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Global Stability to Steady Supersonic Rayleigh Flows in One-Dimensional Duct*

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Abstract Heat exchange plays an important role in hydrodynamical systems, which is an interesting topic in theory and application. In this paper, the authors consider the global stability of steady supersonic Rayleigh flows for the one-dimensional compressible Euler equations with heat exchange, under the small perturbations of initial and boundary conditions in a finite rectilinear duct.

Keywords Compressible Euler equations, Heat exchange, Supersonic Rayleigh flow, Steady solution, Classical solution
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1 Introduction

The problem of supersonic flows passing a duct is well-known in gas dynamics, which can be described by hyperbolic systems of conservation laws. Such flow usually is governed by the compressible Euler equations, which is one of the most fundamental equations in fluid dynamics and atmospheric dynamics. Engineers usually use various ducts to transport gas and control their movement on the engineering. Therefore, it is quite interesting to understand what are the stabilization effect for gas flows in ducts. The effects are considered by engineers such as geometry, friction and heat transfer.

We first give some references about the stability effects of regional geometry on non-isentropic Euler flows. The authors established the stability of a class of cylindrical symmetric transonic shocks for two-dimensional complete compressible steady Euler system in [1]. Yuan et al. showed stability of spherically symmetric subsonic flows and transonic shocks in space R^3 under multidimensional perturbations of boundary conditions in [2]. The main conclusion is that almost all spherically symmetric transonic shock waves are stable, under perturbations of the upcoming supersonic flows and back pressure at the exit of the ducts. Recently, Fang and Gao [3] got the existence of transonic shocks for steady Euler flows in a 3-D axisymmetric cylindrical nozzle, which are governed by the Euler equations with the slip boundary condition on the wall of the nozzle and a receiver pressure at the exit. Fang and Xin got the existence of

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transonic shock solutions to the 2-D steady compressible Euler system in an almost flat finite nozzle (in the sense that it is a generic small perturbation of a flat one), under physical boundary conditions, in which the receiver pressure is prescribed at the exit of the nozzle in [4]. The more related results can be found in [5–7] and references therein.

In engineering, if heat transfer is not considered, the gas flow in a constant area duct with friction is called Fanno flow in aerodynamics (see [8]). For the three-dimensional steady non-isentropic compressible Euler system with friction, Zhao and Yuan [9] showed existence of a class of symmetric subsonic, supersonic and transonic-shock solutions in a straight duct with constant square-section. They further formulated a boundary value problem of subsonic flows, and considered their stability under small perturbations of boundary conditions in [10]. For the unsteady flow, the related stability problem in multidimensional case is very difficulty. Recently, the authors of [11] showed that the global stability of steady supersonic Fanno flows under small perturbations of initial-boundary values in a one-dimensional rectilinear finite duct with constant cross-sections.

The gas flow in a constant area duct with heat transfer and without friction is called Rayleigh flows (see [12]). It is very important to study the hydrodynamic system with heat transfer, both in practical application and theory. There are numerous monographs and textbooks available which investigate three dimensional real fluid mechanics with heat transfer from the engineering point of view (see [13] and references therein). A special global-in-time solution to the isothermal Euler system with heat transport in the whole space was given by Dyson [14]. Global solutions to the compressible Euler equations with heat transport by convection in three space dimensions are shown to exist through perturbations of Dysons isothermal affine solutions in [15]. Yuan et al. studied the steady Rayleigh flows, namely, the effects of heat exchange, on stabilization of transonic shock for steady compressible Euler equations in two-dimensional rectilinear ducts in [16]. The authors got that for given heat exchange per unit mass of gas, almost all the associated one-dimensional transonic shock are stable, while for given heat exchange per unit volume of gas, the resultant one-dimensional transonic shocks are not stable, provided the perturbations of the upstream supersonic flow and downstream back pressure satisfy some symmetry requirements.

In this paper, we consider the steady Rayleigh flows, namely, the steady effects of heat exchange, on stabilization of supersonic solutions in one-dimensional rectilinear ducts under the perturbations of initial and boundary data. We will consider the stability of supersonic solution by utilizing the method of characteristics (see [17–18]). The main ingredient is to obtain the priori uniform C^1 estimates of the classical solutions by wave decomposition. In one-dimensional duct, the motion of flow with heat exchange is governed by the full compressible Euler equations as following

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho E)_t + (\rho E u + pu)_x = \rho Q(x), \end{cases} \tag{1.1}$$

where $\rho(t,x), u(t,x),\ p(t,x)$ denote the density of mass, the velocity, the pressure of the gas, respectively, and $E = \frac{1}{2}|u|^2 + \frac{1}{\gamma-1}\frac{p}{\rho}$ is the total energy per unit mass of the polytropic gas. $Q(x)$ is a given function of x, representing heat produced (absorbed) per unit volume of the gas per unit time. $Q(x) > 0$ stands for heat addition and $Q(x) < 0$ stands for heat rejection. In this paper we only consider the polytropic gas with state function

$$p = p(\rho, S) = A(S)\rho^\gamma,$$

where $\gamma > 1$ is the adiabatic index, S is entropy of the flow. Let the length of the duct is L, which is less than L_m. Here, L_m is the maximal length that the gas is supersonic in $[0, L_m)$, which will be determined later.

We first consider the following special while physically significant case to system (1.1). Suppose that the flow only depends on x-variable, and $u \geq 0$. Then, system (1.1) is reduced to

$$\begin{cases} (\rho u)_x = 0, \\ (\rho u^2 + p)_x = 0, \\ \Big(\rho\Big(\dfrac{1}{2}u^2 + \dfrac{\gamma}{\gamma-1}\dfrac{p}{\rho}\Big)u\Big)_x = \rho Q(x). \end{cases} \tag{1.2}$$

Suppose that ρ, u, p are C^1 solutions. Then, we have

$$\begin{cases} u\rho_x + \rho u_x = 0, \\ \rho u u_x + p_x = 0, \\ -\dfrac{\gamma}{\gamma-1}\dfrac{p}{\rho}u\rho_x + \rho u^2 u_x + \dfrac{\gamma}{\gamma-1}u p_x = \rho Q(x). \end{cases} \tag{1.3}$$

Thus, we can recast the above system with Mach number $M > 1$ as

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}x} = (\gamma-1)\dfrac{Q(x)}{c^2}\dfrac{1}{1-M^2}, \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}x} = -(\gamma-1)\dfrac{\rho Q(x)}{c^2 u}\dfrac{1}{1-M^2}, \\ \dfrac{\mathrm{d}p}{\mathrm{d}x} = -(\gamma-1)\dfrac{\rho Q(x)}{u}\dfrac{M^2}{1-M^2}. \end{cases} \tag{1.4}$$

Here, $M = \frac{u}{c}$ is Mach number, and $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sonic speed. The flow is called supersonic flow if $u > c$ at a point, i.e., $M > 1$. Noting $c^2 = \frac{\gamma p}{\rho}$ and (1.4), we can also get that

$$2cc_x = \gamma\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{p}{\rho}\Big) = -\gamma(\gamma-1)\frac{Q(x)}{u}\frac{1}{1-M^2}\Big(M^2 - \frac{1}{\gamma}\Big), \tag{1.5}$$

namely,

$$\frac{\mathrm{d}c}{\mathrm{d}x} = -\frac{\gamma-1}{2}\frac{Q(x)}{c^2}\frac{\gamma M^2 - 1}{M(1-M^2)}. \tag{1.6}$$

Furthermore, by $M_x = \frac{cu_x - uc_x}{c^2}$, we can derive that

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{(\gamma-1)}{2}\frac{Q(x)}{c^3}\frac{\gamma M^2 + 1}{1-M^2}. \tag{1.7}$$

If the heat exchange $Q(x)$ is a positive C^1 function, we can easily get

$$\frac{\mathrm{d}u}{\mathrm{d}x} < 0, \quad \frac{\mathrm{d}\rho}{\mathrm{d}x} > 0, \quad \frac{\mathrm{d}p}{\mathrm{d}x} > 0, \quad \frac{\mathrm{d}M}{\mathrm{d}x} < 0, \quad \frac{\mathrm{d}c}{\mathrm{d}x} > 0, \tag{1.8}$$

that is, the density, pressure and sound speed are increasing functions, while the velocity, Mach number are decreasing functions with respect to x for supersonic flow.

Let Mach number of the flow at entry $\{x = 0\}$ and the exit $\{x = L\}$ be $M_0 > 1$ and M_L, and ρ_0, u_0, p_0 are the density, velocity and pressure at the entry, respectively. By the third equation of (1.4) and (1.7), and dividing $Q(x)$, we can get

$$\frac{2\gamma M}{\gamma M^2+1}\frac{\mathrm{d}M}{\mathrm{d}x} = -\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{1.9}$$

Integrating the above equation (1.9) with respect to x, we have

$$\frac{p}{p_0} = \frac{\gamma M_0^2+1}{\gamma M^2+1}. \tag{1.10}$$

Through direct computation, we can also obtain

$$\frac{u}{u_0} = \Big(\frac{M}{M_0}\Big)^2\frac{\gamma M_0^2+1}{\gamma M^2+1}, \quad \frac{\rho}{\rho_0} = \Big(\frac{M_0}{M}\Big)^2\frac{\gamma M^2+1}{\gamma M_0^2+1}. \tag{1.11}$$

Furthermore, by (1.7) and (1.9)–(1.11), we can get

$$\frac{\gamma M^2(1-M^2)}{(\gamma M^2+1)^4}\frac{\mathrm{d}M^2}{\mathrm{d}x} = \frac{Q(x)}{Q_0}, \quad \text{where } Q_0 = \frac{p_0u_0(\gamma M_0^2+1)^3}{(\gamma-1)\rho_0 M_0^4} > 0. \tag{1.12}$$

Integrating the above equation from 0 to l ($l \le L$) with respect to x yields that

$$\begin{aligned} &-\frac{M_l^2(1-M_l^2)}{3(\gamma M_l^2+1)^3} - \frac{(1-2M_l^2)}{6\gamma(\gamma M_l^2+1)^2} + \frac{1}{3\gamma^2(\gamma M_l^2+1)} \\ &= -\frac{M_0^2(1-M_0^2)}{3(\gamma M_0^2+1)^3} - \frac{(1-2M_0^2)}{6\gamma(\gamma M_0^2+1)^2} + \frac{1}{3\gamma^2(\gamma M_0^2+1)} + \frac{1}{Q_0}\int_0^l Q(s)\mathrm{d}s. \end{aligned} \tag{1.13}$$

Let

$$F(M_l) \triangleq F(M_0) + \frac{1}{Q_0}\int_0^l Q(s)\mathrm{d}s, \tag{1.14}$$

where

$$F(M) = -\frac{M^2(1-M^2)}{3(\gamma M^2+1)^3} - \frac{(1-2M^2)}{6\gamma(\gamma M^2+1)^2} + \frac{1}{3\gamma^2(\gamma M^2+1)}.$$

For supersonic flow, note that

$$F'(M) = \frac{2\gamma M^3(1-M^2)}{(\gamma M^2+1)^4} < 0. \tag{1.15}$$

Assume that $Q(x)$ is a non-negative C^1 integrable function, and the antiderivative function is $\Theta(x)$, which is a strict monotonic function. From (1.14), we can get

$$F(M_l) - F(M_0) = \frac{1}{Q_0}(\Theta(l) - \Theta(0)), \tag{1.16}$$

and we obtain that

$$l = \Theta^{-1}\{Q_0[F(M_l) - F(M_0)] + \Theta(0)\}. \tag{1.17}$$

Therefore, the maximal length L_m of a duct for supersonic flow is given by

$$\begin{aligned}L_m = \Theta^{-1}\Big\{Q_0\Big[&\frac{3\gamma+2}{6\gamma^2(\gamma+1)^2}+\frac{M_0^2(1-M_0^2)}{3(\gamma M_0^2+1)^3}+\frac{(1-2M_0^2)}{6\gamma(\gamma M_0^2+1)^2}\\ &-\frac{1}{3\gamma^2(\gamma M_0^2+1)}\Big]+\Theta(0)\Big\}\end{aligned} \tag{1.18}$$

for which the Mach number of the flow decelerating from $M_0 > 1$ at entry to the sonic case $M_L = 1$ at the exit. For any given $M_0 > 1$, if the flow is continuous on $x \in [0, L_m)$, then it is always supersonic, and the maximal length L_m of the duct is given by (1.18).

We note that the length $L_m > 0$. Since the length of a duct is longer than L_m, chocking phenomena shall occur and actually such steady supersonic flow model is impossible. Therefore, throughout this paper, we assume that L is strictly less than the maximal length L_m. For given $\gamma > 1$, $M_0 > 1$ and $L < L_m$, we can determine a unique steady supersonic flow in $[0, L]$. In the following we call such a steady solution as background solution, denoted by $(\widetilde{\rho}(x), \widetilde{u}(x), \widetilde{p}(x))$ for $x \in [0, L]$. We remark that such a special solution plays a crucial role in gas dynamics and engineering to understand flow field in ducts with frictions, heat exchange and related numerical simulations. However, it seems that little stability analysis had been carried out before by considering the corresponding initial-boundary value problems of the Euler systems.

In this paper, we are interested in the stability of such background solution in one space dimensional case for system (1.1). It is assumed that the steady solution is supersonic even after small perturbation. We can prescribe boundary conditions on the entry as follows:

$$x = 0: \quad \rho(t, 0) = \rho_1(t), \quad u(t, 0) = u_1(t), \quad p(t, 0) = p_1(t), \quad t \geq 0. \tag{1.19}$$

The initial datum are

$$t = 0: \quad \rho(0, x) = \rho_0(x), \quad u(0, x) = u_0(x), \quad p(0, x) = p_0(x), \qquad x \in [0, L]. \tag{1.20}$$

The purpose of this article is to study that if $\rho_1(t) - \widetilde{\rho}(0)$, $u_1(t) - \widetilde{u}(0)$, $p_1(t) - \widetilde{p}(0)$ and $\rho_0(x) - \widetilde{\rho}(x)$, $u_0(x) - \widetilde{u}(x)$, $p_0(x) - \widetilde{p}(x)$, are small in some sense, can we have a classical solution to the initial-boundary value problem (1.1) and (1.19)–(1.20) in $\{(t, x) \mid (t, x) \in [0, \infty) \times [0, L]\}$ which is still close to the background solution?

This paper is organized as follows. In Section 2 we will introduce the wave decomposition for non-isentropic compressible Euler equations with heat exchange. In Section 3, we will give the stability theorem. In Section 4 we prove Theorem 3.1 by using the method of characteristics curves, on stability of classical solution of steady supersonic Rayleigh flows.

2 Wave Decomposition

In this section, we will introduce the wave decomposition for Euler system (1.1) under the small perturbation of steady supersonic background states. The one-dimensional non-isentropic compressible Euler equations with heat exchange is the following

$$\begin{cases}\rho_t + u\rho_x + \rho u_x = 0,\\ u_t + uu_x + \dfrac{1}{\rho}p_x = 0,\\ p_t + \gamma p u_x + u p_x = (\gamma - 1)\rho Q(x).\end{cases} \tag{2.1}$$

Let

$$\begin{cases} \rho(t,x) = \overline{\rho}(t,x) + \widetilde{\rho}(x), \\ u(t,x) = \overline{u}(t,x) + \widetilde{u}(x), \\ p(t,x) = \overline{p}(t,x) + \widetilde{p}(x), \end{cases} \tag{2.2}$$

where $\widetilde{U} = (\widetilde{\rho}(x), \widetilde{u}(x), \widetilde{p}(x))^\top$ is the steady supersonic background state, and $\overline{U} = (\overline{\rho}(t,x), \overline{u}(t,x), \overline{p}(t,x))^\top$ is the perturbation of the background state. Here, suppose that $\widetilde{\rho}(x) > 0$ without vacuum in $x \in [0, L]$.

Substituting (2.2) into (2.1), we can get

$$\begin{cases} \overline{\rho}_t + u\overline{\rho}_x + \rho\overline{u}_x + \overline{u}\widetilde{\rho}_x + \overline{\rho}\widetilde{u}_x + \widetilde{u}\widetilde{\rho}_x + \widetilde{\rho}\widetilde{u}_x = 0, \\ \overline{u}_t + u\overline{u}_x + \overline{u}\widetilde{u}_x + \widetilde{u}\widetilde{u}_x + \dfrac{1}{\rho}(\overline{p}_x + \widetilde{p}_x) = 0, \\ \overline{p}_t + \gamma p\overline{u}_x + u\overline{p}_x + \gamma\overline{p}\widetilde{u}_x + \gamma\widetilde{p}\widetilde{u}_x + \overline{u}\widetilde{p}_x + \widetilde{u}\widetilde{p}_x = (\gamma-1)(\overline{\rho} + \widetilde{\rho})Q(x). \end{cases} \tag{2.3}$$

By (1.3), we obtain

$$\begin{cases} \overline{\rho}_t + u\overline{\rho}_x + \rho\overline{u}_x = -\overline{u}\widetilde{\rho}_x - \overline{\rho}\widetilde{u}_x, \\ \overline{u}_t + u\overline{u}_x + \dfrac{1}{\rho}\overline{p}_x = -\overline{u}\widetilde{u}_x - H(\rho,\widetilde{\rho})\overline{\rho}\widetilde{p}_x, \\ \overline{p}_t + \gamma p\overline{u}_x + u\overline{p}_x = -\gamma\overline{p}\widetilde{u}_x - \overline{u}\widetilde{p}_x + (\gamma-1)\overline{\rho}Q(x), \end{cases} \tag{2.4}$$

where $H(\rho,\widetilde{\rho}) = -\frac{1}{\rho\widetilde{\rho}}$.

Then system (2.4) can be rewritten as the following quasi-linear hyperbolic system

$$\overline{U}_t + A(U)\overline{U}_x + B(\widetilde{U})\overline{U} = C(\overline{U}), \tag{2.5}$$

where $U(t,x) = \overline{U}(t,x) + \widetilde{U}(x)$, and

$$A(U) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \dfrac{1}{\rho} \\ 0 & \gamma p & u \end{pmatrix}, \quad B(\widetilde{U}) = \begin{pmatrix} \widetilde{u}_x & \widetilde{\rho}_x & 0 \\ H(\rho,\widetilde{\rho})\widetilde{p}_x & \widetilde{u}_x & 0 \\ 0 & \widetilde{p}_x & \gamma\widetilde{u}_x \end{pmatrix}, \quad C(\overline{U}) = \begin{pmatrix} 0 \\ 0 \\ (\gamma-1)\overline{\rho}Q(x) \end{pmatrix}.$$

Remark 2.1 For 1-D quasilinear hyperbolic systems, the dissipation term maybe help to prevent the formation of singularity. With the appropriate small initial data or other structure assumptions, the global and blowup of classical solution to Cauchy problem of hyperbolic system have been obtained, see [19–22] and references therein.

Through simple computations, we can get the eigenvalues of system (2.5) are

$$\lambda_1(U) = u - c, \quad \lambda_2(U) = u, \quad \lambda_3(U) = u + c. \tag{2.6}$$

We can choose the right eigenvectors as follows

$$\begin{cases} r_1(U) = \dfrac{c^2}{\sqrt{\rho^2 + c^2 + \rho^2 c^4}}\Big(\dfrac{\rho}{c^2}, -\dfrac{1}{c}, \rho\Big)^\top, \\ r_2(U) = (1, 0, 0)^\top, \\ r_3(U) = \dfrac{c^2}{\sqrt{\rho^2 + c^2 + \rho^2 c^4}}\Big(\dfrac{\rho}{c^2}, \dfrac{1}{c}, \rho\Big)^\top. \end{cases} \tag{2.7}$$

Then,

$$r_i(U)^\top r_i(U) \equiv 1, \quad i = 1, 2, 3. \tag{2.8}$$

Without loss of generality, we can also choose that

$$l_i(U) r_j(U) \equiv \delta_{ij}, \quad i, j = 1, 2, 3, \tag{2.9}$$

where δ_{ij} stands for the Kronecker's symbol, $l_i(U)$ is the ith left eigenvector which has the same regularity as $r_i(U)$.

Denote

$$V_i = l_i(U)\overline{U}, \quad W_i = l_i(U)\overline{U}_x, \quad i = 1, 2, 3. \tag{2.10}$$

By (2.8)–(2.10), it is easy to see that

$$\overline{U} = \sum_{i=1}^{3} V_i r_i(U), \quad \overline{U}_x = \sum_{i=1}^{3} W_i r_i(U). \tag{2.11}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(U) \frac{\partial}{\partial x} \tag{2.12}$$

be the directional derivative along the ith characteristic. Noting (2.5) and (2.10)–(2.11), by wave decomposition (see [17–18, 23–24]), we have

$$\begin{aligned}
\frac{\mathrm{d}V_i}{\mathrm{d}_i t} &= \frac{\partial V_i}{\partial t} + \lambda_i(U) \frac{\partial V_i}{\partial x} \\
&= \sum_{j,k=1}^{3} (\lambda_j - \lambda_i) l_i(U) (\nabla_U r_j(U))^\top r_k(U) W_j V_k + l_i(U) B(\widetilde{U}) \sum_{j,k=1}^{3} (\nabla_U r_j(U))^\top r_k(U) V_j V_k \\
&\quad - \sum_{k=1}^{3} l_i(U) (\nabla_U r_k(U))^\top C(\overline{U}) V_k - \sum_{k=1}^{3} \lambda_i \widetilde{U}_x^\top l_i(U)^\top (\nabla_U r_k(U))^\top V_k \\
&\quad - l_i(U) B(\widetilde{U}) \sum_{k=1}^{3} r_k V_k + l_i(U) C(\overline{U}) \\
&\triangleq \sum_{j,k=1}^{3} \Phi_{ijk}(U) W_j V_k + \sum_{j,k=1}^{3} \widetilde{\Phi}_{ijk}(U) V_j V_k - \sum_{k=1}^{3} \widetilde{\widetilde{\Phi}}_{ik}(U) V_k + l_i(U) C(\overline{U}),
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
&\Phi_{ijk}(U) = (\lambda_j(U) - \lambda_i(U)) l_i(U) (\nabla_U r_j(U))^\top r_k(U), \\
&\widetilde{\Phi}_{ijk}(U) = l_i(U) B(\widetilde{U}) (\nabla_U r_j(U))^\top r_k(U), \\
&\widetilde{\widetilde{\Phi}}_{ik}(U) = l_i(U) (\nabla_U r_k(U))^\top C(\overline{U}) + \lambda_i(U) \widetilde{U}_x^\top l_i(U)^\top (\nabla_U r_k(U))^\top + l_i(U) B(\widetilde{U}) r_k(U).
\end{aligned}$$

Notice that

$$\Phi_{iik}(U) \equiv 0, \quad \forall\ k = 1, 2, 3.$$

Using similar procedures, we can get (see [17–18, 23–24])

$$\begin{aligned}&\frac{\partial W_i}{\partial t}+\frac{\partial(\lambda_i W_i)}{\partial x}\\ =&\sum_{j,k=1}^{3}(\lambda_j-\lambda_k)l_i(U)(\nabla_U r_k(U))^\top r_j(U)W_jW_k-\sum_{k=1}^{3}\lambda_k\widetilde{U}_x^\top l_i(U)^\top(\nabla_U r_k(U))^\top W_k\\ &+\sum_{k=1}^{3}l_i(U)(\nabla_U r_k)^\top(B(\widetilde{U})\overline{U}-C(\overline{U}))W_k+l_i(U)(C(\overline{U})-B(\widetilde{U})\overline{U})_x.\end{aligned}\tag{2.14}$$

Suppose that $\widetilde{U}$ is a C^2 function. Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}W_i}{\mathrm{d}_i t}=&\frac{\partial W_i}{\partial t}+\lambda_i(U)\frac{\partial W_i}{\partial x}\\ =&\sum_{j,k=1}^{3}(\lambda_j-\lambda_k)l_i(U)(\nabla_U r_k(U))^\top r_j(U)W_jW_k-\sum_{k=1}^{3}(\nabla_U\lambda_i(U))r_k(U)W_iW_k\\ &-(\nabla_U\lambda_i(U))\widetilde{U}_xW_i+\sum_{j,k=1}^{3}l_i(U)(\nabla_U r_k(U))^\top B(\widetilde{U})r_j(U)V_jW_k\\ &-\sum_{k=1}^{3}l_i(U)(\nabla_U r_k(U))^\top C(\overline{U})W_k\\ &-\sum_{k=1}^{3}\lambda_k\widetilde{U}_x^\top l_i(U)^\top(\nabla_U r_k(U))^\top W_k-l_i(U)B(\widetilde{U})\sum_{j=1}^{3}r_j(U)W_j+l_i(U)C(\overline{U})_x\\ &-l_i(U)B(\widetilde{U})_x\overline{U}\\ \triangleq&\sum_{j,k=1}^{3}\Psi_{ijk}(U)W_jW_k+\sum_{j,k=1}^{3}\widetilde{\Psi}_{ijk}(U)V_jW_k\\ &+\sum_{j,k=1}^{3}\widetilde{\widetilde{\Psi}}_{ijk}(U)W_k+l_i(U)C(\overline{U})_x-l_i(U)B(\widetilde{U})_x\overline{U},\end{aligned}\tag{2.15}$$

where

$$\begin{aligned}&\Psi_{ijk}(U)=\{(\lambda_j(U)-\lambda_k(U))l_i(U)(\nabla_U r_k(U))^\top r_j(U)-(\nabla_U\lambda_k(U))r_j(U)\delta_{jk}+(j|k)\},\\ &\widetilde{\Psi}_{ijk}(U)=l_i(U)(\nabla_U r_k(U))^\top B(\widetilde{U})r_j(U),\\ &\widetilde{\widetilde{\Psi}}_{ijk}(U)=-\lambda_k(U)\widetilde{U}_x^\top l_i(U)^\top(\nabla_U r_k(U))^\top-l_i(U)(\nabla_U r_k(U))^\top C(\overline{U})\\ &\qquad-(\nabla_U\lambda_i(U))\widetilde{U}_x-l_i(U)B(\widetilde{U})r_j(U),\end{aligned}$$

and $(j|k)$ stands for all terms obtained by changing j and k in the previous terms (see [25–27]). Hence

$$\Psi_{ijj}(U)\equiv 0,\quad \forall\ i\neq j,\ i=1,2,3.$$

Remark 2.2 Here, in order to get our desired result, we must change the order of the variables t and x. Then the Euler system (2.4) can be rewritten as

$$\overline{U}_x+A^{-1}(U)\overline{U}_t+A^{-1}(U)B(\widetilde{U})\overline{U}=A^{-1}(U)C(\overline{U}).\tag{2.16}$$

Using similar procedures, we can get the characteristic form as (2.13) and (2.15).

Since the matrices $A(U)$ and $A^{-1}(U)$ have the same left eigenvectors, we can still define the variables $\widehat{V}_i$ and $\widehat{W}_i$ by the same formula (2.10). Suppose that $\widehat{\lambda}_i(U)(i=1,2,3)$ are the eigenvalues of matrix $A^{-1}(U)$, and $\widehat{l}_i(U)$, $\widehat{r}_i(U)$ $(i=1,\ 2,\ 3)$ are the left eigenvectors and right eigenvectors with respect to $\widehat{\lambda}_i(U)$, respectively. Let

$$\widehat{V}_i=\widehat{l}_i(U)\overline{U},\quad \widehat{W}_i=\widehat{l}_i(U)\overline{U}_t,\quad i=1,2,3. \tag{2.17}$$

By (2.16)–(2.17), we can get

$$\begin{aligned}
&\frac{\mathrm{d}\widehat{V}_i}{\mathrm{d}_i x}=\frac{\partial\widehat{V}_i}{\partial x}+\widehat{\lambda}_i(U)\frac{\partial\widehat{V}_i}{\partial t}\\
=&\sum_{j,k=1}^{3}(\widehat{\lambda}_j-\widehat{\lambda}_i)\widehat{l}_i(U)(\nabla_U\widehat{r}_j(U))^\top\widehat{r}_k(U)\widehat{W}_j\widehat{V}_k+\widehat{l}_i(U)B(\widetilde{U})\sum_{j,k=1}^{3}\widehat{\lambda}_j(\nabla_U\widehat{r}_j(U))^\top\widehat{r}_k\widehat{V}_j\widehat{V}_k\\
&-\widehat{\lambda}_j\sum_{k=1}^{3}\widehat{l}_i(U)(\nabla_U\widehat{r}_k(U))^\top C(\overline{U})\widehat{V}_k-\sum_{k=1}^{3}\widetilde{U}_x^\top\widehat{l}_i(U)^\top(\nabla_U\widehat{r}_k(U))^\top\widehat{V}_k\\
&+\widehat{\lambda}_i\widehat{l}_i(U)C(\overline{U})-\widehat{\lambda}_i\widehat{l}_i(U)B(\widetilde{U})\sum_{k=1}^{3}(\nabla_U\widehat{r}_k(U))^\top\widehat{V}_k\\
\triangleq&\sum_{j,k=1}^{3}\widehat{\Phi}_{ijk}(U)\widehat{W}_j\widehat{V}_k+\sum_{j,k=1}^{3}\widehat{\widehat{\Phi}}_{ijk}(U)\widehat{V}_j\widehat{V}_k-\sum_{k=1}^{3}\widehat{\widehat{\widehat{\Phi}}}_{ik}(U)\widehat{V}_k+\widehat{\lambda}_i\widehat{l}_i(U)C(\overline{U}),
\end{aligned}\tag{2.18}$$

where

$$\begin{aligned}
&\widehat{\Phi}_{ijk}(U)=(\widehat{\lambda}_j(U)-\widehat{\lambda}_i(U))\widehat{l}_i(U)(\nabla_U\widehat{r}_j(U))^\top\widehat{r}_k(U),\\
&\widehat{\widehat{\Phi}}_{ijk}(U)=\widehat{\lambda}_j(U)\widehat{l}_i(U)B(\widetilde{U})(\nabla_U\widehat{r}_j(U))^\top\widehat{r}_k(U),\\
&\widehat{\widehat{\widehat{\Phi}}}_{ik}(U)=\widehat{\lambda}_j(U)\widehat{l}_i(U)(\nabla_U\widehat{r}_k(U))^\top C(\overline{U})+\widetilde{U}_x^\top\widehat{l}_i(U)^\top(\nabla_U\widehat{r}_k(U))^\top+\widehat{\lambda}_i(U)\widehat{l}_i(U)B(\widetilde{U})\widehat{r}_k(U).
\end{aligned}$$

Similarly, we can also get the equations of $\widehat{W}_i$ as follows

$$\begin{aligned}
&\frac{\mathrm{d}\widehat{W}_i}{\mathrm{d}_i x}=\frac{\partial\widehat{W}_i}{\partial x}+\widehat{\lambda}_i(U)\frac{\partial\widehat{W}_i}{\partial t}\\
=&\sum_{j,k=1}^{3}(\widehat{\lambda}_j-\widehat{\lambda}_k)\widehat{l}_i(U)(\nabla_U\widehat{r}_k(U))^\top\widehat{r}_j(U)\widehat{W}_j\widehat{W}_k-\sum_{k=1}^{3}\widetilde{U}_x^\top\widehat{l}_i(U)^\top(\nabla_U\widehat{r}_k(U))^\top\widehat{W}_k\\
&+\sum_{k=1}^{3}\widehat{l}_i(U)(\nabla_U\widehat{r}_k(U))^\top B(\widetilde{U})\widehat{r}_j(U)\widehat{W}_k\widehat{V}_j-\sum_{k=1}^{3}\widehat{l}_i(U)(\nabla_U\widehat{r}_k(U))^\top A^{-1}(U)C(\overline{U})\widehat{W}_k\\
&-\sum_{i,k=1}^{3}(\nabla_U\widehat{\lambda}_i(U))\widehat{r}_k(U)\widehat{W}_i\widehat{W}_k+\widehat{l}_i(U)[A^{-1}(U)C(\overline{U})-A^{-1}(U)B(\widetilde{U})\overline{U}]_t\\
\triangleq&\sum_{j,k=1}^{3}\widehat{\Psi}_{ijk}(U)\widehat{W}_j\widehat{W}_k+\sum_{j,k=1}^{3}\widehat{\widehat{\Psi}}_{ijk}(U)\widehat{V}_j\widehat{W}_k+\sum_{j,k=1}^{3}\widehat{\widehat{\widehat{\Psi}}}_{ijk}(U)\widehat{W}_k\\
&+\widehat{l}_i(U)(A^{-1}(U)C(\overline{U}))_t-\widehat{l}_i(U)(A^{-1}(U)B(\widetilde{U}))_t\overline{U},
\end{aligned}\tag{2.19}$$

where

$$\begin{aligned}
&\widehat{\Psi}_{ijk}(U) = \{(\widehat{\lambda}_j(U) - \widehat{\lambda}_k(U))\widehat{l}_i(U)(\nabla_U \widehat{r}_k(U))^\top \widehat{r}_j(U) - (\nabla_U \widehat{\lambda}_k(U))\widehat{r}_j(U)\delta_{jk} + (j|k)\},\\
&\widehat{\widehat{\Psi}}_{ijk}(U) = \widehat{l}_i(U)(\nabla_U \widehat{r}_k(U))B(\widetilde{U})\widehat{r}_j(U),\\
&\widehat{\widehat{\widehat{\Psi}}}_{ijk}(U) = -\widetilde{U}_x^\top \widehat{l}_i(U)^\top (\nabla_U \widehat{r}_k(U))^\top - \widehat{l}_i(U)(\nabla_U \widehat{r}_k(U))^\top A^{-1}(U)C(\overline{U})\\
&\qquad\qquad - \widehat{l}_i(U)A^{-1}(U)B(\widetilde{U})\widehat{r}_j(U).
\end{aligned}$$

3 Stability of Steady Supersonic Solutions

In this section, we will get the global stability of steady supersonic solutions for compressible Euler equations with heat transfer in one-dimensional duct with the length L, which is strictly less than L_m.

For supersonic flows in the duct, the perturbations of initial data and boundary condition have quite different effects on the stability of the solutions. Therefore, we consider the mixed initial-boundary value problem of system (1.1) with the initial data

$$t = 0: \begin{cases} \rho = \overline{\rho}_0(x) + \widetilde{\rho}(x), \\ u = \overline{u}_0(x) + \widetilde{u}(x), \quad x \in [0, L] \\ p = \overline{p}_0(x) + \widetilde{p}(x), \end{cases} \tag{3.1}$$

and boundary data on the entry

$$x = 0: \begin{cases} \rho = \overline{\rho}_l(t) + \widetilde{\rho}(0), \\ u = \overline{u}_l(t) + \widetilde{u}(0), \quad t \geq 0, \\ p = \overline{p}_l(t) + \widetilde{p}(0), \end{cases} \tag{3.2}$$

where $\overline{\rho}_0$, $\overline{u}_0$, $\overline{p}_0$, $\overline{\rho}_l$, $\overline{u}_l$, $\overline{p}_l$ are C^1 functions. Without loss of generality, the conditions of C^1 compatibility are supposed to be satisfied at the point $(0,0)$, i.e.,

$$\overline{\rho}_0(0) = \overline{\rho}_l(0), \quad \overline{u}_0(0) = \overline{u}_l(0), \quad \overline{p}_0(0) = \overline{p}_l(0), \tag{3.3}$$

and

$$\begin{cases} \overline{\rho}_l'(0) + (\overline{u}_0(0) + \widetilde{u}(0))(\overline{\rho}_0'(0) + \widetilde{\rho}'(0)) + (\overline{\rho}_0(0) + \widetilde{\rho}(0))(\overline{u}_0'(0) + \widetilde{u}'(0)) = 0, \\ (\overline{\rho}_l(0) + \widetilde{\rho}(0))\overline{u}_l'(0) + (\overline{\rho}_0(0) + \widetilde{\rho}(0))(\overline{u}_0(0) + \widetilde{u}(0))(\overline{u}_0'(0) + \widetilde{u}'(0)) \\ +(\overline{p}_0'(0) + \widetilde{p}'(0)) = 0, \\ \overline{p}_l'(0) + \gamma(\overline{p}_0(0) + \widetilde{p}(0))(\overline{u}_0'(0) + \widetilde{u}'(0)) + (\overline{u}_0(0) + \widetilde{u}(0))(\overline{p}_0'(0) + \widetilde{p}'(0)) \\ -(\gamma - 1)(\overline{\rho}_0(0) + \widetilde{\rho}(0))Q(0) = 0. \end{cases} \tag{3.4}$$

Moreover, on the domain under consideration, we also suppose that the background solution satisfies

$$k_1 \leq \lambda_1(\widetilde{U}) = \widetilde{u} - \widetilde{c} < \lambda_2(\widetilde{U}) = \widetilde{u} < \lambda_3(\widetilde{U}) = \widetilde{u} + \widetilde{c} \leq k_2, \tag{3.5}$$

where k_1, k_2 are two positive constants.

Under the above assumptions, we can get the following stability result.

Theorem 3.1 *Suppose that* $\|(\overline{\rho}_0(x), \overline{u}_0(x), \overline{p}_0(x))\|_{C^1[0,L]}$, $\|(\overline{\rho}_l(t), \overline{u}_l(t), \overline{p}_l(t))\|_{C^1(\mathbb{R}^+)}$ *are sufficiently small, and* $\widetilde{\rho}(x), \widetilde{u}(x), \widetilde{p}(x)$ *are* C^2 *function on* $[0, L]$ *satisfying* (3.5). *Furthermore, the conditions of* C^1 *compatibility* (3.3)–(3.4) *are satisfied and* $Q(x)$ *is a positive* C^2 *function on* $[0, L]$. *Then, the mixed initial-boundary value problem* (1.1) *and* (3.1)–(3.2) *admits a unique* C^1 *solution* $U = (\rho(t, x), u(t, x), p(t, x))^\top$ *with small* C^1 *norm on the domain*

$$D = \{(t, x) \mid t \geq 0,\ 0 \leq x \leq L\}.$$

Remark 3.1 Here we only give the boundary condition on $x = 0$. The reason is that, by the supersonic condition $u > c$ in the duct, the flow at $x = L$ is completely determined by the initial data in $[0, L]$ and boundary data on $x = 0$.

4 Proof of Theorem

According to the local existence and uniqueness of C^1 solutions (see [28]), in order to get the global existence and uniqueness of C^1 solution (see [17, 24–25, 29–30]), it suffices to establish a C^1 uniform prior estimate of the classical solutions. As a matter of fact, to prove the global stability C^1 solutions of system (1.1) and (3.1)–(3.2), it suffices to get the bounded $\|V(t, \cdot)\|_0$ and $\|W(t, \cdot)\|_0$.

Without loss of generality, we suppose that

$$\|(\overline{\rho}_0(x), \overline{u}_0(x), \overline{p}_0(x))\|_{C^1[0,L]} < \varepsilon, \quad \|(\overline{\rho}_l(t), \overline{u}_l(t), \overline{p}_l(t))\|_{C^1(\mathbb{R}^+)} < \varepsilon, \tag{4.1}$$

and in the region D,

$$|V_i(t, x)| \leq C\varepsilon, \quad |W_i(t, x)| \leq C\varepsilon, \quad \forall\ i = 1, 2, 3 \text{ and } (t, x)\ \in D, \tag{4.2}$$

where C is a suffciently large constant which can be determined by the estimates in this section, and $\varepsilon > 0$ is a suitably small constant. Then by (2.9)–(2.10), it is easy to get

$$|\overline{U}(t, x)| \leq C\varepsilon, \quad \forall\ (t, x) \in D. \tag{4.3}$$

Here and hereafter, C, C_i and C_i^* etc. denote positive constants only depending on ε, L, $\|(\widetilde{\rho}, \widetilde{u}, \widetilde{p})\|_{C^2[0,L]}$ as well as T_i defined by

$$T_i = \min_{t \geq 0,\, x \in [0,L]} \frac{L}{\lambda_i(U(t, x_i^*(t)))} > 0, \quad i = 1, 2, 3. \tag{4.4}$$

In the end, we will show the validity of hypothesis (4.2).

Let

$$V(\tau) = \sup_{0 \leq t \leq \tau} \|V(t, \cdot)\|_0, \quad W(\tau) = \sup_{0 \leq t \leq \tau} \|W(t, \cdot)\|_0, \tag{4.5}$$

where $\|\cdot\|_0$ stands for the C^0 norm on $[0, L]$. Furthermore, let $x = x_i^*(t)$ be the characteristic passing through the origin, on which it holds that

$$\begin{cases} \dfrac{\mathrm{d}x_i^*(t)}{\mathrm{d}t} = \lambda_i(U(t, x_i^*(t))), \\ x_i^*(0) = 0. \end{cases} \tag{4.6}$$

For any given point $(t,x)\in D$, we draw down the ith characteristic passing through (t,x), noting (2.6) and (4.4), there are four possibilities:

Case 1 The region (see Figure 1)

$$D_1=\{(t,x)\ \mid 0\le t\le T_1,\ x\ge x_3^*(t)\}.$$

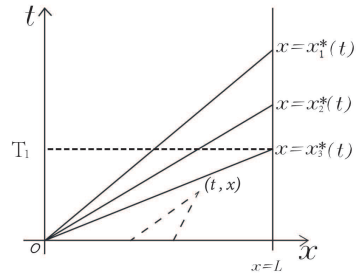

Figure 1

The ith characteristic intersects the x-axis at a point $(0,\alpha_i)$. Integrating the ith equation in (2.13) along this characteristic curve with respect to τ from 0 to t, and noting (4.2) and (4.4), we get

$$\begin{aligned}|V_i(t,x)|&=|V_i(0,\alpha_i)|+\Big|\int_0^t\sum_{j,k=1}^3\Phi_{ijk}(U)W_jV_k\,\mathrm{d}\tau\Big|+\Big|\int_0^t\sum_{j,k=1}^3\widetilde{\Phi}_{ijk}(U)V_jV_k\,\mathrm{d}\tau\Big|\\&\quad+\Big|-\int_0^t\sum_{k=1}^3\widetilde{\widetilde{\Phi}}_{ik}(U)V_k\,\mathrm{d}\tau\Big|+\Big|\int_0^t l_i(U)C(\overline{U})\,\mathrm{d}\tau\Big|\\&\le|V_i(0,\alpha_i)|+C_1\int_0^{T_1}V(\tau)\,\mathrm{d}\tau+\int_0^{T_1}\frac{(\gamma-1)}{c}|Q(x)|\Big[|V_2|+\frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\mathrm{d}\tau\\&\le\|V_i(0,\cdot)\|_{C^0}+C_1\int_0^{T_1}V(\tau)\,\mathrm{d}\tau+C_1^*\int_0^{T_1}V(\tau)\,\mathrm{d}\tau.\end{aligned}\tag{4.7}$$

By Gronwall's inequality, we can get

$$|V(t,x)|\le C\|V(0,\cdot)\|_{C^0},\quad\forall\ (t,x)\in D_1.\tag{4.8}$$

Similarly, integrating the ith equation in (2.15) along the ith characteristic curve, we have

$$\begin{aligned}|W_i(t,x)|&=|W_i(0,\alpha_i)|+\Big|\int_0^t\sum_{j,k=1}^3\Psi_{ijk}(U)W_jW_k\Big|\,\mathrm{d}\tau+\Big|\int_0^t\sum_{j,k=1}^3\widetilde{\Psi}_{ijk}(U)V_jW_k\Big|\,\mathrm{d}\tau\\&\quad+\Big|\int_0^t\sum_{j,k=1}^3\widetilde{\widetilde{\Psi}}_{ijk}(U)W_k\Big|\,\mathrm{d}\tau+\Big|\int_0^t l_i(U)C(\overline{U})_x-l_i(U)B(\widetilde{U})_x\overline{U}\Big|\,\mathrm{d}\tau\\&\le\|W_i(0,\alpha_i)\|_{C^0}+C_2\int_0^{T_1}W(\tau)\,\mathrm{d}\tau+\int_0^{T_1}\frac{(\gamma-1)}{c}\Big\{|Q(x)|\Big[|W_2|+\frac{\rho}{c^2}(|W_1|+|W_3|)\Big]\\&\quad+|Q'(x)|\Big[|V_2|+\frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\Big\}\,\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{T_1}\Big\{\Big|\frac{\rho^2(H\widetilde{p}_x)_x-(1\pm\gamma)\rho c\widetilde{u}_{xx}\pm\widetilde{p}_{xx}}{c^2}\Big||V_1|\\
&+|\rho(H\widetilde{p}_x)_x||V_2|+\Big|\frac{\rho^2(H\widetilde{p}_x)_x+(1\pm\gamma)\rho c\widetilde{u}_{xx}\mp\widetilde{p}_{xx}}{c^2}\Big||V_3|\Big\}\,\mathrm{d}\tau\\
\le{}& \|W_i(0,\cdot)\|_{C^0}+C_2\int_0^{T_1}W(\tau)\,\mathrm{d}\tau+C_2^*\int_0^{T_1}W(\tau)\,\mathrm{d}\tau+C_2^{**}\int_0^{T_1}V(\tau)\,\mathrm{d}\tau. \qquad (4.9)
\end{aligned}
$$

Then noting (4.8), Gronwall's inequality implies that

$$
|W(t,x)|\le C(\|V(0,\cdot)\|_{C^0}+\|W(0,\cdot)\|_{C^0}),\quad \forall\ (t,x)\in D_1. \qquad (4.10)
$$

Case 2 The region (see Figure 2)

$$
D_2=\{(t,x)\,|\,0\le t\le T_2,\ x_2^*(t)\le x\le x_3^*(t)\}.
$$

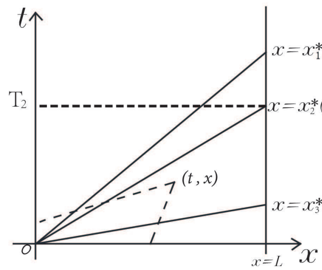

Figure 2

For any given point $(t,x)\in D_2$, integrating the 2nd equations in (2.13) and (2.15) along the second characteristic curve that intersects the x-axis at a point $(0,\alpha_2)$, we can get

$$
\begin{aligned}
|V_2(t,x)|\le{}& |V_2(0,\alpha_2)|+C_3\int_0^t V(\tau)\,\mathrm{d}\tau+\int_0^t\frac{(\gamma-1)}{c}|Q(x)|\Big[|V_2|+\frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\,\mathrm{d}\tau\\
\le{}& \|V_2(0,\cdot)\|_{C^0}+C_3\int_0^{T_2}V(\tau)\,\mathrm{d}\tau+C_3^*\int_0^{T_2}V(\tau)\,\mathrm{d}\tau. \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
|W_2(t,x)|\le{}& |W_2(0,\alpha_2)|+C_4\int_0^t W(\tau)\,\mathrm{d}\tau+\int_0^t\frac{(\gamma-1)}{c}\Big\{|Q(x)|\Big[|W_2|+\frac{\rho}{c^2}(|W_1|+|W_3|)\Big]\\
&+|Q'(x)|\Big[|V_2|+\frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\Big\}\,\mathrm{d}\tau+\int_0^t\Big\{\Big|\frac{c^2\widetilde{\rho}_{xx}+(\gamma-1)\rho c\widetilde{u}_{xx}-\widetilde{p}_{xx}}{c^2}\Big||V_1|\\
&+|c\widetilde{u}_{xx}||V_2|+\Big|\frac{-c^2\widetilde{\rho}_{xx}+(\gamma-1)\rho c\widetilde{u}_{xx}+\widetilde{p}_{xx}}{c^2}\Big||V_3|\Big\}\,\mathrm{d}\tau\\
\le{}& \|W_2(0,\cdot)\|_{C^0}+C_4\int_0^{T_2}W(\tau)\,\mathrm{d}\tau+C_4^*\int_0^{T_2}W(\tau)\,\mathrm{d}\tau+C_4^{**}\int_0^{T_2}V(\tau)\,\mathrm{d}\tau. \qquad (4.12)
\end{aligned}
$$

Then summing up (4.11)–(4.12), by Gronwall's inequality, we have

$$
|V_2(t,x)|+|W_2(t,x)|\le C(\|V_2(0,\cdot)\|_{C^0}+\|W_2(0,\cdot)\|_{C^0}),\quad \forall\ (t,x)\in D_2. \qquad (4.13)
$$

Similarly, integrating the 3rd equations in (2.13) and (2.15) along the third characteristic curve that intersects the t-axis at a point $(\tau_3,0)$, one has

$$
|V_3(t,x)|\le|V_3(\tau_3,0)|+C_5\int_{\tau_3}^t V(\tau)\,\mathrm{d}\tau+\int_{\tau_3}^t\frac{(\gamma-1)}{c}|Q(x)|\Big[|V_2|+\frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\,\mathrm{d}\tau
$$

$$\leq \|V_3(\cdot,0)\|_{C^0} + C_5\int_{\tau_3}^{T_2} V(\tau)\,\mathrm{d}\tau + C_5^*\int_{\tau_3}^{T_2} V(\tau)\,\mathrm{d}\tau. \tag{4.14}$$

$$\begin{aligned}|W_3(t,x)| &\leq |W_3(\tau_3,0)| + C_6\int_{\tau_3}^{t} W(\tau)\,\mathrm{d}\tau + \int_{\tau_3}^{t}\frac{(\gamma-1)}{c}\Big\{|Q(x)|\Big[|W_2| + \frac{\rho}{c^2}(|W_1|+|W_3|)\Big]\\ &\quad + |Q'(x)|\Big[|V_2| + \frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\Big\}\,\mathrm{d}\tau + \int_{\tau_3}^{t}\Big\{\Big|\frac{\rho^2(H\widetilde{p}_x)_x - (1\pm\gamma)\rho c\widetilde{u}_{xx} \pm \widetilde{p}_{xx}}{c^2}\Big||V_1|\\ &\quad + |\rho(H\widetilde{p}_x)_x||V_2| + \Big|\frac{\rho^2(H\widetilde{p}_x)_x + (1\mp\gamma)\rho c\widetilde{u}_{xx} \mp \widetilde{p}_{xx}}{c^2}\Big||V_3|\Big\}\,\mathrm{d}\tau\\ &\leq \|W_3(\cdot,0)\|_{C^0} + C_6\int_{\tau_3}^{T_2} W(\tau)\,\mathrm{d}\tau + C_6^*\int_{\tau_3}^{T_2} W(\tau)\,\mathrm{d}\tau + C_6^{**}\int_{\tau_3}^{T_2} V(\tau)\,\mathrm{d}\tau. \end{aligned}\tag{4.15}$$

Because the boundary data is sufficiently small. Using the Gronwall's inequality, the combination (4.14)–(4.15) leads to

$$|V_3(t,x)| + |W_3(t,x)| \leq C(\|V_3(\cdot,0)\|_{C^0} + \|W_3(\cdot,0)\|_{C^0}), \quad \forall\ (t,x)\in D_2. \tag{4.16}$$

Case 3 The region (see Figure 3)

$$D_3 = \{(t,x)\mid 0\leq t\leq T_3,\ x_1^*(t)\leq x\leq x_2^*(t)\}.$$

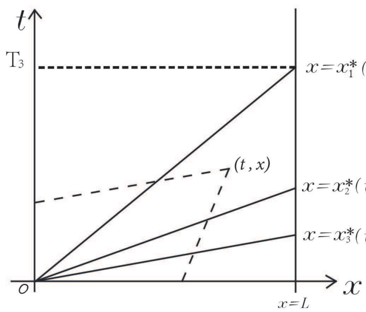

Figure 3

For any given point $(t,x)\in D_3$, integrating the 1st equations in (2.13) and (2.15) along the first characteristic curve that intersects the x-axis at a point $(0,\alpha_1)$, we can get

$$\begin{aligned}|V_1(t,x)| &\leq |V_1(0,\alpha_1)| + C_7\int_0^t V(\tau)\,\mathrm{d}\tau + \int_0^t \frac{(\gamma-1)}{c}|Q(x)|\Big[|V_2| + \frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\,\mathrm{d}\tau\\ &\leq \|V_1(0,\cdot)\|_{C^0} + C_7\int_0^{T_3} V(\tau)\,\mathrm{d}\tau + C_7^*\int_0^{T_3} V(\tau)\,\mathrm{d}\tau.\end{aligned}\tag{4.17}$$

$$\begin{aligned}|W_1(t,x)| &\leq |W_1(0,\alpha_1)| + C_8\int_0^t W(\tau)\,\mathrm{d}\tau + \int_0^t \frac{(\gamma-1)}{c}\Big\{|Q(x)|\Big[|W_2| + \frac{\rho}{c^2}(|W_1|+|W_3|)\Big]\\ &\quad + |Q'(x)|\Big[|V_2| + \frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\Big\}\,\mathrm{d}\tau\\ &\quad + \int_0^t\Big\{\Big|\frac{\rho^2(H\widetilde{p}_x)_x - (1\pm\gamma)\rho c\widetilde{u}_{xx} \pm \widetilde{p}_{xx}}{c^2}\Big||V_1|\\ &\quad + |\rho(H\widetilde{p}_x)_x||V_2| + \Big|\frac{\rho^2(H\widetilde{p}_x)_x + (1\mp\gamma)\rho c\widetilde{u}_{xx} \mp \widetilde{p}_{xx}}{c^2}\Big||V_3|\Big\}\,\mathrm{d}\tau\end{aligned}$$

$$\leq \|W_1(0,\cdot)\|_{C^0} + C_8 \int_0^{T_3} W(\tau)\,\mathrm{d}\tau + C_8^* \int_0^{T_3} W(\tau)\,\mathrm{d}\tau + C_8^{**} \int_0^{T_3} V(\tau)\,\mathrm{d}\tau. \quad (4.18)$$

Then combining (4.17)–(4.18) and using Gronwall's inequality, we can get

$$|V_1(t,x)| + |W_1(t,x)| \leq C(\|V_1(0,\cdot)\|_{C^0} + \|W_1(0,\cdot)\|_{C^0}), \quad \forall\ (t,x) \in D_3. \quad (4.19)$$

Similarly, integrating the 2nd equations in (2.13) and (2.15) along the second characteristic curve that intersects the t-axis at a point $(\tau_2, 0)$, one has

$$\begin{aligned}|V_2(t,x)| &\leq |V_2(\tau_2,0)| + C_9 \int_{\tau_2}^{t} V(\tau)\,\mathrm{d}\tau + \int_{\tau_2}^{t} \frac{(\gamma-1)}{c}|Q(x)|\Big[|V_2| + \frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\,\mathrm{d}\tau \\ &\leq \|V_2(\cdot,0)\|_{C^0} + C_9 \int_{\tau_2}^{T_3} V(\tau)\,\mathrm{d}\tau + C_9^* \int_{\tau_2}^{T_3} V(\tau)\,\mathrm{d}\tau, \qquad (4.20)\end{aligned}$$

$$\begin{aligned}|W_2(t,x)| &\leq |W_2(\tau_2,0)| + C_{10} \int_{\tau_2}^{t} W(\tau)\,\mathrm{d}\tau + \int_{\tau_2}^{t} \frac{(\gamma-1)}{c}\Big\{|Q(x)|\Big[|W_2| + \frac{\rho}{c^2}(|W_1|+|W_3|)\Big] \\ &\quad + |Q'(x)|\Big[|V_2| + \frac{\rho}{c^2}(|V_1|+|V_3|)\Big]\Big\}\,\mathrm{d}\tau \\ &\quad + \int_{\tau_2}^{t} \Big\{\Big|\frac{c^2\widetilde{\rho}_{xx} + (\gamma-1)\rho c\widetilde{u}_{xx} - \widetilde{p}_{xx}}{c^2}\Big||V_1| \\ &\quad + |c\widetilde{u}_{xx}||V_2| + \Big|\frac{-c^2\widetilde{\rho}_{xx} + (\gamma-1)\rho c\widetilde{u}_{xx} + \widetilde{p}_{xx}}{c^2}\Big||V_3|\Big\}\,\mathrm{d}\tau \\ &\leq \|W_2(\cdot,0)\|_{C^0} + C_{10} \int_{\tau_2}^{T_3} W(\tau)\,\mathrm{d}\tau + C_{10}^* \int_{\tau_2}^{T_3} W(\tau)\,\mathrm{d}\tau + C_{10}^{**} \int_{\tau_2}^{T_3} V(\tau)\,\mathrm{d}\tau. \quad (4.21)\end{aligned}$$

Because the boundary data is small enough. Using the Gronwall's inequality, summing up (4.20)–(4.21), we get

$$|V_2(t,x)| + |W_2(t,x)| \leq C(\|V_2(\cdot,0)\|_{C^0} + \|W_2(\cdot,0)\|_{C^0}), \quad \forall\ (t,x) \in D_3. \quad (4.22)$$

Case 4 In the region (see Figure 4)

$$D_4 = \{(t,x) \mid 0 \leq t,\ x \leq x_1^*(t)\},$$

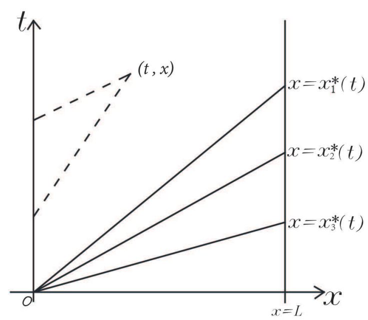

Figure 4

we want to exchange the order of the variable t and x, and rewrite the wave decomposition equations as (2.18)–(2.19). Denote

$$\widehat{V}(\xi) = \sup_{D_4\cap\{0\leq x\leq\xi\}} |\widehat{V}(t,x)|, \quad \widehat{W}(\xi) = \sup_{D_4\cap\{0\leq x\leq\xi\}} |\widehat{W}(t,x)|. \quad (4.23)$$

For any given point $(t,x)\in D_4$, integrating the ith equation in (2.18) with respect to x along the ith characteristic curve, which is assumed to intersect the t-axis at a point $(t_i,0)$, we find that

$$\begin{aligned}
|\widehat{V}_i(t,x)| &= |\widehat{V}_i(t_i,0)| + \Big|\int_0^x \sum_{j,k=1}^3 \widehat{\Phi}_{ijk}(U)\widehat{W}_j\widehat{V}_k\mathrm{d}\xi\Big| + \Big|\int_0^x \sum_{j,k=1}^3 \widehat{\widehat{\Phi}}_{ijk}(U)\widehat{V}_j\widehat{V}_k\mathrm{d}\xi\Big| \\
&\quad - \Big|\int_0^x \sum_{k=1}^3 \widehat{\widehat{\widehat{\Phi}}}_{ik}(U)\widehat{V}_k\mathrm{d}\xi\Big| + \Big|\int_0^x \widehat{\lambda}_i\widehat{l}_i(U)C(\overline{U})\mathrm{d}\xi\Big| \\
&\le |\widehat{V}_i(t_i,0)| + C_{11}\int_0^x \widehat{V}(\xi) + C_{11}^*\int_0^x \widehat{\lambda}_i\frac{(\gamma-1)}{c}|Q(x)|\Big[|\widehat{V}_1| \\
&\quad + \frac{\rho}{c^2}(|\widehat{V}_2|+|\widehat{V}_3|)\Big]\,\mathrm{d}\xi \\
&\le \|\widehat{V}_i(\cdot,0)\|_{C^0} + C_{11}\int_0^x \widehat{V}(\xi)\,\mathrm{d}\xi + C_{11}^*\int_0^x \widehat{V}(\xi)\,\mathrm{d}\xi.
\end{aligned} \tag{4.24}$$

Summing up for $i=1,2,3$ and using the Gronwall's inequality, we get

$$|\widehat{V}(t,x)| \le C\|\widehat{V}(\cdot,0)\|_{C^0}, \quad \forall\ (t,x)\in D_4. \tag{4.25}$$

Similarly, integrating the ith equation in (2.19) along the ith characteristic curve, it follows that

$$\begin{aligned}
|\widehat{W}_i(t,x)| &= |\widehat{W}_i(t_i,0)| + \Big|\int_0^x \sum_{j,k=1}^3 \widehat{\Psi}_{ijk}(U)\widehat{W}_j\widehat{W}_k\mathrm{d}\xi\Big| + \Big|\int_0^x \sum_{j,k=1}^3 \widehat{\widehat{\Psi}}_{ijk}(U)\widehat{V}_j\widehat{W}_k\mathrm{d}\xi\Big| \\
&\quad + \Big|\int_0^x \sum_{j,k=1}^3 \widehat{\widehat{\widehat{\Psi}}}_{ijk}(U)\widehat{W}_k\mathrm{d}\xi\Big| + \Big|\int_0^x \widehat{l}_i(U)(A^{-1}(U)C(\overline{U}))_t\mathrm{d}\xi\Big| \\
&\quad - \Big|\int_0^x \widehat{l}_i(U)(A^{-1}(U)B(\widetilde{U}))_t\overline{U}\mathrm{d}\xi\Big| \\
&\le |\widehat{W}_i(t_i,0)| + C_{12}\int_0^x \widehat{W}(\xi)\,\mathrm{d}\xi + C_{12}^*\int_0^x (\gamma-1)\rho u|Q(x)|(|\widehat{W}_i|+|\widehat{V}_i|+|\widehat{V}_iW_i|)\,\mathrm{d}\xi \\
&\quad + C_{12}^{**}\int_0^x (|\widetilde{\rho}_x+\widetilde{u}_x+\widetilde{p}_x|)|\widehat{V}_i\widehat{W}_i|\,\mathrm{d}\xi \\
&\le \|\widehat{W}_i(\cdot,0)\|_{C^0} + C_{12}\int_0^x \widehat{W}(\xi)\,\mathrm{d}\xi + C_{12}^*\int_0^x \widehat{W}(\xi)\,\mathrm{d}\xi + C_{12}^{**}\int_0^x \widehat{V}(\xi)\,\mathrm{d}\xi.
\end{aligned} \tag{4.26}$$

Using the Gronwall's inequality after summing up the above inequality for $i=1,2,3$, and thanks to (4.25), we have

$$|\widehat{W}(t,x)| \le C(\|\widehat{W}(\cdot,0)\|_{C^0} + \|\widehat{V}(\cdot,0)\|_{C^0}), \quad \forall\ (t,x)\in D_4. \tag{4.27}$$

Hence, both $\|\overline{U}\|_0$ and $\|D_{t,x}\overline{U}\|_0$ are small if ε is chosen to be sufficiently small, provided that $\|(\widetilde{\rho},\widetilde{u})\|_{C^2[0,L]}$ are bounded. Then by (4.8), (4.10) (4.13), (4.16), (4.19), (4.22), (4.25) and (4.27), $|V(t,x)|$, $|W(t,x)|$ must be sufficiently small. This implies the validity of hypothesis (4.2). Therefore, we obtained a uniform C^1 a priori estimate for the global stability of steady supersonic solution. The proof of Theorem 3.1 is completed.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Liu, L. and Yuan, H. R., Stability of cylindrical transonic shocks for the two-dimensional steady compressible Euler system, *J. Hyperbolic Differ. Equ.*, **5**(2), 2008, 347–379.

[2] Liu, L., Xu, G. and Yuan, H. R., Stability of spherically symmetric subsonic flows and transonic shocks under multidimensional perturbations, *Adv. Math.*, **291**, 2016, 696–757.

[3] Fang, B. X. and Gao, X., On admissible positions of transonic shocks for steady Euler flows in a 3-D axisymmetric cylindrical nozzle, *J. Differential Equations*, **288**, 2021, 62–117.

[4] Fang, B. X. and Xin, Z. P., On admissible locations of transonic shock fronts for steady Euler flows in an almost flat finite nozzle with prescribed receiver pressure, *Comm. Pure Appl. Math.*, **74**(7), 2021, 1493–1544.

[5] Chen, G. Q. and Feldman, M., Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, *J. Amer. Math. Soc.*, **16**(3), 2003, 461–494.

[6] Li, J., Xin, Z. P. and Yin, H. C., Transonic shocks for the full compressible Euler system in a general two-dimensional de Laval nozzle, *Arch. Ration. Mech. Anal.*, **207**(2), 2013, 533–581.

[7] Weng, S. K., Xie, C. J. and Xin, Z. P., Structural stability of the transonic shock problem in a divergent three-dimensional axisymmetric perturbed nozzle, *SIAM J. Math. Anal.*, **53**(1), 2021, 279–308.

[8] Shapiro, A. H., The Dynamics and Thermodynamics of Compressible Fluid Flow, **1**, New York, Wiley, 1953.

[9] Zhao, Q. and Yuan, H. R., Subsonic flow passing a duct for three-dimensional steady compressible Euler system with friction, *Sci. Sin. Math.*, **51**, 2021, 1–23 (in Chinese).

[10] Zhao, Q. and Yuan, H. R., Stabilization effect of frictions for transonic shocks in steady compressible Euler flows passing three-dimensional ducts, *Acta Math. Sci.* Ser. B, **40**(2), 2020, 470–502.

[11] Wei, F. L., Liu, J. L. and Yuan, H. R., Global stability to steady supersonic solutions of the 1-D compressible Euler equations with frictions, *J. Math. Anal. Appl.*, **495**(2), 2021, 14 pp.

[12] Rathakrishnan, E., Applied Gas Dynamics, John Wiley & Sons Ltd, Singapore, 2010.

[13] Tannehill, J. C., Anderson, D. A. and Pletcher, R. H., Computational Fluid Mechanics and Heat Transfer, Washington, Taylor and Francis, 1987.

[14] Dyson, F. J., Dynamics of a spinning gas cloud, *J. Math. Mech.*, **18**(1), 1968, 91–101.

[15] Rickard, C., Global solutions to the compressible Euler equations with heat transport by convection around Dysons isothermal affine solutions, *Arch. Ration. Mech. Anal.*, **241**(2), 2021, 947–1007.

[16] Gao, J. L., Liu, L. and Yuan, H. R., On stability of transonic shocks for steady Rayleigh flows in two-dimensional duct, *SIAM J. Math. Anal.*, 2020, 5287–5337.

[17] Li, T. T., Global Classical Solutions for Quasilinear Hyperbolic Systems, RAM: Research in App. Math., **32**, Masson, Paris, John Wiley & Sons, Ltd., Chichester, 1994.

[18] Kong, D. X., Cauchy Problem for Quasilinear Hyperbolic Systems, MSJ Memoirs, **6**, Math. Society of Japan, Tokyo, 2000.

[19] Li, D. Q. and Qin, T. H., Global smooth solutions for a class of quasilinear hyperbolic systems with dissipative terms, A Chinese summary appears in Chinese Ann. Math. Ser. A, **6**(2), 1985, 255, *Chinese Ann. Math. Ser. B*, **6**(2), 1985, 199–210.

[20] Sun, F. Q. and Wang, M. X., Global and blow-up solutions for a system of nonlinear hyperbolic equations with dissipative terms, *Nonlinear Anal.*, **64**(4), 2006, 739–761.

[21] Yang, T., Zhu, C. J. and Zhao, H. J., Global smooth solutions for a class of quasilinear hyperbolic systems with dissipative terms, *Proc. Roy. Soc. Edinburgh Sect. A*, **127**(6), 1997, 1311–1324.

[22] Liu, C. M. and Qu, P., Global classical solution to partially dissipative quasilinear hyperbolic systems, *J. Math. Pures Appl.*, **97**(3), 2012, 262–281.

[23] John, F., Formation of singularities in one-dimensional nonlinear wave propagation, *Commu. Pure Appl. Math.*, **27**, 1974, 377–405.

[24] Li, T. T., Zhou, Y. and Kong, D. X., Global classical solutions for general quasilinear hyperbolic systems with decay initial data, *Nonlinear Anal.*, **28**(8), 1997, 1299–1332.

[25] Wang, Z. Q., Exact controllability for non-autonomous first order quasilinear hyperbolic systems, *Chin. Ann. Math. Ser. B*, **27**(6), 2006, 643–656.

[26] Wang, Z. Q. and Yu, L. X., Exact boundary controllability for one-dimensional adiabatic flow system, *Appl. Math. J. Chinese Univ. Ser. A*, **23**(1), 2008, 35–40.

[27] Yu, L. X., Semi-global C^1 solution to the mixed initial-boundary value problem for a kind of quasilinear hyperbolic systems, *Chin. Ann. Math. Ser. A*, **25**(5) 2004, 549–560 (in Chinese).

[28] Li, T. T. and Yu, W. C., Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke University Math., Series V, Duke Univ. Press, Durham, 1985.

[29] Li, T. T. and Rao, B. P., Local exact boundary controllability for a class of quasilinear hyperbolic systems, *Chin. Ann. Math. Ser. B*, **23**(2), 2002, 209–218.

[30] Li, T. T. and Jin, Y., Semi-global C^1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems, *Chin. Ann. Math. Ser. B*, **22**(3), 2001, 325–336.