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# Weak Graph Map Homotopy and Its Applications*

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**Abstract** The authors introduce a notion of a weak graph map homotopy (they call it $M$-homotopy), discuss its properties and applications. They prove that the weak graph map homotopy equivalence between graphs coincides with the graph homotopy equivalence defined by Yau et al in 2001. The difference between them is that the weak graph map homotopy transformation is defined in terms of maps, while the graph homotopy transformation is defined by means of combinatorial operations. They discuss its advantages over the graph homotopy transformation. As its applications, they investigate the mapping class group of a graph and the 1-order $MP$-homotopy group of a pointed simple graph. Moreover, they show that the 1-order $MP$-homotopy group of a pointed simple graph is invariant up to the weak graph map homotopy equivalence.

**Keywords** Weak graph map homotopy, Trivial vertex, Strong deformation retract, Mapping class group, $MP$-Homotopy group
**2000 MR Subject Classification** 55P10, 55Q70, 05C99, 05C38

## 1 Introduction

Molecular spaces play an important role in digital topology for image processing, computer graphics and pattern recognition etc (see [16]). And the topological interpretation of the induced intersection graph from the molecular spaces are helpful to understand molecular spaces. So, the homotopy theories of graphs have been constructed in a series of previous papers (see for example, [1–2, 4–7, 10, 20]). In [4], Chen, Yau and Yeh have studied graph homotopy equivalence based on contractible graph transformations introduced in [13–14]. Ivashchenko and Yeh have showed that contractible graph transformations do not change the Euler characteristic and the homology groups of graphs (see [13–17, 21]). In [8], Espinoza, Fras-Armenta and Hernndez have had an application of contractible graph transformation to the computation of

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the persistent homology of the filtered Vietoris-Rips complex, which is an important tool of topological data analysis. Moreover, Boulet, Fieux and Jouve have defined the $s$-homotopy for graphs in [3] by introducing the notion of $s$-dismantlability, which is closely related to the graph homotopy.

In order to deeply study the homotopy transformations of graphs, we introduce notions of weak graph map (see Definition 2.1) and weak graph map homotopy (we call it $M$-homotopy) for graphs (see Definition 2.2). The main results consist of Theorems 2.1–2.5, 3.1 and 4.3. The idea comes from algebraic topology in which topological homotopy and homotopic invariants are studied in [11]. It turns out that the $M$-homotopy equivalence between graphs coincides with the graph homotopy equivalence defined by Yau et al. in [4] (see Theorem 3.1). The difference between them is that the $M$-homotopy transformation is defined in terms of maps, while the graph homotopy transformation in [4] is defined by means of combinatorial operations. One of advantages of $M$-homotopy transformation is that it reflects the transformation process more accurately than the graph homotopy transformation in [4]. Based on the advantage, we investigate some applications of $M$-homotopy transformation, including the mapping class group of a graph and the 1-order $MP$-homotopy group of a pointed simple graph. Some theorems in Sections 2 and 4 (see, Theorems 2.1–2.5, Corollary 2.2, Propositions 4.1–4.2, Theorems 4.2–4.4) show advantages of the $M$-homotopy transformation over the graph homotopy transformation. Moreover, we show that the $M$-homotopy equivalence has close relationships with the $s$-homotopy type in [3], and that the 1-order $MP$-homotopy group of a pointed simple graph and homology groups of a graph are invariant up to the $M$-homotopy equivalence (see Theorem 4.3).

The rest of the paper is organized as follows. In Section 2, we introduce notions of weak graph map and weak graph map homotopy (we call it $M$-homotopy) for graphs, study properties of the $M$-homotopy. In Section 3, we discuss relations between the $M$-homotopy transformation and other homotopy transformations on graphs. Section 4 gives applications of $M$-homotopy to the mapping class group of a graph and the 1-order $MP$-homotopy group of a pointed simple graph.

## 2 Weak Graph Map Homotopy

Let $G = (V, E)$ be a simple graph, i.e., a graph without loops and multiple edges, where $V$ is the vertex set and $E$ is the edge set. We always regard $V$ as a finite set in present paper. The edge set $E$ is also called an adjacency relation. If $u, v \in V$ are adjacent vertices, we just write $uv \in E$. For a subset $V' \subset V$ and for $E' = \{uv \in E \mid u, v \in V'\}$, the graph $G' = (V', E')$ is called the induced subgraph of $G$ (see [19]).

Let $G = (V, E)$ be a simple graph and for each vertex $v \in G$,

$$AN_G(v) = \{v' \in G \mid v'v \in E\} \cup \{v\}.$$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. A mapping of the vertex sets $f : V_1 \to V_2$ is called a graph map if $f(AN_{G_1}(v)) \subset AN_{G_2}(f(v))$ for all $v \in V_1$ (see [6]). $G_1$ and

$G_2$ are called graph isomorphic if there are graph maps $f : G_1 \to G_2$ and $g : G_2 \to G_1$ such that $g \circ f = 1_{G_1}$ and $f \circ g = 1_{G_2}$, $f$ is called a graph isomorphism between $G_1$ and $G_2$, and in the case $G_1 = G_2 = G$, $f$ is called an automorphism of the graph $G$. All automorphisms of $G$ form a group under composition of graph maps, which is denoted by $Aut(G)$.

For our purpose, we review some notions firstly.

Two vertices $u$ and $v$ of a simple graph $G = (V, E)$ are semi-adjacent if $uv \notin E$ and there is a vertex $w$ such that $uw, wv \in E$ (see [9]). Notice that $w$ is not necessarily unique. Let

$$SN_G(v) = \{v' \in G \mid v' \text{ is semi-adjacent to } v\}$$

and

$$SAN_G(v) = AN_G(v) \cup SN_G(v).$$

Contractible graphs (see [4]) are defined inductively by gluing and deleting vertices and edges as follows: (1) The graph of a single vertex is a contractible graph; (2) a graph is called a contractible graph if it can be obtained from a contractible graph by a sequence of the following graph operations:

(GO1) Deleting a vertex: A vertex $v$ of a simple graph $G$ can be deleted if $L_G(v)$ is a contractible graph, where $L_G(v)$ is a subgraph of $G$ induced by $AN_G(v) \setminus \{v\}$.

(GO2) Gluing a vertex: If $G'$ is a contractible subgraph of $G$, then a vertex $v$ not in $G$ can be glued to $G$ to produce a new graph $G''$ so that $L_{G''}(v)$ is $G'$.

(GO3) Deleting an edge: An edge $uv$ of $G$ can be deleted if $L_G(u) \cap L_G(v)$ is a contractible graph.

(GO4) Gluing an edge: For two non-adjacent vertices $u$ and $v$ of $G$, the edge $uv$ can be glued to $G$ if $L_G(u) \cap L_G(v)$ is a contractible graph.

Two graphs are called graph homotopy equivalent (see [4]) if one can be obtained from the other by a sequence of graph operations (GO1-GO4), which is called a graph homotopy transformation. Then we introduce the notions of the weak graph map and the weak graph map homotopy.

**Definition 2.1** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. A map of the vertex sets $f : V_1 \to V_2$ is called a weak graph map denoted by $f : G_1 \to G_2$ if it satisfies the follows: For all $v \in V_1$,*

(i) *if $L_{G_1}(v)$ is not a contractible graph, then $f(AN_{G_1}(v)) \subset AN_{G_2}(f(v))$;*

(ii) *if $L_{G_1}(v)$ is a contractible graph, then $f(AN_{G_1}(v)) \subset SAN_{G_2}(f(v))$.*

It is easy to check that a graph map is a weak graph map. But the converse doesn't hold. For example, as shown in Figure 1, define $r : G \to G \setminus \{a\}$ by $r(a) = b$ and $r(v) = v$ for all $v \neq a$. Then $r$ is not a graph map though it is a weak graph map.

In addition, notice that the composite of weak graph maps does not need to be a weak graph map. For example, as shown in Figure 2, let $f : G_1 \to G_2$ be a weak graph map such that $f(a) = 2$, $f(b) = 4$ and $f(c) = 6$, $g : G_2 \to G_2$ be a weak graph map such that $g(4) = 5$ and $g(i) = i$ for all $i \neq 4$. Then $g \circ f : G_1 \to G_2$ is not a weak graph map since $b \in AN_{G_1}(a)$ and $g \circ f(b) = 5 \notin SAN_{G_2}(2) = SAN_{G_2}(g \circ f(a))$.

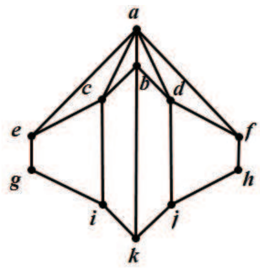


Figure 1 An example of a weak graph map but not a graph map

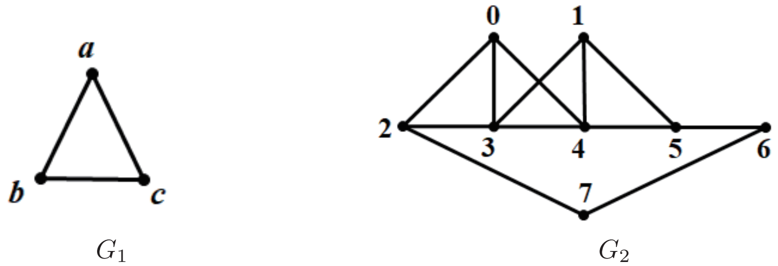


Figure 2 An example showing that the composite of two weak graph maps is not a weak graph map

In order to study the weak graph map homotopy, we need to recall the notion of the strong product (see [12]) of two graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The strong product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is defined as follows: The vertex set $V(G_1 \times G_2)$ is the Cartesian product $V_1 \times V_2$, two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in $G_1 \times G_2$ if one of the following conditions is satisfied: (i) $u_1v_1 \in E_1$ and $u_2 = v_2$; (ii) $u_1 = v_1$ and $u_2v_2 \in E_2$; (iii) $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$ (see [12]), i.e., $AN_{G_1\times G_2}((v_1, v_2)) = AN_{G_1}(v_1) \times AN_{G_2}(v_2)$.

Let $[0, m]_Z$ denote a simple graph whose set of vertices is $\{0, 1, \cdots, m\}$ and the set of edges is $\{i(i + 1) \mid i = 0, 1, \cdots, m - 1\}$.

**Definition 2.2** *Let $G_1$ and $G_2$ be two simple graphs and $f, g : G_1 \to G_2$ be weak graph maps. Suppose that there exists a simple graph $[0, m]_Z$ and a function $F : G_1 \times [0, m]_Z \to G_2$ such that for all $(v, t) \in G_1 \times [0, m]_Z$,*

(i) *$F(v, 0) = f(v)$ and $F(v, m) = g(v)$ for all $v \in G_1$;*

(ii) *if $L_{G_1}(v)$ is not a contractible graph, then $F(AN_{G_1\times[0,m]_Z}((v, t))) \subset AN_{G_2}(F(v, t))$;*

(iii) *if $L_{G_1}(v)$ is a contractible graph, then $F((AN_{G_1}(v)\backslash\{v\})\times AN_{[0,m]_Z}(t)) \subset SAN_{G_2}(F(v, t))$ and $F(\{v\} \times AN_{[0,m]_Z}(t)) \subset AN_{G_2}(F(v, t))$.*

*Then we say that $F$ is a weak graph map homotopy (denoted by $M$-homotopy) between $f$ and $g$, and $f$ and $g$ are $M$-homotopic, denoted by $f \simeq_M g$.*

We say that a graph $G$ is $M$-contractible if the identity map $1_G$ is $M$-homotopic to a constant map $c : G \to G$.

It is not hard to prove that the $M$-homotopy relation is an equivalence relation.

**Definition 2.3** *Let $(G_1, G_1')$ be a pair of simple graphs (i.e., $G_1'$ is an induced subgraph of $G_1$), $G_2$ be a simple graph and $f, g : G_1 \to G_2$ be two weak graph maps. Suppose that there exists an $M$-homotopy $F : G_1 \times [0, m]_Z \to G_2$ between $f$ and $g$ such that for all $t \in [0, m]_Z$, $F_t(v) = f(v) = g(v)$ for all $v \in G_1'$. Then we call $F$ an $M$-homotopy relative to $G_1'$ between $f$ and $g$, and we say that $f$ and $g$ are $M$-homotopic relative to $G_1'$, denoted by $f \simeq_{M.rel\ G_1'} g$.*

A retraction of a simple graph $G$ onto an its induced subgraph $G'$ is a weak graph map $r : G \to G$ such that $r(G) = G'$ and $r \mid_{G'}$ is the identity map $1_{G'}$. We need to notice that, for a retraction of $G$ onto an its induced subgraph $G'$, $r : G \to G'(\subset G)$ does not need to be a weak graph map but the inclusion $i : G' \hookrightarrow G$ and $r = i \circ r : G \to G' \hookrightarrow G$ are weak graph maps. For example, for the simple graph $G_2$ in Figure 2 and its induced subgraph $G_2'$ with the vertex set $V' = \{0, 1, 2, 3, 5, 6, 7\}$, let $r : G_2 \to G_2'$ be a map such that $r(4) = 5$ and $r(i) = i$ for all $i \neq 4$. Then $r$ is not a weak graph map since $0 \in AN_{G_2}(4)$ and $r(0) = 0 \notin SAN_{G_2'}(5) = SAN_{G_2'}(r(4))$. However, the inclusion $i : G_2' \hookrightarrow G_2$ and $r = i \circ r : G_2 \to G_2' \hookrightarrow G_2$ are weak graph maps. If the identity map $1_G$ is $M$-homotopic to a retraction $r : G \to G$ relative to $r(G) \subset G$, then we say that $r(G)$ is an $M$-strong deformation retract of $G$. As a special case, if there is a vertex $v_0 \in G$ such that $v_0$ is an $M$-strong deformation retract of $G$, then we say that $(G, v_0)$ is pointed $M$-contractible.

By $M$-strong deformation retracts, one could define the notion of $M$-homotopy equivalence as follows.

**Definition 2.4** *Two simple graphs $G_1$ and $G_2$ are called $M$-homotopy equivalent if there exists a sequence of simple graphs $G_1 = G_0', G_1', \cdots, G_n' = G_2$ such that one of the two simple graphs $G_i'$ and $G_{i+1}'$ is an $M$-strong deformation retract of the other for every $0 \leq i \leq n-1$.*

Clearly this defines an equivalence relation. In addition, if $(G, v_0)$ is pointed $M$-contractible, then $G$ is $M$-homotopy equivalent to a graph with a single vertex. A transformation of a graph up to $M$-homotopy equivalence is called an $M$-homotopy transformation.

**Definition 2.5** *A vertex $v$ of a simple graph $G$ is a trivial vertex if $L_G(v)$ is a contractible graph.*

**Proposition 2.1** *Let $G$ be a simple graph and $v \in G$. If there is a vertex $u$ in the set $AN_G(v) \setminus \{v\}$ that is adjacent to any vertex $z \in AN_G(v) \setminus \{v, u\}$, then $v$ is a trivial vertex.*

**Proof** If there is a vertex $u$ in the set $AN_G(v) \setminus \{v\}$ that is adjacent to any vertex $z \in AN_G(v) \setminus \{v, u\}$, then $L_G(v)$ is a cone graph which is a contractible graph by [4, Lemma 3.3]. So $v$ is a trivial vertex of $G$.

However, the converse doesn't hold. For example, as Figure 1, the vertex $a$ is a trivial vertex and there is no vertex $u$ in $AN_G(a) \setminus \{a\}$ that is adjacent to any vertex $z \in AN_G(a) \setminus \{a, u\}$.

Now we have the following theorem.

**Theorem 2.1** *Let $G$ be a simple graph and $v_0, v_1, \cdots, v_n$ be trivial vertices of $G$. If $(AN_G(v_i) \setminus \{v_i\}) \cap (AN_G(v_j) \setminus \{v_j\}) = \emptyset$ $(0 \le i, j \le n$ and $i \ne j)$, then $G \setminus \{v_0, v_1, \cdots, v_n\}$ is an $M$-strong deformation retract of $G$.*

**Proof** Assume that $v_i$ $(0 \le i \le n)$ is a trivial vertex. For any $u_i \in L_G(v_i)$, define a map $r: G \to G$ by $r(v_i) = u_i$ and $r(v') = v'$ for all $v' \notin \{v_0, v_1, \cdots, v_n\}$. Then $r$ is a retraction of $G$ onto $G \setminus \{v_0, v_1, \cdots, v_n\}$. In fact, for all $w_i \in L_G(v_i)$, $r(w_i) = w_i \in SAN_G(u_i) = SAN_G(r(v_i))$, so $r$ is a weak graph map. It's immediate that $1_G \simeq_{M.rel\ (G\setminus\{v_0,v_1,\cdots,v_n\})} r$.

**Corollary 2.1** *Let $G$ be a simple graph and $v \in G$ a trivial vertex. Then $G \setminus \{v\}$ is an $M$-strong deformation retract of $G$.*

**Remark 2.1** Let $G$ be a simple graph and $v \in G$. If there is a vertex $u$ in the set $AN_G(v) \setminus \{v\}$ that is adjacent to any vertex $z \in AN_G(v) \setminus \{v, u\}$, then a map $f: G \to G \setminus \{v\}$ given by $f(v) = u$ and $f(v') = v'$ for all $v' \ne v$, is called a folding of $G$ at the vertex $v$ (see [6]). In terms of $M$-homotopy, $f$ is a retraction of $G$ onto $G \setminus \{v\}$ by Proposition 2.1. However, the converse doesn't hold. For example, as Figure 1, $r: G \to G$ given by $r(a) = b$ and $r(v') = v'$ for all $v' \ne a$, then $r$ is not a folding of $G$ at the vertex $a$ though it is a retraction of $G$ onto $G \setminus \{a\}$.

Later, we will show that the converse of Corollary 2.1 does also hold (see Corollary 2.3). For our purpose, we give the following theorem which provides a useful property of two weak graph maps which are already known to be $M$-homotopic.

**Theorem 2.2** *Let $G_1$ and $G_2$ be two simple graphs and $f, g: G_1 \to G_2$ be weak graph maps. If $f \simeq_M g$, then there exists a sequence $f = f_0, f_1, \cdots, f_n = g$ such that for every $0 \le i \le n-1$, $f_i$ is a weak graph map and there is a vertex $v_i \in G_1$ with the following properties:*

*(1) $f_i(v)$ and $f_{i+1}(v)$ coincide in $G_1 \setminus \{v_i\}$, and*

*(2) $f_i(v_i) \in AN_{G_2}(f_{i+1}(v_i))$.*

**Proof** Since $f \simeq_M g$, there is a simple graph $[0, m]_Z$ and a mapping $F: G_1 \times [0, m]_Z \to G_2$ such that for all $(v, t) \in G_1 \times [0, m]_Z$, (i) $F(v, 0) = f(v)$ and $F(v, m) = g(v)$ for all $v \in G_1$; (ii) if $L_{G_1}(v)$ is not a contractible graph, then $F(AN_{G_1 \times [0,m]_Z}((v, t))) \subset AN_{G_2}(F(v, t))$; (iii) if $L_{G_1}(v)$ is a contractible graph, then $F(AN_{G_1}(v) \setminus \{v\} \times AN_{[0,m]_Z}(t)) \subset SAN_{G_2}(F(v, t))$ and $F(\{v\} \times AN_{[0,m]_Z}(t)) \subset AN_{G_2}(F(v, t))$.

Let $i_0 = \min\{j \mid \exists\ v \in G_1 \text{such that } f_0(v) \ne F(v, j)\}$ and $A_0 = \{v \in G_1 \mid f_0(v) \ne F(v, i_0)\} (\ne \emptyset)$. Choose $v_0 \in A_0$, we may define $f_1: G_1 \to G_2$ by $f_1(v)|_{G_1\setminus\{v_0\}} = f_0(v)|_{G_1\setminus\{v_0\}}$

and $f_1(v_0) = F(v_0, i_0)$.

For all $v' \in AN_{G_1}(v_0) \setminus \{v_0\}$, then

$$(v', i_0 - 1) \in AN_{G_1}(v_0) \times AN_{[0,m]_Z}(i_0) = AN_{G_1 \times [0,m]_Z}((v_0, i_0)).$$

If $L_{G_1}(v_0)$ is not a contractible graph, then $f_1(v') = f_0(v') = F(v', i_0 - 1) \in AN_{G_2}(F(v_0, i_0)) = AN_{G_2}(f_1(v_0))$; if $L_{G_1}(v_0)$ is a contractible graph, then

$$f_1(v') = f_0(v') = F(v', i_0 - 1) \in SAN_{G_2}(F(v_0, i_0)) = SAN_{G_2}(f_1(v_0)).$$

Therefore, $f_1$ is a weak graph map.

By induction and finiteness of $G_1$ and $G_2$, we arrive at the result.

**Definition 2.6** *Let $G$ be a simple graph and $G'$ be an induced subgraph of $G$.*

(i) *If $G$ has no trivial vertices, then $G$ is called a minimal simple graph.*

(ii) *If all the trivial vertices of $G$ are in $G'$, then the pair $(G, G')$ of simple graphs is called a minimal pair.*

**Theorem 2.3** *Let $G$ be a simple graph and $(G, G')$ be a minimal pair. A weak graph map $f : G \to G$ is $M$-homotopic to the identity $1_G$ relative to $G'$ if and only if $f = 1_G$.*

**Proof** According to Theorem 2.2, if $f \simeq_{M.rel\ G'} 1_G$, then there is a sequence $1_G = f_0, f_1, \cdots, f_n = f$ such that for every $0 \le i \le n-1$, $f_i$ is a weak graph map and there is a vertex $v_i \in G$ such that $f_i(v)$ and $f_{i+1}(v)$ coincide in $G \setminus \{v_i\}$ and $f_i(v_i) \in AN_G(f_{i+1}(v_i))$. We suppose that $f_1|_{G \setminus \{v_0\}} = 1_{G \setminus \{v_0\}}$ and $f_1(v_0) \neq v_0$. Then $v_0 \notin G'$, hence $L_G(v_0)$ is not a contractible graph. Then $v' = f_1(v') \in AN_G(f_1(v_0))$ for all $v' \in AN_G(v_0) \setminus \{v_0\}$ since $f_1$ is a weak graph map. Since $f_1(v_0) \in AN_G(v_0)$, $v_0$ is a trivial vertex by Proposition 2.1, which is a contradiction.

**Corollary 2.2** *Let $G$ be a minimal simple graph. A weak graph map $f : G \to G$ is $M$-homotopic to the identity $1_G$ if and only if $f = 1_G$.*

**Proof** Since $G$ is a minimal simple graph, $G' = \emptyset$ in Theorem 2.3. So $f : G \to G$ is $M$-homotopic to the identity $1_G$ if and only if $f$ is $M$-homotopic to $1_G$ relative to $\emptyset$ if and only if $f = 1_G$.

**Theorem 2.4** *Let $G$ be a simple graph and $(G, G')$ $(G' \neq \emptyset)$ be a pair of graphs. If $G \setminus G'$ is an $M$-strong deformation retract of $G$, then there is a trivial vertex $v$ of $G$ such that $v \in G'$.*

**Proof** Suppose that there is no trivial vertex of $G$ in $G'$. Then $(G, G \setminus G')$ is a minimal pair. Since $G \setminus G'$ is an $M$-strong deformation retract of $G$, $1_G \simeq_{M.rel\ (G \setminus G')} r$, where $r : G \to G$ is a retraction of $G$ onto $G \setminus G'$. According to Theorem 2.3, $r = 1_G$, which is a contradiction.

**Corollary 2.3** *Let $G$ be a simple graph and $v \in G$. If $G \setminus \{v\}$ is an $M$-strong deformation retract of $G$, then $v$ is a trivial vertex of $G$.*

Now we arrive at one of the main theorems.

**Theorem 2.5** *Let $G$ be a simple graph and $\{v_0, v_1, \cdots, v_n\} \subset V(G)$. If $G \setminus \{v_0, v_1, \cdots, v_n\}$ is an $M$-strong deformation retract of $G$, then there exists a sequence*

$$G = G_0, G_1 = G_0 \setminus \{v_{i_0}\}, G_2 = G_1 \setminus \{v_{i_1}\}, \cdots, G_{n+1} = G_n \setminus \{v_{i_n}\}$$

*($\{v_{i_0}, v_{i_1}, \cdots, v_{i_n}\} = \{v_0, v_1, \cdots, v_n\}$) such that $G_{j+1}$ is an $M$-strong deformation retract of $G_j$ ($0 \le j \le n$).*

**Proof** According to Theorem 2.4, there is $v_{i_0} \in \{v_0, v_1, \cdots, v_n\}$ such that $v_{i_0}$ is a trivial vertex of $G$ since $G \setminus \{v_0, v_1, \cdots, v_n\}$ is an $M$-strong deformation retract of $G$. By Corollary 2.1, $G_1 = G_0 \setminus \{v_{i_0}\}$ is also an $M$-strong deformation retract of $G$. So we have the following diagram:

$$\begin{tikzcd}
G \arrow[r, shift left, "r_0^1"] \arrow[dr, shift left, "r_0^{n+1}"] & G_1 \arrow[l, shift left, "i_1^0"] \arrow[d, shift left, "r_1^{n+1}"] \\
& G_{n+1} \arrow[ul, shift left, "i_{n+1}^0"] \arrow[u, shift left, "i_{n+1}^1"]
\end{tikzcd}$$

where $r_0^1 : G \to G$ and $r_0^{n+1} : G \to G$ are retractions of $G$ onto its induced subgraphs $G_1$ and $G_{n+1}$, respectively; $r_1^{n+1} : G_1 \to G_1$ is a retraction of $G_1$ onto an its induced subgraph $G_{n+1}$; $i_1^0 : G_1 \to G$, $i_{n+1}^0 : G_{n+1} \to G$, $i_{n+1}^1 : G_{n+1} \to G_1$ are the inclusions. Because $G_1$ and $G_{n+1}$ are both $M$-strong deformation retracts of $G$, $r_0^1 \circ i_1^0 = 1_{G_1}$, $i_{n+1}^0 \circ r_0^{n+1} \simeq_{M.rel\ G_{n+1}} 1_G$. Since

$$r_1^{n+1} = r_0^{n+1} \circ i_1^0,$$

and

$$i_{n+1}^1 = r_0^1 \circ i_{n+1}^0,$$

$$r_1^{n+1} = i_{n+1}^1 \circ r_1^{n+1} = r_0^1 \circ i_{n+1}^0 \circ r_0^{n+1} \circ i_1^0 \simeq_{M.rel\ G_{n+1}} r_0^1 \circ 1_G \circ i_1^0 = r_0^1 \circ i_1^0 = 1_{G_1}.$$

Therefore, $G_{n+1}$ is also an $M$-strong deformation retract of $G_1$. By Theorem 2.4, there is $v_{i_1} \in \{v_0, v_1, \cdots, v_n\} \setminus \{v_{i_0}\}$ such that $v_{i_1}$ is a trivial vertex of $G_1$. The result holds by the mathematical induction.

From Theorem 2.5, one can find all $M$-strong deformation retracts of a simple graph by removing trivial vertices one by one. For example, if a simple graph $G$ is a connected tree then there exists a $v_0 \in G$ such that $(G, v_0)$ is pointed $M$-contractible, i.e., an $M$-strong deformation retract of $G$ can be obtained by removing trivial vertices one by one until there is a single vertex.

## 3 Relations Between the $M$-Homotopy Transformation and Other Homotopy Transformations on Graphs

In this section we mainly compare the $M$-homotopy transformation defined in terms of maps with the graph homotopy transformation in [4] and the $s$-homotopy transformation in [3] defined by means of combinatorial operations. Some theorems (see, Theorems 2.1–2.5, Corollary 2.2) in Section 2 have shown the advantages of the $M$-homotopy transformation, in next section we further discuss its applications.

By definitions, the trivial vertex is coincident with the deletable vertex in [4, 14–17]. Furthermore, we have another main theorem.

**Theorem 3.1** *Let $G_1$ and $G_2$ be two simple graphs. $G_1$ and $G_2$ are $M$-homotopy equivalent if and only if they are graph homotopy equivalent* (*in the sense of Yau et al.* [4]).

**Proof** [4, Lemma 3.4] proves that the edge deletion (gluing) can be realized by the composition of a vertex gluing (deletion) and a vertex deletion (gluing). And its proof shows that a vertex $v$ gluing or deletion satisfies that $L_G(v)$ is a contractible graph, i.e., $v$ is a trivial vertex.

If $G_1$ and $G_2$ are graph homotopy equivalent, then one of the two simple graphs can be obtained from the other by a sequence of vertex deletion operations (GO1) and vertex gluing operations (GO2). By Corollary 2.1, a vertex $v$ gluing or deletion satisfies that one of the two simple graphs $G_v^0$ and $G_v^1$ is an $M$-strong deformation retract of the other, where $G_v^0$ is the simple graph before changing the vertex $v$ and $G_v^1$ is the simple graph after changing the vertex $v$. So there exists a sequence of simple graphs $G_1 = G'_0, G'_1, \cdots, G'_n = G_2$ such that one of the two simple graphs $G'_i$ and $G'_{i+1}$ is an $M$-strong deformation retract of the other for every $0 \le i \le n-1$, i.e., $G_1$ and $G_2$ are $M$-homotopy equivalent.

Conversely, if $G_1$ and $G_2$ are $M$-homotopy equivalent, then there exists a sequence of simple graphs $G_1 = G'_0, G'_1, \cdots, G'_n = G_2$ such that one of the two simple graphs $G'_i$ and $G'_{i+1}$ is an $M$-strong deformation retract of the other for every $0 \le i \le n-1$. Without loss of generality, suppose $G'_{i+1}$ is an $M$-strong deformation retract of $G'_i$. According to Theorem 2.5, there exists a sequence

$$G'_i = G''_0, G''_1 = G''_0 \setminus \{v_{i_0}\}, G''_2 = G''_1 \setminus \{v_{i_1}\}, \cdots, G'_{i+1} = G''_n \setminus \{v_{i_n}\}$$

($\{v_{i_0}, v_{i_1}, \cdots, v_{i_n}\} \subset G'_i$) such that $G''_{j+1}$ is an $M$-strong deformation retract of $G''_j$ ($0 \le j \le n$). Therefore, one of the two simple graphs $G_1$ and $G_2$ can be obtained from the other by removing or gluing trivial vertices one by one, i.e., by a sequence of graph operations (GO1-GO2). Therefore, $G_1$ and $G_2$ are graph homotopy equivalent.

Therefore, the $M$-homotopy equivalence coincides with the graph homotopy equivalence.

**Remark 3.1** The clique complex of a simple graph $G$ is an abstract simplicial complex with all complete subgraphs of $G$ as its faces (see [18]). Homology groups of $G$ are defined as homology groups of the clique complex of $G$. If $v \in G$ is a trivial vertex, then $G \setminus \{v\}$ has the same homology groups as $G$ by [14], i.e., the $M$-homotopy transformation preserves the invariance of homology groups. In next section, we will introduce the 1-order homotopy group and show the invariance up to the $M$-homotopy transformation (see Theorem 4.3 for more details).

Let us investigate the relations between the $M$-homotopy transformation and $s$-homotopy transformation in [3].

We recall the notion of $s$-dismantlable in [3] as follows: Let $G$ be a simple graph. A vertex $v$ of $G$ is called dismantlable if there is another vertex $v' \neq v$ such that $AN_G(v) \subset AN_G(v')$. A graph $G$ is called dismantlable if one can write $V(G) = \{v_1, v_2, \cdots, v_n\}$ such that $v_i$ is dismantlable in the subgraph induced by $\{v_1, v_2, \cdots, v_i\}$ for $2 \leq i \leq n$. A vertex $v$ of a simple graph $G$ is called $s$-dismantlable in $G$ if $L_G(v)$ is dismantlable (see [3]). Two simple graphs $G$ and $G'$ have the same $s$-homotopy type if there is a sequence $G = G_0, G_1, \cdots, G_n = G'$ of simple graphs such that $G = G_0 \xrightarrow{s} G_1 \xrightarrow{s} \cdots \xrightarrow{s} G_n = G'$, where each arrow $\xrightarrow{s}$ represents the suppression or the addition of an $s$-dismantlable vertex (see [3]). The above process is called an $s$-homotopy transformation.

From [3, Proposition 5.1(1)], we know that if two simple graphs $G_1$ and $G_2$ have the same $s$-homotopy type, then they are $M$-homotopy equivalent.

In summary, the $M$-homotopy transformation is defined in terms of maps, and the graph homotopy transformation and the $s$-homotopy transformation are defined by combinatorial operations. Simple graphs $G_1$ and $G_2$ are $M$-homotopy equivalent if and only if they are graph homotopy equivalent. If simple graphs $G_1$ and $G_2$ have the same $s$-homotopy type, then they are $M$-homotopy equivalent.

In next section, we further show some advantages of the $M$-homotopy over the graph homotopy and $s$-homotopy by investigating applications of the $M$-homotopy.

## 4 Applications of $M$-Homotopy

In this section, as applications of $M$-homotopy, we investigate the mapping class group of a simple graph and the 1-order $MP$-homotopy group of a pointed simple graph.

### 4.1 Mapping class group up to $M$-homotopy

As an application of $M$-homotopy, we introduce the mapping class group of a graph $G = (V, E)$. It is defined by

$$MCG(G) = Aut(G)/\left(M\text{-homotopy}\right) = Aut(G)/Aut_0(G),$$

namely, the group of $M$-homotopy classes of all automorphisms of $G$, where $Aut_0(G)$ is the subgroup of $Aut(G)$ consisting of elements that are $M$-homotopic to the identity $1_G$.

In general, there is a short exact sequence of groups:

$$1 \to Aut_0(G) \to Aut(G) \to MCG(G) \to 1.$$

The study of the mapping class group as a quotient group of the automorphism group is helpful to understand the structure of the automorphism group, which has a close relation to symmetries of graphs. Moreover, the mapping class group is invariant up to graph isomorphism.

By definition, it is immediate to have $MCG(G) = 0$ if $G$ is a complete graph.

**Example 4.1** The mapping class group $MCG(G)$ of a graph $G$ as shown in Figure 3.

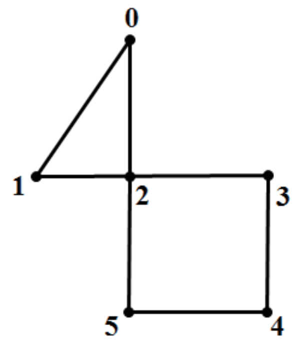


Figure 3 A graph $G$ with $MCG(G) = Z_2$

For the simple graph $G$ in Figure 3,

$$Aut(G) = \langle e, a, b, ab \mid a^2 = b^2 = e, ab = ba\rangle, \quad Aut_0(G) = \langle e, a \mid a^2 = e\rangle,$$

where

$$e = 1_G, \quad a = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(i.e., exchange of two vertices 0 and 1),

$$b = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 5 & 4 & 3 \end{pmatrix}.$$

So,

$$MCG(G) = Aut(G)/Aut_0(G) = \langle e, b \mid b^2 = e\rangle = Z_2,$$

which reflects the symmetry of the graph $G$ with a symmetric axis determined by the two vertices 2 and 4.

**Definition 4.1** *Let $G = (V, E)$ be a simple graph. $G$ is called a semi-complete graph if any two vertices $u$ and $v$ of $G$ are semi-adjacent and $L_G(v)$ is a contractible graph for every $v \in G$.*

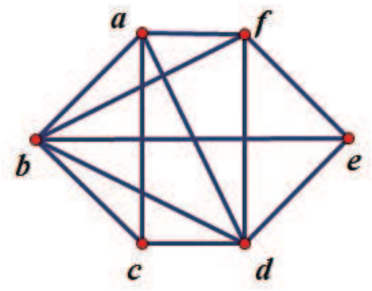


Figure 4 An example of a semi-complete graph $G$

For example, as shown in Figure 4, $G$ is a semi-complete graph.

Then we have immediately the following proposition.

**Proposition 4.1** *Let $G = (V, E)$ be a semi-complete graph. Then $MCG(G) = 0$.*

By Corollary 2.2, we have the following proposition.

**Proposition 4.2** *If $G$ is a minimal simple graph, then $MCG(G) = Aut(G)$.*

The above propositions show some advantages of the $M$-homotopy transformation over the graph homotopy transformation and $s$-homotopy transformation.

### 4.2 $MP$-homotopy group

A pointed simple graph $G^*$ is a simple graph $G$ with a fixed base vertex $*_G \in V(G)$. A pointed weak graph map $f : G^* \to H^*$ between pointed simple graphs is a weak graph map $f : G \to H$ such that $f(*_G) = *_H$.

Two pointed weak graph maps $f, g : G^* \to H^*$ are called $M$-homotopic if they are $M$-homotopic relative to $*_G$.

#### 4.2.1 Construction of $\pi_0$

Let $G^*$ be a pointed simple graph, $V_2^* = \{0, 1\}$ be the pointed simple graph consisting of two isolated vertices with the base vertex $*_{V_2} = 0$, and $[V_2^*, G^*]$ be the set of pointed weak graph maps from $V_2^*$ to $G^*$. It is clear that there is a one to one correspondence between the set of such maps and the set of vertices of the simple graph $G$.

For $\varphi \in [V_2^*, G^*]$, we denote by $[\varphi]_M$ the $M$-homotopy class of the element $\varphi$, and $\pi_0(G^*) = \{[\varphi]_M \mid \varphi \in [V_2^*, G^*]\}$. The set $\pi_0(G^*)$ coincides with the set of path connected components of $G$. In particular, $G$ is path connected if and only if $\pi_0(G^*) = \{[c^*]_M\}$, where $c^* : V_2^* \to *_G$ is the constant map.

**Theorem 4.1** *Let $G^*$ and $H^*$ be two pointed simple graphs. Any pointed weak graph map $f : G^* \to H^*$ induces a map $\pi_0(f) : \pi_0(G^*) \to \pi_0(H^*)$. If $f, g : G^* \to H^*$ are $M$-homotopic, then $\pi_0(f) = \pi_0(g)$.*

**Proof** For $[\varphi]_M$ being presented by a pointed weak graph map $\varphi : V_2^* \to G^*$, we define $\pi_0(f)([\varphi]_M) = [f \circ \varphi]_M \in \pi_0(H^*)$. If there is a pointed weak graph map $\phi : V_2^* \to G^*$ such that $\phi \simeq_M \varphi$, it is not hard to check that $f \circ \phi \simeq_M f \circ \varphi$, that is, $[f \circ \phi]_M = [f \circ \varphi]_M$. So $\pi_0(f)$ is well-defined.

For the $M$-homotopic maps $f, g : G^* \to H^*$, we have $f \circ \varphi \simeq_M g \circ \varphi$ for any pointed weak graph map $\varphi : V_2^* \to G^*$, that is $[f \circ \varphi]_M = [g \circ \varphi]_M$. So, $\pi_0(f) = \pi_0(g)$.

### 4.2.2 $MP$-Homotopy and 1-order $MP$-homotopy group $\pi_1$

For $[0, m]_Z$, a pointed simple graph $[0, m]_Z^*$ always has the base vertex $* = 0$.

**Definition 4.2** *A pointed path-map on a pointed simple graph $G^*$ is a pointed weak graph map (in fact, pointed graph map) $\varphi : [0, m]_Z^* \to G^*$. A loop on $G^*$ is a pointed path-map $\varphi : [0, m]_Z^* \to G^*$ such that $\varphi(m) = *_G$.*

**Definition 4.3** *Let $G^*$ be a pointed simple graph and $\varphi : [0, m]_Z^* \to G^*$, $\phi : [0, m']_Z^* \to G^*$ be two pointed path-maps. We say that $\varphi$ and $\phi$ are elementary $MP$-homotopic, if*

*(1) for $m \geq m'$, there is a vertex $v_0 \in [0, m]_Z^*$ and a monotonically increasing graph map $h : [0, m]_Z^* \to [0, m']_Z^*$ such that* (i) *$h(0) = 0$, $h(m) = m'$, and $\varphi \simeq_M \phi \circ h$;* (ii) *$\varphi(v)$ and $\phi(h(v))$ coincide in $[0, m]_Z^* \setminus \{v_0\}$;* (iii) *$\varphi(v_0) \in AN_G(\phi(h(v_0)))$, or*

*(2) for $m' \geq m$, there is a vertex $v_0 \in [0, m']_Z^*$ and a monotonically increasing graph map $h : [0, m']_Z^* \to [0, m]_Z^*$ such that* (i) *$h(0) = 0$, $h(m') = m$, and $\phi \simeq_M \varphi \circ h$;* (ii) *$\phi(v)$ and $\varphi(h(v))$ coincide in $[0, m']_Z^* \setminus \{v_0\}$;* (iii) *$\phi(v_0) \in AN_G(\varphi(h(v_0)))$.*

**Example 4.2** An example of elementary $MP$-homotopy is shown in Figure 5. The loops $\varphi$ and $\phi$ are elementary $MP$-homotopic.

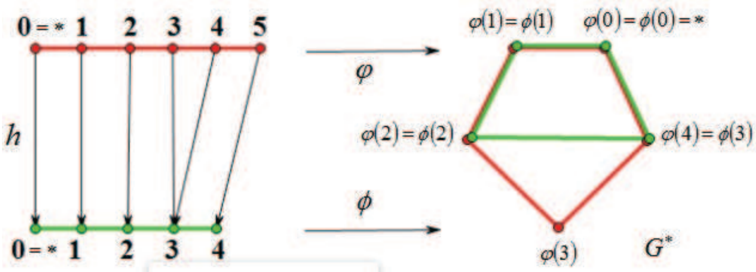


Figure 5 The loops $\varphi : [0, 5]_Z^* \to G^*$ and $\phi : [0, 4]_Z^* \to G^*$

**Definition 4.4** *Let $G^*$ be a pointed simple graph. Two pointed path-maps $\varphi$ and $\phi$ in $G^*$ are called $MP$-homotopic and write $\varphi \simeq_{MP} \phi$ if there exists a finite sequence $\varphi = \varphi_0, \varphi_1, \cdots, \varphi_n = \phi$ in $G^*$ such that $\varphi_i$ and $\varphi_{i+1}$ are elementary $MP$-homotopic for any $i = 0, 1, \cdots, n-1$.*

It is not hard to prove that the $MP$-homotopy is an equivalence relation.

Let $[\varphi]_{MP}$ be the $MP$-homotopy class of the loop $\varphi$ of a pointed simple graph $G^*$ and $A_1(G^*)$ be the set of all $[\varphi]_{MP}$. Then we will define a group structure on $A_1(G^*)$.

For two simple graphs $[0,m]_Z$ and $[0,m']_Z$, $[0,m]_Z \vee [0,m']_Z$ denotes a simple graph defined by identification of the vertices $m \in [0,m]_Z$ and $0 \in [0,m']_Z$.

(i) For a path-map $\varphi : [0,m]_Z \to G$, the inverse path-map $\varphi^- : [0,m]_Z \to G$ is defined by $\varphi^-(i) = \varphi(m-i)$ for all $0 \le i \le m$.

(ii) For two path-maps $\varphi : [0,m]_Z \to G$ and $\phi : [0,m']_Z \to G$ with $\varphi(m) = \phi(0)$, the concatenation path-map $\varphi \vee \phi : [0, m+m']_Z \to G$ is defined by

$$\varphi \vee \phi(i) = \begin{cases} \varphi(i), & 0 \le i \le m, \\ \phi(i-m), & m \le i \le m+m'. \end{cases}$$

Clearly, if $\varphi$ is a loop in $G^*$ then $\varphi^-$ is also a loop, and the concatenation of two loops is also a loop. Let us define a product in $A_1(G^*)$ as follows.

**Definition 4.5** *Let $G^*$ be a pointed simple graph. For any two loops $\varphi : [0,m]_Z^* \to G^*$, $\phi : [0,m']_Z^* \to G^*$, a product of $[\varphi]_{MP}$ and $[\phi]_{MP}$ is defined by $[\varphi]_{MP} \cdot [\phi]_{MP} = [\varphi \vee \phi]_{MP}$.*

It is obvious that the product in $A_1(G^*)$ is well-defined. In fact, if there are two loops $\varphi' : [0,n]_Z^* \to G^*$, $\phi' : [0,n']_Z^* \to G^*$ such that $\varphi' \simeq_{MP} \varphi$ and $\phi' \simeq_{MP} \phi$, then $\varphi' \vee \phi' \simeq_{MP} \varphi' \vee \phi \simeq_{MP} \varphi \vee \phi$, i.e., $[\varphi' \vee \phi']_{MP} = [\varphi \vee \phi]_{MP}$.

**Proposition 4.3** *Let $G^*$ be a pointed simple graph. Then $A_1(G^*)$ is a group under the product in Definition* 4.5.

**Proof** (i) The product in Definition 4.5 satisfies the associative law.

(ii) The $MP$-homotopy class $[e]_{MP}$ of the loop $e : \{0\}^* \to G^*$ satisfies the condition of an identity element.

(iii) For any $MP$-homotopy class $[\varphi]_{MP}$ of the loop $\varphi : [0,m]_Z^* \to G^*$, $[\varphi^-]_{MP}$ is the inverse of $[\varphi]_{MP}$.

This group is called 1-order $MP$-homotopy group of $G^*$, denoted by $\pi_1(G^*)$.

**Theorem 4.2** *Let $G^*$ denote a simple graph $G$ with a base vertex $*_G$ and $G^{*'}$ denote a simple graph $G$ with a base vertex $*'_G$. If $\gamma : [0,h]_Z^* \to G^*$ is a pointed path-map with $\gamma(h) = *'_G$, then $\gamma$ induces an isomorphism of* 1*-order $MP$-homotopy groups*

$$\beta_\gamma : \pi_1(G^*) \to \pi_1(G^{*'}).$$

**Proof** For any loop $\varphi$ in $G^*$, define

$$\beta_\gamma([\varphi]_{MP}) = [\gamma^- \vee \varphi \vee \gamma]_{MP}.$$

It is not hard to prove that $\beta_\gamma$ is well-defined. Further, $\beta_\gamma$ is a homomorphism since

$$\begin{aligned}\beta_\gamma([\varphi]_{MP} \cdot [\phi]_{MP}) &= \beta_\gamma([\varphi \vee \phi]_{MP})\\ &= [\gamma^- \vee \varphi \vee \phi \vee \gamma]_{MP}\\ &= [\gamma^- \vee \varphi \vee \gamma \vee \gamma^- \vee \phi \vee \gamma]_{MP}\\ &= [\gamma^- \vee \varphi \vee \gamma]_{MP} \cdot [\gamma^- \vee \phi \vee \gamma]_{MP}\\ &= \beta_\gamma([\varphi]_{MP}) \cdot \beta_\gamma([\phi]_{MP}).\end{aligned}$$

Finally, $\beta_\gamma$ is an isomorphism with inverse $\beta_{\gamma^-}$ since

$$\begin{aligned}\beta_{\gamma^-} \circ \beta_\gamma([\varphi]_{MP}) &= \beta_{\gamma^-}([\gamma^- \vee \varphi \vee \gamma]_{MP})\\ &= [\gamma \vee \gamma^- \vee \varphi \vee \gamma \vee \gamma^-]_{MP}\\ &= [\varphi]_{MP},\end{aligned}$$

and similarly $\beta_\gamma \circ \beta_{\gamma^-}([\varphi']_{MP}) = [\varphi']_{MP}$ for any loop $\varphi'$ in $G^{*'}$.

Thus if $G$ is path connected, the group $\pi_1(G^*)$ is, up to isomorphism, independent of the choice of base vertex $*_G$.

In order to prove the 1-order $MP$-homotopy group is invariant up to the $M$-homotopy equivalence, we introduce the following definition.

**Definition 4.6** *A core of a simple graph $G$ is an $M$-strong deformation retract of $G$ which is a minimal simple graph.*

**Remark 4.1** The cores of a simple graph $G$ might not be unique. For example, let $G$ be a simple graph as in Figure 6$(a)$, then $(b)$ and $(c)$ are both the cores of $G$. However, the cores of a simple graph $G$ have the same homology groups as $G$ (see Remark 3.1 for details).

**Theorem 4.3** *Let $G_1^*$ and $G_2^*$ be two path connected pointed simple graphs. If $G_1$ and $G_2$ are $M$-homotopy equivalent, then there is an isomorphism*

$$\pi_1(G_1^*) \approx \pi_1(G_2^*).$$

**Proof** For $G_1^*$, one can obtain, by Theorem 4.2, a path connected pointed simple graph $G_1^{*'}$ such that $*'$ belongs to the core of $G_1$ and $\pi_1(G_1^*) \approx \pi_1(G_1^{*'})$. Suppose $v \in G_1^{*'}$ is a trivial vertex of $G_1$. Then $v \neq *'$ and $G_1 \setminus \{v\}$ is an $M$-strong deformation retract of $G_1$. For a loop $\varphi : [0,n]_Z^* \to G_1^{*'}$ such that $v \in \varphi([0,n]_Z^*)$, then there is an $i_0 = \min\{i \mid 0 < i < n \text{ and } \varphi(i) = v\}$, hence $\varphi(i_0 - 1) \neq v$ and $\varphi(i_0 - 1) \in AN_{G_1}(v)$. Since $L_{G_1}(v)$ is a contractible graph, one could choose $\varphi_v : [0, n_v]_Z \to L_{G_1}(v)$ such that $\varphi_v(0) = \varphi(i_0 - 1)$ and $\varphi_v(n_v) = \varphi(i_0 + l_v)$, where $l_v > 0$

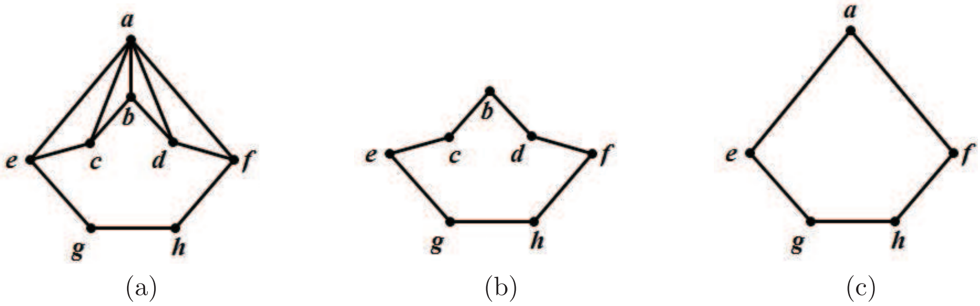


Figure 6 An example of non-uniqueness of the cores

is the minimal integer such that $\varphi(i_0 + l_v) \neq v$. So $\varphi_1 \vee \varphi_v \vee \varphi_2 \simeq_{MP} \varphi$, where $\varphi_1 = \varphi_{[0,i_0]_Z^*}$ and $\varphi_2 : [0, n - i_0 - l_v]_Z \to G_1$ satisfies that $\varphi_2(j) = \varphi(j + i_0 + l_v)$. By finiteness of the length of the loop $\varphi$, one could obtain a loop $\varphi' : [0, n']_Z^* \to G_1^{*'}$ such that $\varphi'([0, n']_Z^*) \subset (G_1 \setminus \{v\})^{*'}$ and $\varphi' \simeq_{MP} \varphi$, i.e., $\varphi \in [\varphi']_{MP}$. So $\pi_1((G_1 \setminus \{v\})^{*'}) \approx \pi_1(G_1^{*'})$. Therefore, every $M$-strong deformation retract of $G_1$ has, by Theorem 2.5 and induction, the same 1-order $MP$-homotopy group as $G_1^{*'}$.

Since $G_1$ and $G_2$ are $M$-homotopy equivalent, there exists a sequence of simple graphs $G_1 = G'_0, G'_1, \cdots, G'_m = G_2$ such that one of the two simple graphs $G'_i$ and $G'_{i+1}$ is an $M$-strong deformation retract of the other for every $0 \leq i \leq m - 1$. Therefore, we have

$$\pi_1(G_1^*) = \pi_1(G_0^{'*}) \approx \pi_1(G_1^{'*}) \approx \cdots \approx \pi_1(G_m^{'*}) = \pi_1(G_2^*).$$

Therefore, the 1-order $MP$-homotopy group of a pointed simple graph $G^*$ is invariant up to $M$-homotopy equivalence.

**Theorem 4.4** *For a simple graph $G$, there is a unique $MP$-homotopy class of path-maps connecting any two vertices in $G$ if and only if it is path connected and has trivial 1-order $MP$-homotopy group.*

**Proof** The existence of path-maps connecting every pair of vertices of $G$ means path connectedness, so it suffices to concern only with the uniqueness of $MP$-homotopy class of path-maps. Suppose $\pi_1(G) = 0$. If $\varphi$ and $\phi$ are two path-maps from $u$ to $v$ in $G$, then $\varphi \simeq_{MP} \varphi \vee \phi^- \vee \phi \simeq_{MP} \phi$ since $\phi^- \vee \phi \simeq_{MP} e_v$ and $\varphi \vee \phi^- \simeq_{MP} e_u$, where $e_v : \{0\} \to G$ such that $e_v(0) = v$ and $e_u : \{0\} \to G$ such that $e_u(0) = u$.

Conversely, since there is only one $MP$-homotopy class of path-maps connecting a base vertex $*_G$ to itself, all loops at $*_G$ are $MP$-homotopic to the constant loop $e : \{0\}^* \to G^*$, that is,

$$\pi_1(G^*) = 0.$$

## 5 Summary and Further Works

This paper has developed a new concept of a weak graph map homotopy (we call it $M$-homotopy) on graphs. It turns out that the $M$-homotopy equivalence between graphs coincides with the graph homotopy equivalence defined in [4]. The difference between them is that the $M$-homotopy transformation is defined in terms of maps, while the graph homotopy transformation in [4] is defined by means of combinatorial operations. As its applications, we investigate the mapping class group of a graph and the 1-order $MP$-homotopy groups of a pointed simple graph. In addition, we show that the $M$-homotopy transformation has its advantages over the graph homotopy transformation, that the $M$-homotopy equivalence has close relationships with the $s$-homotopy type in [3], and that the 1-order $MP$-homotopy group of a pointed simple graph and homology groups of a graph are invariant up to the $M$-homotopy equivalence.

As a further work, we attempt to deeply probe into the mapping class groups of graphs up to $M$-homotopy, and investigate higher order $MP$-homotopy group of a pointed simple graph.

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## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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