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∞-Tilting Subcategories in Extriangulated Categories*

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Abstract In this paper, the authors introduce a new definition of ∞-tilting (resp. cotilting) subcategories with infinite projective dimensions (resp. injective dimensions) in an extriangulated category. They give a Bazzoni characterization of ∞-tilting (resp. cotilting) subcategories. Also, they obtain a partial Auslander-Reiten correspondence between ∞-tilting (resp. cotilting) subcategories and coresolving (resp. resolving) subcategories with an $\mathbb{E}$-projective generator (resp. $\mathbb{E}$-injective cogenerator) in an extriangulated category.

Keywords Extriangulated category, ∞-Tilting subcategory, Auslander-Reiten correspondence, Bazzoni characterization
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1 Introduction

The notion of extriangulated categories was introduced by Nakaoka and Palu in [6] as a simultaneous generalization of exact categories and triangulated categories, see also [3, 8, 10]. Exact categories and extension closed subcategories of an extriangulated category are extriangulated categories, while there exist some other examples of extriangulated categories which are neither exact nor triangulated (see [1, 9–10]).

In [10], Zhu and Zhuang introduced tilting subcategories in an extriangulated category and studied their properties. This enables us to treat the tilting theory and its generalizations appeared before in a uniform way. More precisely, they obtained Bazzoni's characterization of tilting (resp. cotilting) subcategories and the Auslander-Reiten correspondence between tilting (resp. cotilting) subcategories and coresolving covariantly (resp. resolving contravariantly) finite subcatgories in the extriangulated category.

Motivated by this idea, we introduce ∞-tilting (resp. cotilting) subcategories with infinite projective dimensions (resp. injective dimensions) in an extriangulated category to generalized some results about tilting (resp. cotilting) subcategory in [10]. More precisely, we also obtain Bazzoni's characterization and the Auslander-Reiten correspondence between ∞-tilting (resp. cotilting) subcategories and coresolving (resp. resolving) subcategories with an $\mathbb{E}$-projective generator (resp. $\mathbb{E}$-injective cogenerator).

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The paper is constructed as follows. In Section 2, we recall the definition of an extriangulated category and outline some basic properties that will be used later. In Section 3, we define ∞-tilting (resp. cotilting) subcategories in an extriangulated category, obtain Bazzoni's characterization and get an one-to-one correspondence between ∞-tilting (resp. cotilting) subcategories and coresolving (resp. resolving) subcatgories with an $\mathbb{E}$-projective generator (resp. $\mathbb{E}$-injective cognerator) which are closed under direct summands and satisfy some conditions. This bijection is the Auslander-Reiten correspondence established in [5] when the extriangulated category is the module category of finite generated left modules over an Artin algebra.

2 Preliminaries

Throughout the article, $\mathscr{C}$ denotes an additive category. All subcategories considered are full additive subcategories closed under isomorphisms. We denote by $\mathscr{C}(A, B)$ the set of morphisms from A to B in $\mathscr{C}$. If $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$, we denote composition of f and g by gf. We recall the definition and some basic properties of extriangulated categories from [6, 10].

Suppose that $\mathscr{C}$ is equipped with a biadditive functor $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$, where Ab is the category of abelian groups. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-extension. Zero element $\delta \in \mathbb{E}(C, A)$ is called the spilt $\mathbb{E}$-extension.

For any $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, since $\mathscr{C}$ and $\mathbb{E}$ are additive, we can define the $\mathbb{E}$-extension

$$\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A').$$

Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $\mathbb{E}$-extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A'), \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviate $\mathbb{E}(C, a)(\delta)$ and $\mathbb{E}(c, A)(\delta)$ to $a_*\delta$ and $c^*\delta$, respectively.

Definition 2.1 (see [6, Definition 2.3]) *A morphism from an $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ to another $\mathbb{E}$-extension $\delta' \in \mathbb{E}(C', A')$ is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ satisfying $a_*\delta = c^*\delta'$.*

We simply denote it as $(a, c) : \delta \to \delta'$.

Let $A, C \in \mathscr{C}$ be any pair of objects. Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathscr{C}$ are said to be equivalent if there exists an isomorphism $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "b", "\cong"'] & C \arrow[d, equal] \\ A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C \end{tikzcd}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A, C \in \mathscr{C}$, we denote as $0 = [A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus C \xrightarrow{[0\ 1]} C]$.

Definition 2.2 (see [6, Definition 2.9]) *Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a realization of $\mathbb{E}$ if it satisfies the following condition.*

Let $\delta \in \mathbb{E}(C, A)$ *and* $\delta' \in \mathbb{E}(C', A')$ *be any two* $\mathbb{E}$*-extension with* $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ *and* $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. *Then, for any morphism* $(a, c) : \delta \to \delta'$, *there exists a morphism* $b \in \mathscr{C}(B, B')$ *which makes the following diagram commutative.*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"'] \\ A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

In this case, we say the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ.

Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say the triplet (a, b, c) realizes (a, c).

Definition 2.3 (see [6, Definition 2.10]) *Let* $\mathscr{C}$, $\mathbb{E}$ *be as above. A realization of* $\mathbb{E}$ *is said to be additive, if it satisfies the following conditions.*

(1) *For any* $A, C \in \mathscr{C}$, *the split* $\mathbb{E}$*-extension* $0 \in \mathbb{E}(C, A)$ *satisfies* $\mathfrak{s}(0) = 0$;

(2) *for any pair of* $\mathbb{E}$*-extension* $\delta \in \mathbb{E}(C, A)$ *and* $\delta' \in \mathbb{E}(C', A')$, $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Definition 2.4 (see [6, Definition 2.12]) *We call the triplet* $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ *an externally triangulated category, or for short, extriangulated category if it satisfies the following conditions*:

(ET1) $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ *is a biadditive functor.*

(ET2) $\mathfrak{s}$ *is an additive realization of* $\mathbb{E}$.

(ET3) *Let* $\delta \in \mathbb{E}(C, A)$ *and* $\delta' \in \mathbb{E}(C', A')$ *be any pair of* $\mathbb{E}$*-extensions, realized as* $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ *and* $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. *For any commutative square in* $\mathscr{C}$,

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

there exists a morphism $(a, c) : \delta \to \delta'$ *which is realized by* (a, b, c).

(ET3)$^{\mathrm{op}}$ *Dual of* (ET3).

(ET4) *Let* (A, δ, D) *and* (B, δ', F) *be two* $\mathbb{E}$*-extensions realized by* $A \xrightarrow{f} B \xrightarrow{f'} D$ *and* $B \xrightarrow{g} C \xrightarrow{g'} F$, *respectively. Then there exists an object* $\mathbb{E} \in \mathscr{C}$, *a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[d, "d"] \\ A \arrow[r, "h"] & C \arrow[r, "h'"] \arrow[d, "g'"] & E \arrow[d, "e"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

in $\mathscr{C}$ *and an* $\mathbb{E}$*-extension* $\delta'' \in \mathscr{C}(E, A)$ *realized by* $A \xrightarrow{h} C \xrightarrow{h'} E$, *which satisfy the following compatibilities*:

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ *realizes* $\mathbb{E}(F, f')(\delta')$,

(ii) $\mathbb{E}(d, A)(\delta'') = \delta$,

(iii) $\mathbb{E}(E,f)(\delta'')=\mathbb{E}(e,B)(\delta')$.

(ET4)$^{\rm op}$ *Dual of* (ET4).

For an extriangulated category $\mathscr{C}$, we use the following notation (see [6, 10]).

• A sequence $A\xrightarrow{a}B\xrightarrow{b}C$ is called conflation if it realizes some $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$, in which case, the morphism a is called a inflation, the morphism b is called a deflation and we call $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ a $\mathbb{E}$-triangle and denote it by (C,δ,A).

• Let $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle, A is called the CoCone of the deflation b: $B\to C$, and we denote it by CoCone(b); C is called the Cone of the inflation a: $A\to B$, and we denote it by Cone(a). Note that the CoCone of a deflation and the Cone of an inflation are well-defined by [6, Remark 3.10].

Remark 2.1 Let $\mathscr{C}$ be an extriangulated category.

(1) (see [6, Remark 2.16]) Both inflations and deflations are closed under composition.

(2) A subcategory $\mathscr{T}$ of $\mathscr{C}$ is called extension-closed if for any $\mathbb{E}$-triangle $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ with $A,\ C\in\mathscr{T}$, we have $B\in\mathscr{T}$.

Definition 2.5 (see [6, Definition 3.23]) *Let $\mathscr{C}$ be an extriangulated category. An object I is called injective if for any $\mathbb{E}$-triangle $A\xrightarrow{x}B\xrightarrow{y}C\overset{\delta}{\dashrightarrow}$ and any morphism $c\in\mathscr{C}(A,I)$, there exists a morphism $b\in\mathscr{C}(B,I)$, satisfying $b\circ x=c$.*

Projective objects are defined dually. The subcategory consisting of injective (resp. projective) objects in $\mathscr{C}$ is denoted by Inj$(\mathscr{C})$(resp.Proj$(\mathscr{C})$).

By [6, Proposition 3.24], an object E is injective if and only if it satisfies $\mathbb{E}(A,E)=0$ for any $A\in\mathscr{C}$. The dual property holds for projective objects in $\mathscr{C}$.

Definition 2.6 *A subcategory $\mathscr{X}\subset\mathscr{C}$ is called coresolving if it contains* Inj$(\mathscr{C})$*, closed under extensions and cones of inflations. Resolving subcategory can be defined dually.*

Definition 2.7 (see [6, Definition 3.25]) *Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. If for any object $A\in\mathscr{C}$, there exists an $\mathbb{E}$-triangle $A\to I\to A_1\overset{\delta}{\dashrightarrow}$, with $I\in$* Inj$(\mathscr{C})$*, then we say the extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough injectives. Dually, if for any object $C\in\mathscr{C}$, there exists an $\mathbb{E}$-triangle $C_1\to P\to C\overset{\sigma}{\dashrightarrow}$, with $P\in$* Proj$(\mathscr{C})$*, then we say the extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough projectives.*

Liu and Nakaoka (see [4, 5.1–5.2]) defined the higher extension groups in an extriangulated category having enough projectives and injectives. They showed the following result.

Lemma 2.1 (see [4, Proposition 5.2]) *Let $A\xrightarrow{f}B\xrightarrow{g}C\overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. For any object $X\in\mathscr{C}$, there are long exact sequences*

$$\cdots\to\mathbb{E}^i(X,A)\xrightarrow{f_*}\mathbb{E}^i(X,B)\xrightarrow{g_*}\mathbb{E}^i(X,C)\to\mathbb{E}^{i+1}(X,A)\xrightarrow{f_*}\mathbb{E}^{i+1}(X,B)\xrightarrow{g_*}\mathbb{E}^{i+1}(X,C)\to\cdots$$

$$\cdots\to\mathbb{E}^i(C,X)\xrightarrow{f^*}\mathbb{E}^i(B,X)\xrightarrow{g^*}\mathbb{E}^i(A,X)\to\mathbb{E}^{i+1}(C,X)\xrightarrow{f^*}\mathbb{E}^{i+1}(B,X)\xrightarrow{g^*}\mathbb{E}^{i+1}(C,X)\to\cdots$$

In particularly, there exist long exact sequences

$$\mathscr{C}(X,A)\xrightarrow{\mathscr{C}(X,f)}\mathscr{C}(X,B)\xrightarrow{\mathscr{C}(X,g)}\mathscr{C}(X,C)\xrightarrow{(\delta_\sharp)_X}\mathbb{E}(X,A)\xrightarrow{f_*}\mathbb{E}(X,B)\xrightarrow{g_*}\mathbb{E}(X,C)\to\cdots$$

$$\mathscr{C}(C,X)\xrightarrow{\mathscr{C}(g,X)}\mathscr{C}(B,X)\xrightarrow{\mathscr{C}(f,X)}\mathscr{C}(A,X)\xrightarrow{(\delta^\sharp)_X}\mathbb{E}(C,X)\xrightarrow{g^*}\mathbb{E}(B,X)\xrightarrow{f^*}\mathbb{E}(A,X)\to\cdots$$

For a subcategory $\mathscr{X}\subseteq\mathscr{C}$, we define

$$\mathscr{X}^{\perp} = \{Y \in \mathscr{C} \mid \mathbb{E}^i(X,Y) = 0,\ \forall i \geq 1,\ X \in \mathscr{X}\}$$

and

$$^{\perp}\mathscr{X} = \{Y \in \mathscr{C} \mid \mathbb{E}^i(Y,X) = 0,\ \forall i \geq 1,\ X \in \mathscr{X}\}.$$

Definition 2.8 *A subcategory $\mathscr{X} \subseteq \mathscr{C}$ is called self-orthogonal provided that*

$$\mathscr{X} \subseteq \mathscr{X}^{\perp}.$$

Definition 2.9 *Let $\mathcal{W}$ be a class of objects in $\mathscr{C}$. An $\mathbb{E}$-triangle $A \to B \to C \overset{\delta}{\dashrightarrow}$ is called to be $\mathscr{C}(\mathcal{W},-)$-exact (resp. $\mathscr{C}(-,\mathcal{W})$-exact) if for any $W \in \mathcal{W}$, the induced sequence of abelian group $\mathscr{C}(W,A) \to \mathscr{C}(W,B) \to \mathscr{C}(W,C) \to 0$ (resp. $\mathscr{C}(C,W) \to \mathscr{C}(B,W) \to \mathscr{C}(A,W) \to 0$) is exact in Ab.*

Definition 2.10 (see [10, lemma 2]) *An $\mathbb{E}$-triangle sequence in $\mathscr{C}$ is defined as a sequence*

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

such that for any n, there are $\mathbb{E}$-triangles $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta_n}{\dashrightarrow}$ and the differential $d_n = g_{n-1}f_n$, where g_n is an inflation and f_n is a deflation.

3 ∞-Tilting (resp. Cotilting) Subcategories

In this section, we begin with the definitions of ∞-tilting (resp. cotilting) subcategories in an extriangulated category $\mathscr{C}$ with enough projectives and injectives. Then we formulated the Bazzoni characterization for ∞-tilting (resp. cotilting) subcategories.

For a subcategory $\mathscr{T}$ of $\mathscr{C}$, denote by $\mathrm{Pres}^{\infty}(\mathscr{T})$(resp.$\mathrm{Copres}^{\infty}(\mathscr{T})$) the subcategory of all objects $A \in \mathscr{C}$ such that there is an infinite $\mathbb{E}$-triangle sequence in $\mathscr{C}$,

$$\cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} A \quad (\text{resp. } A \xrightarrow{d_0} T_0 \xrightarrow{d_1} T_1 \xrightarrow{d_2} T_2 \to \cdots)$$

with $T_i \in \mathscr{T}$ for $i \geq 0$.

If moreover $A \in \mathscr{T}^{\perp}$ and $\mathrm{CoCone}(d_i) \in \mathscr{T}^{\perp}$ (resp. $A \in {}^{\perp}\mathscr{T}$ and $\mathrm{Cone}(d_i) \in {}^{\perp}\mathscr{T}$), the subcategory of all objects $A \in \mathscr{C}$ is denoted by ${}_{\mathscr{T}}\mathcal{X}$ (resp. $\mathcal{X}_{\mathscr{T}}$).

Let $\mathscr{X}$ and $\mathscr{Y}$ be two subcategories of $\mathscr{C}$. We call a subcategory $\mathscr{X}$ an $\mathbb{E}$-projective generator (resp. $\mathbb{E}$-injective cogenerator) of $\mathscr{Y}$ if $\mathscr{X} \subseteq \mathscr{Y}$, $\mathbb{E}(X,Y) = 0$ for any $X \in \mathscr{X}$, $Y \in \mathscr{Y}$ and for any object $Y \in \mathscr{Y}$, there is an $\mathbb{E}$-triangle

$$Y_1 \to X \to Y \overset{\delta}{\dashrightarrow} \quad (\text{resp. } Y \to X \to Y_1 \overset{\delta}{\dashrightarrow})$$

in $\mathscr{C}$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Definition 3.1 *Let $\mathscr{T}$ be a subcategory of an extriangulated category $\mathscr{C}$ closed under direct summands. $\mathscr{T}$ is called an ∞-tilting subcategory if the following conditions are satisfied:*

(1) *$\mathscr{T}$ is an $\mathbb{E}$-projective generator for ${}_{\mathscr{T}}\mathcal{X}$;*

(2) $\mathrm{Inj}(\mathscr{C}) \subseteq {}_{\mathscr{T}}\mathcal{X}$.

Denote add$T$ by the closure of T under finite direct sums and summands. We have that an object $T \in \mathscr{C}$ is called an ∞-tilting object if add$T$ is an ∞-tilting subcategory.

Definition 3.2 *Let $\mathscr{W}$ be a subcategory of an extriangulated category $\mathscr{C}$ closed under direct summands. $\mathscr{T}$ is called an ∞-cotilting subcategory if the following conditions are satisfied:*

(1) *$\mathscr{W}$ is an $\mathbb{E}$-injective cogenerator for $\mathcal{X}_{\mathscr{W}}$;*

(2) $\mathrm{Proj}(\mathscr{C}) \subseteq \mathcal{X}_{\mathscr{W}}$.

An object $W \in \mathscr{C}$ is called an ∞-tilting object if add$W$ is an ∞-cotilting subcategory.

Example 3.1 Let R be an Artin algebra, let mod$R$ be the category of finitely generated left R-modules, and let $T \in \text{mod } R$ be a Wakamatsu tilting (resp. Wakamatsu cotilting) module (see [5]). Then add$T$ is an ∞-tilting (resp. ∞-cotilting) subcategory in our sense. Actually, by [5, Proposition 2.1], we can show that a finitely generated R-module T is a Wakamatsu tilting module if and only if T is an ∞-tilting object in our sense.

Lemma 3.1 *Let $\mathscr{T}$ be a class of objects in $\mathscr{C}$.*

(1) *Consider the following commutative diagram of $\mathbb{E}$-triangles.*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C \arrow[r, dashed, "a_*\delta"] & {} \end{tikzcd}$$

If the first row is $\mathscr{C}(\mathscr{T}, -)$-exact (resp. $\mathscr{C}(-, \mathscr{T})$-exact), then so is the second row.

(2) *Consider the following commutative diagram of $\mathbb{E}$-triangles.*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "c^*\delta"] \arrow[d, "c"] & {} \\ A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta"] & {} \end{tikzcd}$$

If the second row is $\mathscr{C}(\mathscr{T}, -)$-exact (resp. $\mathscr{C}(-, \mathscr{T})$-exact), then so is the first row.

Proof We only prove (1), the proof of (2) is dually. Let $T \in \mathscr{T}$. We have the following commutative diagram in Ab.

$$\begin{tikzcd} \mathscr{C}(T,B) \arrow[r, "\mathscr{C}(T,y)"] \arrow[d, "\mathscr{C}(T,b)"] & \mathscr{C}(T,C) \arrow[r] \arrow[d, equal] & 0 \\ \mathscr{C}(T,B') \arrow[r, "\mathscr{C}(T,y')"] & \mathscr{C}(T,C) & \end{tikzcd}$$

Hence $\mathscr{C}(T, y')$ is epic and the $\mathbb{E}$-triangle $a_*\delta$ is $\mathscr{C}(\mathscr{T}, -)$-exact.

When the $\mathbb{E}$-triangle δ is $\mathscr{C}(-, \mathscr{T})$-exact, the $\mathbb{E}$-triangle $a_*\delta$ is $\mathscr{C}(-, \mathscr{T})$-exact following from [1, Lemma 3.1].

The following proposition indicates the subcategory $\text{Pres}^\infty(\mathscr{C})$ is closed under extensions and cones of inflations in an $\mathbb{E}$-triangle.

A subcategory $\mathscr{T}$ (resp. $\mathscr{W}$) is said to be ∞-quasi-projective (resp. ∞-quasi-injective) if any infinite $\mathbb{E}$-triangle sequence $\cdots \to T_2 \to T_1 \to T_0 \to M$ (resp. $A \to T^0 \to T^1 \to T^2 \to \cdots$) with $T_i \in \mathscr{T}$ (resp. $T^i \in \mathscr{W}$) for $i \geq 0$ is $\mathscr{C}(\mathscr{T}, -)$-exact (resp. $\mathscr{C}(-, \mathscr{W})$-exact).

Proposition 3.1 *Assume that $\mathscr{T}$ is a self-orthogonal and ∞-quasi-projective subcategory of an extriangulated category $\mathscr{C}$. Let $A \to B \to C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle with $A \in \text{Pres}^\infty(\mathscr{T})$. Then the $\mathbb{E}$-triangle (C, δ, A) is $\mathscr{C}(\mathscr{T}, -)$-exact and $C \in \text{Pres}^\infty(\mathscr{T})$ if and only if $B \in \text{Pres}^\infty(\mathscr{T})$.*

Proof ($\Rightarrow$) By the assumption, there exist two $\mathbb{E}$-triangles $K_A \to T_A \xrightarrow{f} A \overset{\delta_A}{\dashrightarrow}$ with $K_A \in \text{Pres}^\infty(\mathscr{T})$, $T_A \in \mathscr{T}$, and $K_C \to T_C \to C \overset{\delta_C}{\dashrightarrow}$ with $K_C \in \text{Pres}^\infty(\mathscr{T})$, $T_C \in \mathscr{T}$. Since

$\mathscr{T}$ is an ∞-quasi-projective subcategory, both $\mathbb{E}$-triangles are $\mathscr{C}(\mathscr{T},-)$-exact. By [6, Corollary 3.12], for any object $T \in \mathscr{T}$, we have the following exact sequence in Ab,

$$\mathscr{C}(T,T_A) \xrightarrow{\mathscr{C}(T,f)} \mathcal{C}(T,A) \xrightarrow{(\delta_A)_\sharp} \mathbb{E}(T,K_A) \to \mathbb{E}(T,T_A).$$

Hence $\mathbb{E}(T,K_A)=0$. By [2, Lemma 3.4], we have the following commutative diagram,

$$\begin{tikzcd}
K_A \arrow[r] \arrow[d] & K_B \arrow[r] \arrow[d] & K_C \arrow[r, dashed, "\delta'"] \arrow[d] & {} \\
T_A \arrow[r, "\binom{1}{0}"] \arrow[d, "f"] & T_A \oplus T_C \arrow[r, "(0\ 1)"] \arrow[d] & T_C \arrow[r, dashed, "0"] \arrow[d] & {} \\
A \arrow[r] \arrow[d, dashed] & B \arrow[r] \arrow[d, dashed] & C \arrow[r, dashed, "\delta"] \arrow[d, dashed] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles. So we get an $\mathbb{E}$-triangle $K_B \to T_A \oplus T_B \to B \dashrightarrow$ with $T_A \oplus T_B \in \mathscr{T}$. Since $\mathbb{E}(T,K_A)=0$, the $\mathbb{E}$-triangle (K_C,δ',K_A) in the first row is $\mathscr{C}(\mathscr{T},-)$-exact. Repeating the same method to the $\mathbb{E}$-triangle (K_C,δ',K_A), and go on, we get $B \in \mathrm{Pres}^{\infty}(\mathscr{T})$.

($\Leftarrow$) Since $B \in \mathrm{Pres}^{\infty}(\mathscr{T})$, there exists an $\mathbb{E}$-triangle $K_B \to T_B \to B \overset{\delta_B}{\dashrightarrow}$ with $T_B \in \mathscr{T}$ and $K_B \in \mathrm{Pres}^{\infty}(\mathscr{T})$. Since $\mathscr{T}$ is an ∞-quasi-projective subcategory, the $\mathbb{E}$-triangle (B,δ_B,K_B) is $\mathscr{C}(\mathscr{T},-)$-exact. Using (ET4), we have the following commutative diagram,

$$\begin{tikzcd}
K_B \arrow[r, equal] \arrow[d] & K_B \arrow[d] & & \\
D \arrow[r] \arrow[d, "m"] & T_B \arrow[r] \arrow[d, "n"] & C \arrow[r, dashed, "\delta'"] \arrow[d, equal] & {} \\
A \arrow[r, "x"] \arrow[d, dashed, "x^*\delta_B"] & B \arrow[r, "y"] \arrow[d, dashed, "\delta_B"] & C \arrow[r, dashed, "\delta"] & {} \\
{} & {} & &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles. By Lemma 3.1, the $\mathbb{E}$-triangle $(A,x^*\delta_B,K_B)$ is $\mathscr{C}(\mathscr{T},-)$-exact. So $D \in \mathrm{Pres}^{\infty}(\mathscr{T})$ by the argument in ($\Rightarrow$). Hence $C \in \mathrm{Pres}^{\infty}(\mathscr{T})$. Since $\mathscr{T}$ is an ∞-quasi-projective subcategory, the $\mathbb{E}$-triangle (C,δ',D) is $\mathscr{C}(\mathscr{T},-)$-exact. Hence the $\mathbb{E}$-triangle (C,δ,A) is $\mathscr{C}(\mathscr{T},-)$-exact also by Lemma 3.1.

Dually, we have following proposition.

Proposition 3.2 *Let $\mathscr{W}$ be an ∞-cotilting subcategory and $A \to B \to C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle with $A \in \mathrm{Copres}^{\infty}(\mathscr{W})$. Then δ is $\mathscr{C}(-,\mathscr{W})$-exact and $C \in \mathrm{Copres}^{\infty}(\mathscr{W})$ if and only if $B \in \mathrm{Copres}^{\infty}(\mathscr{W})$.*

Now, we prove the Bazzoni characterization of ∞-tilting (cotilting) categories in an extriangulated category.

Theorem 3.1 *Let $\mathscr{T}$ be a subcategory of $\mathscr{C}$ closed under direct summands and $\mathscr{T}^{\perp} \subseteq \mathrm{Pres}^{\infty}(\mathscr{T})$. Then $\mathscr{T}$ is an ∞-tilting subcategory which is ∞-quasi-projective if and only if $\mathrm{Pres}^{\infty}(\mathscr{T}) = \mathscr{T}^{\perp}$.*

Proof ($\Leftarrow$) Let (†) $\cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M$ be any infinite $\mathbb{E}$-triangle sequence with $T_i \in \mathscr{T}$, $i \geq 0$. Then there exist $\mathbb{E}$-triangles

$$M_{i+1} \to T_i \to M_i \dashrightarrow$$

with $M_{i+1} = \mathrm{CoCone}(d_i)$ for $i \geq 0$ and $M_0 = M$ by Definition 2.10. So $M_i \in \mathrm{Pres}^{\infty}(\mathscr{T})$ for any $i \geq 0$. Since $\mathrm{Pres}^{\infty}(\mathscr{T}) = \mathscr{T}^{\perp}$, $\mathbb{E}(T, M_i) = 0$ for any $T \in \mathscr{T}$ and $i \geq 0$. Hence the infinite $\mathbb{E}$-triangle sequence (†) is $\mathscr{C}(\mathscr{T}, -)$-exact and $\mathscr{T}$ is an ∞-quasi-projective subcategory. Since $\mathrm{Inj}(\mathscr{C}) \subseteq \mathscr{T}^{\perp} = \mathrm{Pres}^{\infty}(\mathscr{T})$, $\mathrm{Inj}(\mathscr{C}) \subseteq {}_{\mathscr{T}}\mathcal{X}$. Moreover $\mathscr{T}^{\perp} = \mathrm{Pres}^{\infty}(\mathscr{T})$ implies $\mathscr{T}$ is self-orthogonal. So $\mathscr{T}$ is an $\mathbb{E}$-projective generator of ${}_{\mathscr{T}}\mathcal{X}$. Hence $\mathscr{T}$ is an ∞-tilting subcategory by Definition 3.1.

($\Rightarrow$) Let $M \in \mathrm{Pres}^{\infty}(\mathscr{T})$, then there exists an $\mathbb{E}$-triangle $M \to E \to M_1 \overset{\delta_M}{\dashrightarrow}$ with E an injective object. Since $\mathscr{T}$ is an ∞-tilting subcategory, $E \in {}_{\mathscr{T}}\mathcal{X}$ by Definition 3.1. So there exists an $\mathbb{E}$-triangle $K_E \to T_E \to E \overset{\delta_E}{\dashrightarrow}$ with $T_E \in \mathscr{T}$ and $K_E \in \mathscr{T}^{\perp}$, which is clearly $\mathscr{C}(\mathscr{T}, -)$-exact. Using (ET4), we have the following commutative diagram,

$$\begin{tikzcd}
K_E \arrow[r, equal] \arrow[d] & K_E \arrow[d] & & \\
D \arrow[r] \arrow[d, "m"] & T_E \arrow[r] \arrow[d, "n"] & M_1 \arrow[r, dashed, "\delta_D"] \arrow[d, equal] & {} \\
M \arrow[r, "x"] \arrow[d, dashed, "x^*\delta_B"] & E \arrow[r, "y"] \arrow[d, dashed, "\delta_E"] & M_1 \arrow[r, dashed, "\delta_M"] & {} \\
{} & {} & &
\end{tikzcd}$$

where all rows and columns are $\mathbb{E}$-triangles and $\delta_M = m_*\delta_D$. By the assumption that $\mathscr{T}^{\perp} \subseteq \mathrm{Pres}^{\infty}(\mathscr{T})$, we get $K_E \in \mathrm{Pres}^{\infty}(\mathscr{T})$. So $D \in \mathrm{Pres}^{\infty}(\mathscr{T})$ by Proposition 3.1. Thus $M_1 \in \mathrm{Pres}^{\infty}(\mathscr{T})$. Since $\mathscr{T}$ is ∞-quasi-projective, the $\mathbb{E}$-triangle (M_1, δ_M, D) is $\mathscr{C}(\mathscr{T}, -)$-exact. By Lemma 3.1, the $\mathbb{E}$-triangle (M_1, δ_M, M) is also $\mathscr{C}(\mathscr{T}, -)$-exact. By [4, Proposition 5.2], there exists the following exact sequence in Ab:

$$\mathscr{C}(T, E) \to \mathscr{C}(T, M_1) \xrightarrow{(\delta_M)_\sharp} \mathbb{E}(T, M) \to \mathbb{E}(T, E) \quad \text{for any } T \in \mathscr{T}.$$

So $\mathbb{E}(T, M) = 0$. Repeating the same process to M_1, we get $\mathbb{E}(T, M_1) = 0$. Since $\mathbb{E}^{i+1}(T, M) \cong \mathbb{E}^{i}(T, M_1)$ for any $i \geq 1$, we get $M \in \mathscr{T}^{\perp}$. Therefore $\mathrm{Pres}^{\infty}(\mathscr{T}) \subseteq \mathscr{T}^{\perp}$ and $\mathrm{Pres}^{\infty}(\mathscr{T}) = \mathscr{T}^{\perp}$.

Dually, we have the following theorem.

Theorem 3.2 *Let $\mathscr{W}$ be a subcategory of $\mathscr{C}$ closed under direct summands and ${}^{\perp}\mathscr{W} \subseteq \mathrm{Copres}^{\infty}(\mathscr{W})$. Then $\mathscr{W}$ is an ∞-cotilting subcategory which is ∞-quasi-injective if and only if $\mathrm{Copres}^{\infty}(\mathscr{W}) = {}^{\perp}\mathscr{W}$.*

The symbol $\widehat{\mathscr{X}_n}$ (resp. $\check{\mathscr{X}}_n$) denotes the subcategory of objects $A \in \mathscr{C}$ such that there exists an $\mathbb{E}$-triangle sequence

$$X_n \to X_{n-1} \to \cdots \to X_0 \to A \quad (\text{resp. } A \to X^0 \to X^1 \to \cdots \to X^n)$$

with each X_i (resp. X^i) is contained in $\mathcal{X}$.

In [10, Theorem 2], the authors obtained the Auslander-Reiten correspondence for tilting subcategories in an extriangulated category. In that setting, $\mathscr{X}$ is a coresolving subcategory with an $\mathbb{E}$-projective generator such that $\mathscr{C} = \check{\mathscr{X}}_n$ and the latter condition is actually essential to prove the Auslander-Reiten correspondence. In the tilting case, the equality $\mathscr{C} = \check{\mathscr{T}}_n$ follows from the fact that $\mathscr{T}$ has finite projective dimension, which fails in general for the case of the projective dimension is infinite, see [7, Example 3.1] in $\operatorname{mod} R$, where R is an artin algebra.

Conscious of this central difference between the two contexts, we can only prove a partial Auslander-Reiten correspondence for ∞-tilting subcategories in an extriangulated category.

The following lemma ensures that there exists an injective map between the class of ∞-tilting subcategories and coresolving subcategories with an $\mathbb{E}$-projective generator.

Lemma 3.2 *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two subcategories of $\mathscr{C}$, which are both closed under direct summands. If both of them are $\mathbb{E}$-projective generators of a subcategory $\mathscr{X}$, then $\mathscr{T}_1 = \mathscr{T}_2$.*

Proof Since each $\mathscr{T}_i$ is an $\mathbb{E}$-projective generator of $\mathscr{X}$ $\mathscr{T}_i \in \mathscr{X}$ for $i = 1,\ 2$. So there exist two $\mathbb{E}$-triangles $X_1 \to T_2 \to T_1 \overset{\delta_1}{\dashrightarrow}$ and $X_2 \to T_1 \to T_2 \overset{\delta_2}{\dashrightarrow}$ with $T_i \in \mathscr{T}_i$ and $X_i \in \mathscr{X}$ for $i = 1,\ 2$. Clearly $\mathbb{E}(T_i, X_i) = 0$. As $\mathscr{X}$ is closed under direct summands, $\mathscr{T}_1 \subseteq \mathscr{T}_2$ and $\mathscr{T}_2 \subseteq \mathscr{T}_1$. Hence $\mathscr{T}_1 = \mathscr{T}_2$.

Proposition 3.3 *Assume that $\mathscr{T}$ is a subcategory of $\mathscr{C}$ closed under direct summands. Then $\phi : \mathscr{T} \to {}_{\mathscr{T}}\mathcal{X}$ is an injective map between the class of ∞-tilting subcategories and coresolving subcategories with an $\mathbb{E}$-projective generator.*

Proof Assume that $\mathscr{T}$ is an ∞-tilting subcategory. Then $\operatorname{Inj}(\mathscr{C}) \subseteq {}_{\mathscr{T}}\mathcal{X}$ and $\mathscr{T}$ is an $\mathbb{E}$-projective generator of ${}_{\mathscr{T}}\mathcal{X}$ by Definition 3.1. Moreover, by Proposition 3.1, ${}_{\mathscr{T}}\mathcal{X}$ is closed under extensions and Cones of inflations. Hence ${}_{\mathscr{T}}\mathcal{X}$ is a coresolving subcategory. By Lemma 3.2, ϕ is an injective map.

Proposition 3.4 *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Then ψ: $\mathscr{X} \to {}^{\perp}\mathscr{X} \cap \mathscr{X}$ is a surjective map between the class of coresolving subcategories $\mathscr{X}$ with an $\mathbb{E}$-projective generator and the class of ∞-tilting subcategories.*

Proof Let $\mathscr{T}$ be an $\mathbb{E}$-projective generator of $\mathscr{X}$. Then $\mathscr{T} \subseteq {}^{\perp}\mathscr{X} \cap \mathscr{X}$. Assume that $A \in {}^{\perp}\mathscr{X} \cap \mathscr{X}$. Then there exists an $\mathbb{E}$-triangle $B \to T_0 \to A \overset{\delta}{\dashrightarrow}$ with $T_0 \in \mathscr{T}$ and $B \in \mathscr{X}$. Since $A \in {}^{\perp}\mathscr{X}$, $\mathbb{E}(A, B) = 0$. So $A \in \mathscr{T}$. Hence $\mathscr{T} = {}^{\perp}\mathscr{X} \cap \mathscr{X}$.

Since $\mathcal{X}$ is a coresolving subcategory, $\operatorname{Inj}(\mathscr{C}) \subseteq {}_{\mathscr{T}}\mathcal{X}$. Clearly $\mathbb{E}^i(\mathscr{T}, \mathscr{T}) = 0$ for any $i \geq 1$. So $\mathscr{T} \subseteq \mathscr{T}^{\perp}$ is an $\mathbb{E}$-projective generator of ${}_{\mathscr{T}}\mathcal{X}$ and thus $\mathscr{T}$ is an ∞-tilting subcategory. By Proposition 3.3, ψ is a surjective map.

Since $\mathscr{T}$ is an $\mathbb{E}$-projective generator of $\mathscr{X}$, $\mathscr{X} \subseteq \mathcal{T}^{\perp}$. Thus $\mathscr{X} \subseteq {}_{\mathscr{T}}\mathcal{X}$. By Proposition 3.3, we get $\mathscr{X} \subseteq (\phi \circ \psi)(\mathscr{X})$.

We can collect the results in Propositions 3.3–3.4 and obtain the following partial Auslander-Reiten correspondence for ∞-tilting subcategories in an extriangulated category.

Theorem 3.3 *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Then*

(1) *there is an inverse bijection between classes of ∞-tilting subcategories $\mathscr{T}$ and coresolving subcategories $\mathscr{X}$ with an $\mathbb{E}$-projective generator, maximal among those with the same*

$\mathbb{E}$-projective generator, and the assignments are ϕ: $\mathscr{T} \mapsto {}_{\mathscr{T}}\mathcal{X}$ and ψ: $\mathscr{X} \mapsto {}^{\perp}\mathscr{X} \cap \mathscr{X}$.

(2) *There is an inverse bijection between classes of ∞-cotilting subcategories $\mathscr{W}$ and resolving subcategories $\mathscr{Y}$ with an $\mathbb{E}$-injective cogenerator, maximal among those with the same $\mathbb{E}$-injective generator, and the assignments are ϕ: $\mathscr{W} \mapsto \mathcal{X}_{\mathscr{W}}$ and ψ: $\mathscr{Y} \mapsto \mathscr{Y}^{\perp} \cap \mathscr{Y}$.*

Proof We only prove (1) and the proof of (2) is dually.

Let $\mathscr{X}$ be any coresolving subcategory with an $\mathbb{E}$-projective generator $\mathscr{T}$, then $\mathscr{X} \subseteq (\phi \circ \psi)(\mathscr{X}) = {}_{\mathscr{T}}\mathcal{X}$ by Proposition 3.4. Thus, for any ∞-tilting subcategories $\mathscr{T}$, $\phi(\mathscr{T})$ is maximal among those coresolving subcategories with the same $\mathbb{E}$-projective generator $\mathscr{T}$.

Conversely, if $\mathscr{X}$ is a subcategory maximal among those with the previous properties, then $\mathscr{T} = {}^{\perp}\mathscr{X} \cap \mathscr{X}$ is an $\mathbb{E}$-projective generator of $\mathscr{X}$ and $\psi(\mathscr{X}) = \mathscr{T}$ is an ∞-tilting subcategory by Proposition 3.3. So $\mathscr{T}$ is an $\mathbb{E}$-projective generator of ${}_{\mathscr{T}}\mathcal{X}$. Hence ${}_{\mathscr{T}}\mathcal{X} \subseteq \mathscr{X}$ and $\mathscr{X} = (\phi \circ \psi)(\mathscr{X})$.

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Declarations

Conflicsts of interest The authors declare no conflicts of interest.

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