a version of the weak formulation (1.9) that makes use of time-dependent test functions. The next result supplies that formulation.

Lemma 4.1 (space-time weak formulation) *Let ρ be a weak L^2 solution of* (1.2) *with initial datum $\rho|_{t=0} = \rho_0$. Suppose ρ is renormalizable in the sense of Definition* 1.3. *Fix $F \in C^2(\mathbb{R})$*

with $F, F', F'' \in L^\infty(\mathbb{R})$. *For any* $\psi \in C^\infty([0,T] \times M)$, *the following equation holds* $\mathbb{P}$-*a.s., for any* $t \in [0,T]$,

$$
\begin{aligned}
&\int_M F(\rho(t))\psi(t)\,\mathrm{d}V_h - \int_M F(\rho_0)\psi(0)\,\mathrm{d}V_h \\
&= \int_0^t \int_M F(\rho(s))\partial_t\psi\,\mathrm{d}V_h\,\mathrm{d}s + \int_0^t \int_M F(\rho(s))\,u(\psi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \sum_{i=1}^N \int_0^t \int_M F(\rho(s))\,a_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}W^i(s) + \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M F(\rho(s))\,a_i(a_i(\psi))\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \int_0^t \int_M G_F(\rho(s))\,\mathrm{div}_h\,u\,\psi\,\mathrm{d}V_h\,\mathrm{d}s - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s))\,\mathrm{div}_h\,a_i\,\psi\,\mathrm{d}V_h\,\mathrm{d}W^i(s) \\
&\quad - \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M \Lambda_i(1)\,G_F(\rho(s))\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M F''(\rho(s))(\rho(s)\,\mathrm{div}_h\,a_i)^2\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s))\overline{a}_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}s. \qquad (4.1)
\end{aligned}
$$

Proof It is sufficient to consider test functions of the form $\psi(t,x) = \theta(t)\phi(x)$, where $\theta \in C_c^1((-1, T+1))$ and $\phi \in C^\infty(M)$, because the general result will then follow from a density argument for the tensor product. We start off from the following space-weak formulation, see (1.9):

$$
\begin{aligned}
&\int_M F(\rho(t))\phi\,\mathrm{d}V_h \\
&= \int_M F(\rho_0)\phi\,\mathrm{d}V_h + \int_0^t \int_M F(\rho(s))\,u(\phi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \sum_{i=1}^N \int_0^t \int_M F(\rho(s))\,a_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}W^i(s) + \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M F(\rho(s))\,a_i(a_i(\phi))\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \int_0^t \int_M G_F(\rho(s))\,\mathrm{div}_h\,u\,\phi\,\mathrm{d}V_h\,\mathrm{d}s - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s))\,\mathrm{div}_h\,a_i\,\phi\,\mathrm{d}V_h\,\mathrm{d}W^i(s) \\
&\quad - \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M \Lambda_i(1)\,G_F(\rho(s))\,\phi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^N \int_0^t \int_M F''(\rho(s))(\rho(s)\,\mathrm{div}_h\,a_i)^2\,\phi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s))\overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s, \quad \mathbb{P}\text{-a.s., for any } t \in [0,T].
\end{aligned}
$$

We multiply this equation by $\dot{\theta}(t)$ and integrate the result over $t \in [0, \overline{t}]$. All the time-integrals are absolutely continuous by definition, and thus we can integrate them by parts. For example,

$$\begin{aligned}
&\int_0^{\bar{t}} \dot{\theta}(t) \int_0^t \int_M F(\rho(s))\, u(\phi)\, \mathrm{d}V_h\, \mathrm{d}s\, \mathrm{d}t \\
&\quad = \theta(\bar{t}) \int_0^{\bar{t}} \int_M F(\rho(s))\, u(\phi)\, \mathrm{d}V_h\, \mathrm{d}s - \int_0^{\bar{t}} \theta(t) \int_M F(\rho(t))\, u(\phi)\, \mathrm{d}V_h\, \mathrm{d}t
\end{aligned}$$

and so forth. We can also integrate by parts the stochastic integrals. For example,

$$\begin{aligned}
&\int_0^{\bar{t}} \dot{\theta}(t) \int_0^t \int_M F(\rho(s))\, a_i(\phi)\, \mathrm{d}V_h\, \mathrm{d}W^i(s)\, \mathrm{d}t \\
&\quad = \theta(\bar{t}) \int_0^{\bar{t}} \int_M F(\rho(s))\, a_i(\phi)\, \mathrm{d}V_h\, \mathrm{d}W^i(s) - \int_0^{\bar{t}} \theta(t) \int_M F(\rho(s))\, a_i(\phi)\, \mathrm{d}V_h\, \mathrm{d}W^i(t)
\end{aligned}$$

and so forth. Finally,

$$\begin{aligned}
&\int_0^{\bar{t}} \dot{\theta}(t) \Big(\int_M F(\rho(t))\phi\, \mathrm{d}V_h - \int_M F(\rho_0)\phi\, \mathrm{d}V_h \Big)\, \mathrm{d}t \\
&\quad = \int_0^{\bar{t}} \int_M F(\rho(t))\dot{\theta}(t)\phi\, \mathrm{d}V_h\, \mathrm{d}t + \int_M F(\rho_0)\theta(0)\phi\, \mathrm{d}V_h - \int_M F(\rho_0)\theta(\bar{t})\phi\, \mathrm{d}V_h,
\end{aligned}$$

where the last term is aggregated together with the other "$\theta(\bar{t}) \int_0^{\bar{t}}(\cdots)$" terms that appear, eventually leading to $\int_M F(\rho(\bar{t}))\theta(\bar{t})\phi\, \mathrm{d}V_h$. Therefore, after many straightforward rearrangements of terms, we arrive at (now replacing $\bar{t}$ by t)

$$\begin{aligned}
&\int_M F(\rho(t))\theta(t)\phi\, \mathrm{d}V_h - \int_M F(\rho_0)\theta(0)\phi\, \mathrm{d}V_h \\
&= \int_0^t \int_M F(\rho(s))\dot{\theta}(s)\phi\, \mathrm{d}V_h\, \mathrm{d}s \\
&\quad + \int_0^t \int_M F(\rho(s))\, u(\theta(s)\phi)\, \mathrm{d}V_h\, \mathrm{d}s + \sum_{i=1}^N \int_0^t \int_M F(\rho(s))\, a_i(\theta(s)\phi)\, \mathrm{d}V_h\, \mathrm{d}W^i(s) \\
&\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M F(\rho(s))\, a_i(a_i(\theta(s)\phi))\, \mathrm{d}V_h\, \mathrm{d}s - \int_0^t \int_M G_F(\rho(s))\, \mathrm{div}_h\, u\, \theta(s)\phi\, \mathrm{d}V_h\, \mathrm{d}s \\
&\quad - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s))\, \mathrm{div}_h\, a_i\, \theta(s)\phi\, \mathrm{d}V_h\, \mathrm{d}W^i(s) \\
&\quad - \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M \Lambda_i(1)\, G_F(\rho(s))\, \theta(s)\phi\, \mathrm{d}V_h\, \mathrm{d}s \\
&\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_M F''(\rho(s))(\rho(s)\, \mathrm{div}_h\, a_i)^2\, \theta(s)\phi\, \mathrm{d}V_h\, \mathrm{d}s \\
&\quad - \sum_{i=1}^N \int_0^t \int_M G_F(\rho(s))\overline{a}_i(\theta(s)\phi)\, \mathrm{d}V_h\, \mathrm{d}s.
\end{aligned}$$

By density of tensor products (see [12]), this equation continues to hold for any test function $\psi \in C_c^\infty((-1, T+1) \times M)$ and thus for any $\psi \in C^\infty([0,T] \times M)$.

5 Irregular Test Functions

We need to insert into the weak formulation (4.1) test functions $\psi(t,x)$ that are non-smooth. Clearly, in view of our assumptions, the stochastic integrals in (4.1) are zero-mean martingales. Hence, after taking the expectation in (4.1), we obtain

$$\begin{aligned}
&\mathbb{E}\int_M F(\rho(t))\psi(t)\,\mathrm{d}V_h - \mathbb{E}\int_M F(\rho_0)\psi(0)\,\mathrm{d}V_h \\
&= \mathbb{E}\int_0^t\int_M F(\rho(s))\partial_t\psi\,\mathrm{d}V_h\,\mathrm{d}s + \mathbb{E}\int_0^t\int_M F(\rho(s))\,u(\psi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^t\int_M F(\rho(s))\,a_i(a_i(\psi))\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \mathbb{E}\int_0^t\int_M G_F(\rho(s))\,\mathrm{div}_h\, u\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^t\int_M \Lambda_i(1)\,G_F(\rho(s))\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^t\int_M F''(\rho(s))(\rho(s)\,\mathrm{div}_h\, a_i)^2\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \sum_{i=1}^N \mathbb{E}\int_0^t\int_M G_F(\rho(s))\overline{a}_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}s,
\end{aligned} \tag{5.1}$$

which holds for any test function $\psi \in C^\infty([0,T]\times M)$.

The main result of this section is in the following.

Lemma 5.1 (non-smooth test functions) *Let ρ be a weak L^2 solution of* (1.2) *with initial datum $\rho|_{t=0} = \rho_0$ and assume that ρ is renormalizable. Fix $F \in C^2(\mathbb{R})$ with $F, F', F'' \in L^\infty(\mathbb{R})$. Fix a time $t_0 \in (0,T]$ and consider* (5.1) *evaluated at $t = t_0$. Then* (5.1) *continues to hold for any $\psi \in W^{1,2,p}([0,t_0]\times M)$ with $p > d+2$.*

Proof By Proposition 2.1, $W^{1,2,p}([0,t_0]\times M)$ compactly embeds into $C^0([0,t_0]\times M)$ (since $p > d+2$). Moreover, the first order x-derivatives of a $W^{1,2,p}$ function belong to $C^0([0,t_0]\times M)$. Therefore, given a function

$$\psi \in W^{1,2,p}([0,t_0]\times M),$$

the very definition of $W^{1,2,p}$ implies the existence of a sequence $\{\psi_j\}_{j\ge 1} \subset C^\infty([0,t_0]\times M)$ such that $\psi_j \to \psi$ in $W^{1,2,p}([0,t_0]\times M)$. Besides, we have

$$\psi_j \to \psi, \quad \nabla\psi_j \to \nabla\psi \quad \text{uniformly on } [0,t_0]\times M.$$

We extend the functions ψ_j to $C^\infty([0,T]\times M)$ by means of Proposition 10.1. These extensions are also denoted by ψ_j. Consequently, we can insert ψ_j into (5.1).

Equipped with the above convergences and the assumptions $\rho \in L_t^\infty L_{\omega,x}^2$ and $u \in L_t^1 \overrightarrow{W_x^{1,2}}$, it is straightforward (repeated applications of Hölder's inequality) to verify that (5.1) holds for test functions ψ that belong to $W^{1,2,p}([0,t_0]\times M)$.

6 On the Ellipticity of $\sum_i a_i(a_i)$, Proof of Lemma 1.1

In this section we will prove Lemma 1.1. Before doing that, however, let us explain why the second order differential operator $\sum_i a_i(a_i(\cdot))$, in general, fails to be non-degenerate (elliptic). To this end, we introduce the following (smooth) sections of the endomorphisms over TM:

$$\mathcal{A}_i(x)X := (X, a_i(x))_h \, a_i(x), \quad x \in M, \ X \in T_xM, \ i = 1, \cdots, N. \tag{6.1}$$

It is clear that these sections are symmetric with respect to h, namely

$$(\mathcal{A}_i(x)X, Y)_h = (X, \mathcal{A}_i(x)Y)_h, \quad x \in M, \ X, Y \in T_xM.$$

Set

$$\mathcal{A} := \mathcal{A}_1 + \cdots + \mathcal{A}_N, \tag{6.2}$$

which is still a smooth section of the symmetric endomorphisms over TM. Given the sections $\mathcal{A}_1, \cdots, \mathcal{A}_N$ and $\mathcal{A}$, we define the following second order linear differential operators in divergence form:

$$\begin{aligned} &C^2(M) \ni \psi \mapsto \mathrm{div}_h(\mathcal{A}_i \nabla_h \psi), \quad i = 1, \cdots, N, \\ &C^2(M) \ni \psi \mapsto \mathrm{div}_h(\mathcal{A} \nabla_h \psi) = \sum_{i=1}^N \mathrm{div}_h(\mathcal{A}_i \nabla_h \psi). \end{aligned}$$

Observe that the following identity holds trivially:

$$a_i(a_i(\psi)) = \mathrm{div}_h(\mathcal{A}_i \nabla_h \psi) - \overline{a}_i(\psi),$$

thus

$$\sum_{i=1}^N a_i(a_i(\psi)) = \mathrm{div}_h(\mathcal{A} \nabla_h \psi) - \sum_{i=1}^N \overline{a}_i(\psi), \tag{6.3}$$

where $\overline{a}_i$ is short-hand for the first order differential operator $(\mathrm{div}_h\, a_i)\, a_i$. Thus $\sum_{i=1}^N a_i(a_i(\cdot))$ is non-degenerate (elliptic) if and only if $\mathrm{div}_h(\mathcal{A}\nabla_h \cdot)$ is so.

In view of (6.3), let us see why the induced differential operator $\sum_{i=1}^N a_i(a_i(\cdot))$ may degenerate. From the very definition of $\mathcal{A}$, we have

$$(\mathcal{A}(x)X, X)_h = \sum_{i=1}^N (\mathcal{A}_i(x)X, X)_h = \sum_{i=1}^N (X, a_i(x))_h^2, \quad x \in M, \ X \in T_xM,$$

and the last expression may be zero unless we can find vector fields $a_{i_1}(x), \cdots, a_{i_d}(x)$ that constitute a basis for T_xM. Note that this can also happen in the "ideal" case $N = d$, that is, one can always find suitable $x \in M$ and $X \in T_xM$ such that $(\mathcal{A}(x)X, X)_h = 0$. The explanation for this fact is geometric in nature. In general, given an arbitrary d-dimensional smooth manifold M, it is not possible to construct a global frame, i.e., smooth vector fields $E_1, \cdots, E_d$ forming a basis for T_xM for all $x \in M$. If this happens, the manifold is called parallelizable. Examples of parallelizable manifolds are Lie groups (like $\mathbb{R}^d$, $\mathbb{T}^d$) and $\mathbb{S}^d$ with $d \in \{1, 3, 7\}$.

Nevertheless, by compactness of M, one can always find vector fields $a_1, \cdots, a_N$ with $N \geq d$, depending on the geometry of M, such that the resulting operator $\mathrm{div}_h(\mathcal{A}\nabla_h \cdot)$ becomes the Laplace-Beltrami operator (and hence elliptic). In other words, to implement our strategy of using noise to avoid density concentrations, we will add to the original SPDE (1.2) as many independent Wiener processes and first order differential operators $\overline{a}_1, \cdots, \overline{a}_N$ as deemed necessary by the geometry of the manifold itself. Note that in the Euclidean case (see [3, 6, 19]) one can always resort to the canonical differential operators $a_i = \partial_i$ and thus $\sum_{i=1}^{N} a_i(a_i(\cdot)) = \mathrm{div}_h(\mathcal{A}\nabla_h \cdot) = \Delta \cdot$ (with $N = d$). This simple approach does not work for us because of the Riemannian structure of the underlying domain M.

Having said all of that, let us now return to the proof of Lemma 1.1, which will be a trivial consequence of the following crucial result.

Lemma 6.1 *There exist $N = N(M)$ smooth vector fields $a_1, \cdots, a_N$ on M such that the corresponding section $\mathcal{A}$, see* (6.1) *and* (6.2)*, satisfies*

$$(\mathcal{A}(x)X, Y)_h = 2(X, Y)_h, \quad \forall x \in M,\ \forall X, Y \in T_xM.$$

Consequently, $\mathcal{A}(x) = 2\, I_{T_xM}$ for all $x \in M$.

Proof Let $p \in M$. Then, by means of the Gram-Schmidt algorithm, we can easily construct a local orthonormal frame near p, that is, a local frame $E_{p,1}, \cdots, E_{p,d}$ defined in an open neighborhood $\mathcal{U}_p$ of p that forms an orthonormal basis for the tangent space at each point of the neighborhood (see [32, p.24] for details). Since $\{\mathcal{U}_p\}_{p\in M}$ forms an open covering of M, the compactness of M ensures the existence of $p_1, \cdots, p_L \in M$ such that $\bigcup_{j=1}^{L} \mathcal{U}_{p_j} = M$ and a collection of locally smooth vector fields $\{E_{p_j,i}\}_{\substack{i=1,\cdots,d\\ j=1,\cdots,L}}$ with the aforementioned property. Let us now consider a smooth partition of unity subordinate to $\{\mathcal{U}_{p_j}\}_{j=1}^{L}$, which we may write as $\{\alpha_j^2\}_{j=1}^{L}$, where $\alpha_j \in C^\infty(M)$ and $\sum_{j=1}^{L} \alpha_j^2 = 1$. Set $\widetilde{E}_{p_j,i} := \alpha_j E_{p_j,i}$ for $i = 1, \cdots, d$ and $j = 1, \cdots, L$. Extending these vector fields by zero outside their supports, we obtain global smooth vector fields on M.

Observe that if $\alpha_j(x) \neq 0$, then $x \in (\operatorname{supp}\alpha_j)^\circ = (\operatorname{supp}\alpha_j^2)^\circ \subset \mathcal{U}_{p_j}$. As a result, $E_{p_j,1}(x), \cdots, E_{p_j,d}(x)$ constitute an orthonormal basis for T_xM. For convenience, we rename the vector fields $\{\widetilde{E}_{p_j,i}\}_{\substack{i=1,\cdots,d\\ j=1,\cdots,L}}$ as $\beta_1, \cdots, \beta_N$, where $N := d \cdot L$.

As before, we define sections $\mathcal{B}_i$ of the endomorphisms over TM by setting

$$\mathcal{B}_i(x)X := (X, \beta_i(x))_h\, \beta_i(x), \quad x \in M,\ X \in T_xM$$

for $i = 1, \cdots, N$, and $\mathcal{B} := \mathcal{B}_1 + \cdots + \mathcal{B}_N$.

For an arbitrary $x \in M$ and $X \in T_x M$, we compute

$$\begin{aligned}(\mathcal{B}(x)X, X)_h &= \sum_{k=1}^{N} (X, \beta_k(x))_h^2 = \sum_{j=1}^{L} \sum_{i=1}^{d} (X, \widetilde{E}_{p_j,i}(x))_h^2 \\ &= \sum_{j:\alpha_j(x)\neq 0} \sum_{i=1}^{d} (X, E_{p_j,i}(x))_h^2 \alpha_j^2(x) \\ &= \sum_{j:\alpha_j(x)\neq 0} \alpha_j^2(x) \sum_{i=1}^{d} (X, E_{p_j,i}(x))_h^2 \\ &= \sum_{j:\alpha_j(x)\neq 0} \alpha_j^2(x) |X|_h^2 = |X|_h^2.\end{aligned}$$

By the polarization identity for inner products and the symmetry of $\mathcal{B}$, this last equality implies that

$$(\mathcal{B}(x)X, Y)_h = (X, Y)_h, \quad \forall x \in M, \ \forall X, Y \in T_x M,$$

and thus $\mathcal{B}(x) = I_{T_x M}$.

Setting $a_i := \sqrt{2}\beta_i$, $i = 1, \cdots, N$, concludes the proof of the lemma.

Proof of Lemma 1.1 Fix $\psi \in C^2(M)$. In view of Lemma 6.1, the identity (6.3) becomes

$$\sum_{i=1}^{N} a_i(a_i(\psi)) = 2 \operatorname{div}_h(\nabla_h \psi) - \sum_{i=1}^{N} \overline{a}_i(\psi) = 2\Delta_h \psi - \sum_{i=1}^{N} \overline{a}_i(\psi),$$

where $\overline{a}_i = (\operatorname{div}_h a_i)\, a_i$.

From now on, we will be using the vector fields $a_1, \cdots, a_N$ constructed in Lemma 6.1, in which case the Itô SPDE (1.3) becomes

$$\mathrm{d}\rho + \operatorname{div}_h \Big(\rho\Big[u - \frac{1}{2}\sum_{i=1}^{N} \overline{a}_i\Big]\Big)\,\mathrm{d}t + \sum_{i=1}^{N} \operatorname{div}_h(\rho\, a_i)\,\mathrm{d}W^i(t) - \Delta_h \rho\,\mathrm{d}t = 0. \tag{6.4}$$

The space-weak formulation of this SPDE is

$$\begin{aligned}\int_M \rho(t)\psi\,\mathrm{d}V_h &= \int_M \rho_0 \psi\,\mathrm{d}V_h + \int_0^t \int_M \rho(s)\Big[\,u(\psi) - \frac{1}{2}\sum_{i=1}^{N} \overline{a}_i(\psi)\Big]\,\mathrm{d}V_h\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{N} \int_0^t \int_M \rho(s)\, a_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}W^i(s) + \int_0^t \int_M \rho(s)\,\Delta_h \psi\,\mathrm{d}V_h\,\mathrm{d}s,\end{aligned}$$

see Definition 1.2 and (1.4).

7 Test Function for Duality Method

In this section we first construct a solution to the following parabolic Cauchy problem on the manifold M: Given $0 < t_0 \le T$, solve

$$\begin{cases} \partial_t v - \Delta_h v + b(t,x)v = f(x,t) & \text{on } [0,t_0] \times M, \\ v(0,x) = 0 \quad \text{on } M, \end{cases} \tag{7.1}$$

where b and f are given irregular functions in $L^p([0,t_0]\times M)$ (with $p\geq 1$ to be fixed later). We follow the strategy outlined in [4, p.131] (for smooth b,f), making use of Fredholm theory and anisotropic Sobolev spaces. Toward the end of this section, we utilize the solution of (7.1) to construct a test function that will form the core of a duality argument given in an upcoming section.

Consider the space $W_0^{1,2,p}([0,t_0]\times M)$, which is the subspace of functions in the anisotropic Sobolev space $W^{1,2,p}([0,t_0]\times M)$ vanishing at $t=0$. Let L designate the heat operator on M, namely $L=\partial_t-\Delta_h$. According to [4, Thm.4.45], L is an isomorphism of $W_0^{1,2,p}([0,t_0]\times M)$ onto $L^p([0,t_0]\times M)$ for $1\leq p<\infty$. Consider the multiplication operator

$$K_b: W_0^{1,2,p}([0,t_0]\times M)\to L^p([0,t_0]\times M), \quad v\overset{K_b}{\mapsto} bv.$$

To guarantee that this operator is well-defined, we must assume $p>d+2$. In this way, in view of Proposition 2.1, $W_0^{1,2,p}([0,t_0]\times M)$ compactly embeds into $C^0([0,t_0]\times M)$ and the first order space-derivatives of $v\in W_0^{1,2,p}([0,t_0]\times M)$ are continuous on $[0,t_0]\times M$. It then follows that

$$\int_0^{t_0}\int_M |bv|^p\,\mathrm{d}V_h\,\mathrm{d}t\leq \|v\|_{C^0}^p\int_0^{t_0}\int_M |b|^p\,\mathrm{d}V_h\,\mathrm{d}t,$$

guaranteeing that K_b is well-defined.

Claim K_b is compact.

First of all, K_b is continuous:

$$\|K_b v\|_{L^p}\leq \|v\|_{C^0}\|b\|_{L^p}\leq C\|v\|_{W_0^{1,2,p}}\|b\|_{L^p},$$

where $C>0$ is a constant coming from the anisotropic Sobolev embedding, consult Proposition 2.1. Clearly, $W_0^{1,2,p}([0,t_0]\times M)$ is reflexive, being a closed subspace of $W^{1,2,p}([0,t_0]\times M)$. Hence, to arrive at the claim, it is enough to prove that K_b is completely continuous. Recall that a bounded linear operator $T:X\to Y$ between Banach spaces is called completely continuous if weakly convergent sequences in X are mapped to strongly converging sequences in Y. Let $\{v_n\}_{n\geq 1}$ be a sequence in $W_0^{1,2,p}([0,t_0]\times M)$ such that $v_n\rightharpoonup v\in W_0^{1,2,p}$. By the compact embedding $W_0^{1,2,p}\subset\subset C^0$, $v_n\to v$ in C^0. Hence,

$$\|K_b v_n-K_b v\|_{L^p}\leq \|v_n-v\|_{C^0}\|b\|_{L^p}\to 0,$$

and so K_b is completely continuous. This concludes the proof of the claim.

Next, being an isomorphism, L is a Fredholm operator from $W_0^{1,2,p}$ to L^p. This implies that $L+K_b$ is a Fredholm operator, with index $\mathrm{Ind}(L+K_b)=\mathrm{Ind}(L)$, where trivially $\mathrm{Ind}(L)=0$ (L is invertible). Thus our goal is to verify the following claim.

Claim Either $\ker(L+K_b)$ is trivial or $\mathrm{codim}(R(L+K_b))=0$.

If this claim holds, then we will be able to conclude that (7.1) is solvable for any $f\in L^p([0,t_0]\times M)$. The proof of the claim is divided into three main steps.

Step 1 $b \in C^\infty([0,t_0]\times M)$.

Our aim is to show that $\ker(L+K_b)$ is trivial. Let $v \in W_0^{1,2,p}([0,t_0]\times M)$ solve $(L+K_b)v = 0$. Since $p > d+2$ and b is smooth, it follows from parabolic regularity theory that v is (at least) in $C^{1,2}([0,t_0]\times M)$. Indeed, by the anisotropic Sobolev embedding (Proposition 2.1), $v \in C^{0,\gamma}([0,t_0]\times M)$ with $\gamma = 1 - \frac{1+d}{p}$. Therefore,

$$Lv = -bv \in C^{0,\gamma}([0,t_0]\times M),$$

and $v(0,\cdot) = 0$ on M. Parabolic regularity theory (see e.g. [4, p. 130]) implies that $\partial_t v$ and the second derivatives of v with respect to x are Hölder continuous.

By the chain rule, the function $\psi := \frac{v^2}{2}$ satisfies

$$L\psi = -|\nabla_h v|_h^2 - 2b\psi \le -2b\psi.$$

Since b is bounded and $\psi(0,x) = 0$, the maximum principle (see [10, Prop.4.3]) implies that $\psi \le 0$ everywhere. On the other hand, $\psi \ge 0$ by definition. It follows that $\psi \equiv 0$, and so $v \equiv 0$.

Hence, given any $b \in C^\infty([0,t_0]\times M)$, the Cauchy problem (7.1) admits a unique solution for any $f \in L^p([0,t_0]\times M)$.

Step 2 A priori estimates (smooth data).

Let us consider the more general problem

$$\begin{cases} \partial_t v - \Delta_h v + b(t,x)v = g(x,t) & \text{on } [0,t_0]\times M, \\ v(0,x) = c(x) & \text{on } M, \end{cases} \tag{7.2}$$

where $b, g \in C^\infty([0,t_0]\times M)$ and $c \in C^\infty(M)$. This problem admits a unique solution $v \in C^{1,2}([0,t_0]\times M)$, given by $v = \widetilde{v} + c$, where $\widetilde{v}$ solves (7.1) with right-hand side $f = g - cb + \Delta_h c \in L^p([0,t_0]\times M)$.

From known a priori estimates for the heat equation on manifolds (see [4, Thm.4.45]), there is a constant $C_0 = C_0(p,M)$ such that ($\widetilde{v} = v - c$),

$$\begin{aligned} \|v\|_{W^{1,2,p}} = \|\widetilde{v} + c\|_{W^{1,2,p}} &\le \|\widetilde{v}\|_{W^{1,2,p}} + T\|c\|_{W^{2,p}(M)} \\ &\le C_0\|g - bc + \Delta_h c\|_{L^p} + T\|c\|_{W^{2,p}(M)} \\ &\le C_0[\|g\|_{L^p} + \|b\|_{L^p}\|c\|_{C^0(M)} + T\|\Delta_h c\|_{L^p(M)}] + T\|c\|_{W^{2,p}(M)}, \end{aligned}$$

where $W^{2,p}(M)$ denotes the standard Sobolev space on (M,h), which embeds into $C^0(M)$ (recall $p > 2+d$). Therefore, for a constant $C = C(p,M,T)$, we infer

$$\|v\|_{W^{1,2,p}} \le C[\|g\|_{L^p} + \|c\|_{W^{2,p}(M)}(\|b\|_{L^p} + 1)]. \tag{7.3}$$

Summarizing, the general Cauchy problem (7.2) with $b, g \in C^\infty([0,t_0]\times M)$ and $c \in C^\infty(M)$ admits a unique solution $v \in C^{1,2}([0,t_0]\times M)$ satisfying (7.3).

Step 3 Well-posedness of (7.2), non-smooth b, g.

The aim is to prove the well-posedness of (7.2)—and thus (7.1)—for irregular b and g in $L^p([0,t_0]\times M)$. Since $C^\infty([0,t_0]\times M)$ is dense in $L^p([0,t_0]\times M)$ (see[4, Thm.2.9]), there exist sequences $\{b_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ of smooth functions such that

$$b_n \xrightarrow{L^p} b, \quad g_n \xrightarrow{L^p} g.$$

From the previous step, there exists a unique solution $v_n \in W^{1,2,p}([0,t_0]\times M)$ of

$$\begin{cases} \partial_t v_n - \Delta_h v_n + b_n(t,x)v_n = g_n(x,t) \quad \text{on } [0,t_0]\times M, \\ v_n(0,x) = c(x) \quad \text{on } M. \end{cases}$$

In view of (7.3), $\{v_n\}_{n\geq 1}$ is bounded in $W^{1,2,p}([0,t_0]\times M)$. Therefore, up to a subsequence, we may assume that

$$\begin{cases} v_n \rightharpoonup v \in W^{1,2,p}([0,t_0]\times M), \\ v_n \to v \in C^0([0,t_0]\times M). \end{cases}$$

Given these convergences, it is easy to conclude that v solves the Cauchy problem (7.2) with $b,g\in L^p([0,t_0]\times M)$ and $c\in C^\infty(M)$.

We summarize our findings so far in the following proposition.

Proposition 7.1 (well-posedness of parabolic Cauchy problem, non-smooth data) *Suppose b and g belong to $L^p([0,t_0]\times M)$. Then there exists a unique solution $v\in W^{1,2,p}([0,t_0]\times M)$ to the Cauchy problem* (7.2) *with initial data $c\in C^\infty(M)$. Furthermore, the a priori estimate* (7.3) *holds.*

Proof The only assertion that remains to be verified is the one about uniqueness, but uniqueness of the solution is an immediate consequence of (7.3).

Remark 7.1 The "non-smooth" quantifier in Proposition 7.1 refers to the functions b and g in (7.2). In upcoming applications it is essential that b,g are allowed to be irregular (but a smooth initial function c is fine, like $c\equiv 1$).

Let us now consider the special Cauchy problem

$$\begin{cases} \partial_t v - \Delta_h v + b(t,x)v = -b(x,t) \quad \text{on } [0,t_0]\times M, \\ v(0,x) = 0 \quad \text{on } M \end{cases} \tag{7.4}$$

with $b\in C^\infty([0,t_0]\times M)$ and $b\leq 0$. This problem corresponds to (7.2) with a nonnegative smooth source g (namely, $g=-b\geq 0$).

By the previous discussion, there exists a unique solution $v\in C^{1,2}([0,t_0]\times M)$ to (7.4). Clearly, we have

$$\partial_t v - \Delta_h v \geq -b(t,x)v,$$

where $b\geq -C$ for some positive constant C (since b is smooth). Thanks to the maximum principle (see [10, Prop. 4.3]), this implies that $v\geq 0$ on $[0,t_0]\times M$.

Next, suppose that b is irregular with $b\in L^p([0,t_0]\times M)$ ($p>d+2$) and $b\leq 0$ almost everywhere. Let $v\in W_0^{1,2,p}([0,t_0]\times M)$ be the unique solution of the Cauchy problem (7.4), as supplied by Proposition 7.1. We would like to conclude that v is nonnegative. To this end, approximate b in $L^p([0,t_0]\times M)$ by $\{b_n\}_{n\geq 1}\subset C^\infty([0,t_0]\times M)$ with $b_n\leq 0$ for all n, and let v_n be the corresponding (unique) solution in $C^{1,2}([0,t_0]\times M)$ of

$$\begin{cases}\partial_t v_n-\Delta_h v_n+b_n(t,x)v_n=-b_n(x,t) & \text{on } [0,t_0]\times M,\\ v_n(0,x)=0 \quad \text{on } M.\end{cases}$$

Then $v_n\geq 0$. By the a priori estimate (7.3), which now reads

$$\|v_n\|_{W^{1,2,p}}\leq C\|b_n\|_{L^p},$$

and the previous discussion, we infer that $v_n \xrightarrow{C^0} w$ (up to a subsequence), for some limit function $0\leq w\in W_0^{1,2,p}([0,t_0]\times M)$ that solves (7.4) with $b\in L^p([0,t_0]\times M)$. By uniqueness, we conclude that $v=w\geq 0$.

To summarize, we have proved that for $0\geq b\in L^p([0,t_0]\times M)$ (with $p>d+2$), there exists a unique solution $0\leq v\in W_0^{1,2,p}([0,t_0]\times M)$ of (7.4), satisfying

$$\|v\|_{W^{1,2,p}}\leq C\|b\|_{L^p}.$$

We are now in a position to prove the main result of this section.

Proposition 7.2 (test function for duality method) *Suppose $b\in L^p([0,t_0]\times M)$ with $p>d+2$ and $b\leq 0$. Then the terminal value problem*

$$\begin{cases}\partial_t\phi+\Delta_h\phi-b(t,x)\phi=0 & \text{on } [0,t_0]\times M,\\ \phi(t_0,x)=1 \quad \text{on } M\end{cases}\tag{7.5}$$

admits a unique solution $\phi\in W^{1,2,p}([0,t_0]\times M)\cap C^0([0,t_0]\times M)$ with continuous first order spatial derivatives. Moreover, $\phi\geq 1$ everywhere and the following a priori estimates hold:

$$\|\phi\|_{W^{1,2,p}([0,t_0]\times M)}\leq T+C(p,M,T)\|b\|_{L^p([0,t_0]\times M)}\tag{7.6}$$

and (*consequently*)

$$\|\phi\|_{C^0([0,t_0]\times M)}+\|\nabla\phi\|_{C^0([0,t_0]\times M)}\lesssim_{d,M,p,T} 1+\|b\|_{L^p([0,t_0]\times M)}.\tag{7.7}$$

Proof The solution ϕ of (7.5) is obtained by setting $\phi(t,x):=1+v(t_0-t,x)$, where $v\in W_0^{1,2,p}([0,t_0]\times M)$ is the unique solution of the Cauchy problem

$$\begin{cases}\partial_t v-\Delta_h v+\widetilde{b}(t,x)v=-\widetilde{b}(x,t) & \text{on } [0,t_0]\times M,\\ v(0,x)=0 \quad \text{on } M,\end{cases}$$

where $\widetilde{b}(t,x):=b(t_0-t,x)$. Proposition 2.1 therefore supplies the existence and uniqueness of ϕ, estimate (7.6), and also the lower bound $\phi\geq 1$. The final estimate (7.7) follows from the anisotropic Sobolev inequality (Proposition 2.1) and (7.6).

Remark 7.2 Observe that the right-hand side of (7.6) is non-decreasing in $\|b\|_{L^p}$, a fact that will be exploited in Section 9.

8 L^2 Estimate and Uniqueness for Weak Solutions

The main outcome of this section is an a priori estimate that is valid for arbitrary weak L^2 solutions of the SPDE (1.2), with a rough velocity field u satisfying in particular $\operatorname{div}_h u \in L^p_{t,x}$ for some $p > d+2$. The proof relies fundamentally on the special noise vector fields a_i constructed in Lemma 1.1, the renormalization result provided by Theorem 1.1, and a duality method that makes use of the test function constructed in Proposition 7.2.

Theorem 8.1 (L^2 estimate and uniqueness) *Let ρ be an arbitrary weak L^2 solution of the stochastic continuity equation* (1.2), *with initial datum $\rho_0 \in L^2(M)$, velocity vector field u satisfying* (1.5), (1.6) *and* (1.7), *and noise vector fields $a_1, \cdots, a_N$ given by Lemma* 1.1. *Then*

$$\sup_{0\le t\le T} \|\rho(t)\|^2_{L^2(\Omega\times M)} \le C\|\rho_0\|^2_{L^2(M)}, \tag{8.1}$$

where $C = C(d, M, p, T, a_i, \|\operatorname{div}_h u\|_{L^p([0,T]\times M)}, \|u\|_\infty)$ is a constant that is non-decreasing in $\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$ and $\|u\|_\infty$; here, for convenience, we have set $\|u\|_\infty := \|u\|_{L^\infty([0,T];\overrightarrow{L^\infty(M)})}$.

Furthermore, weak L^2 solutions are uniquely determined by their initial data.

Proof Since $u \in L^1([0,T]; \overrightarrow{W^{1,2}(M)})$, the weak solution ρ is renormalizable, in view of Theorem 1.1. However, Theorem 1.1 asks for bounded nonlinearities F. To handle $F(\xi) = \xi^2$, we must employ an approximation (truncation) procedure.

We pick any increasing function $\chi \in C^\infty([0,\infty))$ such that $\chi(\xi) = \xi$ for $\xi \in [0,1]$, $\chi(\xi) = 2$ for $\xi \ge 2$, $\chi(\xi) \in [1,2]$ for $\xi \in (1,2)$, and $A_0 := \sup_{\xi\ge 0} \chi'(\xi) > 1$. Set $A_1 := \sup_{\xi\ge 0} |\chi''(\xi)|$. We define the rescaled function $\chi_\mu(\xi) = \mu\,\chi\big(\frac{\xi}{\mu}\big)$ for $\mu > 0$. The relevant approximation of $F(\xi) = \xi^2$ is $F_\mu(\xi) := \chi_\mu(\xi^2)$ for $\xi \in \mathbb{R}$, $\mu > 0$.

Some tedious computations will reveal that

$$\begin{aligned}
&F_\mu \in C^\infty(\mathbb{R}), \quad \lim_{\mu\to\infty} F_\mu(\xi) = \xi^2, \quad \sup_{\xi\in\mathbb{R}} F_\mu(\xi) \le 2\mu, \quad \sup_{\mu>0} F_\mu(\xi) \le 2\xi^2,\\
&\sup_{\xi\in\mathbb{R}} |F'_\mu(\xi)| \le 2\sqrt{2}A_0\sqrt{\mu}, \quad \sup_{\mu>0} |F'_\mu(\xi)| \le 2\sqrt{2}A_0|\xi|, \quad \lim_{\mu\to\infty} F'_\mu(\xi) = 2\xi,\\
&\lim_{\mu\to\infty} F''_\mu(\xi) = 2, \quad |F''_\mu(\xi)| \le 8A_1 + 2A_0.
\end{aligned} \tag{8.2}$$

Furthermore, the function $G_{F_\mu}(\xi) = \xi F'_\mu(\xi) - F_\mu(\xi)$ satisfies

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}} |G_{F_\mu}(\xi)| \le (4A_0 + 2)\mu, \quad \sup_{\mu>0} |G_{F_\mu}(\xi)| \le 2(\sqrt{2}A_0 + 1)\xi^2,\\
&\text{and} \quad \lim_{\mu\to\infty} G_{F_\mu}(\xi) = \xi^2,
\end{aligned}$$

and the following estimate:

$$|G_{F_\mu}(\xi)| \le C_\chi F_\mu(\xi), \quad |\xi^2 F''_\mu(\xi)| \le C_\chi \begin{cases} F_\mu(\xi) & \text{for } |\xi| \le \sqrt{\mu},\\ \xi^2 & \text{for } |\xi| \in [\sqrt{\mu}, \sqrt{2\mu}]\\ O(\mu) & \text{for } |\xi| > \sqrt{2\mu} \end{cases} \tag{8.3}$$

for some constant $C_\chi > 0$ independent of μ.

Fix $t_0 \in (0, T]$ and consider (5.1) evaluated at $t = t_0$ and with $F = F_\mu$. Then, in view of the choice of noise vector fields a_i (see Lemma 1.1), the following equation holds for any $\psi \in W^{1,2,p}([0, t_0] \times M)$ (as long as $p > d + 2$):

$$
\begin{aligned}
&\mathbb{E}\int_M F_\mu(\rho(t_0))\psi(t_0)\,\mathrm{d}V_h - \mathbb{E}\int_M F_\mu(\rho_0)\psi(0)\,\mathrm{d}V_h \\
&= \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\partial_t\psi\,\mathrm{d}V_h\,\mathrm{d}s + \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,u(\psi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,\Delta_h\psi\,\mathrm{d}V_h\,\mathrm{d}s - \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,\overline{a}_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \mathbb{E}\int_0^{t_0}\int_M G_{F_\mu}(\rho(s))\,\mathrm{div}_h\,u\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^{t_0}\int_M \Lambda_i(1)\,G_{F_\mu}(\rho(s))\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^{t_0}\int_M F_\mu''(\rho(s))(\rho(s)\,\mathrm{div}_h\,a_i)^2\,\psi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\quad - \sum_{i=1}^N \mathbb{E}\int_0^{t_0}\int_M G_{F_\mu}(\rho(s))\overline{a}_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}s,
\end{aligned}
\tag{8.4}
$$

where we have applied Theorem 1.1 to (6.4) and the time-space weak formulation with non-smooth test functions $\psi(t, x)$, see Proposition 5.1. Let ϕ be the unique solution of (7.5) with $b = -C_\chi|\,\mathrm{div}_h\,u|$, where $C_\chi > 0$ is the constant appearing in (8.3). The existence of ϕ is guaranteed by Proposition 7.2. Moreover, ϕ belongs to $W^{1,2,p}([0, t_0] \times M) \cap C^0([0, t_0] \times M)$, the estimates (7.6)–(7.7) hold, and ϕ is lower bounded by 1 everywhere in $[0, t_0] \times M$. Thanks to Proposition 5.1, we can use ϕ as test function in (8.4).

Making use of (8.3), we obtain

$$
\begin{aligned}
-\mathbb{E}\int_0^{t_0}\int_M G_{F_\mu}(\rho(s))\,\mathrm{div}_h\,u\,\phi\,\mathrm{d}V_h\,\mathrm{d}s &\le \mathbb{E}\int_0^{t_0}\int_M |G_{F_\mu}(\rho(s))||\,\mathrm{div}_h\,u|\,\phi\,\mathrm{d}V_h\,\mathrm{d}s \\
&\le \mathbb{E}\int_0^{t_0}\int_M C_\chi F_\mu(\rho(s))|\,\mathrm{div}_h\,u|\,\phi\,\mathrm{d}V_h\,\mathrm{d}s \\
&= -\mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,b\,\phi\,\mathrm{d}V_h\,\mathrm{d}s.
\end{aligned}
$$

Now, recalling that the test function ϕ is the unique solution of the PDE problem (7.5), (8.4) (with $\psi = \phi$) simplifies to

$$
\begin{aligned}
\mathbb{E}\int_M F_\mu(\rho(t_0))\phi(t_0)\,\mathrm{d}V_h \le \mathbb{E}&\int_M F_\mu(\rho_0)\phi(0)\,\mathrm{d}V_h \\
&+ \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,u(\phi)\,\mathrm{d}V_h\,\mathrm{d}s \\
&- \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,\overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M \Lambda_i(1)\, G_{F_\mu}(\rho(s))\,\phi\,\mathrm{d}V_h\,\mathrm{d}s\\
&+ \frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M F_\mu''(\rho(s))(\rho(s)\operatorname{div}_h a_i)^2\,\phi\,\mathrm{d}V_h\,\mathrm{d}s\\
&- \sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M G_{F_\mu}(\rho(s))\overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s. \qquad (8.5)
\end{aligned}
$$

Using the fourth property in (8.2) and the estimate (7.7) satisfied by the solution ϕ of (7.5) with $b = -C_\chi|\operatorname{div}_h u|$, we obtain

$$
\begin{aligned}
&- \frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,\overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s\\
&\le C(a_i)\|\nabla\phi\|_{C^0([0,t_0]\times M)}\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\,\mathrm{d}V_h\,\mathrm{d}s\\
&\le C(a_i,\|\operatorname{div}_h u\|_{L^p([0,T]\times M)})\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\,\mathrm{d}V_h\,\mathrm{d}s\\
&\le C(a_i,\|\operatorname{div}_h u\|_{L^p([0,T]\times M)})\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s,
\end{aligned}
$$

where we have also exploited that $\phi \ge 1$. Observe that the constant C is non-decreasing in $\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$, see Remark 7.2, and we do not write its dependency on the d, M, p, T. Similarly, using also (8.3), we have

$$
-\sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M G_{F_\mu}(\rho(s))\overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s \le C\,\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s
$$

for a possibly different constant $C = C(a_i,\|\operatorname{div}_h u\|_{L^p([0,T]\times M)})$, still non-decreasing in

$$
\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}.
$$

Similar bounds can be derived for the terms on the third and fourth lines of (8.5):

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M F_\mu''(\rho(s))(\rho(s)\operatorname{div}_h a_i)^2\,\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s \le C\,\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi\,\mathrm{d}V_h\,\mathrm{d}s,\\
&-\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\int_0^{t_0}\int_M \Lambda_i(1)\,G_{F_\mu}(\rho(s))\,\phi\,\mathrm{d}V_h\,\mathrm{d}s \le C\,\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s.
\end{aligned}
$$

Therefore (8.5) becomes

$$
\begin{aligned}
\mathbb{E}\int_M F_\mu(\rho(t_0))\phi(t_0)\,\mathrm{d}V_h &\le \mathbb{E}\int_M F_\mu(\rho_0)\phi(0)\,\mathrm{d}V_h + \mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,u(\phi)\,\mathrm{d}V_h\,\mathrm{d}s\\
&\quad + C(a_i,\|\operatorname{div}_h u\|_{L^p([0,T]\times M)})\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s.
\end{aligned}
$$

Arguing as above, since $u \in L^\infty([0,T];\overrightarrow{L^\infty(M)})$,

$$
\mathbb{E}\int_0^{t_0}\int_M F_\mu(\rho(s))\,u(\phi)\,\mathrm{d}V_h\,\mathrm{d}s \le C(a_i,\|\operatorname{div}_h u\|_{L^p([0,T]\times M)},\|u\|_\infty)\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s,
$$

where the constant C is non-decreasing in $\|u\|_\infty$ as well.

In conclusion, we have obtained

$$\mathbb{E}\int_M F_\mu(\rho(t_0))\phi(t_0)\,\mathrm{d}V_h \le \mathbb{E}\int_M F_\mu(\rho_0)\phi(0)\,\mathrm{d}V_h + C\,\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s, \tag{8.6}$$

where C depends in particular on $a_1,\cdots,a_N$, $\|u\|_\infty$ and $\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$ but not on μ; C is non-decreasing in $\|u\|_\infty$ and $\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$.

By the dominated convergence theorem (ϕ is continuous, $\rho_0 \in L^2(M)$), we obtain

$$\mathbb{E}\int_M F_\mu(\rho_0)\phi(0)\,\mathrm{d}V_h \to \int_M \rho_0^2\,\phi(0)\,\mathrm{d}V_h$$

as $\mu \to \infty$. On the other hand, by Fatou's lemma, we can send $\mu \to \infty$ in the term on the left-hand side of (8.6), arriving at

$$\mathbb{E}\int_M \rho^2(t_0)\phi(t_0)\,\mathrm{d}V_h \le \int_M \rho_0^2\,\phi(0)\,\mathrm{d}V_h + C\,\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\phi(s)\,\mathrm{d}V_h\,\mathrm{d}s. \tag{8.7}$$

Since ϕ is lower bounded by 1, we can replace the term on the left-hand side by $\mathbb{E}\int_M \rho^2(t_0)\,\mathrm{d}V_h$. On the other hand, in view of (7.7), we can bound (remove) the ϕ part from the terms on the right-hand side of (8.7) by $\|\phi\|_{C^0([0,t_0]\times M)} \lesssim 1 + \|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$, where $\lesssim$ does not depend on t_0. As a result, (8.7) becomes

$$\mathbb{E}\int_M \rho^2(t_0)\,\mathrm{d}V_h \le K\int_M \rho_0^2\,\mathrm{d}V_h + K\,\mathbb{E}\int_0^{t_0}\int_M \rho^2(s)\,\mathrm{d}V_h\,\mathrm{d}s,$$

where K depends in particular on $a_1,\cdots,a_N$, $\|u\|_\infty$ and $\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$, still non-decreasing in $\|u\|_\infty$ and $\|\operatorname{div}_h u\|_{L^p([0,T]\times M)}$.

Setting $\Phi(t_0) := \mathbb{E}\int_M \rho^2(t_0)\,\mathrm{d}V_h \in [0,\infty)$ and $\Phi(0) := C\int_M \rho_0^2\,\mathrm{d}V_h \in [0,\infty)$, the last inequality reads as

$$\Phi(t_0) \le \Phi(0) + K\int_0^{t_0}\Phi(s)\,\mathrm{d}s, \quad 0 < t_0 \le T.$$

The integrability properties of weak solutions implies $\Phi \in L^1([0,t_0])$ for any $t_0 \le T$. Hence, by Grönwall's inequality,

$$\Phi(t) \le \Phi(0)\mathrm{e}^{Kt}, \quad t \in [0,T].$$

This concludes the proof of (8.1), which also implies the uniqueness assertion.

Remark 8.1 Regarding the uniqueness assertion in Proposition 8.1, we mention that it is possible to prove uniqueness without an additional assumption on $\operatorname{div}_h u$. This follows from the renormalized formulation (1.8) with $F(\cdot) = |\cdot|$, modulo an approximation argument. Since the existence of weak solutions (which asks that $\operatorname{div}_h u \in L^p$) holds in the L^2 setting, we have chosen not to focus on L^1 uniqueness.

9 Proof of Main Result, Theorem 1.2

We divide the proof of Theorem 1.2 into four parts (subsections), starting with the procedure for smoothing the irregular velocity vector field u, yielding $u_\tau \in C^\infty$ such that $u_\tau \approx u$ for $\tau > 0$ small. In the second subsection we rely on the L^2 estimate in Proposition 8.1 to ensure weak compactness of a sequence $\{\rho^\tau\}_{\tau>0}$ of approximate solutions, obtained by solving the SPDE (1.2) with smooth initial datum ρ_0 and smooth velocity field u_τ. The limit of a weakly converging subsequence is easily shown to be a solution of the SPDE. In the third subsection we remove the assumption that ρ_0 is smooth. Finally, we prove a technical lemma utilized in the second subsection.

9.1 Smoothing of velocity vector field u

We extend the vector field u outside of $[0,T]$ by setting $u(t,\cdot) \equiv 0$ for $t < 0$ and $t > T$, yielding $u \in L^\infty(\mathbb{R}; \overrightarrow{L^\infty(M)})$.

Let $\{\mathcal{E}_\tau\}_{\tau\geq 0}$ denote the de Rham-Hodge semigroup on 1-forms, associated to the de Rham-Hodge Laplacian on (M,h). We refer to Section 10 for a collection of properties of the heat kernel on forms.

For a.e. $t \in \mathbb{R}$ and all $\tau > 0$, $\mathcal{E}_\tau u(t)$ is a smooth vector field on M and

$$\|\mathcal{E}_\tau u(t)\|_{\overrightarrow{L^\infty(M)}} \leq \mathrm{e}^{\varepsilon^2\tau}\|u(t)\|_{\overrightarrow{L^\infty(M)}},$$

where ε is a constant such that $\mathrm{Ric}_M \geq -\varepsilon^2 h$. By assumption, we clearly have $u(t) \in \overrightarrow{L^r(M)}$ for a.e. t and thus

$$\mathcal{E}_\tau u(t) \xrightarrow{\tau\downarrow 0} u(t) \quad \text{in } \overrightarrow{L^r(M)}, \quad r \in [1,\infty),$$

where the null-set is r-independent.

Let η be a standard mollifier on $\mathbb{R}$, and set

$$\eta_\tau(t) := \tau^{-1}\eta\Big(\frac{t}{\tau}\Big), \quad t \in \mathbb{R}.$$

We now define the following vector field:

$$u_\tau(t,x) := \int_{\mathbb{R}} \mathcal{E}_\tau u(t',x)\eta_\tau(t-t')\,\mathrm{d}t' \in T_xM,$$

which is well-defined because

$$\begin{aligned}
|u_\tau(t,x)|_h &\leq \int_{\mathbb{R}} |\mathcal{E}_\tau u(t',x)|_h \eta_\tau(t-t')\,\mathrm{d}t' \\
&\leq \int_{\mathbb{R}} \|\mathcal{E}_\tau u(t)\|_{\overrightarrow{L^\infty(M)}}\eta_\tau(t-t')\,\mathrm{d}t' \\
&\leq \mathrm{e}^{\varepsilon^2\tau}\int_{\mathbb{R}} \|u(t')\|_{\overrightarrow{L^\infty(M)}}\eta_\tau(t-t')\,\mathrm{d}t' < \infty
\end{aligned}$$

for any $t \in \mathbb{R}$ and $x \in M$. Clearly, $u_\tau : \mathbb{R}\times M \to TM$ is smooth in both variables,

$$\|u_\tau\|_{L^\infty(\mathbb{R};\overrightarrow{L^\infty(M)})} \leq \mathrm{e}^{\varepsilon^2\tau}\|u\|_{L^\infty(\mathbb{R};\overrightarrow{L^\infty(M)})} \tag{9.1}$$

and $\operatorname{supp} u_\tau \subset [-1, T+1] \times M$ for all $\tau \ll 1$.

For a.e. $t \in \mathbb{R}$,

$$\operatorname{div}_h(\mathcal{E}_\tau u(t,x)) = P_\tau \operatorname{div}_h u(t,x), \quad x \in M, \tag{9.2}$$

where P_τ is the heat kernel on functions. Indeed, fixing $\phi \in C^1(M)$, we compute (see [17, eq. 4.3])

$$\begin{aligned}
\int_M \operatorname{div}_h(\mathcal{E}_\tau u(t,x))\phi \, \mathrm{d}V_h &= -\int_M (\mathcal{E}_\tau u(t,x), \nabla\phi)_h \, \mathrm{d}V_h \\
&= -\int_M (u(t,x), \mathcal{E}_\tau \nabla\phi)_h \, \mathrm{d}V_h = -\int_M (u(t,x), \nabla P_\tau \phi)_h \, \mathrm{d}V_h \\
&= \int_M \operatorname{div}_h u(t,x) P_\tau \phi \, \mathrm{d}V_h = \int_M P_\tau \operatorname{div}_h u(t,x)\phi \, \mathrm{d}V_h,
\end{aligned}$$

where we have used the relation [17],

$$\mathcal{E}_\tau \nabla\phi = \nabla P_\tau \phi, \quad \phi \in C^1(M),$$

and so the identity (9.2) follows.

The next lemma expresses $\operatorname{div}_h u_\tau$ in terms of $\operatorname{div}_h u$.

Lemma 9.1 (formula for $\operatorname{div}_h u_\tau$) *For any $t \in \mathbb{R}$ and $x \in M$,*

$$\operatorname{div}_h u_\tau(t,x) := \int_{\mathbb{R}} \operatorname{div}_h(\mathcal{E}_\tau u(t',x))\eta_\tau(t-t') \, \mathrm{d}t',$$

where $\operatorname{div}_h(\mathcal{E}_\tau u(t',x))$ can be computed in terms of $\operatorname{div}_h u$ and the heat kernel on functions, see (9.2).

Proof Locally expressing $\mathcal{E}_\tau u(t,x)$ as

$$\mathcal{E}_\tau u(t,x) = \Big(\int_M e(\tau,x,y)_{ij} u^j(t,y) \, \mathrm{d}V_h(y) \Big) h^{ik}(x) \, \partial_k$$

for a.e. $t \in \mathbb{R}$, see Section 10, we obtain (temporarily dropping Einstein's summation convention for k)

$$\begin{aligned}
\partial_k(\mathcal{E}_\tau u(t,x))^k &= \int_M \partial_k e(\tau,x,y)_{ij} u^j(t,y) \, \mathrm{d}V_h(y) \, h^{ik}(x) \\
&\quad + \int_M e(\tau,x,y)_{ij} u^j(t,y) \, \mathrm{d}V_h(y) \, \partial_k h^{ik}(x)
\end{aligned}$$

and thus

$$|\partial_k(\mathcal{E}_\tau u(t,x))^k| \le C(M,\tau) \|u\|_{\overrightarrow{L^\infty(M)}}.$$

Therefore we are allowed to interchange $\int_{\mathbb{R}}$ and ∂_k to obtain

$$\partial_k(u_\tau(t,x))^k = \int_{\mathbb{R}} \partial_k(\mathcal{E}_\tau u(t',x))^k \eta_\tau(t-t') \, \mathrm{d}t'.$$

From here, recalling the local expression for div_h (see Section 2), it is now immediate to conclude that locally

$$\operatorname{div}_h u_\tau(t,x) = \int_{\mathbb{R}} \operatorname{div}_h(\mathcal{E}_\tau u(t',x))\eta_\tau(t-t')\,\mathrm{d}t'.$$

Fix $x \in M$. In view of Lemma 9.1 and basic convolution estimates on $\mathbb{R}$, $\|\operatorname{div}_h u_\tau(\cdot,x)\|_{L^p(\mathbb{R})} \leq \|\operatorname{div}_h \mathcal{E}_\tau u(\cdot,x)\|_{L^p(\mathbb{R})}$ for any $\tau > 0$, and thus

$$\|\operatorname{div}_h u_\tau\|_{L^p(\mathbb{R}\times M)} \leq \|\operatorname{div}_h \mathcal{E}_\tau u\|_{L^p(\mathbb{R}\times M)}.$$

As a result, via (9.2), we obtain

$$\begin{aligned}\|\operatorname{div}_h u_\tau\|_{L^p(\mathbb{R}\times M)} &\leq \|P_\tau \operatorname{div}_h u\|_{L^p(\mathbb{R}\times M)}\\ &\leq \|\operatorname{div}_h u\|_{L^p(\mathbb{R}\times M)} = \|\operatorname{div}_h u\|_{L^p([0,T]\times M)}.\end{aligned} \tag{9.3}$$

9.2 Weak compactness of approximate solutions

Let ρ^τ be the unique weak L^2 solution of the SPDE (1.2) with initial datum $\rho_0 \in C^\infty(M)$, noise vector fields a_i given by Lemma 1.1, and irregular velocity field u (satisfying the assumptions of Theorem 1.2) replaced by the smooth vector field u_τ.

We refer to Propositions 3.1 and 8.1 for the existence, uniqueness, and properties of the solution, which satisfies the Itô SPDE

$$\mathrm{d}\rho^\tau + \operatorname{div}_h(\rho^\tau u_\tau)\,\mathrm{d}t + \sum_{i=1}^N \operatorname{div}_h(\rho^\tau a_i)\,\mathrm{d}W^i(t) - \Delta_h \rho^\tau\,\mathrm{d}t - \frac{1}{2}\sum_{i=1}^N \operatorname{div}_h(\rho^\tau \overline{a}_i)\,\mathrm{d}t = 0$$

weakly in x, $\mathbb{P}$-a.s.,

that is, for any $\psi \in C^\infty(M)$, the following equation holds $\mathbb{P}$-a.s.:

$$\begin{aligned}\int_M \rho^\tau(t)\psi\,\mathrm{d}V_h &= \int_M \rho_0\psi\,\mathrm{d}V_h + \int_0^t\int_M \rho^\tau(s)\,u_\tau(\psi)\,\mathrm{d}V_h\,\mathrm{d}s\\ &\quad + \sum_{i=1}^N \int_0^t\int_M \rho^\tau(s)\,a_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}W^i(s) + \int_0^t\int_M \rho^\tau(s)\,\Delta_h\psi\,\mathrm{d}V_h\,\mathrm{d}s\\ &\quad - \frac{1}{2}\sum_{i=1}^N \int_0^t\int_M \rho^\tau(s)\,\overline{a}_i(\psi)\,\mathrm{d}V_h\,\mathrm{d}s, \quad t \in [0,T].\end{aligned} \tag{9.4}$$

In view of (9.1), (9.3) and (8.1), recalling the "monotonicity properties" of the constant C, we obtain the τ-independent L^2 estimate

$$\sup_{0\leq t\leq T} \|\rho^\tau(t)\|_{L^2(\Omega\times M)} \leq C(T, a_i, \|\operatorname{div}_h u\|_{L^p([0,T]\times M)}, \|u\|_\infty)\|\rho_0\|_{L^2(M)}.$$

In other words, $\{\rho^\tau\}_{\tau\in(0,1)}$ is bounded in $L^\infty([0,T];L^2(\Omega\times M))$.

Since $(L^2(\Omega\times M))^\star$ is separable and $([0,T],\mathrm{d}t)$ is a finite measure space, we know that $L^\infty([0,T];L^2(\Omega\times M))$ is the dual of $L^1([0,T];L^2(\Omega\times M))$. Therefore, there exist $\{\tau_n\}_{n\geq 1} \subset (0,1)$ with $\tau_n \downarrow 0$ and $\rho \in L^\infty([0,T];L^2(\Omega\times M))$ such that

$$\rho^{\tau_n} \overset{\star}{\rightharpoonup} \rho \quad \text{in } L^\infty([0,T];L^2(\Omega\times M))$$

as $n \to \infty$, which means that

$$\int_0^T \int_\Omega \int_M (\rho^{\tau_n} - \rho)\, \theta \; \mathbb{P} \otimes \mathrm{d}V_h \otimes \mathrm{d}t \overset{n\uparrow\infty}{\longrightarrow} 0, \quad \forall \theta \in L^1([0,T]; L^2(\Omega \times M)).$$

We follow the arguments in [37]. Fix $\phi \in C^\infty(M)$. The process $\int_M \rho^{\tau_n}(t)\phi \,\mathrm{d}V_h$ is adapted by definition and converges weakly in $L^2(\Omega_T)$ to the process $\int_M \rho(t)\phi\,\mathrm{d}V_h$. Since the space of adapted processes is a closed subspace of $L^2(\Omega_T)$, it is weakly closed, and hence the limit process is adapted.

For the same reason, the processes $\int_M \rho^{\tau_n}(t) a_i(\phi)\,\mathrm{d}V_h$, $i = 1, \cdots, N$ are adapted and their Itô integrals are well defined. Since the Itô integral is linear and continuous from the space of adapted $L^2(\Omega_T)$ processes to $L^2(\Omega_T)$, it is also weakly continuous. As a result,

$$\int_0^\cdot \int_M \rho^{\tau_n}(s) a_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}W_s^i \overset{n\uparrow\infty}{\rightharpoonup} \int_0^\cdot \int_M \rho(s) a_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}W_s^i \quad \text{in } L^2(\Omega_T).$$

Exploiting the weak continuity of the time-integrals,

$$\int_0^\cdot \int_M \rho^{\tau_n}(s) \Delta_h \phi\,\mathrm{d}V_h\,\mathrm{d}s \overset{n\uparrow\infty}{\rightharpoonup} \int_0^\cdot \int_M \rho(s) \Delta_h \phi\,\mathrm{d}V_h\,\mathrm{d}s \quad \text{in } L^2(\Omega_T)$$

and, for $i = 1, \cdots, N$,

$$\int_0^\cdot \int_M \rho^{\tau_n}(s) \overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s \overset{n\uparrow\infty}{\rightharpoonup} \int_0^\cdot \int_M \rho(s) \overline{a}_i(\phi)\,\mathrm{d}V_h\,\mathrm{d}s \quad \text{in } L^2(\Omega_T).$$

It remains to pass to the limit in the term involving the velocity field u_τ in (9.4). The proof of the next lemma is postponed to the end of this section.

Lemma 9.2 *For any $r \in [1, \infty)$, $u_\tau \to u$ in $L^r(\mathbb{R}; \overrightarrow{L^r(M)})$ as $\tau \downarrow 0$.*

Lemma 9.2 immediately implies

$$u_{\tau_n}(\phi) \overset{n\uparrow\infty}{\longrightarrow} u(\phi) \quad \text{in } L^r(\mathbb{R} \times M), \quad r \in [1, \infty).$$

Using this, the goal is to verify that

$$\int_M \rho^{\tau_n} u_{\tau_n}(\phi)\,\mathrm{d}V_h \overset{n\uparrow\infty}{\rightharpoonup} \int_M \rho u(\phi)\,\mathrm{d}V_h \quad \text{in } L^2(\Omega_T). \tag{9.5}$$

Fix an arbitrary $\psi \in L^2(\Omega_T)$. Then

$$\begin{aligned} I(n) &:= \int_{\Omega_T} \Big(\int_M \rho^{\tau_n} u_{\tau_n}(\phi)\,\mathrm{d}V_h \Big) \psi \,\mathbb{P} \otimes \mathrm{d}s - \int_{\Omega_T} \Big(\int_M \rho u(\phi)\,\mathrm{d}V_h \Big) \psi \,\mathbb{P} \otimes \mathrm{d}s \\ &= \int_{\Omega_T} \Big(\int_M \rho^{\tau_n} (u_{\tau_n}(\phi) - u(\phi))\,\mathrm{d}V_h \Big) \psi \,\mathbb{P} \otimes \mathrm{d}s \\ &\quad + \int_{\Omega_T} \Big(\int_M (\rho^{\tau_n} - \rho) u(\phi)\,\mathrm{d}V_h \Big) \psi \,\mathbb{P} \otimes \mathrm{d}s =: I_1(n) + I_2(n). \end{aligned}$$

By repeated applications of the Cauchy-Schwarz inequality,

$$|I_1(n)| \le \int_{\Omega_T} |\psi| \|\rho^{\tau_n}(s)\|_{L^2(M)} \|(u_{\tau_n} - u)(\phi)\|_{L^2(M)} \,\mathbb{P} \otimes \mathrm{d}s$$

$$
\begin{aligned}
&\leq \int_0^T \|\psi(s)\|_{L^2(\Omega)} \|\rho^{\tau_n}(s)\|_{L^2(\Omega\times M)} \|(u_{\tau_n} - u)(\phi)\|_{L^2(M)} \,\mathrm{d}s \\
&\leq \|\rho^{\tau_n}\|_{L^\infty([0,T];L^2(\Omega\times M))} \int_0^T \|\psi(s)\|_{L^2(\Omega)} \|(u_{\tau_n} - u)(\phi)\|_{L^2(M)} \,\mathrm{d}s \\
&\leq C\|\psi\|_{L^2(\Omega_T)} \|(u_{\tau_n} - u)(\phi)\|_{L^2([0,T]\times M)} \overset{n\uparrow\infty}{\longrightarrow} 0.
\end{aligned}
$$

For the I_2 term it is enough to check that $u(\phi)\psi \in L^1([0,T]; L^2(\Omega \times M))$, because in that case we would get $I_2(n) \to 0$ directly from the definition of the weak convergence $\rho^\tau \overset{n\uparrow\infty}{\rightharpoonup} \rho$. In point of fact, we have

$$
\begin{aligned}
&\int_0^T \Big(\int_{\Omega\times M} |u(\phi)\psi|^2 \,\mathrm{d}V_h \,\mathrm{d}\mathbb{P} \Big)^{\frac{1}{2}} \,\mathrm{d}s \\
&= \int_0^T \|\psi(s)\|_{L^2(\Omega)} \Big(\int_M |u(\phi)|^2 \,\mathrm{d}V_h \Big)^{\frac{1}{2}} \,\mathrm{d}s \\
&\leq \int_0^T \|\psi(s)\|_{L^2(\Omega)} \|\phi\|_{C^1(M)} \|u(s)\|_{\overrightarrow{L^\infty(M)}} \,\mathrm{d}s \\
&\leq \|\phi\|_{C^1(M)} \|u\|_{L^\infty([0,T];\overrightarrow{L^\infty(M)})} \int_0^T \|\psi(s)\|_{L^2(\Omega)} \,\mathrm{d}s \\
&\leq \|\phi\|_{C^1(M)} \|u\|_{L^\infty([0,T];\overrightarrow{L^\infty(M)})} \sqrt{T} \|\psi\|_{L^2(\Omega_T)} \,\mathrm{d}s < \infty.
\end{aligned}
$$

Therefore $I_2(n) \overset{n\uparrow\infty}{\longrightarrow} 0$, and thus $I(n) \overset{n\uparrow\infty}{\longrightarrow} 0$. This concludes the proof of (9.5).

We may now pass to the limit in the SPDE (9.4) with $\tau = \tau_n$, to conclude that ρ satisfies (1.4) for a.e. $(\omega, t) \in \Omega_T$. Since the right-hand-side of (1.4) clearly defines a continuous stochastic process, the process $\int_M \rho(\cdot, x)\phi(x) \,\mathrm{d}V_h(x)$ has a continuous modification. In other words, we have constructed a weak L^2 solution to (1.2) under the assumption that $\rho_0 \in C^\infty(M)$.

9.3 General initial datum, $\rho_0 \in L^2(M)$

To finish off the proof, we must remove the smoothness assumption on the initial datum ρ_0. We follow the same strategy as above, but this time it is simpler since we have to regularize functions (not vector fields) defined on the manifold M.

Given $\rho_0 \in L^2(M)$, we employ the heat semigroup $\{P_\tau\}_{\tau>0}$ on functions to regularize ρ_0, see Section 10 for details. The following properties are known:

$$
\begin{aligned}
&P_\tau \rho_0 \in C^\infty(M), \quad \|P_\tau \rho_0\|_{L^2(M)} \leq \|\rho_0\|_{L^2(M)}, \\
&\text{and} \quad P_\tau \rho_0 \overset{L^2(M)}{\longrightarrow} \rho_0 \quad \text{as } \tau \downarrow 0.
\end{aligned}
$$

According to the previous subsection, there exists a unique weak L^2 solution ρ^τ of (1.2) with initial datum $P_\tau \rho_0 \in C^\infty(M)$, irregular velocity field u satisfying the assumptions listed in Theorem 1.2, and noise vector fields a_i given by Lemma 1.1. As before, Proposition 8.1 supplies the estimate $\|\rho^\tau(t)\|_{L^2(\Omega\times M)} \leq C_T \|\rho_0\|_{L^2(M)}$ for all $t \in [0,T]$, where the constant C_T is independent of τ. This implies that u^τ is weakly compact, i.e., there exists a subsequence $\{\tau_n\}_{n\geq 1} \subset (0,1)$ with $\tau_n \overset{n\uparrow\infty}{\longrightarrow} 0$ and a limit $\rho \in L^\infty([0,T]; L^2(\Omega \times M))$ such that

$$
\rho^{\tau_n} \overset{\star}{\rightharpoonup} \rho \quad \text{in } L^\infty([0,T]; L^2(\Omega \times M)).
$$

For any $\phi \in C^\infty(M)$, we have trivially that

$$\int_M P_{\tau_n}\rho_0\,\phi\,\mathrm{d}V_h \overset{n\uparrow\infty}{\rightharpoonup} \int_M \rho_0\,\phi\,\mathrm{d}V_h \quad \text{in } L^2(\Omega_T).$$

The limit of the remaining terms in (9.4) can be computed as before, which in the end leads to the conclusion that ρ is a weak L^2 solution of (1.2).

9.4 Proof of Lemma 9.2

To conclude the proof of Theorem 1.2, we need to verify the validity of Lemma 9.2. Define for convenience

$$\mathcal{J}_\tau(t,x) := \int_{\mathbb{R}} u(t',x)\eta_\tau(t-t')\,\mathrm{d}t', \quad t\in\mathbb{R},\ x\in M.$$

We have

$$|u_\tau(t,x)-\mathcal{J}_\tau(t,x)|_h \le \int_{\mathbb{R}} |\mathcal{E}_\tau u(t',x)-u(t',x)|_h\eta_\tau(t-t')\,\mathrm{d}t'.$$

By basic convolution estimates on $\mathbb{R}$, for any $r\in[1,\infty)$,

$$\|\,|u_\tau(\cdot,x)-\mathcal{J}_\tau(\cdot,x)|_h\,\|_{L^r(\mathbb{R})} \le \|\,|\mathcal{E}_\tau u(\cdot,x)-u(\cdot,x)|_h\,\|_{L^r(\mathbb{R})}, \quad x\in M,$$

where $\|\,|\cdot|_h\,\|^r_{L^r(\mathbb{R})} = \int_{\mathbb{R}} |\cdot|_h^r\,\mathrm{d}t$. Thus,

$$\begin{aligned}&\|u_\tau-\mathcal{J}_\tau\|_{L^r(\mathbb{R};\overrightarrow{L^r(M)})}\\ &\le \|\mathcal{E}_\tau u-u\|_{L^r(\mathbb{R};\overrightarrow{L^r(M)})} = \Big(\int_{\mathbb{R}} \|\mathcal{E}_\tau u(t,\cdot)-u(t,\cdot)\|^r_{\overrightarrow{L^r(M)}}\,\mathrm{d}t\Big)^{\frac{1}{r}}.\end{aligned}$$

Observe that the integrand in the $\mathrm{d}t$-integral converges to zero as $\tau\downarrow 0$ for a.e. $t\in\mathbb{R}$. Furthermore, see Section 10,

$$\|\mathcal{E}_\tau u(t,\cdot)-u(t,\cdot)\|_{\overrightarrow{L^r(M)}} \le \Big(\exp\Big(\varepsilon^2\Big|1-\frac{2}{r}\Big|\tau\Big)+1\Big)\|u(t,\cdot)\|_{\overrightarrow{L^r(M)}},$$

which is integrable on $\mathbb{R}$ by assumption on u (here $-\varepsilon$ is a lower bound of the Ricci tensor on M). Therefore, by means of the dominated convergence theorem, we conclude that $u_\tau-\mathcal{J}_\tau\to 0$ in $L^r(\mathbb{R};\overrightarrow{L^r(M)})$ as $\tau\downarrow 0$.

Hence, with an error term $o(1)\to 0$ as $\tau\downarrow 0$,

$$u_\tau - u = \mathcal{J}_\tau - u + o(1),$$

so it remains to verify that $\mathcal{J}_\tau - u$ converges to zero in $L^r_t\overrightarrow{L^r_x}$. Locally we have $|\mathcal{J}_\tau(t,x)-u(t,x)|_h \le C(M,h)|\mathcal{J}_\tau(t,x)-u(t,x)|_{\mathrm{eucl}}$. Since the right-hand side converges to zero in $L^r(\mathbb{R})$ for all $x\in M$, it follows that the same holds for the left-hand side. We have

$$\begin{aligned}\int_{\mathbb{R}}\int_M |\mathcal{J}_\tau(t,x)-u(t,x)|_h^r\mathrm{d}V_h(x)\,\mathrm{d}t &= \int_M\int_{\mathbb{R}} |\mathcal{J}_\tau(t,x)-u(t,x)|_h^r\,\mathrm{d}t\,\mathrm{d}V_h(x)\\ &= \sum_\kappa \int_M \alpha_\kappa(z)\Big(\int_{\mathbb{R}} |\mathcal{J}_\tau(t,z)-u(t,z)|_h^r\,\mathrm{d}t\Big)\,|h_\kappa(z)|^{\frac{1}{2}}\,\mathrm{d}z,\end{aligned}$$

where $(\alpha_\kappa)_\kappa$ is an arbitrary smooth partition of unity. Arguing as we did above,

$$\|\,|\mathcal{J}_\tau(\cdot,x)-u(\cdot,x)|_h\,\|^r_{L^r(\mathbb{R})} \le 2^r\|\,|u(\cdot,x)|_h\,\|^r_{L^r(\mathbb{R})}$$

for any $x\in M$, and hence, by means of the dominated convergence theorem, $\mathcal{J}_\tau - u\to 0$ in $L^r(\mathbb{R};\overrightarrow{L^r(M)})$. This concludes the proof of Lemma 9.1.

10 Appendix

10.1 Heat kernel on functions

We collect here some relevant properties of the heat kernel H on (M,h), that is, the fundamental solution of the heat operator

$$L = \partial_t - \Delta_h.$$

(1) The mapping $(x,y,t) \mapsto H(x,y,t)$ belongs to $C^\infty(M \times M \times (0,\infty))$, is symmetric in x and y for any $t>0$ and is positive.

(2) For any function $w \in L^r(M)$, $r \in [1,\infty]$, setting

$$P_t w(x) := \int_M H(x,y,t) w(y) \, \mathrm{d}V_h(y), \quad x \in M, \ t > 0, \tag{10.1}$$

we have $P_t w \in C^\infty(M)$. Moreover,

$$\|P_t w\|_{L^r(M)} \leq \|w\|_{L^r(M)}, \quad t > 0,$$

and, for any finite $r \geq 1$,

$$P_t w \xrightarrow{L^r(M)} w \quad \text{as } t \to 0^+.$$

For proofs of these basic results, see [26].

10.2 Heat kernel on forms

During the proof of Theorem 1.2, we also make use of the heat kernel on forms. We recall here its most salient properties without proofs, referring to [5, 12, 16, 21] for details. Firstly, we define the space $L^2(M,h)$ as the closure of the space of smooth 1-forms on M with respect to the norm

$$\Big(\int_M |\zeta|_h^2 \, \mathrm{d}V_h \Big)^{\frac{1}{2}}, \quad \text{where } |\zeta|_h^2 = h^{ab} \zeta_a \zeta_b \text{ locally.}$$

Denote by $\{\mathcal{E}_\tau\}_{\tau \geq 0}$ the de Rham-Hodge semigroup on 1-forms, associated to the de Rham-Hodge Laplacian, which by elliptic regularity has a kernel $e(\tau,\cdot,\cdot)$. More precisely, for any $\tau > 0$, $e(\tau,\cdot,\cdot)$ is a double form on $M \times M$, such that for any 1-form $\zeta \in L^2(M,h)$ and any $P \in M$,

$$(\mathcal{E}_\tau \zeta)(P) = \int_M e(\tau, P, Q) \wedge \star_Q \, \zeta(Q),$$

where $\star$ is the Hodge star operator, Q is a point in M, and $\wedge$ is the wedge product between forms. Concretely, in a coordinate patch $(U,(x^i))$ around P and in a coordinate patch $(U',(y^j))$ around Q, if we write the double form $e(\tau,\cdot,\cdot)$ as

$$e(\tau,x,y) = (e(\tau,x,y)_{ij} \, \mathrm{d}x^i) \, \mathrm{d}y^j$$

and ζ as $\zeta(y) = \zeta_k(y) \, \mathrm{d}y^k$, then the above integral becomes

$$(\mathcal{E}_\tau \zeta)(x) = \Big(\int_M e(\tau,x,y)_{ij} \, h^{jk}(y) \, \zeta_k(y) \, \mathrm{d}V_h(y) \Big) \, \mathrm{d}x^i.$$

For a vector field V, we denote by $V^\flat$ the 1-form obtained by lowering an index via the metric h; analogously, for a 1-form ζ, we denote by $\zeta^\sharp$ the vector field obtained by raising an index via the metric.

We define for a vector field V the following quantity

$$\mathcal{E}_\tau V := ((\mathcal{E}_\tau V^\flat))^\sharp.$$

Let $\varepsilon \geq 0$ be a constant such that $\mathrm{Ric}_M \geq -\varepsilon^2 h$, where Ric_M denotes the Ricci tensor of (M,h) (the constant ε clearly exists because M is compact). We have the following remarkable properties: For any $V \in \overrightarrow{L^p(M)}$, $p \in [1,\infty]$,

- $\mathcal{E}_\tau V$ is a smooth vector field for any $\tau > 0$,
- $\mathcal{E}_\tau V \to V$ in $\overrightarrow{L^p(M)}$ as $\tau \downarrow 0$ for any finite p,
- $\|\mathcal{E}_\tau V\|_{\overrightarrow{L^p(M)}} \leq \mathrm{e}^{\varepsilon^2|1-\frac{2}{p}|\tau} \|V\|_{\overrightarrow{L^p(M)}}$ for any $\tau \geq 0$ (see [5]).

Furthermore, in analogy with (10.1), the following local expression holds:

$$(\mathcal{E}_\tau V)(x) = \Big(\int_M e(\tau,x,y)_{ij}\, V^j(y)\, \mathrm{d}V_h(y)\Big) h^{ik}(x)\, \partial_k.$$

Finally, one can show that (see [21] for details)

$$\begin{aligned}
\mathrm{div}\ \mathcal{E}_\tau V(x) = &\int_M \partial_k e(\tau,x,y)_{ij}\, V^j(y)\, \mathrm{d}V_h(y)\, h^{ik}(x) \\
&+ \int_M e(\tau,x,y)_{ij}\, V^j(y)\, \mathrm{d}V_h(y)\, \partial_k h^{ik}(x) \\
&+ \Gamma^\rho_{\rho k}(x) \int_M e(\tau,x,y)_{ij}\, V^j(y)\, \mathrm{d}V_h(y)\, h^{ik}(x)
\end{aligned}$$

in local coordinates x (differentiation is carried out in x).

10.3 Proof of Proposition 2.1

Let $\{G_i\}_{i=1}^R$ be a finite covering of M and $\{(G_i,\phi_i)\}_{i=1}^R$ be the corresponding charts. Without loss of generality, we may assume that $\phi_i(G_i) = B$ for all i, where B is the unit ball in $\mathbb{R}^d$. Let $\{\alpha_i\}_{i=1}^R$ be a smooth partition of unity subordinate to $\{G_i\}_{i=1}^R$. On $\operatorname{supp}\alpha_i$ the metric tensor h and its derivatives of all orders are bounded in the system of coordinates corresponding to the chart (G_i,ϕ_i). Define, for $i = 1,\cdots,R$,

$$\psi_i : [0,T]\times G_i \to [0,T]\times B, \quad (t,P) \mapsto (t,\phi_i(P)),$$

which is a finite smooth atlas for $[0,T]\times M$. Observe that $[0,T]\times G_i$ is diffeomorphic to $[0,T]\times B$. Moreover, $\widetilde{\alpha}_i : [0,T]\times M \to [0,1]$, $\widetilde{\alpha}_i(t,P) := \alpha_i(P)$ is a smooth partition of unity subordinate to $\{[0,T]\times G_i\}_{i=1}^R$.

Let $w \in W^{1,2,p}([0,T]\times M)$. Then clearly, in view of the discussion above,

$$\begin{aligned}
w \in W^{1,2,p}([0,T]\times M) &\iff \widetilde{\alpha}_i w \in W^{1,2,p}([0,T]\times M), \quad \forall i \\
&\iff (\widetilde{\alpha}_i w)\circ \psi_i^{-1} \in W^{1,2,p}([0,T]\times B), \quad \forall i,
\end{aligned}$$

where $W^{1,2,p}([0,T]\times B)$ denotes the more familiar Euclidean anisotropic Sobolev space (see [8]), which can be defined similarly via (2.1) with $\mathrm{d}V_h = \mathrm{d}x$ and $\nabla^k = \nabla^k_{\mathrm{eucl}}$.

For this space we have the compact embedding

$$W^{1,2,p}([0,T]\times B)\subset\subset C^{0,1-\frac{1+d}{p}}([0,T]\times\overline{B})$$

and

$$\partial_{x_j}(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\in C^{0,1-\frac{1+d}{p}}([0,T]\times\overline{B}),\quad j=1,\cdots,d,$$

provided $p>d+2$, see [41] for example. In particular, for all i,

$$\begin{aligned}&\|(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\|_{C^0([0,T]\times\overline{B})}+\|\nabla_{\mathrm{eucl}}(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\|_{C^0([0,T]\times\overline{B})}\\ &\le C(p,d,B)\|(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\|_{W^{1,2,p}([0,T]\times B)}.\end{aligned}$$

Exploiting the boundedness of the metric tensor, we get

$$\begin{aligned}&\|\widetilde{\alpha}_i w\|_{C^0([0,T]\times M)}+\|\nabla(\widetilde{\alpha}_i w)\|_{C^0([0,T]\times M)}\\ &=\|\widetilde{\alpha}_i w\|_{C^0([0,T]\times G_i)}+\|\nabla(\widetilde{\alpha}_i w)\|_{C^0([0,T]\times G_i)}\\ &\le\|(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\|_{C^0([0,T]\times\overline{B})}+C_i\|\nabla_{\mathrm{eucl}}(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\|_{C^0([0,T]\times\overline{B})}\\ &\le C(p,d,B,i)\|(\widetilde{\alpha}_i w)\circ\psi_i^{-1}\|_{W^{1,2,p}([0,T]\times B)}\\ &\le C'(p,d,B,i)\|\widetilde{\alpha}_i w\|_{W^{1,2,p}([0,T]\times M)}.\end{aligned}$$

Therefore, by the triangle inequality and summing over i,

$$\begin{aligned}&\|w\|_{C^0([0,T]\times M)}+\|\nabla w\|_{C^0([0,T]\times M)}\\ &\le C(p,d,M)\sum_{i=1}^{R}\|\widetilde{\alpha}_i w\|_{W^{1,2,p}([0,T]\times M)}\le C'(p,d,M)\|w\|_{W^{1,2,p}([0,T]\times M)},\end{aligned}$$

where in the last passage we have used the fact that the derivatives of $\widetilde{\alpha}_i$ are bounded. The compactness of the embedding is now evident.

10.4 An auxiliary result

We now prove a useful result about the extension of smooth functions, which is used during the proof of Proposition 5.1.

Proposition 10.1 (extension of C^∞ functions) *Let $0<S<T$ and consider $w\in C^\infty([0,S]\times M)$. Then we can extend w to a function $v\in C^\infty([0,T]\times M)$.*

Proof Let $\{G_i\}_{i=1}^R$ be a finite covering of M and $\{(G_i,\phi_i)\}_{i=1}^R$ be the corresponding charts. Without loss of generality, we may assume that $\phi_i(G_i)=B$ for all i, where B is the unit ball in $\mathbb{R}^d$. Let $\{\alpha_i\}_{i=1}^R$ be a squared smooth partition of unity subordinate to $\{G_i\}_{i=1}^R$, such that $\sum_{i=1}^{R}\alpha_i^2=1$. Define, for $i=1,\cdots,R$,

$$\psi_i:[0,T]\times G_i\to[0,T]\times B,\quad (t,P)\mapsto(t,\phi_i(P)),$$

which is a finite smooth atlas for $[0,T]\times M$. Observe that $[0,T]\times G_i$ is diffeomorphic to $[0,T]\times B$. Besides, $\widetilde{\alpha}_i:[0,T]\times M\to[0,1]$, $\widetilde{\alpha}_i(t,P):=\alpha_i(P)$ is a squared smooth partition of unity subordinate to $\{[0,T]\times G_i\}_{i=1}^R$.

Given $w \in C^\infty([0,S] \times M)$, we define $\widetilde{w}_i \in C^\infty([0,S] \times \mathbb{R}^d)$, $i = 1, \cdots, R$ by

$$\widetilde{w}_i(t,x) := \begin{cases} (\widetilde{\alpha}_i w) \circ \psi_i^{-1} & \text{for } (t,x) \in [0,S] \times B, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that for any $t \in [0,S]$, $\operatorname{supp} \widetilde{w}_i(t,\cdot) \subset \operatorname{supp} \alpha_i \circ \phi_i^{-1}$. Seeley's extension theorem (see [44]) supplies an extension operator

$$\mathcal{E} : C^\infty([0,S] \times \mathbb{R}^d) \to C^\infty(\mathbb{R} \times \mathbb{R}^d).$$

Thanks to this, we can build an extension $\mathcal{E}\widetilde{w}_i$ of $\widetilde{w}_i$ in $C^\infty(\mathbb{R} \times \mathbb{R}^d)$. Set

$$\overline{w}_i := (\alpha_i \circ \phi_i^{-1}) \mathcal{E}\widetilde{w}_i \in C^\infty(\mathbb{R} \times \mathbb{R}^d),$$

and notice that for any $t \in \mathbb{R}$, $\operatorname{supp} \overline{w}_i(t,\cdot) \subset \operatorname{supp} \alpha_i \circ \phi_i^{-1}$.

We may lift this function to M by setting

$$w_i(t,P) := \begin{cases} \overline{w}_i(t,\phi(P)) & \text{for } (t,P) \in \mathbb{R} \times \operatorname{supp} \alpha_i, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $w_i \in C^\infty(\mathbb{R} \times M)$ and for $t \in [0,S]$ we have

$$w_i(t,P) = \begin{cases} \alpha_i^2(P) w(t,P) & \text{for } P \in \operatorname{supp} \alpha_i \\ 0, & \text{otherwise.} \end{cases}$$

Setting $v := \sum_{i=1}^{R} w_i \in C^\infty(\mathbb{R} \times M) \subset C^\infty([0,T] \times M)$ we have

$$v(t,P) = \sum_{i : P \in \operatorname{supp} \alpha_i} \alpha_i^2(P) w(t,P) = w(t,P), \quad (t,P) \in [0,S] \times M,$$

and thus the desired extension is established.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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