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Two Applications of the $\partial\overline{\partial}$-Hodge Theory*

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Abstract Using Hodge theory and Banach fixed point theorem, Liu and Zhu developed a global method to deal with various problems in deformation theory. In this note, the authors generalize Liu-Zhu's method to treat two deformation problems for non-Kähler manifolds. They apply the $\partial\overline{\partial}$-Hodge theory to construct a deformation formula for (p,q)-forms of compact complex manifold under deformations, which can be used to study the Hodge number of complex manifold under deformations. In the second part of this note, by using the $\partial\overline{\partial}$-Hodge theory, they provide a simple proof of the unobstructed deformation theorem for the non-Kähler Calabi-Yau $\partial\overline{\partial}$-manifolds.

Keywords Complex structures, Deformations, (p,q)-Forms, Non-Kähler Calabi-Yau manifolds
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1 Introduction

Various problems in deformation theory can be reduced to solving certain $\overline{\partial}$-equations on complex manifolds. In the classical deformation theory developed by Kodaira-Spencer [9], in order to construct a complex analytic family of a compact complex manifold, the starting point is to solve the Maurer-Cartan equation. Kodaira et al. [3, 9] introduced the method of formal power series, and showed the convergence of this series through a beautiful majorant series. Recently, a global method which significantly simplifies the procedure was established by Liu and Zhu [7–8]. They also provide a simple proof of the unobstructed theorem for Calabi-Yau manifold originally due to [16–17]. Another interesting question is how to extend the holomorphic canonical forms of compact Kähler manifold under the deformations. In [6], Liu-Rao-Yang introduced the extension equation whose solution would provide the desired holomorphic forms under deformations. Their method was further simplified by [7–8] with intrinsic use of the Hodge theory on compact Kähler manifold.

In this note, we generalize Liu-Zhu's method to treat two deformation problems for non-Kähler manifolds by using the $\partial\overline{\partial}$-Hodge theory essentially. In the first part, we construct an extension formula for d-closed (p,q)-forms of compact complex manifold under deformations. In the second part, we provide a simple proof of the unobstructed deformation theorem for the non-Kähler Calabi-Yau $\partial\overline{\partial}$-manifolds.

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1.1 Deformations of (p, q)-forms

In order to study the dimension of certain cohomology group of complex manifold under the deformation of complex structure, it is natural to consider the deformation of (p, q)-form of complex manifold.

Let's explain the main idea and fix some notations. Let X be a compact complex manifold satisfying the $(p, q+1)$-type $\partial\overline{\partial}$-formula (3.9), let $\{X_t\}_{|t|<\varepsilon}$ be a holomorphic family of compact complex manifolds with $X_0 := X$. We use $\varphi(t)$ to denote the Betrami differential corresponding to X_t, and we let $\mathrm{e}^{i_{\varphi(t)}}$ be the linear map given by formula (3.2), which maps a form on X to a form on X_t. Given a d-closed form σ_0 on X, in order to construct a d-closed form $\mathrm{e}^{i_{\varphi(t)}}(\sigma(t))$ on X_t, the form $\sigma(t)$ on X needs to satisfy certain differential equations, i.e., (3.8).

The key observation is that if we change the differential equation (3.8) into a corresponding integral equation (3.11), then the solution can be constructed directly. By using Banach fixed point theorem, one knows this integral equation has a unique solution for small t. By using the map e^{i_φ}, we get a d-closed form on X_t, whose (p, q)-part is $\overline{\partial}_t$-closed.

Specifically, we obtain the following theorem.

Theorem 1.1 *Let X be a compact complex manifold satisfying the $(p, q+1)$-type $\partial\overline{\partial}$-formula. Given a* d*-closed $\sigma_0 \in A^{p,q}(X)$, there is $\sigma(t) \in A^{p,q}(X)$ satisfying* (3.8) *and that $P_{\varphi(t)}(\mathrm{e}^{i_{\varphi(t)}}(\sigma(t)))$ is $\overline{\partial}_t$-closed in $A^{p,q}(X_t)$.*

1.2 Unobstructed deformation theorem for Calabi-Yau $\partial\overline{\partial}$-manifolds

A Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. The celebrated Bogomolov-Tian-Todorov theorem (see [16–17]) said that every Calabi-Yau manifold has unobstructed deformation. It is easy to see that the proof shown in [16] is still valid if the Kähler assumption is weakened to the $\partial\overline{\partial}$-lemma, see [10] for a brief description of the proof for (non-Kähler) Calabi-Yau $\partial\overline{\partial}$-manifold, see Definition 4.1. Later on, [5, 15] proved much more general unobstructedness theorems for non-Kähler Calabi-Yau manifolds than Popovici's and their proofs are rather different. In the rest of this note, we will use the global method in [7–8] to give a simple and complete proof for the obstructedness theorem for Calabi-Yau $\partial\overline{\partial}$-manifold.

Let's explain the main idea briefly. Suppose X is a Calabi-Yau $\partial\overline{\partial}$-manifold. In order to show the unobstructedness, we need to solve the following Maurer-Cartan equation

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi], \quad \varphi \in A^{0,1}(X, T^{1,0}X). \tag{1.1}$$

Choosing a nowhere vanishing holomorphic section Ω of the canonical bundle K_X, for any given class $\eta \in H^1(X, T^{1,0}X)$ and $\varphi_1 \in \eta$, let $\varphi(t)$ be a power series with $\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}|_{t=0} = \varphi_1$. We change the above differential equation (1.1) for $\varphi(t)$ into the following integral equation

$$(\varphi(t) - \varphi_1 t)\lrcorner\Omega = \frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi(t), \varphi(t)]\lrcorner\Omega). \tag{1.2}$$

Then, using Banach fixed point theorem and standard regularity of elliptic operator, we show that the above integral equation has a unique solution $\varphi(t) \in A^{0,1}(X, T^{1,0}X)$ which is holomorphic with respect to t in a neighborhood of 0.

More precisely, we have the following theorem.

Theorem 1.2 *Given any $\eta \in H^1(X, T^{1,0}X)$, we can choose $\varphi_1 \in \eta$, such that* $\mathrm{d}(\varphi_1 \lrcorner \Omega) = 0$. *Furthermore, there is $\varepsilon > 0$, such that for $|t| < \varepsilon$, there is a unique $\varphi(t) \in A^{0,1}(X, T^{1,0}X)$ which is holomorphic in t and satisfies $(\varphi(t) - \varphi_1 t) \lrcorner \Omega = \frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi(t), \varphi(t)] \lrcorner \Omega)$, and $\varphi(t)$ satisfies*

(1) $\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$;

(2) $(\varphi(t) - \varphi_1 t) \lrcorner \Omega$ *is ∂-exact and* $\partial(\varphi(t) \lrcorner \Omega) = 0$.

2 Preliminaries

2.1 Hodge theory

At first, we recall the classical Hodge theory on compact complex manifold, which can be found in [9].

Let (X, h) be a compact complex manifold X with a hermitian metric h which induces an L^2 inner product on the space $A^{p,q}(X)$ of smooth (p, q)-forms on X. Let $\Box_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, and Hodge theory implies that there exists a Green operator G and a harmonic projection H, and the following identities hold:

$$\Box_{\overline{\partial}} G = G\Box_{\overline{\partial}} = \mathrm{id} - H, \quad \overline{\partial}G = G\overline{\partial}, \quad \overline{\partial}^* G = G\overline{\partial}^*,$$
$$HG = GH = 0, \quad \overline{\partial}H = H\overline{\partial} = 0, \quad \overline{\partial}^* H = H\overline{\partial}^* = 0.$$

Furthermore, the Laplacian $\Box_{\overline{\partial}}$ is second-order elliptic differential operator and we have the following estimate (cf. [9]).

Lemma 2.1 (cf. [9, Page 160]) *With respect to the Hölder norm $\|\cdot\|_{k,\alpha}$, $G_{\overline{\partial}}$ is a bounded operator of order -2, i.e.,*

$$\|G_{\overline{\partial}}\varphi\|_{k,\alpha} \le C\|\varphi\|_{k-2,\alpha}, \quad \forall \varphi \in A^{p,q}(X),$$

where $k \ge 2$, $C = C(k, \alpha)$ is constant.

Similarly, if we consider the differential operator $\partial\overline{\partial}$ instead of $\overline{\partial}$, for the $\partial\overline{\partial}$-Laplacian $\Box_{\partial\overline{\partial}} = \partial\overline{\partial}(\partial\overline{\partial})^* + (\partial\overline{\partial})^*(\partial\overline{\partial})$, we also have the $\partial\overline{\partial}$-Hodge theory as follows.

Proposition 2.1 *The following identities hold*:

$$\Box_{\partial\overline{\partial}} G_{\partial\overline{\partial}} = G_{\partial\overline{\partial}}\Box_{\partial\overline{\partial}} = \mathrm{id} - H_{\partial\overline{\partial}}, \quad \partial\overline{\partial}G_{\partial\overline{\partial}} = G_{\partial\overline{\partial}}\partial\overline{\partial},$$
$$(\partial\overline{\partial})^* G_{\partial\overline{\partial}} = G_{\partial\overline{\partial}}(\partial\overline{\partial})^*, \quad H_{\partial\overline{\partial}}G_{\partial\overline{\partial}} = G_{\partial\overline{\partial}}H_{\partial\overline{\partial}} = 0,$$
$$(\partial\overline{\partial})H_{\partial\overline{\partial}} = H_{\partial\overline{\partial}}(\partial\overline{\partial}) = 0, \quad (\partial\overline{\partial})^* H_{\partial\overline{\partial}} = H_{\partial\overline{\partial}}(\partial\overline{\partial})^* = 0.$$

We refer to [11] for the using of $\partial\overline{\partial}$-Hodge theory. In addition, since $\partial\overline{\partial}$-Laplacian $\Box_{\partial\overline{\partial}}$ is a fourth order elliptic differential operator, then we have the following estimate similar to Lemma 2.1.

Lemma 2.2 *With respect to the Hölder norm $\|\cdot\|_{k,\alpha}$, $G_{\partial\overline{\partial}}$ is a bounded operator of order -4, i.e.,*

$$\|G_{\partial\overline{\partial}}\varphi\|_{k,\alpha} \le C\|\varphi\|_{k-4,\alpha}, \quad \forall \varphi \in A^{p,q}(X),$$

where $k \ge 4$, $C = C(k, \alpha)$ is constant.

Remark 2.1 Since $\overline{\partial}\partial = -\partial\overline{\partial}$ on any complex manifold, we have $(\overline{\partial}\partial)^* = -(\partial\overline{\partial})^*$. Then $\square_{\overline{\partial}\partial} = \square_{\partial\overline{\partial}}$, $G_{\overline{\partial}\partial} = G_{\partial\overline{\partial}}$ and $H_{\overline{\partial}\partial} = H_{\partial\overline{\partial}}$. In the following, sometimes we will also use the Hodge theory for $\overline{\partial}\partial$-Laplacian $\square_{\overline{\partial}\partial}$ alternatively.

2.2 Banach fixed-point theorem

The Banach fixed-point theorem which can be found in any standard textbooks of functional analysis states that, for any contraction mapping f from a closed subset F of a Banach space E into F, there exists a unique $x \in F$ such that $x = f(x)$.

For the convenience of our application, we write down the Banach fixed-point theorem in the following form (cf. [8]).

Theorem 2.1 *Let $(E, \|\cdot\|)$ be a Banach space and suppose F is a closed subset of E. Given $y \in F$, let K be a contraction mapping defined on F, i.e., for any $x_1, x_2 \in F$, we have*

$$\|K(x_1) - K(x_2)\| \le \gamma\|x_1 - x_2\|, \quad \textit{where } \gamma \in (0,1). \tag{2.1}$$

Moreover, K satisfies

$$y + K(x) \in F \quad \textit{for any } x \in F. \tag{2.2}$$

Then, the following equation

$$x = y + K(x) \tag{2.3}$$

has a unique solution $x \in F$.

3 Deformation of (p, q)-Forms and Applications

3.1 Extension equation

Suppose X is a compact complex manifold of dimension n. Let $\varphi \in A^{0,1}(X, T^{1,0}X)$ be an integrable Beltrami differential, and X_φ be the complex manifold with the complex structure determined by φ. We introduce the filtration

$$F^{p,q}(X) = A^{p,q}(X) \oplus A^{p+1,q-1}(X) \oplus \cdots \oplus A^{p+q,0}(X).$$

For any $\sigma \in A^{p,q}(X)$, in local coordinate (U, z), we write

$$\sigma = \sum_{I,J} \sigma_{I,\overline{J}} \mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q}, \tag{3.1}$$

then we define a linear map $\mathrm{e}^{i_\varphi} : A^{p,q}(X) \to F^{p,q}(X_\varphi)$ as follows

$$\begin{aligned} \mathrm{e}^{i_\varphi}(\sigma) &= \sum_{I,J} \sigma_{I,\overline{J}} \mathrm{e}^{i_\varphi}(\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p}) \wedge \mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q} \\ &= \sum_{I,J} \sigma_{I,\overline{J}} (\mathrm{d}z^{i_1} + \varphi \mathrm{d}z^{i_1}) \wedge \cdots \wedge (\mathrm{d}z^{i_p} + \varphi \mathrm{d}z^{i_p}) \wedge \mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q}. \end{aligned} \tag{3.2}$$

Since the space $T^{0,1}(X_\varphi)$ of anti-holomorphic tangent vector fields is spanned by the basis

$$\Big\{X_{\overline{i}} = \frac{\partial}{\partial \overline{z}^i} - \varphi^j_{\overline{i}}\frac{\partial}{\partial z^j}\Big\}_{i=1}^n. \tag{3.3}$$

Denote its dual basis by $\{Y^{\overline{i}}\}_{i=1}^n$ which spans the space $A^{0,1}(X_\varphi) = (T^{0,1}(X_\varphi))^*$. Then $\{\overline{Y^{\overline{i}}}\}_{i=1}^n$ spans the space $A^{1,0}(X_\varphi) = (T^{1,0}(X_\varphi))^*$.

Since $A^{0,1}(X_\varphi) \subset A^1(X_\varphi)$ and $A^1(X_\varphi) = A^1(X)$, we let

$$P_\varphi : A^1(X) \to A^{0,1}(X_\varphi)$$

be the natural projection map. For $\mathrm{d}\overline{z}^i \in A^{0,1}(X) \subset A^1(X)$, there are coefficients a^i_j, b^i_k such that $\mathrm{d}\overline{z}^i = a^i_j Y^{\overline{j}} + b^i_k \overline{Y^{\overline{k}}}$. It is obvious that

$$a^i_j = \mathrm{d}\overline{z}^i(X_{\overline{j}}) = \delta^i_j \quad \text{and} \quad b^i_k = \mathrm{d}\overline{z}^i(\overline{X_{\overline{k}}}) = -\overline{\varphi^i_{\overline{j}}}. \tag{3.4}$$

Therefore, $P_\varphi(\mathrm{d}\overline{z}^i) = Y^{\overline{j}}$, and we have

$$\begin{aligned} P_\varphi(\mathrm{e}^{i_\varphi}(\sigma)) = \sum_{I,J} \sigma_{I,\overline{J}}(\mathrm{d}z^{i_1} + \varphi \mathrm{d}z^{i_1}) \wedge \cdots \wedge (\mathrm{d}z^{i_p} + \varphi \mathrm{d}z^{i_p}) \\ \wedge Y^{\overline{j}_1} \wedge \cdots \wedge Y^{\overline{j}_q} \in A^{p,q}(X_\varphi). \end{aligned} \tag{3.5}$$

Recall the following extension formula (cf. [6])

$$\mathrm{e}^{-i_\varphi} \circ \mathrm{d} \circ \mathrm{e}^{i_\varphi} = \mathrm{d} - \mathcal{L}^{1,0}_\varphi = \mathrm{d} + \partial i_\varphi - i_\varphi \partial. \tag{3.6}$$

Therefore, $\mathrm{e}^{i_\varphi}(\sigma)$ is d-closed on X (or X_φ) if and only if

$$(\mathrm{d} + \partial i_\varphi - i_\varphi\partial)\sigma = 0. \tag{3.7}$$

By comparing the type, one obtains that the above equation is equivalent to

$$\begin{cases} \partial\sigma = 0, \\ \overline{\partial}\sigma = -\partial(\varphi \lrcorner \sigma). \end{cases} \tag{3.8}$$

3.2 Deformation of (p,q)-form

In the following, we consider compact complex manifolds with mild conditions.

Definition 3.1 *A compact complex manifold is said to satisfy the (p,q)-type $\partial\overline{\partial}$-formula, if the following formula*

$$\mathrm{Ker}(\overline{\partial}) \cap \mathrm{Im}(\partial) = \mathrm{Im}(\partial\overline{\partial}) \tag{3.9}$$

holds for any (p,q)-forms lying in $A^{p,q}(X)$.

Let $\{X_t\}_{|t|\le\varepsilon}$ be a holomorphic family of deformations of X satisfying the (p,q)-type $\partial\overline{\partial}$-formula. There exists a family of Betrami differentials $\{\varphi(t)\}_{|t|\le\varepsilon}$ which depends on t holomorphically, such that $X_t = X_{\varphi(t)}$. Hence, $\varphi(t) \in A^{0,1}(X, T^{1,0}X)$ and satisfies the Maurer-Cartan equation

$$\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]. \tag{3.10}$$

Given any d-closed (p,q)-form $\sigma_0 \in A^{p,q}(X)$, we consider the following integral equation

$$\sigma = \sigma_0 - \partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\partial(\varphi(t)\lrcorner\sigma). \tag{3.11}$$

Then we will show that, when ε is small enough, the integral equation (3.11) has a unique solution $\sigma(t) \in A^{p,q}(X_t)$ for $|t| < \varepsilon$.

Proposition 3.1 *There exists $\varepsilon > 0$, such that if $|t| < \varepsilon$, for any* d*-closed (p,q)-form σ_0 on X,* (3.11) *has a unique solution $\sigma(t)$ which is C^k-continuous and holomorphically depends on t.*

Proof Consider the norm space $(A^{p,q}(X), \|\cdot\|_{k+\alpha})$, and denote its completion by E. For brevity, we introduce the operator $K_{\varphi(t)}$ which acts on σ by

$$K_{\varphi(t)}(\sigma) = -\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\partial(\varphi(t)\lrcorner\sigma). \tag{3.12}$$

By the standard estimates for Green's operator $G_{\overline{\partial}\partial}$, there is a constant C independent of σ such that

$$\|K_{\varphi(t)}(\sigma)\|_{k+\alpha} \leq C\|\varphi(t)\|_{k+\alpha} \cdot \|\sigma\|_{k+\alpha}. \tag{3.13}$$

We can choose ε small enough, such that if $|t| < \varepsilon$, then $\|\varphi(t)\|_{k+\alpha} < \frac{1}{2C}$.

Now the integral equation (3.11) becomes

$$\sigma = \sigma_0 + K_{\varphi(t)}(\sigma). \tag{3.14}$$

Let $F = E$ in Banach fixed-point Theorem 2.1. Therefore the condition (2.2) is obviously. For condition (2.1), let $\sigma, \sigma' \in E$, then by formula (3.13), we have

$$\begin{aligned} \|K_{\varphi(t)}(\sigma) - K_{\varphi(t)}(\sigma')\|_{k+\alpha} &= \|K_{\varphi(t)}(\sigma - \sigma')\|_{k+\alpha} \\ &\leq \frac{1}{2}\|\sigma - \sigma'\|_{k+\alpha} \end{aligned} \tag{3.15}$$

for any $|t| < \varepsilon$.

Therefore, the Banach fixed-point Theorem 2.1 implies that there is a unique solution $\sigma(t)$ of the integral equation (3.11) in E for any $|t| \leq \varepsilon$.

Then we will show that $\sigma(t)$ depends holomorphically on t. From the Banachi-fixed point Theorem 2.1, we know that the solution $\sigma(t)$ is constructed as the limit of the following sequence:

$$\begin{aligned} \sigma_1(t) &= \sigma_0 + K_{\varphi(t)}(\sigma_0), \\ \sigma_n(t) &= \sigma_0 + K_{\varphi(t)}(\sigma_{n-1}) \quad \text{for } n \geq 1. \end{aligned}$$

Then, for any $|t| \leq \varepsilon$, the solution $\sigma(t)$ is given by

$$\sigma(t) = \lim_{n\to\infty} \sigma_n(t) = \sigma_0 + \sum_{k=1}^{\infty} K^k_{\varphi(t)}(\sigma_0). \tag{3.16}$$

Since $\varphi(t)$ is the Betrami differential from the holomorphic family of complex manifold X, it follows that $\varphi(t)$ is holomorphic depends on t. Then $\varphi(t)$ can be written as a convergent power series in t. From the above expression (3.16) for $\sigma(t)$, we obtain that $\sigma(t)$ is also a power series in t for $|t| < \varepsilon$. Finally, it is easy to see that the $\|\cdot\|_{k,\alpha}$-norm of the $\sigma(t)$ is finite which implies that $\sigma(t)$ is convergent for $|t| < \varepsilon$. Therefore, $\sigma(t)$ is holomorphic in t for $|t| < \varepsilon$.

According to Proposition 3.1, $\sigma(t)$ is just C^k-continuous. Similar to [9], by using the standard regularity theory for elliptic differential operator, we obtain that the solution $\sigma(t)$ obtained in above Proposition 3.1 is a smooth form for small t.

Theorem 3.1 *Suppose X is a compact complex manifold which satisfies the $(p, q+1)$-type $\partial\overline{\partial}$-formula* (3.9), *and let σ_0 be a* d*-closed (p, q)-form on X, then the solution $\sigma(t)$ obtained in Proposition* 3.1 *satisfies*

$$\overline{\partial}\sigma(t) + \partial(\varphi(t)\lrcorner\sigma(t)) = 0. \tag{3.17}$$

Proof Since both $\varphi(t)$ and $\sigma(t)$ are holomorphic in t, we consider the Taylor expansions $\varphi(t) = \sum\limits_{i\geq 1} \varphi_i t^i$ and $\sigma(t) = \sum\limits_{j\geq 0} \sigma_j t^j$. Then (3.10) implies $\overline{\partial}\varphi_1 = 0$ and

$$\overline{\partial}\varphi_k = \sum_{\substack{i+j=k\\ i,j\geq 1}} \frac{1}{2}[\varphi_i, \varphi_j] \tag{3.18}$$

for $k \geq 2$.

From (3.11), we obtain

$$\sigma_k = -\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\partial\Big(\sum_{\substack{i+j=k\\ i\geq 1, j\geq 0}} \varphi_i\lrcorner\sigma_j\Big) \tag{3.19}$$

for $k \geq 1$. Then we only need to show that

$$\overline{\partial}\sigma_k = -\partial\sum_{\substack{i+j=k\\ i\geq 1, j\geq 0}} \varphi_i\lrcorner\sigma_j. \tag{3.20}$$

For $k = 1$, we have

$$\begin{aligned}\overline{\partial}\sigma_1 &= -\overline{\partial}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\partial(\varphi_1\lrcorner\sigma_0)\\ &= -(\Box_{\overline{\partial}\partial}G_{\overline{\partial}\partial} - (\overline{\partial}\partial)^*\overline{\partial}\partial G_{\overline{\partial}\partial})\partial(\varphi_1\lrcorner\sigma_0)\\ &= -\partial(\varphi_1\lrcorner\sigma_0) - H_{\overline{\partial}\partial}\partial(\varphi_1\lrcorner\sigma_0)\\ &= -\partial(\varphi_1\lrcorner\sigma_0).\end{aligned} \tag{3.21}$$

Indeed, since $\overline{\partial}\partial(\varphi_1\lrcorner\sigma_0) = -\partial\overline{\partial}(\varphi_1\lrcorner\sigma_0) = -\partial(\overline{\partial}\varphi_1\lrcorner\sigma_0 + \varphi_1\lrcorner\overline{\partial}\sigma_0) = 0$, with the help of $(p, q+1)$-type $\partial\overline{\partial}$-lemma, we have $\partial(\varphi_1\lrcorner\Omega_0) \in \mathrm{Im}(\overline{\partial}\partial)$ which implies $H_{\overline{\partial}\partial}\partial(\varphi_1\lrcorner\sigma_0) = 0$.

By hypothesis induction, we assume (3.20) holds for $k \leq l-1$. Now for $k = l$, by using Hodge theory for $\Box_{\overline{\partial}\partial}$, we obtain

$$\begin{aligned}\overline{\partial}\sigma_l &= -\overline{\partial}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\partial\sum_{\substack{i+j=l\\ i\geq 1, j\geq 0}} \varphi_i\lrcorner\sigma_j\\ &= -\partial\sum_{\substack{i+j=l\\ i\geq 1, j\geq 0}} \varphi_i\lrcorner\sigma_j - H_{\overline{\partial}\partial}\partial\sum_{\substack{i+j=l\\ i\geq 1, j\geq 0}} \varphi_i\lrcorner\sigma_j.\end{aligned} \tag{3.22}$$

Then we need to show $H_{\overline{\partial}\partial}\partial \sum\limits_{\substack{i+j=l\\ i\geq 1,j\geq 0}} \varphi_i\lrcorner\sigma_j = 0$. By straightforward computations, we have

$$\begin{aligned}\overline{\partial}\partial \sum_{\substack{i+j=l\\ i\geq 1,j\geq 0}} \varphi_i\lrcorner\sigma_j &= -\partial\overline{\partial} \sum_{\substack{i+j=l\\ i\geq 1,j\geq 0}} \varphi_i\lrcorner\sigma_j \\ &= -\partial \sum_{\substack{i+j=l\\ i\geq 1,j\geq 0}} (\overline{\partial}\varphi_i\lrcorner\sigma_j + \varphi_i\lrcorner\overline{\partial}\sigma_j) \\ &= -\partial \sum_{\substack{i+j=l\\ i\geq 1,j\geq 0}} \Big(\frac{1}{2}\sum_{a+b=i}[\varphi_a,\varphi_b]\lrcorner\sigma_j - \sum_{a+b=j}\varphi_i\lrcorner\varphi_a\lrcorner\sigma_b\Big) \\ &= -\partial \sum_{a+b+c=l}(-\partial(\varphi_a\lrcorner\varphi_b\lrcorner\sigma_c)) = 0. \end{aligned} \tag{3.23}$$

By using the (p,q)-type $\partial\overline{\partial}$-formula (3.9), we obtain that $\partial \sum\limits_{\substack{i+j=l\\ i\geq 1,j\geq 0}} \varphi_i\lrcorner\sigma_j$ is $\overline{\partial}\partial$-exact, hence

$$H_{\overline{\partial}\partial}\partial \sum_{\substack{i+j=l\\ i\geq 1,j\geq 0}} \varphi_i\lrcorner\sigma_j = 0.$$

From the previous analysis, $\mathrm{e}^{i_{\varphi(t)}}(\sigma(t))$ is d-closed in $F^{p,q}(X_t)$. We write

$$\begin{aligned}\mathrm{e}^{i_{\varphi(t)}}(\sigma(t)) &= \alpha^{p,q} + \alpha^{p+1,q-1} + \cdots + \alpha^{p+q,0} \\ &\in A^{p,q}(X_t)\oplus A^{p+1,q-1}(X_t)\oplus\cdots\oplus A^{p+q,0}(X_t),\end{aligned} \tag{3.24}$$

where $\alpha^{p,q} = P_{\varphi(t)}(\mathrm{e}^{i_{\varphi(t)}}\sigma(t))$. Since $\mathrm{d}(\mathrm{e}^{i_{\varphi(t)}}\sigma(t)) = 0$ and $\mathrm{d} = \partial_t + \overline{\partial}_t$, by comparing types, we obtain $\overline{\partial}_t(\alpha^{p,q}) = 0$.

Hence we have the following theorem.

Theorem 3.2 *Given a* d*-closed* $\sigma_0 \in A^{p,q}(X)$*, there is* $\sigma(t) \in A^{p,q}(X)$ *satisfying* (3.8)*, and*

$$P_{\varphi(t)}(\mathrm{e}^{i_{\varphi(t)}}(\sigma(t)))$$

is $\overline{\partial}_t$*-closed in* $A^{p,q}(X_t)$*.*

In [14], Rao and Zhao introduced a natural method to deform the (p,q)-form on a given compact complex manifold. Based on their computations, they found an extension equation for $\overline{\partial}_t$-closed (p,q)-form under the deformed complex manifold X_t. In our approach, we observe that the original extension equation (cf. [2, 6]) is still workable in this case. It provides a simple method to construct the (p,q)-forms under deformation. Theorem 3.2 was directly used to prove the invariance of Hodge numbers over complex manifolds (cf. [14]). See also the recent work in [13] for more applications.

Another potential application of Theorem 3.2 is to construct the explicit Kähler forms ω_t under the deformations. Although the deformation stability of Kähler manifold was proved firstly in [4, 9] with highly nontrivial arguments from the theory of elliptic differential equations, it is still interesting to find an elementary proof of the Kähler stability by constructing the Kähler forms ω_t explicitly under the deformations. Such elementary proof was first given in

[12]. Our method could provide a much simpler approach. By Theorem 3.2, we first get a $\overline{\partial}_t$-closed $(1,1)$-form $\widetilde{\omega_t}$ on X_t. Since X_t also satisfies $\partial\overline{\partial}$-lemma according to [1], $\widetilde{\omega_t}$ can be represented by a d-closed form on X_t, still denoted by $\widetilde{\omega_t}$. Then we set $\omega_t := \frac{1}{2}(\widetilde{\omega_t} + \overline{\widetilde{\omega_t}})$ to get a real d-closed $(1,1)$-form on X_t. However, in order to show the positivity of ω_t, up to now, we still need a highly nontrivial argument from [9] which shows ω_t is differentially depends on t. A challenge question is how to prove this statement without using this argument in [9].

4 Unobstructed Deformation Theorem for Calabi-Yau $\partial\overline{\partial}$-Manifolds

A Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. It is well-known that deformations of Calabi-Yau manifolds are always unobstructed. This fact was referred to as the Bogomolov-Tian-Todorov unobstructedness theorem (cf. [16–17]), see also [7–8] for a simple approach. In fact, the Kähler condition here can be weakened to the validity of $\partial\overline{\partial}$-lemma. In the rest of this note, we will apply the global method developed in [7–8] to give a simple proof of the obstructedness theorem for (non-Kähler) Calabi-Yau $\partial\overline{\partial}$-manifold.

Definition 4.1 *A compact complex manifold X is said to be a Calabi-Yau $\partial\overline{\partial}$-manifold if* (i) *the $\partial\overline{\partial}$-lemma holds on X*; (ii) *the canonical bundle K_X is trivial.*

Theorem 4.1 *The deformation of Calabi-Yau $\partial\overline{\partial}$-manifold is unobstructed.*

In the following, we will provide a simple and complete proof for this theorem. Let us begin with some basic results.

Lemma 4.1 *Let X be a compact complex manifold of dimension n with trivial canonical bundle K_X. Let Ω be a non-vanishing holomorphic section of K_X. Then, for $q = 0, \cdots, n$, we have an isomorphism*

$$T_\Omega : A^{0,q}(X, T^{1,0}X) \to A^{n-1,q}(X) \tag{4.1}$$

given by $T_\Omega(\alpha) = \alpha \lrcorner \Omega$ for any $\alpha \in A^{0,q}(X, T^{1,0}X)$.

Proof Suppose $z^1, \cdots, z^n$ are the local holomorphic coordinates on some open subset $U \subset X$, we write $\Omega = f \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$, where f is a holomorphic function on U without zeros. It is obvious that T_Ω is injective. On the other hand, given any $\sigma \in A^{n-1,q}(X)$, we write $\sigma = \sum\limits_j \sigma^j \mathrm{d}z^1 \wedge \cdots \wedge \widehat{\mathrm{d}z^j} \wedge \mathrm{d}z^n$, where $\sigma^j \in A^{0,q}(X)$. If we let $\varphi = \sum\limits_j (-1)^{j-1} \frac{\sigma^j}{f} \frac{\partial}{\partial z^j} \in A^{0,q}(X, T^{1,0}X)$, then

$$\varphi \lrcorner \Omega = \sum_j \sigma^j \mathrm{d}z^1 \wedge \cdots \wedge \widehat{\mathrm{d}z^j} \wedge \cdots \wedge \mathrm{d}z^n = \sigma. \tag{4.2}$$

Hence T_Ω is surjective.

Lemma 4.2 (cf. [10, Lemma 3.3]) *Let X be a compact complex manifold such that K_X is trivial. Then, for $q = 1$, the above isomorphism T_Ω satisfies*

$$T_\Omega(\operatorname{Ker} \overline{\partial}) = \operatorname{Ker} \overline{\partial} \quad \textit{and} \quad T_\Omega(\operatorname{Im} \overline{\partial}) = \operatorname{Im} \overline{\partial}. \tag{4.3}$$

Hence T_Ω induces an isomorphism in cohomology

$$T_{[\Omega]} : H^{0,1}(X, T^{1,0}X) \to H^{n-1,1}(X) \tag{4.4}$$

defined by $T_{[\Omega]}([\varphi]) = [\varphi \lrcorner \Omega]$ *for all* $[\varphi] \in H^{0,1}(X, T^{1,0}X)$.

Proof By a straightforward computation, we have

$$\overline{\partial}(\varphi \lrcorner \Omega) = \overline{\partial}\varphi \lrcorner \Omega \tag{4.5}$$

for any $\varphi \in A^{0,q}(X, T^{1,0}X)$, then (4.3) follows by the isomorphism of T_Ω.

We take the identity $T_\Omega(\operatorname{Im}\overline{\partial}) = \operatorname{Im}\overline{\partial}$ for example. Given any $\overline{\partial}\theta \in \operatorname{Im}(\overline{\partial})$, where $\theta \in A^0(X, T^{1,0}X)$, then we have $(\overline{\partial}\theta)\lrcorner\Omega = \overline{\partial}(\theta\lrcorner\Omega)$ which implies $T_\Omega(\operatorname{Im}\overline{\partial}) \subseteq \operatorname{Im}\overline{\partial}$. On the other hand, given any $\overline{\partial}\tau \in \operatorname{Im}\overline{\partial}$ for some $\tau \in A^{n-1,0}(X)$, there is $\varphi_\tau \in A^0(X, T^{1,0}X)$ such that $\varphi_\tau \lrcorner \Omega = \tau$. Hence $\overline{\partial}\varphi_\tau \in \operatorname{Im}\overline{\partial}$ satisfies $(\overline{\partial}\varphi_\tau)\lrcorner\Omega = \overline{\partial}(\varphi_\tau\lrcorner\Omega) = \overline{\partial}\tau$, it follows that $\operatorname{Im}\overline{\partial} \subseteq T_\Omega(\operatorname{Im}\overline{\partial})$.

Lemma 4.3 (Tian-Todorov Lemma, [6, Lemma 3.3]) *Let X be any complex manifold of dimension n. Given $\Omega \in A^{n,0}(X)$ and $\varphi, \varphi' \in A^{0,1}(X, T^{1,0}X)$, then we have*

$$[\varphi, \varphi']\lrcorner\Omega = -\partial(\varphi\lrcorner(\varphi'\lrcorner\Omega)) + \varphi\lrcorner\partial(\varphi'\lrcorner\Omega) + \varphi'\lrcorner\partial(\varphi\lrcorner\Omega). \tag{4.6}$$

In particular, if $\partial(\varphi'\lrcorner\Omega) = 0$ and $\partial(\varphi\lrcorner\Omega) = 0$, then $[\varphi, \varphi']\lrcorner\Omega \in \operatorname{Im}\partial$, where the Lie bracket $[\varphi, \varphi'] \in A^{0,k+k'}(X, T^{1,0}X)$ is defined by

$$[\varphi, \varphi'] = \sum_{i,j=1}^{n} (\varphi^i \wedge \partial_i \varphi'^j - (-1)^{kk'} \varphi'^i \wedge \partial_i \varphi^j) \otimes \partial_j$$

for $\varphi = \sum\limits_i \varphi^i \partial_i \in A^{0,k}(X, T^{1,0}X)$ *and* $\varphi' = \sum\limits_i \varphi'^i \partial_i \in A^{0,k'}(X, T^{1,0}X)$.

4.1 Proof of Theorem 4.1

In the following, we assume X to be a Calabi-Yau $\partial\overline{\partial}$-manifold. Let Ω be a non-vanishing holomorphic section of K_X. Our goal is to solve the following Maurer-Cartan equation on X:

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi], \quad \varphi \in A^{0,1}(X, T^{1,0}X). \tag{4.7}$$

We begin with a simple observation.

Proposition 4.1 *Given any $[\eta] \in H^1(X, T^{1,0}X)$, there is $\varphi_1 \in [\eta]$, such that $\varphi_1\lrcorner\Omega$ is* d*-closed.*

Proof Since $T_{[\Omega]}([\eta]) = [\eta\lrcorner\Omega] \in H^{n-1,1}(X)$, by $\partial\overline{\partial}$-lemma, there is $\tau \in [\eta\lrcorner\Omega]$ such that $\mathrm{d}\tau = 0$. By the isomorphism of $T_\Omega : A^{0,1}(X, T^{1,0}X) \to A^{n-1,1}(X)$, there is $\varphi_1 \in A^{0,1}(X, T^{1,0}X)$ such that $\varphi_1\lrcorner\Omega = \tau$, and $\overline{\partial}\varphi_1 = 0$ since $\overline{\partial}\tau = 0$. Moreover, $[\varphi_1\lrcorner\Omega] = [\tau] = [\eta\lrcorner\Omega]$ which implies $[\varphi_1] = [\eta]$ by the isomorphism of $T_{[\Omega]}$. Therefore, $\varphi_1 \in [\eta]$ and $\mathrm{d}(\varphi_1\lrcorner\Omega) = \mathrm{d}\tau = 0$.

Given any $[\eta] \in H^1(X, T^{1,0}X)$, in order to solve (4.7) we consider the associated integral equation:

$$(\varphi(t) - \varphi_1 t)\lrcorner\Omega = \frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi(t), \varphi(t)]\lrcorner\Omega). \tag{4.8}$$

Here φ_1 is chosen to be in $[\eta]$ with $\varphi_1\lrcorner\Omega$ d-closed, as guaranteed by Proposition 4.1.

Proposition 4.2 *There exists $\varepsilon > 0$, such that for $|t| < \varepsilon$, (4.8) has a unique solution $\varphi(t)$ which is C^k-continuous and depends holomorphically on t.*

Proof Let Ω^* be the dual of Ω such that $\Omega \lrcorner \Omega^* = 1$. The equation (4.8) is equivalent to

$$\varphi(t) - \varphi_1 t = \frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi(t), \varphi(t)] \lrcorner \Omega) \lrcorner \Omega^*. \tag{4.9}$$

Under the Hölder norm $\|\cdot\|_{k+\alpha}$ as in [9], we have the estimates

$$\|\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\varphi\|_{k+\alpha} \le C_1 \|G_{\overline{\partial}\partial}\varphi\|_{k+3+\alpha} \le C_1 C_2 \|\varphi\|_{k-1+\alpha} \tag{4.10}$$

since here the $\partial\overline{\partial}$-Lapalacian $\square_{\overline{\partial}\partial}$ is a differential operator of order four. Then

$$\begin{aligned}\left\|\frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi,\psi] \lrcorner \Omega) \lrcorner \Omega^*\right\|_{k+\alpha} &\le \left\|\frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi,\psi] \lrcorner \Omega)\right\|_{k+\alpha} \cdot \|\Omega^*\|_{k+\alpha} \\ &\le \frac{1}{2} C_1 C_2 \|[\varphi,\psi] \lrcorner \Omega\|_{k-1+\alpha} \cdot \|\Omega^*\|_{k+\alpha} \\ &\le \frac{1}{2} C_1 C_2 C_3 \|\varphi\|_{k+\alpha} \|\psi\|_{k+\alpha} \|\Omega\|_{k-1+\alpha} \|\Omega^*\|_{k+\alpha}. \end{aligned} \tag{4.11}$$

In the following, we denote $C = C_1 C_2 C_3 \|\Omega\|_{k-1+\alpha} \|\Omega^*\|_{k+\alpha}$. Also note that C is a constant independent of φ, ψ.

Next we use Banach fixed-point theorem to get a solution of (4.9). The completion of the normed space $(A^{0,1}(X, T^{1,0}X), \|\cdot\|_{k+\alpha})$ is denoted by E which is a Banach space. Take $\delta = \frac{1}{2C}$, we define a closed subset F of E:

$$F = \{\varphi \in E \mid \|\varphi\|_{k+\alpha} \le \delta\}.$$

For any $\varphi \in E$, we introduce the operator $K(\varphi) = \frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi,\varphi] \lrcorner \Omega) \lrcorner \Omega^*$, then (4.9) becomes

$$\varphi = \varphi_1 t + K(\varphi).$$

For $\varphi, \psi \in F$, by the estimate (4.11), we have

$$\begin{aligned}\|K(\varphi) - K(\psi)\|_{k+\alpha} &= \left\|\frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi,\varphi] - [\psi,\psi] \lrcorner \Omega) \lrcorner \Omega^*\right\|_{k+\alpha} \\ &= \left\|\frac{1}{2}\partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}([\varphi+\psi, \varphi-\psi] \lrcorner \Omega) \lrcorner \Omega^*\right\|_{k+\alpha} \\ &\le \frac{1}{2} C \|\varphi+\psi\|_{k+\alpha} \cdot \|\varphi-\psi\|_{k+\alpha} \\ &\le \frac{1}{2} \|\varphi-\psi\|_{k+\alpha}.\end{aligned}$$

Moreover, if $\|\varphi_1 t\|_{k+\alpha} \le \frac{\delta}{2}$, then for any $\varphi \in F$, we have

$$\begin{aligned}\|\varphi_1 t + K(\varphi)\|_{k+\alpha} &\le \|\varphi_1 t\|_{k+\alpha} + \|K(\varphi)\|_{k+\alpha} \\ &\le \frac{\delta}{2} + \frac{1}{2} C \delta^2 \le \delta,\end{aligned}$$

thus $\varphi_1 t + K(\varphi) \in F$.

Therefore the two conditions of Banach fixed-point Theorem 2.1 are satisfied provided $\|\varphi_1 t\|_{k+\alpha} \le \frac{\delta}{2}$. Choose ε small enough such that $\|\varphi_1 t\|_{k+\alpha} \le \frac{\delta}{2}$ for $\|t\| < \varepsilon$, then as a consequence of Banach fixed-point Theorem 2.1 we get a solution $\varphi(t)$ of (4.9) which lies in

F and therefore C^k continuous. By the standard regularity of elliptic operators we obtain the smoothness of the solution $\varphi(t)$.

It only remains to prove that the solution $\varphi(t)$ depends holomorphically on t. Note that $\varphi(t)$ is the limit of $\varphi_n(t)$ with respect to $\|\cdot\|_{k+\alpha}$, where $\varphi_n(t)$ is obtained by iteration:

$$\varphi_1(t) := \varphi_1 t, \quad \varphi_n(t) := \varphi_1 t + K(\varphi_{n-1}(t)).$$

It's clear that $\varphi_n(t)$ are holomorphic in t for $|t| < \varepsilon$ by induction. On the other hand, the convergence $\varphi_n(t) \to \varphi(t)$ is uniform in t for $|t| < \varepsilon$, since the following estimate

$$\begin{aligned}\|\varphi_{n+1}(t) - \varphi_n(t)\|_{k+\alpha} &= \|K(\varphi_n(t)) - K(\varphi_{n-1}(t))\|_{k+\alpha} \\ &\le \frac{1}{2}\|\varphi_n(t) - \varphi_{n-1}(t)\|_{k+\alpha}\end{aligned}$$

holds for all $|t| < \varepsilon$, from which one can easily deduce that the convergence

$$\|\varphi_n(t) - \varphi_m(t)\|_{k+\alpha} \to 0 \quad \text{as } n, m \to \infty$$

is uniform in t. Thus the limit $\varphi(t)$ is holomorphic in t.

Next we prove that the above solution $\varphi(t)$ is indeed a solution of Maurer-Cartan equation.

Proposition 4.3 *Given any* $\varphi_1 \in A^{0,1}(X, T^{1,0}X)$ *with* $\mathrm{d}(\varphi_1 \lrcorner \Omega) = 0$. *Suppose*

$$\varphi(t) = \sum_{i\ge 1} \varphi_i t^i \quad \textit{where } \varphi_i \in A^{0,1}(X, T^{1,0}X) \tag{4.12}$$

is the holomorphic solution to the integral equation (4.8) *guaranteed by Proposition* 4.2*, then* $\varphi(t)$ *satisfies the Maurer-Cartan equation*

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]. \tag{4.13}$$

Proof We only need to check that the solution $\varphi(t) = \sum\limits_{k\ge 1} \varphi_k t^k$ to (4.8) satisfies $\overline{\partial}\varphi_1 = 0$ and

$$\overline{\partial}\varphi_k = \sum_{\substack{i+j=k \\ i,j\ge 1}} \frac{1}{2}[\varphi_i, \varphi_j] \tag{4.14}$$

for $k \ge 2$.

First, note that $\overline{\partial}\varphi_1 = 0$ follows directly from $\overline{\partial}(\varphi_1 \lrcorner \Omega) = 0$ and Lemma 4.2. Now, we assume (4.14) holds for $k \le l-1$. In order to complete the induction, we need to show that (4.14) holds for $k = l$.

Since $\varphi(t) = \sum\limits_{k\ge 1} \varphi_k t^k$ satisfies (4.8), we have $\varphi_k \lrcorner \Omega$ is ∂-exact for $k \ge 2$, and

$$\varphi_l \lrcorner \Omega = \partial(\overline{\partial}\partial)^* G_{\overline{\partial}\partial}\Big(\sum_{\substack{i+j=l \\ i,j\ge 1}} \frac{1}{2}[\varphi_i, \varphi_j] \lrcorner \Omega\Big). \tag{4.15}$$

Then the Tian-Todorov Lemma 4.3 shows

$$\sum_{\substack{i+j=l \\ i,j\ge 1}} \frac{1}{2}[\varphi_i, \varphi_j] \lrcorner \Omega \in \mathrm{Im}(\partial). \tag{4.16}$$

Moreover, we have

$$\sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}[\varphi_i,\varphi_j]\lrcorner\Omega\in \mathrm{Ker}(\overline{\partial}). \tag{4.17}$$

Indeed, by inductive hypothesis

$$\begin{aligned}\overline{\partial}\Big(\sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}[\varphi_i,\varphi_j]\lrcorner\Omega\Big) &= \Big(\sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}\overline{\partial}[\varphi_i,\varphi_j]\Big)\lrcorner\Omega\\ &= \Big(\sum_{\substack{i+j=l\\ i,j\geq 1}}[\overline{\partial}\varphi_i,\varphi_j]\Big)\lrcorner\Omega\\ &= \frac{1}{2}\sum_{\substack{i+j+k=l\\ i,j,k\geq 1}}[[\varphi_i,\varphi_j],\varphi_k]\lrcorner\Omega=0, \end{aligned}\tag{4.18}$$

where the last " = " is given by Jacobi identity for the bracket $[\cdot,\cdot]$.

Therefore, by $\partial\overline{\partial}$-lemma,

$$\sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}[\varphi_i,\varphi_j]\lrcorner\Omega\in \mathrm{Im}(\partial\overline{\partial}). \tag{4.19}$$

Then, applying the operator $\overline{\partial}$ to both sides of (4.15), and by using Hodge theory for the Lapalacian operator $\square_{\overline{\partial}\partial}=(\overline{\partial}\partial)(\overline{\partial}\partial)^*+(\overline{\partial}\partial)^*(\overline{\partial}\partial)$, we obtain

$$\begin{aligned}\overline{\partial}(\varphi_l\lrcorner\Omega) &= (\overline{\partial}\partial)(\overline{\partial}\partial)^*G_{\overline{\partial}\partial}\Big(\sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}[\varphi_i,\varphi_j]\lrcorner\Omega\Big)\\ &= \square_{\overline{\partial}\partial}G_{\overline{\partial}\partial}\Big(\sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}[\varphi_i,\varphi_j]\lrcorner\Omega\Big)\\ &= \sum_{\substack{i+j=l\\ i,j\geq 1}}\frac{1}{2}[\varphi_i,\varphi_j]\lrcorner\Omega, \end{aligned}\tag{4.20}$$

where the last equality follows from $\square_{\overline{\partial}\partial}G_{\overline{\partial}\partial}=\mathrm{id}-H_{\overline{\partial}\partial}$ and $H_{\overline{\partial}\partial}\overline{\partial}\partial=0$. The proof is complete.

In conclusion, we have proved the following theorem.

Theorem 4.2 *Given any $\eta\in H^1(X,T^{1,0}X)$, we can choose $\varphi_1\in\eta$, such that $\mathrm{d}(\varphi_1\lrcorner\Omega)=0$. Furthermore, there is $\varepsilon>0$, such that for $|t|<\varepsilon$, there is a unique $\varphi(t)\in A^{0,1}(X,T^{1,0}X)$ which is holomorphic in t and satisfies $(\varphi(t)-\varphi_1 t)\lrcorner\Omega=\frac{1}{2}\partial(\overline{\partial}\partial)^*G_{\overline{\partial}\partial}([\varphi(t),\varphi(t)]\lrcorner\Omega)$ and the following*

(1) $\overline{\partial}\varphi(t)=\frac{1}{2}[\varphi(t),\varphi(t)]$;

(2) *$(\varphi(t)-\varphi_1 t)\lrcorner\Omega$ is ∂-exact and $\partial(\varphi(t)\lrcorner\Omega)=0$.*

Clearly, Theorem 4.2 implies the unobstructedness Theorem 4.1.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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