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Small Cycles Property of Some Cremer Rational Maps and Polynomials

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Abstract This paper concerns the linearization problem on rational maps of degree $d \geq 2$ and polynomials of degree $d > 2$ from the perspective of non-linearizability. The authors introduce a set $\mathscr{C}_\infty$ of irrational numbers and show that if $\alpha \in \mathscr{C}_\infty$, then any rational map is not linearizable and has infinitely many cycles in every neighborhood of the fixed point with multiplier $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$. Adding more constraints to cubic polynomials, they discuss the above problems by polynomial-like maps. For the family of polynomials, with the help of Yoccoz's method, they obtain its maximum dimension of the set in which the polynomials are non-linearizable.

Keywords Irrationally indifferent fixed point, Linearization problem, Small cycles property

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1 Introduction

In the study of complex dynamical system, it is very important to study the dynamics of a holomorphic map in some small neighborhood of a fixed point. So far, in most cases, it is well understood, but in some cases, there are still extremely difficult problems.

Choosing a suitable local uniformizing parameter, we can express a holomorphic map with a fixed point as holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ near the origin, where λ is the multiplier of the fixed point. For $\lambda = 0$, f is always locally conjugate to the power map $z \mapsto z^m$ for some $m \geq 2$, according to Böttcher Theorem (cf. [2]). In the case $|\lambda| \neq 0, 1$, f is always locally conjugate to $z \mapsto \lambda z$ by Kœnigs Linearization Theorem (cf. [15]). In the rationally indifferent case where $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ and $\alpha \in \mathbb{Q}$, the dynamics of f in some neighborhood of 0 can be understood due to Parabolic Flower Theorem and Parabolic Linearization Theorem (cf. [7, 13, 16, 20]). The remaining case is $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this cases, the origin is called an irrationally indifferent fixed point and the dynamics near the origin turns out to be much more complicated. The first question here is whether there exists a local conformal map h defined in some neighborhood of 0 satisfying $h(0) = 0$, $h'(0) = 1$ and $h^{-1} \circ f \circ h(z) = \lambda z$, which is called the linearization problem. The fixed point is either a Siegel point or a Cremer point, according to a linearization is possible or not (cf. [20]).

In this paper, we are going to concern the linearization problem on holomorphic maps near irrationally indifferent fixed point. From now on, we always assume $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ and the rotation

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number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

It is now well known that a linearization near irrationally indifferent fixed point is closely tied to number-theoretic properties of α. Let $\frac{p_n}{q_n}$ denote the n-th convergent to the continued fraction of α. The set of Brjuno numbers is defined as

$$\mathscr{B} := \Big\{\alpha \in \mathbb{R} \setminus \mathbb{Q} \;\Big|\; \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty\Big\}.$$

Based on earlier results about linearization problem by Cremer [4] and Siegel [31], Brjuno and Rüssmann proved the following great theorem.

Brjuno's Theorem (cf. [3, 30]) *If $\alpha \in \mathscr{B}$, then any holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ is linearizable.*

Yoccoz proved that Brjuno's condition is optimal as follows.

Yoccoz's Theorem (cf. [34]) *If $\alpha \notin \mathscr{B}$, then the quadratic map $P_\lambda(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha} z + z^2$ is not linearizable at the origin. Moreover, this fixed point has the small cycles property: Every neighborhood of the origin contains infinitely many periodic orbits.*

In 1986, Douady raised the following conjecture for rational maps.

Douady's Conjecture (cf. [5]) *Let f be a rational map of degree $d \geq 2$. Then f is linearizable at a fixed point with multiplier $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ if and only if $\alpha \in \mathscr{B}$.*

This conjecture is open even for cubic polynomials and quadratic rational maps, but it's very encouraging that the conjecture is true under some conditions. We give a partial list of the literature containing some important work in this direction: For polynomials (cf. [1, 8–10, 22, 25, 33–34]) and for rational maps (cf. [17, 27]).

The dynamics of f near a Siegel point is simply corresponding to the dynamics of the linear map $z \mapsto \lambda z$, but it is still complicated near a Cremer point. Yoccoz's theorem also raises the question as to whether every Cremer point has small cycles property. Pérez-Marco contributed many works for this problem (cf. [23–25]) and gave a negative answer.

Pérez-Marco introduced the set $\mathscr{B}'$ as follows

$$\mathscr{B}' := \Big\{\alpha \in \mathbb{R} \setminus \mathbb{Q} \;\Big|\; \sum_{n=0}^{\infty} \frac{\log\log q_{n+1}}{q_n} < +\infty\Big\}$$

and showed that if $\alpha \notin \mathscr{B}'$, there exists a holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ and a neighborhood of the origin such that every forward orbit contained in this neighborhood has the origin as accumulation point. Hence such a germ has no small cycles and is not linearizable. Meanwhile, he obtained the following theorem.

Pérez-Marco's Theorem (cf. [25]) *If $\alpha \in \mathscr{B}'$, then any holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ which is not linearizable near the origin has infinitely many cycles in every neighborhood of the origin.*

When $\alpha \notin \mathscr{B}'$, we still wonder whether every Cremer point of some special holomorphic germs, such as polynomials or rational maps, has small cycles property.

In the case of polynomials, let

$$\mathscr{P}_{\lambda,d} := \{P(z) = \lambda z + a_2 z^2 + \cdots + a_d z^d \mid (a_2, \cdots, a_d) \in \mathbb{C}^{d-1}, a_d \neq 0, \lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}\}.$$

Pérez-Marco (cf. [25]) proved that when $P \in \mathscr{P}_{\lambda,d}$ is J-stable in $\mathscr{P}_{\lambda,d}$, if $\alpha \notin \mathscr{B}$, then P is non-linearizable and has infinitely many cycles in every neighborhood of the origin. In 2005, Petracovici (cf. [28]) imposed restrictions on the coefficients of a holomorphic germ $f(z) = \lambda z + \mathcal{O}(z^2)$ and proved that if α satisfies the condition $\log\log q_{n+1} \geq 3q_n$, then f has infinitely many cycles in every neighborhood of the origin. After these, little attention has been paid to the small cycles property.

With adding more constraints to the rotation number, we have the following theorem.

Theorem 1.1 *Assume that the set $\mathscr{C}_\infty$ as*

$$\mathscr{C}_\infty := \Big\{\alpha \in \mathbb{R} \setminus \mathbb{Q} \;\Big|\; \limsup_{n\to\infty} \frac{\log\log q_{n+1}}{q_n} = +\infty\Big\}.$$

Let f be a rational map of degree $d \geq 2$. If $\alpha \in \mathscr{C}_\infty$ and f has a fixed point with multiplier $\lambda = \mathrm{e}^{2\pi \mathrm{i}\alpha}$, then f is non-linearizable and has infinitely many cycles in every neighborhood of this fixed point.

Here with adding more constraints to the polynomials, not to the rotation number, we have the following results.

Theorem 1.2 *If $\alpha \notin \mathscr{B}$ and a polynomial $P(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha}z + az^2 + z^3$ satisfies one of the following conditions*:

(a) *P has a non-repelling and non-zero periodic orbit which is not parabolic.*

(b) *P has disconnected Julia set.*

Then P is not linearizable near the origin and has infinitely many cycles in every neighborhood of the origin.

Even though the Douady Conjecture is not verified so far, it is interesting to find how big the set in which those maps are non-linearizable. Here we obtain its maximum dimension as follows.

Theorem 1.3 *When $\alpha \notin \mathscr{B}$ and $d > 2$, $\mathscr{P}_{\lambda,d}$ contains a holomorphic subfamily of complex dimension $d-1$ whose elements are non-linearizable and has infinitely many cycles in every neighborhood of the origin.*

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we use polynomial-like maps to prove Theorem 1.2. At the end of Section 3, Theorem 1.3 is a natural promotion of the proof of Theorem 1.2.

2 Case of Rational Maps

The dynamics of $f(z) = \lambda z + \mathcal{O}(z^2)$ near the origin always depends on the arithmetic nature of α. Here we give a brief summary about the continued fraction of α and deduce some useful results.

For $x \in \mathbb{R} \setminus \mathbb{Q}$, we denote by

(1) $[x]$ its enteger part, which is the largest integer $\leq x$;

(2) $\{x\} = x - [x]$ its fractional part.

Given $\alpha \in (0,1)$, define sequences $(a_n)_{n\geq 0}$, $(r_n)_{n\geq 0}$ as follows:

$$\begin{cases} r_0 = \alpha, \\ a_n = \Big[\dfrac{1}{r_{n-1}}\Big], \quad r_n = \Big\{\dfrac{1}{r_{n-1}}\Big\}, \quad n \geq 1. \end{cases}$$

Let $p_{-1}=1$ and $q_{-1}=0$, we define $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ as:

$$\begin{aligned} p_0 &= 0, \quad p_n = a_n p_{n-1} + p_{n-2}, \quad n \geq 1, \\ q_0 &= 1, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1. \end{aligned}$$

It is easy to see that

$$\alpha = \frac{p_n + p_{n-1} r_n}{q_n + q_{n-1} r_n} \tag{2.1}$$

and

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n. \tag{2.2}$$

It is known that the fraction $\frac{p_n}{q_n}$ defined by

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

is called the n-th convergent to the continued fraction of α.

We also have

$$\alpha = \lim_{n\to\infty} \frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \ddots}}}.$$

It follows immediately from (2.1) that we have the following lemma.

Lemma 2.1 (cf. [12]) *For $n > 1$, $q_{n+1} > q_n$.*

Lemma 2.2 *Suppose that $\alpha \in (0,1)$ and that $\frac{p_n}{q_n}$ is the n-th convergent to the continued fraction of α. Then*

$$\frac{1}{2|q_n\alpha - p_n|} < q_{n+1} < \frac{1}{|q_n\alpha - p_n|}.$$

Proof It follows from (2.1) that

$$|q_n\alpha - p_n| = \left| q_n \frac{p_{n+1} + p_n r_{n+1}}{q_{n+1} + q_n r_{n+1}} - p_n \right| = \left| \frac{q_n p_{n+1} - p_n q_{n+1}}{q_{n+1} + q_n r_{n+1}} \right|.$$

This, together with (2.2) shows that

$$|q_n\alpha - p_n| = \frac{1}{q_{n+1} + q_n r_{n+1}}.$$

Thus, we have $\frac{1}{2|q_n\alpha - p_n|} < q_{n+1} < \frac{1}{|q_n\alpha - p_n|}$.

Lemma 2.3 (cf. [12]) *Suppose that $\alpha \in (0,1)$ and that $\frac{p_n}{q_n}$ is the n-th convergent to the continued fraction of α. Then $\frac{p_n}{q_n}$ is closer to α than any fraction $\frac{p}{q}$ with denominator $0 < q < q_n$:*

$$|q\alpha - p| \geq |q_n\alpha - p_n|.$$

We have the following corollaries.

Corollary 2.1 *Suppose that $\alpha \in (0,1)$ and that $\frac{p_n}{q_n}$ is the n-th convergent to the continued fraction of α. Then*

$$|\lambda^{q_n} - 1| < |\lambda^q - 1|, \quad q = 1, 2, \cdots, q_n - 1.$$

Corollary 2.2 *Suppose that $\alpha \in (0,1)$ and that $\frac{p_n}{q_n}$ is the n-th convergent to the continued fraction of α. Then*

$$\frac{2}{q_{n+1}} < |\lambda^{q_n} - 1| < \frac{2\pi}{q_{n+1}}. \tag{2.3}$$

Proof Since $|\lambda^{q_n} - 1| = 2\sin(\pi|q_n\alpha - p_n|)$ and

$$2|q_n\alpha - p_n| < \sin(\pi|q_n\alpha - p_n|) < \pi|q_n\alpha - p_n|,$$

we have

$$4|q_n\alpha - p_n| < |\lambda^{q_n} - 1| < 2\pi|q_n\alpha - p_n|.$$

According to Lemma 2.2, we deduce that (2.3) holds.

With these preparations, we can start to prove Theorem 1.1.

Proof of Theorem 1.1 Since $\limsup\limits_{n\to\infty} \frac{\log\log q_{n+1}}{q_n} = +\infty$, for any $M > 0$, there is a large enough n satisfying that

$$\frac{\log\log q_{n+1}}{q_n} > M$$

or

$$\log q_{n+1} > \mathrm{e}^{Mq_n}.$$

It leads to

$$\frac{\log q_{n+1}}{d^{q_n}} > \frac{\mathrm{e}^{Mq_n}}{d^{q_n}}.$$

Hence

$$\Big(\frac{1}{q_{n+1}}\Big)^{\frac{1}{d^{q_n}}} < \mathrm{e}^{-\frac{\mathrm{e}^{Mq_n}}{d^{q_n}}}.$$

Since $(2\pi)^{\frac{1}{d^{q_n}}} > 0$, we have

$$\Big(\frac{2\pi}{q_{n+1}}\Big)^{\frac{1}{d^{q_n}}} < (2\pi)^{\frac{1}{d^{q_n}}}\mathrm{e}^{-\frac{\mathrm{e}^{Mq_n}}{d^{q_n}}}.$$

From (2.3) , it follows that

$$|\lambda^{q_n} - 1|^{\frac{1}{d^{q_n}}} < (2\pi)^{\frac{1}{d^{q_n}}}\mathrm{e}^{-\frac{\mathrm{e}^{Mq_n}}{d^{q_n}}}.$$

Noting that $2\pi < \mathrm{e}^2$, we have

$$|\lambda^{q_n} - 1|^{\frac{1}{d^{q_n}}} < \mathrm{e}^{\frac{2}{d^{q_n}}}\mathrm{e}^{-\frac{\mathrm{e}^{Mq_n}}{d^{q_n}}} = \mathrm{e}^{\frac{2-\mathrm{e}^{Mq_n}}{d^{q_n}}}.$$

It will be convenient to write q_n as q below, since q_n strictly increases to infinite as $n \to \infty$ by Lemma 2.1. Then the above results are still valid for a sufficiently large q.

The following part motivated by the proof of Cremer's Nonlinearizability Theorem (cf. [4, 20]). First consider a monic polynomial $f(z) = \lambda z + a_2z^2 + a_3z^3 + \cdots + z^d$ of degree $d \geq 2$

with a fixed point of multiplier λ at the origin. Then $f^q(z) = \lambda^q z + \cdots + z^{d^q}$. So the fixed points of $f^q(z)$ are the roots of the equation

$$(\lambda^q - 1)z + \cdots + z^{d^q} = 0.$$

Therefore the product of the $d^q - 1$ non-zero fixed points of $f^q(z)$ is equal to $\pm(\lambda^q - 1)$. Choosing q such that $|\lambda^q - 1| < 1$, it follows that at least one of these fixed point z_q satisfies

$$0 < |z_q|^{d^q} < |z_q|^{d^q - 1} \leq |\lambda^q - 1|.$$

For the previous result $|\lambda^q - 1|^{\frac{1}{d^q}} < \mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}}$, it follows that $|z_q| < \mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}}$.

Now, we can conjugate f so that $f(0) = 0$ and $f'(0) = \lambda$. Then we choose a non-zero z_1 satisfying $f(z_1) = 0$ and carrying z_1 to ∞ by a Möbius transformation, we may assume that $f(\infty) = f(0) = 0$. We set $f(z) = \frac{P(z)}{Q(z)}$. For $f(\infty) = 0$, Q is a polynomial of degree d and P is a polynomial of degree strictly less than d. For $f'(0) = \lambda$, we have $P'(0) = \lambda$. Since $f(0) = 0$, P does not contain a constant term. Therefore we may assume that P and Q have the form

$$P(z) = a_{d-1}z^{d-1} + \cdots + a_2 z^2 + \lambda z, \quad Q(z) = z^d + \cdots + 1, \tag{2.4}$$

where each a_i (for $i = 2, \cdots, d-1$) is a possible non-zero coefficient.

We set $f^q(z) = \frac{P_q(z)}{Q_q(z)}$, so Q_q has degree d^q. According to (2.4), P_q and Q_q have the form

$$\begin{aligned} P_q(z) &= \lambda^q z + \cdots + b_{d^q-1} z^{d^q-1}, \\ Q_q(z) &= z^{d^q} + \cdots + 1, \end{aligned}$$

where $b_2, \cdots, b_{d^q-1}$ are possible non-zero coefficients. Thus the equation for fixed points of $f^q(z)$ has the form

$$zQ_q(z) - P_q(z) = z(z^{d^q} + \cdots + (1 - \lambda^q)) = 0.$$

Then just as in the monic polynomial case we get that f^q has at least one of the non-zero fixed point z_q satisfies $|z_q| < |\lambda^q - 1|^{\frac{1}{d^q}} < \mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}}$.

Since $f(0) = 0$, $f'(0) = \lambda$, there exists $\delta_1 > 0$, when $|z| < \delta_1$, $f(z)$ is univalent. We set

$$F(z) = \delta_1^{-1} f(\delta_1 z).$$

Obviously, F satisfies $F(0) = 0$, $F'(0) = \lambda$ and univalent on $\mathbb{D}$. Koebe's Distortion Theorem (cf. [29]) implies that

$$|F(z)| \leq |F'(0)| \frac{|z|}{(1 - |z|)^2}.$$

Hence

$$|\delta_1^{-1} f(\delta_1 z)| \leq \frac{|z|}{(1 - |z|)^2}.$$

So we get the following upper bound

$$|f(z)| \leq \frac{|z|}{\left(1 - |\frac{z}{\delta_1}|\right)^2}.$$

For any $\varepsilon > 0$, set $\delta = \min\{\delta_1(1 - \mathrm{e}^{-\frac{\varepsilon}{2}}), \varepsilon\}$, when $|z| \leq \delta$, we have

$$|f(z)| \leq \frac{|z|}{\left(1 - |\frac{z}{\delta_1}|\right)^2} \leq |z|\mathrm{e}^{\varepsilon}.$$

Hence, if

$$|z_q| \le \mathrm{e}^{-q\varepsilon}\delta \tag{2.5}$$

holds, we can get

$$|f^k(z_q)| \le \mathrm{e}^{-(q-k)\varepsilon}\delta \le \delta \quad \text{for } k = 1, \cdots, q.$$

In order for (2.5) to hold it is sufficient to have

$$\mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}} < \mathrm{e}^{-q\varepsilon}\delta,$$

that is,

$$\mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}+q\varepsilon} < \delta.$$

We note that

$$\begin{aligned}
&\liminf_{q\to\infty} \frac{2-\mathrm{e}^{Mq}+q\varepsilon d^q}{d^q} \\
&= \liminf_{q\to\infty} \frac{-\mathrm{e}^{Mq}+q\varepsilon d^q}{d^q}.
\end{aligned}$$

A large enough M can be taken to make $d^2 < \frac{1}{2}\mathrm{e}^M$ hold. So there is a large enough q to make $d^2 < \frac{1}{2}\mathrm{e}^M$ and $d^q > d\varepsilon$ both be established. Then

$$\begin{aligned}
&\liminf_{q\to\infty} \frac{-\mathrm{e}^{Mq}+q\varepsilon d^q}{d^q} \\
&\le \liminf_{q\to\infty} \frac{-\mathrm{e}^{Mq}+d^q\cdot d^q}{d^q} \\
&= \liminf_{q\to\infty} \frac{-(\mathrm{e}^M)^q+(d^2)^q}{d^q} \\
&\le \liminf_{q\to\infty} \frac{-(\mathrm{e}^M)^q+\left(\frac{1}{2}\mathrm{e}^M\right)^q}{d^q} \\
&\le \liminf_{q\to\infty} \frac{-(\mathrm{e}^M)^q+\frac{1}{2}(\mathrm{e}^M)^q}{d^q} \\
&= \liminf_{q\to\infty} \frac{-\frac{1}{2}(\mathrm{e}^M)^q}{d^q} \\
&= \liminf_{q\to\infty} -\frac{1}{2}\Big(\frac{\mathrm{e}^M}{d}\Big)^q = -\infty.
\end{aligned}$$

Therefore there is large enough q so that $\mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}+q\varepsilon} < \delta$ can be established.

We conclude that if $\limsup\limits_{n\to\infty} \frac{\log\log q_{n+1}}{q_n} = +\infty$, for any $\varepsilon > 0$, there is a q-periodic point z_q satisfied $|z_q| < \mathrm{e}^{\frac{2-\mathrm{e}^{Mq}}{d^q}} < \mathrm{e}^{-q\varepsilon}\delta < \varepsilon$ and $|f^k(z_q)| < \varepsilon$ for $k = 1, \cdots, q$. The arbitrariness of ε leads to that f has infinitely many cycles in every neighborhood of 0, and then f is not linearizable near 0. Equicontinuity is that f is not linearizable and has infinitely many cycles in every neighborhood of a fixed point with multiplier $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$. The proof is complete.

3 Case of Polynomials

At the beginning of this section, we recall some basic facts about polynomials. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree $d \geq 2$. Let $\mathbf{K}_P$ be its filled Julia set, consisting of all $z \in \mathbb{C}$ for which the orbit of z under P remains bounded.

Theorem 3.1 (cf. [20]) *Let P be a polynomial of degree $d \geq 2$. Then the filled Julia set $\mathbf{K}_P$ contains all of the finite critical points of P if and only if $\mathbf{K}_P$ is connected.*

If P is monic and $\mathbf{K}_P$ is connected, the complement $\mathbb{C} \setminus \mathbf{K}_P$ is isomorphic to the complement of the closed unit disk $\overline{\mathbb{D}}$ under a unique conformal isomorphism

$$\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \mathbf{K}_P,$$

such that $\psi(z^d) = P(\psi(z))$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and that is asymptotic to identity near infinity (cf. [20]). Then the Green's function G for P is defined as

$$G(z) := \begin{cases} \log|\psi(z)| > 0, & z \in \mathbb{C} \setminus \mathbf{K}_P, \\ 0, & z \in \mathbf{K}_P. \end{cases}$$

The curves $G =$constant> 0 is called equipotential curves.

For each angle $t \in \mathbb{R}/\mathbb{Z}$, the external ray R_t is defined by

$$R_t := \{\psi(r\mathrm{e}^{2\pi \mathrm{i} t}), r \in (1, +\infty)\}.$$

Then P maps external ray R_t to external ray R_{dt}. Each external ray R_t has d pre-images under P, that is, $\{R_{\frac{t+j}{d}} \mid j = 0, \cdots, d-1\}$.

The external ray R_t is called a periodic ray if $P^n(R_t) = R_t$ for some positive integer n. In particular, if $P(R_t) = R_t$ we say that the external ray R_t is a fixed ray. P has exactly $d-1$ fixed rays

$$R_0, R_{\frac{1}{d-1}}, \cdots, R_{\frac{d-2}{d-1}}.$$

An external ray R_t is called rational if its angel $t \in \mathbb{R}/\mathbb{Z}$ is a rational number. The external ray R_t is eventually periodic if and only if t is a rational number.

An external ray R_t lands at a point $z \in \partial\mathbf{K}_P$ if

$$\lim_{r \to 1^+} \psi(\{r\mathrm{e}^{2\pi \mathrm{i} t}\}) = z.$$

Every periodic external ray lands at a periodic point which is either repelling or parabolic. Every eventually periodic external ray lands at a point which is eventually periodic (cf. [20]).

The closure $\overline{R_t}$ of an external ray is called a closed ray. If an external ray R_t lands, then the closure $\overline{R_t}$ is the union of the external ray and its landing point.

Let $\mathcal{C}$ be a set of some closed external rays. Denote by $\mathbf{U}_{\mathcal{C}}(z)$ the connected component of $\mathbb{C} \setminus \mathcal{C}$ that contains z.

Goldberg and Milnor [11] showed that the $d-1$ fixed closed rays of P cut the plane into some number of basic regions, each of which contains exactly one fixed point or fixed parabolic basin and contains at least one critical point of P. Kiwi [14] strengthened this result, he showed that after cutting the complex plane along an appropriate collection of eventually periodic closed rays, each Cremer periodic point and periodic Fatou component is separated from the others.

In order to obtain a better appreciation of our result, we briefly review the methods by Kiwi as follows. Let P be a polynomial of degree $d \geq 2$ and the filled Julia set $\mathbf{K}_P$ is connected. Let

m be the minimum common multiple of the following list of integers: Periods of periodic Fatou components and periods of Cremer periodic points. Since the number of non-repelling cycles is finite, m is a finite number. Periodic Fatou components and Cremer periodic points of P and P^m are the same, only their periods differ. Let

$$R_0, R_{\frac{1}{d^m-1}}, \cdots, R_{\frac{d^m-2}{d^m-1}}$$

be the fixed external rays of P^m. Let

$$\mathcal{V}_0 := \overline{R}_0 \cup \overline{R}_{\frac{1}{d^m-1}} \cup \cdots \cup \overline{R}_{\frac{d^m-2}{d^m-1}}.$$

When $k \geq 1$, we define sequences $(\mathcal{V}_k)$ inductively by putting $\mathcal{V}_k := P^{-1}(\mathcal{V}_{k-1})$. Since each external ray R_t has d pre-images under P, we have $\mathcal{V}_{k-1} \subset \mathcal{V}_k$. Kiwi proved that there is a finite and well defined integer $N > 0$ such that $\mathcal{V} := \mathcal{V}_N$ satisfies the following lemma.

Lemma 3.1 (Separation Lemma cf. [14]) *Let P be a polynomial with connected Julia set. Then the following conditions can be satisfied*:

(1) $P(\mathcal{V}) \subset \mathcal{V}$.

(2) *Each component of $\mathbb{C} \setminus \mathcal{V}$ contains at most one non-repelling periodic point or periodic Fatou component of P.*

(3) *If $\widehat{z}$ is a Cremer periodic point and c is a critical point such that $\mathbf{U}_{\mathcal{V}}(\widehat{z}) = \mathbf{U}_{\mathcal{V}}(c)$, then $\mathbf{U}_{\mathcal{V}}(P^n(\widehat{z})) = \mathbf{U}_{\mathcal{V}}(P^n(c))$ for all $n \geq 1$.*

(4) *If V is a bounded periodic Fatou component and c is a critical point such that $\mathbf{U}_{\mathcal{V}}(V) = \mathbf{U}_{\mathcal{V}}(c)$, then $\mathbf{U}_{\mathcal{V}}(P^n(V)) = \mathbf{U}_{\mathcal{V}}(P^n(c))$ for all $n \geq 1$.*

A map with some restrictions may behave like a polynomial. Douady and Hubbard introduced the polynomial-like maps as follows.

Definition 3.1 (cf. [6, 18]) *A polynomial-like map of degree d is a triple (U, U', f) where U and U' are open subsets of $\mathbb{C}$ isomorphic to discs with $\overline{U'} \subset U$ and $f : U' \to U$ is a branched covering map of degree d.*

If (U, U', f) is a polynomial-like map, its filled Julia set is defined by $\mathbf{K}_{(U,U',f)} = \bigcap_{n=0}^{\infty} f^{-n}(U')$. The set $\mathbf{K}_{(U,U',f)}$ is a compact subset of U'. The Julia set $\mathbf{J}_{(U,U',f)}$ of (U, U', f) is the boundary of $\mathbf{K}_{(U,U',f)}$.

We often use the following lemma to construct polynomial-like maps.

Lemma 3.2 (cf. [32]) *Let U and V be open subsets of $\widehat{\mathbb{C}}$ and $f : U \to V$ be holomorphic and continuous map to the boundary. Then f is a branched covering map of finite degree if and only if $f(\partial U) \subset \partial V$.*

Let (U, U', f) and (W, W', g) be polynomial-like maps. We will say that f and g are hybrid equivalent if there is a quasiconformal conjugacy ϕ between f and g, defined on a neighborhood of their respective filled Julia sets, such that $\phi \circ f = g \circ \phi$ and $\overline{\partial}\phi = 0$ on $\mathbf{K}_{(U,U',f)}$ (cf. [6]). Polynomial-like maps can be extended to the complex plane in such a way.

Theorem 3.2 (Straightening Theorem cf. [6, 18]) *Every polynomial-like map (U, U', f) is hybrid equivalent to a polynomial g of the same degree. When $\mathbf{K}_{(U,U',f)}$ is connected, the polynomial g is unique up to affine conjugation.*

So some statements of polynomials are also valid for polynomial-like maps. We will use this to generalize the result of the quadratic polynomial to higher order. First, the idea is to extract from a polynomial of degree $d > 2$ a quadratic-like map. For the case in which $\mathbf{K}_P$ is disconnected, the construction is an immediate consequence of the following lemma. It should be pointed out that Lemmas 3.3–3.4 have been proved in [14] and [35], respectively. For the reader's convenience, we include their detailed proof here.

Lemma 3.3 (cf. [14]) *Let P be a polynomial with disconnected Julia set and M be a connected component of $\mathbf{K}_P$ such that $P(M) = M$. If M is not a repelling fixed point then there exists a polynomial-like map (U, U', P) such that $M = \mathbf{K}_{(U,U',P)}$. Moreover, if there exist $m-1$ critical points of P in M, then (U, U', P) is a polynomial-like map of degree m.*

Proof The Green's function G of P is defined as

$$G(z) := \lim_{n\to\infty} d^{-n} \log_+ |P^{\circ n}(z)|, \quad z \in \mathbb{C},$$

where $\log_+ s = \max\{0, \log s\}$. If $r > 0$ is regular value of G, then each component of $\{z \mid G(z) \leq r\}$ is a closed topological disk with smooth boundary. Let $V(r)$ be the connected component of $\{z \mid G(z) \leq r, r > 0\}$ which contains M. Since the number of the critical points of G is finite, G has regular values arbitrarily close to zero. Therefore, we can get that

$$M = \cap V(r),$$

where the intersection is taken over all the regular values r of G.

Since the number of critical point of P is finite, there exists a small enough regular r_0 such that all the critical points of P contained in $V(r_0)$ belong to M, that is, there is not any critical point in $V(r_0) \setminus M$.

Let $U := V(r_0)$, $U' := V\big(\frac{r_0}{d}\big)$, then it is easy to see that $\overline{U'} \subset U$. Since P maps the curve $\big\{z \mid G(z) = \frac{r_0}{d}\big\}$ to the curve $\{z \mid G(z) = r_0\}$, the map $P : U' \to U$ is a branched covering map of finite degree by Lemma 3.2.

If there is not any critical point of P in U', then $P^{-1} : U \to U'$ would be a branched covering map. Since $\overline{U'} \subset U$, $\{(P^{-1})^{\circ n}|_U\}$ converges uniformly to a constant map $U \to z_0$, where z_0 is a point in M. It follows that z_0 is fixed point of (U, U', P) , $|P'(z_0)| > 1$ and $M = z_0$. Thus we conclude that if M is not a repelling fixed point, there is at least one critical point in M, thus (U, U', P) is a polynomial-like map of degree $d \geq 2$ and $M = \mathbf{K}_{(U,U',P)}$. Moreover, if there exist $m - 1$ critical points of P in M, then (U, U', P) is a polynomial-like map of degree m.

Each cubic polynomial P which has a fixed point with multiplier λ is conjugate to $P(z) = \lambda z + az^2 + z^3$ under affine transformations. Therefore, as follows we consider the case

$$P(z) = \lambda z + az^2 + z^3.$$

Lemma 3.4 (cf. [35]) *Let $P(z) = \mathrm{e}^{2\pi \mathrm{i}\alpha} z + az^2 + z^3$ have disconnected Julia set and M be a connected component of $\mathbf{K}_P$ containing 0. Then there exists a quadratic-like map (U, U', P) such that $M = \mathbf{K}_{(U,U',P)}$. Moreover (U, U', P) is hybrid equivalent to $P_\lambda(z) = \mathrm{e}^{2\pi \mathrm{i}\theta} z + z^2$.*

Proof Obviously, $P(M) = M$. From Lemma 3.3, there exists a polynomial-like map (U, U', P), such that $M = \mathbf{K}_{(U,U',P)}$. Because $\mathbf{K}_p$ is not connected, there exists at least one critical point in the basin of attraction of infinity of P. So there is exactly one critical point in M and (U, U', P) is a quadratic-like map. By the Straightening Theorem 3.2, (U, U', P) is hybrid equivalent to a quadratic polynomial. Because the multiplier of a fixed point of a holomorphic diffeomorphism is a topological invariant when it is of module 1 (cf. [21, 26]). Up to affine conjugation, there is only one quadratic polynomial which has a fixed point with multiplier $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$, so this quadratic-like map hybrid equivalent to $P_\lambda(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha}z + z^2$.

For the case in which $\mathbf{K}_P$ is connected, the construction of polynomial-like map depends on the Separation Lemma 3.1.

Lemma 3.5 *Let $P(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha}z + az^2 + z^3$ have connected Julia set, and have a non-repelling and non-zero periodic orbit which is not parabolic. Then there exists a pair of Jordan domains U' and U with $0 \in U'$ and $\overline{U'} \subset U$, such that (U, U', P) is quadratic-like map hybrid equivalent to $P_\lambda(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha}z + z^2$.*

Proof Let $\mathcal{V}$ be the union of closed rays from Lemma 3.1. Let U be the component of $U_{\mathcal{V}}(0)$ which contains 0, and is cut off by an equipotential curve of P. Let U' be the component of $P^{-1}(U)$ containing 0, and hence $P(U') = U$. Since $P(\mathcal{V}) \subset \mathcal{V}$, that is $\mathcal{V} \subset P^{-1}(\mathcal{V})$, we have $U_{P^{-1}(\mathcal{V})}(0) \subseteq U_{\mathcal{V}}(0)$. Therefore $U' \subset U$. Note that U' necessarily contains at least one critical point of P. Otherwise the Schwarz Lemma and $|P'(0)| = 1$ would imply that $U = U'$, and $P|_{U'} : U' \to U$ is a conformal isomorphism which is conjugate to a rotation, contradicting the fact that U' intersects the basin of attraction of infinity for P. The other critical point of P must stay away from U, since otherwise by Lemma 3.1, its entire orbit lives in U. This would contradict the fact that there must be a critical orbit which is associated to the non-repelling periodic orbit.

Since the non-repelling periodic orbit of P is not parabolic, the landing points of the closed rays in $\mathcal{V}$ are not parabolic. Therefore, by a simple "thickening" procedure (cf. [19]), we can get $\overline{U'} \subset U$. So that $P|_{U'} : U' \to U$ is a quadratic-like map. Up to affine conjugation, there is only one polynomial which has a fixed point with multiplier $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$. So this quadratic-like map is hybrid equivalent to $P_\lambda(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha}z + z^2$.

Now we have all the ingredients to prove Theorems 1.2–1.3.

Proof of Theorem 1.2 According to Lemmas 3.4–3.5, there exists a pair of Jordan domains U' and U with $0 \in U'$ and $\overline{U'} \subset U$ such that (U, U', P) is a quadratic-like map and hybrid equivalent to $P_\lambda(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha}z + z^2$. So, by the Straightening Theorem 3.2, there is a quasi-conformal map $\phi : U \to \mathbb{C}$ such that $\phi(P(z)) = P_\lambda(\phi(z))$, $\phi(\mathbf{K}_{(U,U',P)}) = \mathbf{K}_{P_\lambda}$ and $\phi(0) = 0$.

Let V be any neighborhood of 0 and $V^* := V \cap U'$. Obviously, V^* and $\phi(V^*)$ are still a neighborhood of 0. From the Yoccoz's Theorem, there exists a non-zero cycle $\mathcal{L}$ of P_λ in $\phi(V^*)$. Therefore P has a cycle $\phi^{-1}(\mathcal{L})$ in V^*. Since $V^* \subset V$ and the arbitrariness of V, we have proved that P has infinitely many cycles in every neighborhood of 0 and P is not linearizable near 0. The proof is complete.

Proof of Theorem 1.3 The proof is divided into two steps. The first step is to prove that any normalized $P \in \mathscr{P}_{\lambda,d}$ can be perturbed to a map which is quasi-conformally conjugated to the quadratic polynomial P_λ near the origin. The second step is to prove that there is a polynomial subfamily that satisfies the condition.

Step 1 The proof of this part is almost identical to the one which appears in [34]. For any $P \in \mathscr{P}_{\lambda,d}$, since $P'(0) = \lambda$, there exists an affine transformation $\varphi(z) = \delta z, \delta \neq 0$, by conjugation of which, P can be normalized to a polynomial $Q(z) = \delta P(\delta^{-1}z)$ such that Q is univalent on $\mathbb{D}$, $Q(0) = 0$ and $Q'(0) = \lambda$.

We define $\mathbb{D}_r := \{z \mid |z| < r\}$ for $r > 0$, and put $Q_b(z) := Q(z) + bz^2$ for $b \in \mathbb{C}$. Let $\mathscr{U} := \mathbb{D}_{\frac{7}{12}}$ and

$$\mathscr{U}_{Q,b} := \big\{z \in \mathbb{D}_{\frac{1}{2}} \mid \ Q_b(z) \in \mathscr{U}\big\}.$$

We can show that when $|b| \geq 12$, $(\mathscr{U}, \mathscr{U}_{Q,b}, Q_b)$ is a quadratic-like map.

Koebe's Distortion Theorem (cf. [29]) tells us that when $z \in \mathbb{D}$,

$$|Q(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

In particular when $|z| = \frac{1}{2}$ and $|b| \geq 12$, we can get $|Q(z)| \leq 2$ and $|bz^2| \geq 3$. So when $|z| = \frac{1}{2}$, for any $w \in \mathscr{U}$, we have

$$|Q_b(z) - w - bz^2| = |Q(z) - w| \leq |Q(z)| + |w| < 2 + \frac{7}{12} = \frac{31}{12} < |bz^2|. \tag{3.1}$$

That is $|(Q_b(z) - w) - bz^2| < |bz^2|$ for $|z| = \frac{1}{2}$. According to Rouche's Theorem, $Q_b(z) - w$ and bz^2 have the same number of zeros inside $\mathbb{D}_{\frac{1}{2}}$. It follows that there are exactly two solutions for the equation $Q_b(z) = w$ in $\mathbb{D}_{\frac{1}{2}}$, and then $Q_b : \mathscr{U}_{Q,b} \to \mathscr{U}$ is a holomorphic map of degree 2. By (3.1), we can also have $\overline{\mathscr{U}} \subset Q_b(\mathbb{D}_{\frac{1}{2}})$. Therefore Q_b maps $\partial\mathscr{U}_{Q,b}$ to $\partial\mathscr{U}$. According to Lemma 3.2, $Q_b : \mathscr{U}_{Q,b} \to \mathscr{U}$ is a branched covering map of degree 2.

Suppose that $\mathscr{U}_{Q,b}$ is not connected, then $\mathscr{U}_{Q,b}$ would have two connected components. Denote the connected component of $\mathscr{U}_{Q,b}$ containing 0 by $\mathscr{U}'_{Q,b}$. By Schwarz Lemma, Q_b : $\mathscr{U}'_{Q,b} \to \mathscr{U}$ was a conformal isomorphism conjugate to a rotation $z \mapsto \lambda z$. We would have $\mathscr{U} = \mathscr{U}'_{Q,b} \subset \mathscr{U}_{Q,b}$, yielding a contradiction. Therefore, $\mathscr{U}_{Q,b}$ must be connected, and then $\mathscr{U}_{Q,b}$ must be simply connected by Maximum Modulus Theorem.

Obviously, $\overline{\mathscr{U}}_{Q,b} \subset \mathscr{U}$. By Definition 3.1, we have $(\mathscr{U}, \mathscr{U}_{Q,b}, Q_b)$ is a polynomial-like map of degree 2.

Step 2 According to Straightening Theorem 3.2, $(\mathscr{U}, \mathscr{U}_{Q,b}, Q_b)$ is hybrid equivalent to a quadratic polynomial. The multiplier of an indifferent cycle of a holomorphic map is topological invariant when it is of module 1 (cf. [21, 26]). Up to affine conjugation, there is only one quadratic polynomial which has a fixed point with multiplier $\lambda = \mathrm{e}^{2\pi\mathrm{i}\alpha}$, so there exist qusi-conformal map $\phi : \mathscr{U} \to \mathbb{C}$ such that

$$\phi \circ Q_b \circ \phi^{-1} = P_\lambda.$$

Next, same to the proof of Theorem 1.2, we can let V be a neighborhood of 0 and $V^* := V \cap \mathscr{U}_{Q,b}$. Obviously, V^* and $\phi(V^*)$ is still a neighborhood of 0. From Yoccoz's Theorem, there exists a non-zero cycle $\mathcal{L}$ of P_λ in $\phi(V^*)$. Therefore, Q_b has a non-zero cycle $\phi^{-1}(\mathcal{L})$ in V^*. Since $V^* \subset V$, we have proved that Q_b has infinitely many cycles in every neighborhood of 0 and is not linearizable near 0.

Since $Q(z) = \delta P(\delta^{-1}z)$ and $\delta \neq 0$, the map

$$\Phi : (a_2, a_3, \cdots, a_d) \mapsto (\delta^{-1}a_2 + b, \delta^{-2}a_3, \cdots, \delta^{-(d-1)}a_d)$$

is holomorphic and the Jacobian of the map is not zero. Therefore the holomorphic family of polynomials

$$\mathscr{Q}_{\lambda,b} := \left\{ Q_b \mid Q_b(z) = \delta P(\delta^{-1}z) + bz^2, P \in \mathscr{P}_{\lambda,d}, |b| \geq 12, \delta \neq 0 \right\}$$

is a $d-1$ dimensional holomorphic subfamily of $\mathscr{P}_{\lambda,d}$ and all of the elements of $\mathscr{Q}_{\lambda,b}$ has infinitely many cycles in every neighborhood of 0 and is not linearizable near 0. The proof is complete.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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