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Markovian Quadratic BSDEs with an Unbounded Sub-quadratic Growth*

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Abstract This paper is devoted to the solvability of Markovian quadratic backward stochastic differential equations (BSDEs for short) with bounded terminal conditions. The generator is allowed to have an unbounded sub-quadratic growth in the second unknown variable z. The existence and uniqueness results are given to these BSDEs. As an application, an existence result is given to a system of coupled forward-backward stochastic differential equations with measurable coefficients.

Keywords Markovian BSDE, Quadratic growth, Unbounded sub-quadratic term coefficients, Coupled FBSDE

2000 MR Subject Classification 60H10

1 Introduction

In this paper, we use probabilistic methods to study the Markovian backward stochastic differential equations (BSDEs for short)

$$Y_s = g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x}, Y_u, Z_u)\mathrm{d}u - \int_s^T Z_u \mathrm{d}B_u, \quad s \in [0, T], \tag{1.1}$$

where $X_s^{t,x}$ is the unique solution of the forward SDE

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s b(u, X_u^{t,x})\mathrm{d}u + \int_t^s \sigma(u, X_u^{t,x})\mathrm{d}B_u, \quad s \in [t, T], \\ X_s^{t,x} = x, \quad s \in [0, t]. \end{cases} \tag{1.2}$$

The terminal condition $g : \mathbb{R}^m \to \mathbb{R}^d$ is bounded and the diagonally quadratic generator $f : [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{d\times l} \to \mathbb{R}^d$ has unboundedly sub-quadratic growing terms in its last variable z, i.e., there exist $\varepsilon \in (0, 1]$, positive constants C and γ and nondecreasing function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that f^i, the ith component of f, satisfies

$$|f^i(s, x, y, z)| \le \rho(|y|)(1 + |x|^\gamma)(1 + |z|^{2-\varepsilon}) + C|z^i|^2, \quad i = 1, \cdots, d.$$

Under some mild assumptions, we prove the global existence and uniqueness of strong solutions to these Markovian BSDEs.

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The existence and uniqueness result for the nonlinear BSDEs is established by Pardoux and Peng [20] under a Lipschitz continuity assumption on the generator. Since then, many extensions (see [6]) have been devoted to relaxing the Lipschitz continuity assumption. In 2000, Kobylanski [17] proved the existence and uniquess result for the one-dimensional BSDE whose generator has a quadratic growth in the second unknown variable z. Due to the lack of a comparision property, the existence and uniqueness result for multi-dimensional quadratic BSDEs meets with difficulty in the general case and several existing results (see [5, 9, 15]) are restricted within various assumptions. For example, Tevzadze [21] studied the case under the assmuption of small terminal value. Hu and Tang [14] studied the multi-dimensional quadratic BSDEs with diagonal structure on the quadratic term of z. All of these works rely on the boundness of terminal value and generator coefficients to let $\int_0^\cdot ZdW$ be a BMO martingale, which is not true in the unbounded case.

On the other hand, quadratic BSDEs with unbounded terminal value and unbounded generator coefficients were studied to extend Kobylanski's work [17]. Most of the papers, such as Briand and Hu [2–3], studied the one-dimensional case due to the importance of comparision theorem. For the multi-dimensional case, Fan, Hu and Tang [7] studied the multi-dimensional diagonally quadratic BSDEs with unbounded terminal value and unbounded generator coefficients, and their existence and uniqueness result requires that the generator is convex with respect to z. Using analytic and PDE methods, Xing and Žitkovic [22] studied the Markovian quadratic BSDEs and obtained a general result under weak regularity assumptions of the generator and terminal value by virtue of the Lyapunov functions. Their results can be applied to the unbounded Markovian BSDEs somehow, but the generator and terminal condition have to be sufficiently regular.

Unbounded Markovian BSDEs arise from Markovian Nash equilibriums. For example, Çetin and Danilova [4] studied the one-dimensional Markovian BSDEs with unbounded quadratic term coefficients and special forms. Hamadène and Mu [12–13] studied the multi-dimensional Markovian BSDEs with unbounded linear term coefficients and special structure. In this paper, we study Markovian quadratic BSDEs with unbounded sub-quadratic term coefficients and rather general structure, which seem to be new.

Since the generator is unboundedly growing, $\int_0^\cdot Z\mathrm{d}W$ is not necessarily a BMO martingale and the standard techniques of BMO martingale (see [1, 14]) cannot be used to tackle the quadratic term of z. For the multi-dimensional case, inspired by the $\mathcal{L}^q$-domination method in [11, 13, 18] and the θ-method in [7–8], we prove the existence and uniqueness of Markovian strictly and diagonally quadratic BSDEs with unbounded sub-quadratic term coefficients. Particularly, for the one-dimensional case, by virtue of monotone stability theorem and θ-method in [2–3, 17], we prove the existence and uniqueness results when the generator is not necessary to be strictly quadratic.

The remaining of the paper is organized as follows. In Section 2, we list all the notations used in this paper. In Section 3, we state and prove the existence and uniqueness results for multi-dimensional Markovian strictly and diagonally quadratic BSDEs with unbounded sub-quadratic term coefficients. The special one-dimensional case is discussed in Section 4. Finally, a system of coupled forward-backward stochastic differential equation (FBSDE for short) is illustrated to have a solution in Section 5.

2 Notations

Let $T > 0$ be a deterministic finite terminal time and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete probability space where $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented filtration generated by l-dimensional Brownian motion $(B_t)_{t\in[0,T]}$. Denote by $\mathbb{E}_t$ the conditional expectation conditioned on $\mathcal{F}_t$. The Hadamard product of two vectors $a = (a_1, \cdot, a_d)^{\mathrm{T}}$ and $b = (b_1, \cdots, b_d)^{\mathrm{T}}$ is denoted by $a \circ b := (a_1 b_1, \cdots, a_d b_d)^{\mathrm{T}}$. $|Z|$ represents the Frobenius norm $|Z| := \sqrt{\mathrm{trace}(ZZ^{\mathrm{T}})}$ for a matrix Z. For $i = 1, \cdots, d$, denote by Z^i the ith row (component) of the matrix (vector) Z. For $p \geq 1$,

- $\mathbb{S}^p(\mathbb{R}^d)$ is the space of all the d-dimensional continuous adapted processes Y such that

$$\|Y\|_{\mathbb{S}^p} := \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^p\Big]^{\frac{1}{p}} < \infty;$$

- $\mathbb{H}^p(\mathbb{R}^{d\times l})$ is the space of all the predictable processes Z which takes value in $\mathbb{R}^{d\times l}$ such that

$$\|Z\|_{\mathbb{H}^p} := \mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty;$$

- $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times l})$ is the space of all the $Z \in \mathbb{H}^2(\mathbb{R}^{d\times l})$ such that

$$\|Z\|_{\mathrm{BMO}} := \sup_{\tau\in\mathcal{T}} \Big\|\mathbb{E}\Big[\int_\tau^T |Z_s|^2 \mathrm{d}s \mid \mathcal{F}_\tau\Big]\Big\|_{L^\infty(\Omega)}^{\frac{1}{2}} < \infty,$$

where $\mathcal{T}$ denotes the set of all the stopping times $\tau \in [0, T]$;

- $\mathcal{M}(\mathbb{R}^{d\times l})$ is the intersection of all the $\mathbb{H}^p(\mathbb{R}^{d\times l})$, i.e.,

$$\mathcal{M}(\mathbb{R}^{d\times l}) := \bigcap_{p\geq 1} \mathbb{H}^p(\mathbb{R}^{d\times l}).$$

For $Z \in \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times l})$, we denote by $\mathcal{E}(\int_0^\cdot Z\mathrm{d}B)$ the stochastic exponential of stochastic process $\{\int_0^t Z_s \mathrm{d}B_s,\ 0 \leq t \leq T\}$. From the theory of BMO martingale (see [16, Theorem 2.3, p.31]), $\mathcal{E}(\int_0^\cdot Z\mathrm{d}B)$ is a martingale.

3 Multi-dimensional Case

3.1 Assumptions for forward SDE (1.2) and auxiliary lemmas

First, we make the following two assumptions for the forward SDE (1.2). Let C be a positive constant.

(F1) $(b, \sigma) : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{m\times l}$ are Borel measurable functions such that for all $(t, x, x') \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^m$,

$$\begin{aligned}|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| &\leq C|x - x'|;\\ |b(t,0)| + |\sigma(t,0)| &\leq C.\end{aligned}$$

(F2) There exists a constant $\lambda > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^m$,

$$\lambda I_m \leq \sigma(t,x)\sigma^{\mathrm{T}}(t,x) \leq \lambda^{-1} I_m \quad \text{and} \quad |b(t,x)| \leq C.$$

Following Hamadène and Mu [12, Definition 4.2, pp.97–98], we state the definition of $\mathcal{L}^q$-domination condition.

Definition 3.1 *Let $q \in (1, +\infty)$ and $t \in [0, T]$. We say a family of probability measure $\{\nu(s, \mathrm{d}x)\}_{s\in[t,T]}$ is $\mathcal{L}^q$-dominated by another family of probability measure $\{\bar{\nu}(s, \mathrm{d}x)\}_{s\in[t,T]}$ if for all $\delta \in (0, T-t]$, there exists a function $\phi_t^\delta : [t+\delta, T] \times \mathbb{R}^m \to \mathbb{R}^+$ such that the following conditions are satisfied*: (i) $\nu(s, \mathrm{d}x)\mathrm{d}s = \phi_t^\delta(s, x)\bar{\nu}(s, \mathrm{d}x)\mathrm{d}s, \quad \forall (s, x) \in [t+\delta, T] \times \mathbb{R}^m$; (ii) $\forall k \geq 1,\ \phi_t^\delta \in L^q\big([t+\delta, T] \times [-k, k]^m; \bar{\nu}(s, \mathrm{d}x)\mathrm{d}s\big)$.

Lemma 3.1 (see [12, Corollary 4.4, p.99]) *Assume that* (F1)–(F2) *holds true. Let $(t, x) \in [0, T] \times \mathbb{R}^m, s \in (t, T]$ and fix $x_0 \in \mathbb{R}^m$. Denote by $p(t, x; s, dy)$ the law of $X_s^{t,x}$, i.e.*

$$\forall A \in \mathcal{B}(\mathbb{R}^m), \quad p(t, x; s, A) := \mathbb{P}(X_s^{t,x} \in A).$$

Then for any $q \in (1, +\infty)$, the family of probability laws $\{p(t, x; s, dy)\}_{s\in[t,T]}$ is $\mathcal{L}^q$-dominated by $\{p(0, x_0; s, dy)\}_{s\in[t,T]}$.

Remark 3.1 In Lemma 3.1, assumptions (F1)–(F2) can be replaced with the following one:
(F3) (i) $b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ is a Borel measureble function satisfying

$$\sup_{|x_1 - x_2| \leq 1} |b(t, x_1) - b(t, x_2)| + |b(t, 0)| \leq C, \quad \forall (t, x_1, x_2) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^m;$$

(ii) $\sigma \in \mathbb{R}^{m\times l}$ is a constant matrix with full row rank.
Assumption (F3) allows b to be linearly growing in variable x, but σ needs to be constant. Under the assumption (F3), Nam and Xu [19, Propositions 4.5–4.6, pp.14–15] proved that (1.2) has a unique strong solution $X^{t,x}$ and Lemma 3.1 holds true for $q = 2$. Their arguments also yield the result for any $q \in (1, +\infty)$.

3.2 Main result

For the multi-dimensional case, we assume that Markovian BSDE (1.1) is strictly and diagonally quadratic and make the following assumption first. Let $\varepsilon \in (0, 1]$, γ and C be positive constants, $\alpha \in \mathbb{R}^d$ be a constant vector and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function.

(B1) $(\bar{f}, \widehat{f}) : [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{d\times l} \to \mathbb{R}^d \times \mathbb{R}^{d\times l}$ and $g : \mathbb{R}^m \to \mathbb{R}^d$ are Borel measurable functions such that for each $(t, x) \in [0, T] \times \mathbb{R}^m$, $(\bar{f}, \widehat{f})(t, x, \cdot, \cdot)$ is continuous on $\mathbb{R}^d \times \mathbb{R}^{d\times l}$ and for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{d\times l}$ and $i = 1, \cdots, d$, we have

$$f^i(t, x, y, z) := \bar{f}^i(t, x, y, z) + < z^i, \widehat{f}^i(t, x, y, z) > + \frac{\alpha^i}{2}|z^i|^2$$

and

$$\begin{cases} |\widehat{f}(t, x, y, z)| \leq \rho(|y|)(1 + |x|^\gamma)(1 + |z|^{1-\varepsilon}), \\ |\bar{f}(t, x, y, z)| \leq C(1 + |y|), \\ |g(x)| \leq C. \end{cases}$$

Define

$$\widetilde{f}^i(t, x, y, z) := \bar{f}^i(t, x, y, z) + < z^i, \widehat{f}^i(t, x, y, z) >, \quad i = 1, \cdots, d.$$

We also make the following stronger assumption.

(B2) For $i = 1, \cdots, d$, $\alpha^i \neq 0$ and $\widetilde{f}^i(t,x,y,z)$ varies with (t,x,y) and the ith row z^i of matrix z only. The constant γ lies in $(0,\varepsilon)$ and there exists a nondecreasing function $\bar{\rho} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $(t,x,y,y',z,z') \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d\times l} \times \mathbb{R}^{d\times l}$,

$$|\widetilde{f}(t,x,y,z)| \leq \bar{\rho}(|y|)(1+|x|^{\gamma})(1+|z|^{2-\varepsilon})$$

and

$$|\widetilde{f}(t,x,y,z) - \widetilde{f}(t,x,y',z')| \leq \bar{\rho}(|y| \vee |y'|)\Big[|y-y'| + (1+|x|^{\gamma})(1+|z|^{1-\varepsilon}+|z'|^{1-\varepsilon})|z-z'|\Big].$$

We have the following existence and uniqueness result.

Theorem 3.1 *Let assumptions* (F1)–(F2) *and* (B1) *hold. Then for each* $(t,x) \in [0,T] \times \mathbb{R}^m$,

(i) *Markovian* BSDE (1.1) *has a solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^{d\times l})$.

(ii) *There exists a pair of Borel measurable functions* $(u,v) : [0,T] \times \mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}^{d\times l}$ *such that* $(Y_s^{t,x}, Z_s^{t,x}) = (u(s, X_s^{t,x}), v(s, X_s^{t,x}))$, $\forall s \in [t,T]$, $d\mathbb{P} \otimes ds$ *a.e.*

(iii) *If* (B2) *is further satisfied, the solution is unique.*

Proof of Theorem 3.1 In the following proof, without loss of generality, we only consider the case $(t,x) = (0,x_0)$ and omit the superscript $(0,x_0)$ if there is no confusion.

Step 1 Construction of approximating solutions$\{(Y^n,Z^n)\}_{n\geq 1}$.

We define $\{f^n\}_{n\geq 1}$, the approximating generator sequence, as follows. Let

$$\Phi \in C_c^\infty(\mathbb{R}^{d(l+1)}, \mathbb{R})$$

be a smooth, compactly supported function and $\Psi \in C^\infty(\mathbb{R}^{d(l+1)}, \mathbb{R})$ be a smooth function satisfying

$$\Psi(y,z) = \begin{cases} 1, & \text{if } |y| \vee |z| \leq 1, \\ 0, & \text{if } |y| \wedge |z| \geq 2. \end{cases} \tag{3.1}$$

Under assumption (B1), by virtue of the techniques of mollification and truncation, we define

$$\bar{f}^n(t,x,y,z) := \Psi\Big(\frac{y}{n}, \frac{z}{n}\Big) \int_{\mathbb{R}^{d(l+1)}} n^{d(l+1)} \bar{f}(t,x,u,v)\Phi(t,x,n(y-u),n(z-v))\mathrm{d}u\mathrm{d}v$$

and

$$\widehat{f}^n(t,x,y,z) := \mathbb{1}_{\{|x|\leq n\}}\Psi(\frac{y}{n}, \frac{z}{n}) \int_{\mathbb{R}^{d(l+1)}} n^{d(l+1)} \widehat{f}(t,x,u,v)\Phi\big(t,x,n(y-u),n(z-v)\big)\mathrm{d}u\mathrm{d}v.$$

Then we define the approximating sequence $f^n := (f^{n,1}, \cdots, f^{n,d})^{\mathrm{T}}$ by

$$f^{n,i}(t,x,y,z) := \bar{f}^{n,i}(t,x,y,z) + < z^i, \widehat{f}^{n,i}(t,x,y,z) > + \frac{\alpha^i}{2}|z^i|^2, \quad i = 1, \cdots, d. \tag{3.2}$$

Denote by $\widetilde{f}^n := (\widetilde{f}^{n,1}, \cdots, \widetilde{f}^{n,d})^{\mathrm{T}}$ the linear and sub-quadratic growth part of f^n, i.e.,

$$\widetilde{f}^{n,i}(t,x,y,z) := \bar{f}^{n,i}(t,x,y,z) + < z^i, \widehat{f}^{n,i}(t,x,y,z) >, \quad i = 1, \cdots, d. \tag{3.3}$$

We have

$$f^{n,i}(t,x,y,z) := \widetilde{f}^{n,i}(t,x,y,z) + \frac{\alpha^i}{2}|z^i|^2, \quad i = 1, \cdots, d. \tag{3.4}$$

One can easily verify that

(i) For each $n \geq 1$, $\overline{f}^n$ and $\widehat{f}^n$ are globally bounded by a constant R_n and are globally Lipschitz continuous in the last two variables (y,z) with common Lipschitz coefficient L_n. Note that both R_n and L_n depend on n.

(ii) There exist positive constants C and c_n and a nondecreasing function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{d\times l}$, $n \geq 1$ and $i = 1, \cdots, d$,

$$\begin{cases} |f^{n,i}(t,x,y,z)| \leq \rho(|y|)(1+|x|^\gamma)(1+|z|^{2-\varepsilon}) + \dfrac{\alpha^i}{2}|z^i|^2, \\ |\overline{f}^{n,i}(t,x,y,z)| \leq C(1+|y|), \\ |f^{n,i}(t,x,y,z)| \leq c_n + \dfrac{\alpha^i}{2}|z^i|^2. \end{cases} \tag{3.5}$$

(iii) For any $(t,x) \in [0,T] \times \mathbb{R}^m$ and any compact set $\mathbb{K} \in \mathbb{R}^{d(l+1)}$,

$$\sup_{(y,z)\in\mathbb{K}} |f^n(t,x,y,z) - f(t,x,y,z)| \to 0 \quad \text{as } n \to +\infty. \tag{3.6}$$

For $n \geq 1$, consider (1.1) with generator f^n and terminal condition g. By virtue of the boundness of $\overline{f}^n$ and $\widehat{f}^n$, we can get the unique solution $(Y^n, Z^n) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times l})$ from Hu and Tang [14, Theorem 2.3, p.1072].

Without loss of generality, in the following proof we assume $\alpha^i = \frac{1}{2}$ for all $i = 1, \cdots, d$. The proof of existence when $\alpha^i = 0$ for some $i = 1, \cdots, d$ is discussed in the subsequent Remark 3.2.

Step 2 Uniform estimate of $\{(Y^n, Z^n)\}_{n\geq 1}$.

From (3.5) we know that there exists a positive constant $\widetilde{C}$ such that $|g^i|^2 \leq \widetilde{C}$ and for all $(t,w,x,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{d\times l}$,

$$|w\bar{f}^{n,i}(t,x,y,z)| \leq \widetilde{C}(1+|w|^2+|y|^2), \quad i = 1, \cdots, d. \tag{3.7}$$

Let η be the unique solution of the following ordinary differential equation:

$$\eta(t) = d\widetilde{C} + \int_t^T d\widetilde{C}\mathrm{d}s + \int_t^T (d\widetilde{C}+1)\eta(s)\mathrm{d}s, \quad t \in [0,T].$$

One can easily verify that $\eta(\cdot)$ is a continuous decreasing function and we have

$$\frac{\eta(t)}{d} = \widetilde{C} + \int_t^T \widetilde{C}(1+\eta(s))\mathrm{d}s + \int_t^T \frac{\eta(s)}{d}\mathrm{d}s, \quad t \in [0,T].$$

Define

$$\lambda := \sup_{t\in[0,T]} \eta(t) = \eta(0).$$

By virtue of the fact that $\widehat{f}^n$ is bounded by the constant R_n, from Hu and Tang [14, Theorem 2.3, p.1072] we know that (Y^n, Z^n) is also the unique solution to BSDE (1.1) with generator

$$F^{n,i}(t,x,y,z) := \overline{f}^{n,i}(t,x,y,z) + < z^i, \widehat{f}^{n,i}(t, X_t^{0,x_0}, Y_t^n, Z_t^n) > + \frac{1}{2}|z^i|^2, \quad i = 1, \cdots, d$$

and terminal condition g.

Since $|g(X_T^{t,x})| \le d\widetilde{C} \le \lambda$, from (3.7) and the proof of Hu and Tang [14, Theorem 2.3, pp. 1079–1082] we have

$$|Y_t^n|^2 \le \eta_t \le \lambda, \quad t \in [0,T].$$

Therefore, there exists a positive constant $\overline{C} := \sqrt{\lambda}$ such that

$$\sup_{n\ge 1} \|Y^n\|_{\mathbb{S}^\infty} = \bar{C} < +\infty. \tag{3.8}$$

However, for $\{Z^n\}_{n\ge 1} \in \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times l})$, we cannot get the uniform BMO norm due to the unboundness of the coefficients. Instead, for each $p \ge 1$ we can get the uniform $\mathbb{H}^p$ norm of $\{Z^n\}_{n\ge 1}$ with respect to n as follows.

Define nonnegative function

$$\psi(y) := \mathrm{e}^{|y|} - |y| - 1, \quad \forall y \in \mathbb{R}$$

and stopping time

$$\tau_k := \Big\{ t \in [0,T] : \int_0^t |Z_s|^2 \mathrm{d}s \ge k \Big\} \wedge T.$$

For all $y \in \mathbb{R}$ we have

$$|\psi'(y)| = \mathrm{e}^{|y|} - 1 \quad \text{and} \quad \psi''(y) - |\psi'(y)| = 1. \tag{3.9}$$

For each $i = 1, \cdots, d$, using Itô-Tanaka formula to compute $\psi(Y_t^{n,i})$, we have

$$\begin{aligned}
&\psi(Y_0^{n,i}) + \frac{1}{2}\int_0^{t\wedge\tau_k} \psi''(Y_s^{n,i})|Z_s^{n,i}|^2 \mathrm{d}s \\
=\ &\psi(Y_{t\wedge\tau_k}^{n,i}) + \int_0^{t\wedge\tau_k} \psi'(Y_s^{n,i}) f^{n,i}(s, X_s^{0,x_0}, Y_s, Z_s)\mathrm{d}s - \int_0^{t\wedge\tau_k} \psi'(Y_s^{n,i}) Z_s^{n,i} \mathrm{d}B_s \\
\le\ &\psi(Y_{t\wedge\tau_k}^{n,i}) + \int_0^{t\wedge\tau_k} |\psi'(Y_s^{n,i})| \rho(|Y_s^n|)(1 + |X_s^{0,x_0}|^\gamma)(1 + |Z_s^n|^{2-\varepsilon})\mathrm{d}s \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_k} |\psi'(Y_s^{n,i})||Z_s^{n,i}|^2 \mathrm{d}s - \int_0^{t\wedge\tau_k} \psi'(Y_s^{n,i}) Z_s^{n,i} \mathrm{d}B_s.
\end{aligned}$$

Hence from (3.9), we have

$$\begin{aligned}
\frac{1}{2}\int_0^{t\wedge\tau_k} |Z_s^{n,i}|^2 \mathrm{d}s \le\ &\psi(Y_{t\wedge\tau_k}^{n,i}) + \int_0^{t\wedge\tau_k} |\psi'(Y_s^{n,i})| \rho(|Y_s^n|)(1 + |X_s^{0,x_0}|^\gamma)(1 + |Z_s^n|^{2-\varepsilon})\mathrm{d}s \\
&- \int_0^{t\wedge\tau_k} \psi'(Y_s^{n,i}) Z_s^{n,i} \mathrm{d}B_s.
\end{aligned}$$

Since $\|Y^n\|_{\mathbb{S}^\infty}$ has a uniform bound, taking the supremum over $t \in [0,T]$, we have

$$\begin{aligned}
\int_0^{\tau_k} |Z_u^{n,i}|^2 \mathrm{d}u \le\ &2 \sup_{t\in[0,T]} \psi(Y_{t\wedge\tau_k}^{n,i}) + 2\int_0^{\tau_k} |\psi'(Y_s^{n,i})| \rho(|Y_s^n|)(1 + |X_s^{0,x_0}|^\gamma)(1 + |Z_s^n|^{2-\varepsilon})\mathrm{d}s \\
&+ 2 \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau_k} \psi'(Y_s^{n,i}) Z_s^{n,i} \mathrm{d}B_s \Big| \\
\le\ &C_d \Big(1 + \int_0^T |X_s^{0,x_0}|^{\frac{2\gamma}{\varepsilon}} \mathrm{d}s \Big) + \frac{1}{2d}\int_0^{\tau_k} |Z_s^n|^2 \mathrm{d}s
\end{aligned}$$

$$+2\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau_k}\psi'(Y_s^{n,i})Z_s^{n,i}\mathrm{d}B_s\Big|, \tag{3.10}$$

where C_d is a constant depending on d. Summing i from 1 to d yields

$$\int_0^{\tau_k}|Z_u^n|^2\mathrm{d}u \le 2dC_d\Big(1+T\sup_{t\in[0,T]}|X_t^{0,x_0}|^{\frac{2\gamma}{\varepsilon}}\Big)+4\sum_{i=1}^d\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau_k}\psi'(Y_s^{n,i})Z_s^{n,i}\mathrm{d}B_s\Big|. \tag{3.11}$$

Using B-D-G inequality and Young's inequality, for each $p\ge 1$, we have

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^{\tau_k}|Z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big] &\le \widehat{C}_d\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^{0,x_0}|^{\frac{\gamma p}{\varepsilon}}\Big]+\mathbb{E}\Big[\Big(\int_0^{\tau_k}|Z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{4}}\Big]\Big)\\ &\le C_{d,p}+\frac{1}{2}\mathbb{E}\Big[\Big(\int_0^{\tau_k}|Z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big],\end{aligned} \tag{3.12}$$

where $\widehat{C}_d$ is a constant depending on d and $C_{d,p}$ is a constant depending on d and p.

Hence by Fatou's Lemma, we have

$$\sup_{n\ge 1}\|Z^n\|_{\mathbb{H}^p}\le (2C_{d,p})^{\frac{1}{p}}<+\infty. \tag{3.13}$$

Step 3 Exponential transformation and Markovian representation.

For all $i=1,\cdots,d$, consider the exponential transformation

$$\begin{cases}\widetilde{Y}^{n,i;(t,x)}=\mathrm{e}^{Y^{n,i;(t,x)}},\\ \widetilde{Z}^{n,i;(t,x)}=\widetilde{Y}^{n,i;(t,x)}Z^{n,i;(t,x)}.\end{cases} \tag{3.14}$$

From (3.8) and (3.13), for each $p\ge 1$, there exists a positive constant $\widetilde{C}:=\widetilde{C}(p)$ such that

$$\sup_{n\ge 1}\|\widetilde{Y}^{n;(t,x)}\|_{\mathbb{S}^\infty}+\sup_{n\ge 1}\|\widetilde{Z}^{n;(t,x)}\|_{\mathbb{H}^p}\le\widetilde{C}. \tag{3.15}$$

Moreover, $(\widetilde{Y}^{n,i;(t,x)},\widetilde{Z}^{n,i;(t,x)})$ satisfies the BSDE

$$\begin{aligned}\widetilde{Y}_s^{n,i;(t,x)} &= \mathrm{e}^{g(X_T^{t,x})}+\int_s^T\widetilde{Y}_s^{n,i;(t,x)}\widetilde{f}^{n,i}(u,X_u^{t,x},Y_u^{n,i;(t,x)},Z_u^{n,i;(t,x)})\mathrm{d}u\\ &\quad -\int_s^T\widetilde{Z}_u^{n,i;(t,x)}\mathrm{d}B_u,\quad s\in[0,T],\end{aligned} \tag{3.16}$$

where $\widetilde{f}^{n,i}$ is defined in (3.3).

From Nam and Xu [19, Proposition 4.8, p.17], there exists a pair of Borel measurable functions $(u_n,v_n):[0,T]\times\mathbb{R}^m\to\mathbb{R}^d\times\mathbb{R}^{d\times l}$ such that

$$(Y_s^{n;(t,x)},Z_s^{n;(t,x)})=(u_n(s,X_s^{t,x}),v_n(s,X_s^{t,x})),\quad\forall s\in[0,T]. \tag{3.17}$$

So by virtue of (3.14) and (3.17) we know that for each $n\ge 1$ and $i=1,\cdots,d$ there exists a Borel measurable function $\widetilde{u}_n^i$ such that

$$\widetilde{Y}_s^{n,i;(t,x)}=\widetilde{u}_n^i(s,X_s^{t,x}),\quad\forall s\in[0,T]. \tag{3.18}$$

We define respectively $F^n=(F^{n,1},\cdots,F^{n,d})^{\mathrm{T}}$ and $\widetilde{F}^n=(\widetilde{F}^{n,1},\cdots,\widetilde{F}^{n,d})^{\mathrm{T}}$ by

$$F^{n,i}(t,x):=f^{n,i}(t,x,u_n(t,x),v_n(t,x)),\quad (t,x)\in[0,T]\times\mathbb{R}^m$$

and

$$\widetilde{F}^{n,i}(t,x):=\widetilde{u}_n^i(t,x)\widetilde{f}^{n,i}\big(t,x,u_n(t,x),v_n(t,x)\big),\quad (t,x)\in[0,T]\times\mathbb{R}^m.$$

Let $s=t$, from (3.16) we know that for all $(t,x)\in[0,T]\times\mathbb{R}^m$,

$$u_n(t,x)=Y_t^{n;(t,x)}=\mathbb{E}\Big[g(X_T^{t,x})+\int_t^T F^n(u,X_u^{t,x})\mathrm{d}u\Big] \tag{3.19}$$

and

$$\widetilde{u}_n(t,x)=\widetilde{Y}_t^{n;(t,x)}=\mathbb{E}\Big[\mathrm{e}^{g(X_T^{t,x})}+\int_t^T \widetilde{F}^n(u,X_u^{t,x})\mathrm{d}u\Big]. \tag{3.20}$$

Moreover, by virtue of (3.8), (3.14) and (3.19)–(3.20), we have the uniform boundness of

$$u_n:=(u_n^1,\cdots,u_n^d)^{\mathrm{T}}\quad\text{and}\quad \widetilde{u}_n:=(\widetilde{u}_n^1,\cdots,\widetilde{u}_n^d)^{\mathrm{T}},$$

i.e., there exists a positive constant, independent of n,t and x, still denoted by $\widetilde{C}$ such that

$$|u_n(t,x)|+|\widetilde{u}_n(t,x)|\le\widetilde{C},\quad \forall n\ge1,\ (t,x)\in[0,T]\times\mathbb{R}^m. \tag{3.21}$$

Step 4 Convergence of $\{Y^n\}_{n\ge1}$ in $L^p(\Omega\times[0,T];\mathbb{R}^d)$.

For $n\ge1$ and $q\in\big(1,\frac{2}{2-\varepsilon}\big)$, in view of (3.8),(3.13) and (3.15), using Young's inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T|\widetilde{F}^n(u,X_u^{0,x_0})|^q\mathrm{d}u\Big]&\le C_1\mathbb{E}\Big[\int_0^T|\widetilde{Y}_u^n|^q(\rho(|Y_u^n|))^q(1+|X_u^{0,x_0}|^{q\gamma})(1+|Z_u^n|^{q(2-\varepsilon)})\mathrm{d}u\Big]\\&\le C_2\mathbb{E}\Big[\int_0^T(1+|X_u^{0,x_0}|^{\frac{q^2(2-\varepsilon)\gamma}{q(2-\varepsilon)-1}}+|Z_u^n|^2)\mathrm{d}u\Big]\\&\le C_3\Big(1+T\mathbb{E}\Big[\sup_{u\in[0,T]}|X_u^{0,x_0}|^{\frac{q^2(2-\varepsilon)\gamma}{q(2-\varepsilon)-1}}\Big]+\sup_{n\ge1}\mathbb{E}\Big[\int_0^T|Z_u^n|^2\mathrm{d}u\Big]\Big)\\&\le C_4<\infty,\end{aligned} \tag{3.22}$$

where C_1,C_2,C_3 and C_4 are four positive constants independent of n. Therefore, in the sense of subsequence (without loss of generality, we assume that this subsequence is just the sequence $\{n\}_{n\ge1}$), there exists a Borel measurable function $\widetilde{F}:[0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ such that

$$\lim_{n\to+\infty}\widetilde{F}^n=\widetilde{F},\qquad \text{weakly in } L^q([0,T]\times\mathbb{R}^m;p(0,x_0;s,dy)ds). \tag{3.23}$$

Fix $(t,x,q)\in[0,T]\times\mathbb{R}^m\times\big(1,\frac{2}{2-\varepsilon}\big)$. From (3.20), for any $\delta\in(0,T-t]$, we have

$$\begin{aligned}|\widetilde{u}_{n_1}(t,x)-\widetilde{u}_{n_2}(t,x)|&=\Big|\mathbb{E}\Big[\int_t^T(\widetilde{F}^{n_1}(u,X_u^{t,x})-\widetilde{F}^{n_2}(u,X_u^{t,x}))\mathrm{d}u\Big]\Big|\\&\le I_1^{n_1,n_2,\delta}+I_2^{n_1,n_2,\delta,k}+I_3^{n_1,n_2,\delta,k},\quad \forall n_1,n_2,k\ge1,\end{aligned} \tag{3.24}$$

where

$$I_1^{n_1,n_2,\delta}:=\mathbb{E}\Big[\int_t^{t+\delta}|\widetilde{F}^{n_1}(u,X_u^{t,x})-\widetilde{F}^{n_2}(u,X_u^{t,x})|\mathrm{d}u\Big],$$

$$I_2^{n_1,n_2,\delta,k} := \Big|\mathbb{E}\Big[\int_{t+\delta}^T (\widetilde{F}^{n_1}(u,X_u^{t,x}) - \widetilde{F}^{n_2}(u,X_u^{t,x}))\mathbb{1}_{\{|X_u^{t,x}|\le k\}}\mathrm{d}u\Big]\Big|$$

and

$$I_3^{n_1,n_2,\delta,k} := \Big|\mathbb{E}\Big[\int_{t+\delta}^T (\widetilde{F}^{n_1}(u,X_u^{t,x}) - \widetilde{F}^{n_2}(u,X_u^{t,x}))\mathbb{1}_{\{|X_u^{t,x}|> k\}}\mathrm{d}u\Big]\Big|.$$

First, similar to the proof of (3.22), we have

$$\begin{aligned} I_1^{n_1,n_2,\delta} &\le \delta^{\frac{q}{q-1}}\Big\{\mathbb{E}\Big[\int_0^T |\widetilde{F}^{n_1}(u,X_u^{t,x}) - \widetilde{F}^{n_2}(u,X_u^{t,x})|^q \mathrm{d}u\Big]\Big\}^{\frac{1}{q}} \\ &\le C\delta^{\frac{q}{q-1}}. \end{aligned} \tag{3.25}$$

Second, from Lemma 3.1, there exists a function

$$\phi_{t,x,x_0}^{\delta} \in L^{\frac{q}{q-1}}([t+\delta,T]\times[-k,k];p(0,x_0;s,dy))$$

such that

$$p(t,x;s,dy)ds = \phi_{t,x,x_0}^{\delta}(s,y)p(0,x_0;s,dy)ds, \quad \forall (s,x)\in[t+\delta,T]\times\mathbb{R}^m,$$

where $p(t,x;s,dy)$ is the law of $X_s^{t,x}$. So from the weak convergence (3.23) of $\widetilde{F}^n$, for fixed δ and k, we have

$$\begin{aligned} I_2^{n_1,n_2,\delta,k} &= \Big|\int_{\mathbb{R}^m}\int_{t+\delta}^T (\widetilde{F}^{n_1}(u,y) - \widetilde{F}^{n_2}(u,y))\mathbb{1}_{\{|y|\le k\}}p(t,x;u,dy)\mathrm{d}u\Big| \\ &= \Big|\int_{\mathbb{R}^m}\int_{t+\delta}^T (\widetilde{F}^{n_1}(u,y) - \widetilde{F}^{n_2}(u,y))\mathbb{1}_{\{|y|\le k\}}\phi_{t,x,x_0}^{\delta}(u,y)p(0,x_0;u,dy)\mathrm{d}u\Big| \\ &\to 0 \quad \text{as } n_1,n_2\to+\infty. \end{aligned} \tag{3.26}$$

Third, from (3.22) and Hölder's inequality, for all $n_1,n_2\ge 1$ and $\delta\in(0,T-t]$, we have

$$\begin{aligned} I_3^{n_1,n_2,\delta,k} &\le \Big\{\mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|X_u^{t,x}|>k\}}\mathrm{d}u\Big]\Big\}^{\frac{q-1}{q}}\Big\{\mathbb{E}\Big[\int_0^T |\widetilde{F}^{n_1}(u,X_u^{t,x}) - \widetilde{F}^{n_2}(u,X_u^{t,x})|^q\mathrm{d}u\Big]\Big\}^{\frac{1}{q}} \\ &\le C\Big\{\mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|X_u^{t,x}|>k\}}\mathrm{d}u\Big]\Big\}^{\frac{q-1}{q}} \\ &\to 0 \quad \text{as } k\to+\infty. \end{aligned} \tag{3.27}$$

So for any $\varepsilon_0>0$, there is a sufficiently small $\delta>0$ and a sufficiently large k such that $I_1^{n_1,n_2,\delta}<\frac{\varepsilon_0}{3}$ and $I_3^{n_1,n_2,\delta,k}<\frac{\varepsilon_0}{3}$ for all $n_1,n_2\ge 1$. Then for this fixed δ and k, there is a sufficiently large N such that for $n_1,n_2\ge N$, $I_2^{n_1,n_2,\delta,k}<\frac{\varepsilon_0}{3}$. Thus

$$|\widetilde{u}_{n_1}(t,x) - \widetilde{u}_{n_2}(t,x)| < \varepsilon_0, \quad \forall n_1,n_2\ge N.$$

Therefore, for each $(t,x)\in[0,T]\times\mathbb{R}^m$, $\{\widetilde{u}_n(t,x)\}_{n\ge1}$ is a Cauchy sequence. We denote by $\widetilde{u}(t,x)$ its limit.

Since $\widetilde{u}_n$ is Borel measurable, the limit function $\widetilde{u}$ is also a Borel measurable function on $[0,T]\times\mathbb{R}^m$. Define stochastic process $\widetilde{Y}_t := \widetilde{u}(t,X_t^{0,x_0})$. Taking the pathwise limit of $\widetilde{u}_n(t,X_t^{0,x_0})$, from (3.18) we know

$$\widetilde{Y}_t = \lim_{n\to+\infty}\widetilde{u}_n(t,X_t^{0,x_0}) = \lim_{n\to+\infty}\widetilde{Y}_t^n, \quad \forall t\in[0,T], \quad d\mathbb{P}\otimes dt \ \ a.e.$$

For all $p \ge 1$, bounded convergence theorem indicates that $\widetilde{Y}^n$ converges to $\widetilde{Y}$ in $L^p(\Omega \times [0,T];\mathbb{R}^d)$, i.e.,

$$\mathbb{E}\Big[\int_0^T |\widetilde{Y}_u^n - \widetilde{Y}_u|^2 \mathrm{d}u\Big] \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.28}$$

Notice that for any $C > 0$, we have

$$|a-b| \le \mathrm{e}^C |\mathrm{e}^a - \mathrm{e}^b|, \quad \forall (a,b) \in [-C,C]^2.$$

So by the uniform estimate (3.21), we have

$$|u_{n_1}(t,x) - u_{n_2}(t,x)| \le \mathrm{e}^{\widetilde{C}} |\mathrm{e}^{u_{n_1}(t,x)} - \mathrm{e}^{u_{n_2}(t,x)}| = \mathrm{e}^{\widetilde{C}} |\widetilde{u}_{n_1}(t,x) - \widetilde{u}_{n_2}(t,x)|.$$

Therefore, for each $(t,x) \in [0,T] \times \mathbb{R}^m$, $\{u_n(t,x)\}_{n\ge1}$ is also a Cauchy sequence. Following the same proof, one can deduce that there exists a Borel measurable function u such that the stochastic process $Y_t := u(t, X_t^{0,x_0})$ is the limit of Y^n in $L^p(\Omega \times [0,T])$, for all $p \ge 1$.

Step 5 Convergence of $\{Y^n\}_{n\ge1}$ in $\mathbb{S}^2(\mathbb{R}^d)$.

Take a fixed $q_1 \in \big(1, \frac{2}{2-\varepsilon}\big)$. For $n_1, n_2 \ge 1$, $t \in [0,T]$ and $\lambda > 0$, using Itô's formula to compute $|\widetilde{Y}_t^{n_1} - \widetilde{Y}_t^{n_1}|^2$, from (3.15) and Young's inequality, we have

$$\begin{aligned}
&|\widetilde{Y}_t^{n_1} - \widetilde{Y}_t^{n_2}|^2 + \int_t^T |\widetilde{Z}_u^{n_1} - \widetilde{Z}_u^{n_2}|^2 \mathrm{d}u \\
=& 2\int_t^T (\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2})^{\mathrm{T}} (\widetilde{Y}_u^{n_1} \circ \widetilde{f}^{n_1}(u, X_u^{0,x_0}, Y_u^{n_1}, Z_u^{n_2}) - \widetilde{Y}_u^{n_2} \circ \widetilde{f}^{n_2}(u, X_u^{0,x_0}, Y_u^{n_1}, Z_u^{n_2}))\mathrm{d}u \\
&- 2\int_t^T (\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2})^{\mathrm{T}} (\widetilde{Z}_u^{n_1} - \widetilde{Z}_u^{n_2}) \mathrm{d}B_u \\
\le& \widetilde{C}_1 \int_t^T |\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2}| (1 + (1 + |X_u^{0,x_0}|^\gamma)(1 + |Z_u^{n_1}|^{2-\varepsilon} + |Z_u^{n_2}|^{2-\varepsilon}))\mathrm{d}u \\
&- 2\int_t^T (\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2})^{\mathrm{T}} (\widetilde{Z}_u^{n_1} - \widetilde{Z}_u^{n_2}) \mathrm{d}B_u \\
\le& \widetilde{C}_2 \int_t^T |\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2}| (1 + |X_u^{0,x_0}|^{\frac{\gamma q_1}{q_1 - 1}} + |Z_u^{n_1}|^{(2-\varepsilon)q_1} + |Z_u^{n_2}|^{(2-\varepsilon)q_1})\mathrm{d}u \\
&- 2\int_t^T (\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2})^{\mathrm{T}} (\widetilde{Z}_u^{n_1} - \widetilde{Z}_u^{n_2}) \mathrm{d}B_u \\
\le& \widetilde{C}_3 \Big\{ \frac{1}{\lambda^2} \int_t^T |\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2}|^2 \mathrm{d}u + \lambda^2 \int_t^T (1 + |X_u^{0,x_0}|^{\frac{\gamma q_1}{q_1 - 1}})^2 \mathrm{d}u + \frac{1}{\lambda^{p_2}} \int_t^T |\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2}|^{p_2} \mathrm{d}u \\
&+ \lambda^{q_2} \int_t^T (|Z_u^{n_1}|^2 + |Z_u^{n_2}|^2)\mathrm{d}u \Big\} - 2\int_t^T (\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2})^{\mathrm{T}} (\widetilde{Z}_u^{n_1} - \widetilde{Z}_u^{n_2}) \mathrm{d}B_u,
\end{aligned} \tag{3.29}$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$ are three positive constants independent of n_1 and n_2,

$$q_2 := \frac{2}{(2-\varepsilon)q_1} \in \Big(1, \frac{2}{2-\varepsilon}\Big) \quad \text{and} \quad p_2 := \frac{q_2}{q_2 - 1} = \frac{2}{2 - (2-\varepsilon)q_1} > 2.$$

Taking expectation on both side of (3.29) and using uniform estimate (3.13), we have

$$\mathbb{E}\Big[\int_t^T |\widetilde{Z}_u^{n_1} - \widetilde{Z}_u^{n_2}|^2 \mathrm{d}u\Big] \le \widetilde{C}_4 \Big\{ \Big(\frac{1}{\lambda^2} + \frac{1}{\lambda^{p_2}}\Big) \mathbb{E}\Big[\int_t^T |\widetilde{Y}_u^{n_1} - \widetilde{Y}_u^{n_2}|^2 \mathrm{d}u\Big] + (\lambda^2 + \lambda^{q_2}) \Big\}. \tag{3.30}$$

Since $\lambda > 0$ can be chosen arbitrarily, in view of (3.28) and (3.30), we know that the sequence $\{\widetilde{Z}^n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{H}^2(\mathbb{R}^{d\times l})$ and thus has a limit $\widetilde{Z} \in \mathbb{H}^2(\mathbb{R}^{d\times l})$.

From (3.29), using B-D-G inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|\widetilde{Y}_t^{n_1}-\widetilde{Y}_t^{n_2}|^2\Big] \leq{}& \widetilde{C}_5\Big\{\Big(\frac{1}{\lambda^2}+\frac{1}{\lambda^{p_2}}\Big)\mathbb{E}\Big[\int_0^T|\widetilde{Y}_u^{n_1}-\widetilde{Y}_u^{n_2}|^2\mathrm{d}u\Big]+(\lambda^2+\lambda^{q_2})\Big\}\\ &+\frac{1}{2}\mathbb{E}\Big[\sup_{t\in[0,T]}|\widetilde{Y}_t^{n_1}-\widetilde{Y}_t^{n_2}|^2\Big]+2\mathbb{E}\Big[\int_0^T|\widetilde{Z}_u^{n_1}-\widetilde{Z}_u^{n_2}|^2\mathrm{d}u\Big]. \qquad (3.31)\end{aligned}$$

Since $\lambda > 0$ can be chosen arbitrarily and $\{(\widetilde{Y}^n, \widetilde{Z}^n)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{H}^2(\mathbb{R}^d)\otimes\mathbb{H}^2(\mathbb{R}^{d\times l})$, from (3.31) we know that $\{\widetilde{Y}^n\}_{n\geq 1}$ is also a Cauchy sequence in $\mathbb{S}^2(\mathbb{R}^d)$.

Notice that from (3.8) we have

$$|Y_t^{n_1}-Y_t^{n_2}| \leq \mathrm{e}^{\overline{C}}|\mathrm{e}^{Y_t^{n_1}}-\mathrm{e}^{Y_t^{n_2}}| = \mathrm{e}^{\overline{C}}|\widetilde{Y}_t^{n_1}-\widetilde{Y}_t^{n_2}|.$$

So $\{Y^n\}_{n\geq 1}$ is also a Cauchy sequence in $\mathbb{S}^2(\mathbb{R}^d)$. Therefore, Y has a continuous version in $\mathbb{S}^2(\mathbb{R}^d)$ and we still denote by Y this continuous version. Moreover, from (3.8) we know that $Y \in \mathbb{S}^\infty(\mathbb{R}^d)$.

Step 6 Convergence of $\{Z^n\}_{n\geq 1}$ in $\mathbb{H}^2(\mathbb{R}^{d\times l})$.

By virtue of the inequalities

$$|a_1b_1-a_2b_2| \leq |a_1-a_2||b_2|+|a_1||b_1-b_2|, \quad \forall (a_1,a_2,b_1,b_2)\in\mathbb{R}^4$$

and

$$|\mathrm{e}^{-x}-\mathrm{e}^{-y}|^2 \leq 2C\mathrm{e}^C|x-y|, \quad \forall (x,y)\in[-C,C]^2,$$

we have that for $i = 1, \cdots, d$,

$$\begin{aligned}\mathbb{E}\Big[\int_0^T|Z_u^{n_1,i}-Z_u^{n_2,i}|^2u\Big] ={}& \mathbb{E}\Big[\int_0^T|\mathrm{e}^{-Y_u^{n_1,i}}\widetilde{Z}_u^{n_1,i}-\mathrm{e}^{-Y_u^{n_2,i}}\widetilde{Z}_u^{n_2,i}|^2\mathrm{d}u\Big]\\ \leq{}& 2\mathbb{E}\Big[\int_0^T|\mathrm{e}^{-Y_u^{n_1,i}}-\mathrm{e}^{-Y_u^{n_2,i}}|^2|\mathrm{e}^{Y_u^{n_2,i}}Z_u^{n_2,i}|^2\mathrm{d}u\Big]\\ &+2\mathbb{E}\Big[\int_0^T\mathrm{e}^{-2Y_u^{n_1,i}}|\widetilde{Z}_u^{n_1,i}-\widetilde{Z}_u^{n_2,i}|^2\mathrm{d}u\Big]\\ \leq{}& \widehat{C}\Big\{\mathbb{E}\Big[\int_0^T|Y_u^{n_1,i}-Y_u^{n_2,i}||Z_u^{n_2,i}|^2\mathrm{d}u\Big]+\mathbb{E}\Big[\int_0^T|\widetilde{Z}_u^{n_1,i}-\widetilde{Z}_u^{n_2,i}|^2\mathrm{d}u\Big]\Big\}\\ \leq{}& \widehat{C}\mathbb{E}\Big[\sup_{u\in[0,T]}|Y_u^{n_1,i}-Y_u^{n_2,i}|^2\Big]\Big\}^{\frac{1}{2}}\Big\{\mathbb{E}\Big[\Big(\int_0^T|Z_u^{n_2,i}|^2\mathrm{d}u\Big)^2\Big]\Big\}^{\frac{1}{2}}\\ &+\widehat{C}\mathbb{E}\Big[\int_0^T|\widetilde{Z}_u^{n_1,i}-\widetilde{Z}_u^{n_2,i}|^2\mathrm{d}u\Big], \qquad (3.32)\end{aligned}$$

where $\widehat{C} := 4\overline{C}\mathrm{e}^{\overline{C}} \vee 2\mathrm{e}^{2\overline{C}}$ is a constant which only depends on $\overline{C}$ defined in (3.8). From the convergence of Y^n in $\mathbb{S}^2(\mathbb{R}^d)$, the convergence of $\widetilde{Z}^n$ in $\mathbb{H}^2(\mathbb{R}^{d\times l})$ and the uniform estimate (3.13) of $\{Z^n\}_{n\geq 1}$ in $\mathbb{H}^4(\mathbb{R}^{d\times l})$, (3.32) indicates $\{Z^n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{H}^2(\mathbb{R}^{d\times l})$ and has a limit $Z \in \mathbb{H}^2(\mathbb{R}^{d\times l})$. Moreover, following the similar proof as in (3.13), one can prove that $Z \in \mathbb{H}^p(\mathbb{R}^{d\times l})$ for all $p \geq 1$. Therefore, Z is actually in $\mathcal{M}(\mathbb{R}^{d\times l})$.

Step 7 Verification.

Then, we prove that (Y, Z) is a solution to BSDE (1.1) and has a Markovian representation. By the triangle inequality, we have

$$\mathbb{E}\Big[\int_0^T \big|f^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u, Z_u)\big|\mathrm{d}u\Big] \le I_4^{n,k} + I_5^{n,k} + I_6^n, \tag{3.33}$$

where

$$I_4^{n,k} := \mathbb{E}\Big[\int_0^T |f^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u^n, Z_u^n)| \cdot \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|\le k\}}\mathrm{d}u\Big],$$

$$I_5^{n,k} := \mathbb{E}\Big[\int_0^T |f^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u^n, Z_u^n)| \cdot \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|> k\}}\mathrm{d}u\Big]$$

and

$$I_6^n := \mathbb{E}\Big[\int_0^T |f(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u, Z_u)|\mathrm{d}u\Big].$$

First, from (3.6), for $k \ge 1$, we have

$$|f^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u^n, Z_u^n)| \cdot \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|\le k\}} \to 0 \quad \text{as} \quad n \to +\infty.$$

Moreover, there exists $C_k > 0$ such that

$$|f^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u^n, Z_u^n)| \cdot \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|\le k\}} \le C_k(1 + |X_u^{0,x_0}|^\gamma).$$

From Lebesgue's dominated convergence theorem, for fixed $k \ge 1$, we have

$$I_4^{n,k} \to 0 \quad \text{as } n \to +\infty.$$

Second, for a fixed $q \in \big(1, \frac{2}{2-\varepsilon}\big)$, in view of (3.5), (3.8) and (3.13), using Hölder's inequality and Chebyshev inequality, we have

$$\begin{aligned}
&I_5^{n,k}\\
\le&\Big\{\mathbb{E}\Big[\int_0^T |\widetilde{f}^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - \widetilde{f}(u, X_u^{0,x_0}, Y_u^n, Z_u^n)|^q\mathrm{d}u\Big]\Big\}^{\frac{1}{q}}\Big\{\mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|>k\}}\mathrm{d}u\Big]\Big\}^{\frac{q}{q-1}}\\
\le&\overline{C}_1\Big\{\mathbb{E}\Big[\int_0^T (1 + |X_u^{0,x_0}|^{\frac{2\gamma}{2-(2-\varepsilon)q}} + |Z_u^n|^2)\mathrm{d}u\Big]\Big\}^{\frac{1}{q}}\Big\{\mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|>k\}}\mathrm{d}u\Big]\Big\}^{\frac{q}{q-1}}\\
\le&\overline{C}_2\Big\{\mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|Y_u^n|\vee|Z_u^n|>k\}}\mathrm{d}u\Big]\Big\}^{\frac{q}{q-1}}\\
\le&\overline{C}_3k^{-\frac{2q}{q-1}},
\end{aligned} \tag{3.34}$$

where $\overline{C}_1, \overline{C}_2$ and $\overline{C}_3$ are three positive constants independent of n and k.

Third, since Z^n converges to Z in $\mathbb{H}^2(\mathbb{R}^{d\times l})$, in the sense of subsequence Z^n converges to Z, $d\mathbb{P}\otimes dt$ *a.e.* From the continuity of the last two variables of f, we have

$$|f(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u, Z_u)| \to 0 \quad \text{as } n \to +\infty,\ d\mathbb{P}\otimes dt \ \ a.e.$$

From Kobylanski [17, Lemma 2.5, p.569] we know that in the sense of subsequence, $\sup_{n\geq 1}|Z^n| \in \mathbb{H}^2(\mathbb{R}^{d\times l})$. Notice that,

$$|f(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u, Z_u)| \leq \bar{C}\Big(1 + |X_u^{0,x_0}|^{\frac{2\gamma}{2-\varepsilon}} + \sup_{n\geq 1}|Z_u^n|^2 + |Z_u|^2\Big).$$

Therefore, dominated convergence theorem yields

$$I_6^n \to 0 \quad \text{as } n \to +\infty.$$

In summary, for sufficiently large k first and then sufficiently large n, $I_4^{n,k} + I_5^{n,k} + I_6^n$ is sufficiently small. Thus, we have

$$\mathbb{E}\Big[\int_0^T |f^n(u, X_u^{0,x_0}, Y_u^n, Z_u^n) - f(u, X_u^{0,x_0}, Y_u, Z_u)|\mathrm{d}u\Big] \to 0 \quad \text{as } n \to +\infty. \tag{3.35}$$

By virtue of the construction and the convegence of (Y^n, Z^n) and (3.35), we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|Y_t - \Big(g(X_T^{0,x_0}) + \int_t^T f(u, X_u^{0,x_0}, Y_u, Z_u)\mathrm{d}u - \int_t^T Z_u \mathrm{d}B_u\Big)\Big|\Big] = 0.$$

So (Y, Z) is a solution to Markovian BSDE (1.1).

Moreover, we get the Markovian representation of (Y, Z) from the the Markovian representation of (Y^n, Z^n). From the above proof, without loss of generality, we can assume that (Y^n, Z^n) converges to (Y, Z) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^{1\times l})$. So for all $t \in [0, T]$, we have

$$Y_t^i = \limsup_{n\to+\infty} Y_t^{n,i} \quad \text{and} \quad Z_t^{i,j} = \limsup_{n\to+\infty} Z_t^{n,i,j}, \quad i = 1, \cdots, d,\ j = 1, \cdots, l.$$

In view of (3.17), we define a Borel measurable functions pair $(u, v) : [0, T] \times \mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}^{d\times l}$ by

$$u^i(t, x) := \limsup_{n\to+\infty} u_n^i(t, x), \quad i = 1, \cdots, d$$

and

$$v^{i,j}(t, x) := \limsup_{n\to+\infty} v_n^{i,j}(t, x), \quad i = 1, \cdots, d,\ j = 1, \cdots, l.$$

Therefore, for each $i = 1, \cdots, d$ and $j = 1, \cdots, l$, we have

$$Y_t^i = \limsup_{n\to+\infty} Y_t^{n,i} = \limsup_{n\to+\infty}(u_n^i(t, X_t^{0,x_0})) = (\limsup_{n\to+\infty} u_n^i)(t, X_t^{0,x_0}) = u^i(t, X_t^{0,x_0})$$

and

$$Z_t^{i,j} = \limsup_{n\to+\infty} Z_t^{n,i,j} = \limsup_{n\to+\infty}(v_n^{i,j}(t, X_t^{0,x_0})) = (\limsup_{n\to+\infty} v_n^{i,j})(t, X_t^{0,x_0}) = v^{i,j}(t, X_t^{0,x_0})$$

Step 8 Uniqueness under additional assumption (B2).

Assume that (Y, Z) and (Y', Z') are two solutions to Markovian BSDE (1.1) in $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^{d\times l})$ and denote $M := \|Y\|_{\mathbb{S}^\infty} \vee \|Y'\|_{\mathbb{S}^\infty}$.

Define the truncated generator

$$f^M(t, x, y, z) := f\Big(t, x, \frac{My}{|y| \vee M}, z\Big), \quad \forall (t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^{d\times l}.$$

One can easily verify that f^M is Lipschitz continuous in variable y. In addition, (Y,Z) and (Y',Z') are also solutions to Markovian BSDE (1.1) with generator f^M and terminal condition g.

Denote by $f^{M,i}$ the ith component of f^M. Since f is strictly quadratic in each component f^i, from (B2) and Young's inequality we know that there exists a positive constant $C:=C(M)$ such that for all $i=1,\cdots,d$, we have

$$|f^{M,i}(t,x,y,z)|\le C(1+|x|^{\frac{2\gamma}{\varepsilon}})+\frac{1}{2}|z^i|^2$$

and

$$f^{M,i}(t,x,y,z)\ge -C(1+|x|^{\frac{2\gamma}{\varepsilon}})+\frac{1}{4}|z^i|^2.$$

Since $\frac{2\gamma}{\varepsilon}\in(0,2)$, using Dambis-Dubins-Schwarz representation (see [3, Section 5, pp.563–564]), we have

$$\mathbb{E}\Big[\exp\Big(\lambda\sup_{s\in[0,T]}|X_s^{t,x}|^{\frac{2\gamma}{\varepsilon}}\Big)\Big]<\infty,\quad \forall\lambda\ge 0. \tag{3.36}$$

So by virtue of [8, Proposition 2, p.230], we know that for any $i=1,\cdots,d$ and $\delta\in(0,1]$,

$$\mathbb{E}\Big[\exp\Big(\lambda\int_0^T|Z_s^i|^{2-\delta}\mathrm{d}s\Big)\Big]<+\infty,\quad \forall\lambda\ge 0. \tag{3.37}$$

Notice that $\gamma\in(0,\varepsilon)$, so we can take two fixed constants $p_1\in\big(\frac{2}{\varepsilon},\frac{2}{\gamma}\big)$ and $q_2\in\big(\frac{2}{2-\gamma},\frac{2}{2-\varepsilon}\big)$. From Young's inequality, for $(\theta,x,z,z')\in(0,1)\times\mathbb{R}^m\times\mathbb{R}^{d\times l}\times\mathbb{R}^{d\times l}$, we have

$$\begin{aligned}
&(1+|x|^{\gamma})(1+|z|^{1-\varepsilon}+|z'|^{1-\varepsilon})|z-z'|\\
\le&(1+|x|^{\gamma})(1+|z|^{1-\varepsilon}+|z'|^{1-\varepsilon})(|z-\theta z'|+(1-\theta)|z'|)\\
\le&C_{p_1,q_2}(1-\theta)\Big\{\Big(1+|x|^{\gamma p_1}+|z|^{\varepsilon q_1}+|z'|^{\varepsilon q_1}+\Big|\frac{z-\theta z'}{1-\theta}\Big|^2\Big)\\
&+(1+|x|^{\gamma p_2}+|z|^{(2-\varepsilon)q_2}+|z'|^{(2-\varepsilon)q_2})\Big\},
\end{aligned} \tag{3.38}$$

where q_1 and p_2 are the unique constant satisfying

$$\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{2}=1\quad\text{and}\quad \frac{1}{p_2}+\frac{1}{q_2}=1.$$

Notice that all the four positive constants $\gamma p_1,\varepsilon q_1,\gamma p_2$ and $(2-\varepsilon)q_2$ lie in $(0,2)$. Hence from (3.38) and the fact that the function $z\mapsto|z|^2$ is convex, following the similar proof as in [8, Proposition 3 and Remark 4, pp.231–233], one can verify that for $(\theta,t,x,y,y',z,z')\in(0,1)\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times l}\times\mathbb{R}^{d\times l}$ and

$$f^{M,i}(t,x,y,z)=\widetilde{f}^i\Big(t,x,\frac{My}{|y|\vee M},z^i\Big)+\frac{1}{2}|z^i|^2,\quad i=1,\cdots,d,$$

we have

$$f^{M,i}(t,x,y,z)-\theta f^{M,i}(t,x,y',z')\le C(1-\theta)\Big(\Big|\frac{y-\theta y'}{1-\theta}\Big|+\Big|\frac{z^i-\theta z'^i}{1-\theta}\Big|^2+h(x,y,y',z,z',\delta)\Big),$$

where $C>0$ and $\delta\in(0,1]$ are positive constants and

$$h(x,y,y',z,z',\delta):=|x|^{2-\delta}+|y|+|y'|+|z^i|^{2-\delta}+|z'^i|^{2-\delta}.$$

Therefore, in view of (3.36)–(3.37), we can get the uniqueness using θ-method with similar proof as in [7, Theorem 2.8, pp.22–23].

Remark 3.2 Actually, the above proof of existence still holds true when $\alpha^i = 0$ for some $i = 1, \cdots, d$. The only difference is for each $i \in \Lambda := \{i = 1, \cdots, d : \alpha^i = 0\}$, the exponential transformation (3.14) in Step 3 needs to be replaced with

$$\begin{cases} \widetilde{Y}^{n,i;(t,x)} = Y^{n,i;(t,x)}, \\ \widetilde{Z}^{n,i;(t,x)} = Z^{n,i;(t,x)}. \end{cases} \tag{3.39}$$

Remark 3.3 In Theorem 3.1, the drift term b is required to be uniformly bounded. From Remark 3.1 and the above proof, assumptions (F1)–(F2) of Theorem 3.1 can be replaced with assumption (F3) and b can be unbounded.

4 One-dimensional Case

For $d = 1$, by virtue of the comparision theorem, we can get a stronger existence and uniqueness result for the scalar Markovian quadratic BSDE (1.1) with an unbounded sub-quadratic growth.

For example, the generator can be $f(t, x, y, z) = 1 - y + |x|^4|z|^{\frac{3}{2}} + |z|^2$. It should be noticed that this generator cannot be fully covered by the framework of one-dimensional unbounded quadratic BSDEs (see [3]), since $\int_0^T |B_t|^p \mathrm{d}t$ does not have exponential moments of all orders for $p \geq 2$.

First, we make the following assumption. Let $\varepsilon \in (0, 1]$, γ and C be positive constants and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function with $\phi(0) = 0$.

(H1) $f : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{1\times l} \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ are two Borel measurable functions such that for each $(t, x) \in [0, T]\times\mathbb{R}^m$, $f(t, x, \cdot, \cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^{1\times l}$ and for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{1\times l}$,

$$\begin{cases} |f(t, x, y, z)| \leq C(1 + \phi(|y|) + |x|^\gamma|z|^{2-\varepsilon} + |z|^2), \\ |g(x)| \leq C \end{cases}$$

and

$$y\big(f(t, x, y, z) - f(t, x, 0, z)\big) \leq C|y|^2.$$

We also make the following stronger assumption. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function.

(H2) For each $(t, x, y) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}$, $f(t, x, y, \cdot)$ is a convex function, and constant γ lies in $(0, \varepsilon)$ such that for all $(t, x, y, y', z) \in [0, T] \times \mathbb{R}^m \times (\mathbb{R})^2 \times \mathbb{R}^{1\times l}$,

$$|f(t, x, y, z)| \leq C(1 + |y| + |x|^\gamma|z|^{2-\varepsilon} + |z|^2)$$

and

$$|f(t, x, y, z) - f(t, x, y', z)| \leq \rho(|y| \vee |y'|)|y - y'|.$$

We have the following existence and uniqueness result for one-dimensional Markovian BSDEs.

Theorem 4.1 *Let assumptions* (F1) *and* (H1) *hold true. Then*

(i) *Markovian* BSDE (1.1) *has a solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^\infty(\mathbb{R}) \times \mathcal{M}(\mathbb{R}^{1\times l})$.

(ii) *There exists a pair of Borel measurable functions* $(u, v) : [0, T] \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^{1\times l}$ *such that* $(Y_s^{t,x}, Z_s^{t,x}) = (u(s, X_s^{t,x}), v(s, X_s^{t,x}))$, $\forall s \in [t, T]$, $d\mathbb{P} \otimes ds$ *a.e.*

(iii) *If* σ *is bounded and* (H2) *is further satisfied, the solution is unique.*

Proof For notational convenience, we omit the superscript (t, x) if there is no confusion.

Step 1 Existence for nonnegative f.

For each $n \geq 1$, we define the approximating generator by

$$f^n(t, x, y, z) := \mathbb{1}_{\{|x|\leq n\}} f(t, x, y, z).$$

From assumption (H1), we have

$$|f^n(t, x, y, z)| \leq C(1 + \phi(|y|) + n^\gamma |z|^{2-\varepsilon} + |z|^2).$$

According to [3, Lemma 2, page 549], there exists $(Y^n, Z^n) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^{1\times l})$ such that Y^n is the minimal bounded solution to the Markovian BSDE with generator f^n and terminal condition g.

Define $\kappa := C(4 - \varepsilon)$ and nonnegative function

$$\psi(y) := \kappa^{-2}(\mathrm{e}^{\kappa|y|} - \kappa|y| - 1), \quad \forall y \in \mathbb{R}.$$

For all $y \in \mathbb{R}$, we have

$$|\psi'(y)| = \kappa^{-1}(\mathrm{e}^{\kappa|y|} - 1) \quad \text{and} \quad \psi''(y) - \kappa|\psi'(y)| = 1. \tag{4.1}$$

Let $\tau \in [0, T]$ be a stopping time. Since $Y^n \in \mathbb{S}^\infty(\mathbb{R})$, using Itô-Tanaka formula to compute $\psi(Y_\tau^n)$ and taking the conditional expection with respect to $\mathcal{F}_\tau$, we have

$$\begin{aligned}
&\psi(Y_\tau^n) + \frac{1}{2}\mathbb{E}^{\mathcal{F}_\tau}\Big[\int_\tau^T \psi''(Y_s^n)|Z_s^n|^2 \mathrm{d}s\Big] \\
=& \mathbb{E}^{\mathcal{F}_\tau}[g(X_T^{t,x})] + \mathbb{E}^{\mathcal{F}_\tau}\Big[\int_\tau^T \psi'(Y_s^n) f^n(s, X_s^{t,x}, Y_s^n, Z_s^n)\mathrm{d}s\Big] \\
\leq& C + C\mathbb{E}^{\mathcal{F}_\tau}\Big[\int_\tau^T |\psi'(Y_s^n)|(1 + \phi(Y_s^n) + n^\gamma |Z_s|^{2-\varepsilon} + |Z_s|^2)\mathrm{d}s\Big] \\
\leq& C + C\mathbb{E}^{\mathcal{F}_\tau}\Big[\int_\tau^T |\psi'(Y_s^n)|\Big(\frac{2 + \varepsilon n^{\frac{2\gamma}{\varepsilon}}}{2} + \phi(Y_s^n) + \frac{4-\varepsilon}{2}|Z_s|^2\Big)\mathrm{d}s\Big].
\end{aligned} \tag{4.2}$$

In view of (4.1)–(4.2), we have

$$\begin{aligned}
\frac{1}{2}\mathbb{E}^{\mathcal{F}_\tau}\Big[\int_\tau^T |Z_s^n|^2 \mathrm{d}s\Big] &\leq C + C\mathbb{E}^{\mathcal{F}_\tau}\Big[\int_\tau^T |\psi'(Y_s^n)|\Big(\frac{2 + \varepsilon n^{\frac{2\gamma}{\varepsilon}}}{2} + \phi(Y_s^n)\Big)\mathrm{d}s\Big] \\
&\leq C + CT\psi'(\|Y^n\|_{\mathbb{S}^\infty})\Big(\frac{2 + \varepsilon n^{\frac{2\gamma}{\varepsilon}}}{2} + \phi(\|Y^n\|_{\mathbb{S}^\infty})\Big).
\end{aligned} \tag{4.3}$$

Hence $Z^n \in \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times l})$.

Define stochastic process

$$\kappa_u := (Cn^\gamma |Z_u^n|^{1-\varepsilon} + C|Z_u^n|)\frac{Y_u^n Z_u^n}{|Y_u^n Z_u^n|}. \tag{4.4}$$

For $k \geq 0$, using Itô's formula to compute $\mathrm{e}^{ks}|Y_s^n|^2$, we have

$$\begin{aligned}\mathrm{e}^{ks}|Y_s^n|^2 =& \mathrm{e}^{ks}|g(X_T^{t,x})|^2 - 2\int_s^T \mathrm{e}^{ku} Y_u^n Z_u^n \mathrm{d}B_u \\ & + 2\int_s^T \mathrm{e}^{ku}\Big(Y_u^n f^n(u, X_u^{t,x}, Y_u^n, Z_u^n) - \frac{k}{2}|Y_u|^2 - \frac{1}{2}|Z_u|^2\Big)\mathrm{d}u \\ =& \mathrm{e}^{ks}|g(X_T^{t,x})|^2 - 2\int_s^T \mathrm{e}^{ku} Y_u^n Z_u^n \mathrm{d}\widetilde{B}_u^n + 2\int_s^T \mathrm{e}^{ku} F^n(u, X_u^{t,x}, Y_u^n, Z_u^n)\mathrm{d}u, \end{aligned} \tag{4.5}$$

where $\widetilde{B}_s^n := B_s - \int_0^s (\kappa_u)^{\mathrm{T}}\mathrm{d}u$ and

$$F^n(u, X_u^{t,x}, Y_u^n, Z_u^n) := Y_u^n f^n(u, X_u^{t,x}, Y_u^n, Z_u^n) - \frac{k}{2}|Y_u^n|^2 - \frac{1}{2}|Z_u^n|^2 - Y_u^n Z_u^n (\kappa_u)^{\mathrm{T}}. \tag{4.6}$$

Since $Z^n \in \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times l})$ and $Y^n \in \mathbb{S}^\infty(\mathbb{R})$, we can deduce that $\kappa \in \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times l})$ from Young's inequality. Hence Girsanov theorem indicates that $\{\widetilde{B}_s^n\}_{s\in[0,T]}$ is a Brownian motion under the equivalent probability measure $\mathbb{P}^n$, where

$$\frac{d\mathbb{P}^n}{d\mathbb{P}} := \mathcal{E}\Big(\int_0^\cdot \kappa_u \mathrm{d}B_u\Big)_T. \tag{4.7}$$

Notice that by virtue of (H1), we have

$$\begin{aligned} yf^n(u,x,y,z) &= y\big(f^n(t,x,y,z) - f^n(t,x,0,z)\big) + yf^n(t,x,0,z) \\ &\leq C|y|^2 + C|y|(1 + n^\gamma |z|^{2-\varepsilon} + |z|^2) \\ &\leq C|y|(1+|y|) + |yz|(Cn^\gamma|z|^{1-\varepsilon} + C|z|). \end{aligned} \tag{4.8}$$

From (4.4), (4.6) and (4.8), using Young's inequality, we have

$$\begin{aligned} F^n(u, X_u^{t,x}, Y_u^n, Z_u^n) &\leq C|Y_u^n|(1+|Y_u^n|) - \frac{k}{2}|Y_u^n|^2 - \frac{1}{2}|Z_u^n|^2 \\ &\leq \frac{1}{2} + \frac{1}{2}(C^2 + 2C - k)|Y_u^n|^2. \end{aligned} \tag{4.9}$$

Denote by $\mathbb{E}^n$ the expectation operator with respect to $\mathbb{P}^n$. Setting $k = C^2 + 2C$, taking the conditional expectation $\mathbb{E}_s^n$ on both sides of (4.5), using (B1) and (4.9) we finally have

$$|Y_s^n|^2 \leq C^2\mathrm{e}^{k(T-s)} + \int_s^T \mathrm{e}^{k(u-s)}\mathrm{d}u \leq C^2\mathrm{e}^{kT} + \int_0^T \mathrm{e}^{ku}\mathrm{d}u, \quad \forall s\in[0,T]. \tag{4.10}$$

Therefore, the process Y^n has a bound uniformly in n and thus $\sup_{n\geq 1}\|Y^n\|_{\mathbb{S}^\infty} < +\infty$.

Notice that $f^n \leq f^{n+1}$ and Y^n is a minimal solution, so we have $Y^n \leq Y^{n+1}$. From Briand and [3, Lemma 2, p.549], there exists a solution $(Y,Z) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^{1\times l})$ to BSDE (1.1). Following the similar proof as in (3.13), Z is actually in $\mathcal{M}(\mathbb{R}^{1\times l})$.

From [19, Proposition 4.9, pp.17–18], there exists a pair of Borel measurable functions $(u_n, v_n) : [0,T]\times\mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}^{d\times l}$ such that

$$(Y_s^{n;(t,x)}, Z_s^{n;(t,x)}) = (u_n(s, X_s^{t,x}), v_n(s, X_s^{t,x})), \quad \forall s\in[0,T]. \tag{4.11}$$

So, we can get the Markovian representation of $(Y^{t,x}, Z^{t,x})$ with similar proof as in Theorem 3.1.

Step 2 Existence in the general case.

Define $f^{n,p} := \mathbb{1}_{\{|x|\leq n\}} f^+(t,x,y,z) - \mathbb{1}_{\{|x|\leq p\}} f^-(t,x,y,z)$. Following the proof of Step 1, we can get a sequence of $(Y^{n,p}, Z^{n,p}) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^{1\times l})$ with the property

$$\begin{aligned} &Y^{n,(p+1)} \leq Y^{n,p} \leq Y^{(n+1),p}, \quad \forall n \geq 1, p \geq 1, \\ &\sup_{n,p\geq 1} \|Y^{n,p}\|_{\mathbb{S}^\infty} \leq \bar{C}. \end{aligned} \tag{4.12}$$

Setting first $p \to +\infty$ and then $n \to +\infty$, we get the desired result identically as in Step 1.

Step 3 Uniqueness when σ is bounded and (H2) is further satisfied.

Assume that (Y, Z) and (Y', Z') are two solutions to Markovian BSDE (1.1) in $\mathbb{S}^\infty(\mathbb{R}) \times \mathcal{M}(\mathbb{R}^{1\times l})$ and denote $M := \|Y\|_{\mathbb{S}^\infty} \vee \|Y'\|_{\mathbb{S}^\infty}$.

Define the truncated generator

$$f^M(t,x,y,z) := f\Big(t,x,\frac{My}{|y|\vee M},z\Big), \quad \forall (t,x,y,z) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^{1\times l}.$$

One can easily verify that f^M is Lipschitz continuous in variable y. In addition, (Y, Z) and (Y', Z') are also solutions to Markovian BSDE (1.1) with generator f^M and terminal condition g.

By virtue of Young's inequality, we have

$$|f^M(t,x,y,z)| \leq C(1+|x|^{\frac{2\gamma}{\varepsilon}}+|y|+|z|^2), \quad \forall (t,x,y,z) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^{1\times l}.$$

Notice that $\frac{2\gamma}{\varepsilon} \in (0,2)$. Therefore, the uniqueness follows from (3.36) and [3, Theorem 5, p.554].

5 Application to Coupled FBSDE with Measurable Coefficients

We consider the following system of coupled FBSDE:

$$\begin{cases} X_t = x + \displaystyle\int_0^t \Big[b(s,X_s)+\sigma(s,X_s)h(s,X_s,Y_s,Z_s)\Big]\mathrm{d}s + \int_0^t \sigma(s,X_s)\mathrm{d}B_s, \quad t\in[0,T], \\ Y_t = g(X_T) + \displaystyle\int_t^T f(s,X_s,Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}B_s, \quad t\in[0,T]. \end{cases} \tag{5.1}$$

We assume that X and B take values in the same space $\mathbb{R}^m$ and the coefficients of (5.1) satisfy the following assumptions:

(C1) $b : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,T]\times\mathbb{R}^m \to \mathbb{R}^{m\times m}$ satisfy (F1) and (F2);

(C2) $h : [0,T]\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^{d\times m} \to \mathbb{R}^m$ is a Borel measurable function and there exist a positive constant C and a nondecreasing function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, such that $|h(t,x,y,z)| \leq C\big(1+\rho(|y|)\big)$ for all $(t,x,y,z) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^{d\times m}$;

(C3) $g : [0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ and $f : [0,T]\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^{d\times m} \to \mathbb{R}^d$ satisfy (B1);

(C4) for each $(t,x) \in [0,T]\times\mathbb{R}^m$, $f(t,x,\cdot,\cdot)+zh(t,x,\cdot,\cdot)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^{d\times m}$.

By virtue of Theorem 3.1, we have the following existence result for coupled FBSDE (5.1) with measurable coefficients.

Proposition 5.1 *Under the assmuptions* (C1)–(C4), FBSDE (5.1) *has a strong solution* $(X,Y,Z)\in\mathbb{S}^2(\mathbb{R}^m)\times\mathbb{S}^\infty(\mathbb{R}^d)\times\mathcal{M}(\mathbb{R}^{d\times m})$.

Proof Consider the decoupled FBSDE

$$\begin{cases}\widetilde{X}_t = x+\int_0^t b(s,\widetilde{X}_s)\mathrm{d}s+\int_0^t \sigma(s,\widetilde{X}_s)\mathrm{d}B_s, \quad t\in[0,T],\\ \widetilde{Y}_t^i = g^i(\widetilde{X}_T)+\int_t^T [f^i(s,\widetilde{X}_s,\widetilde{Y}_s,\widetilde{Z}_s)+\widetilde{Z}_s^i h(s,\widetilde{X}_s,\widetilde{Y}_s,\widetilde{Z}_s)]\mathrm{d}s\\ \qquad\quad -\int_t^T \widetilde{Z}_s^i\mathrm{d}B_s, \quad i=1,\cdots,d,\ t\in[0,T].\end{cases}\tag{5.2}$$

From Theorem 3.1, we know that FBSDE (5.2) has a solution $(\widetilde{X},\widetilde{Y},\widetilde{Z})\in\mathbb{S}^2(\mathbb{R}^m)\times\mathbb{S}^\infty(\mathbb{R}^d)\times\mathcal{M}(\mathbb{R}^{d\times m})$. Moreover, there exist two Borel measurable functions u and v on $[0,T]\times\mathbb{R}^m$ such that $(\widetilde{Y}_t,\widetilde{Z}_t)=(u(t,\widetilde{X}_t),v(t,\widetilde{X}_t))$ and u is bounded.

From (C1)–(C2) and [10, Theorem 2.1, p.767], we know that there exists a unique strong solution $X\in\mathbb{S}^2(\mathbb{R}^m)$ to the following SDE with measurable drift:

$$\begin{aligned}X_t =&x+\int_0^t [b(s,X_s)+\sigma(s,X_s)h(s,X_s,u(s,X_s),v(s,X_s))]\mathrm{d}s\\ &+\int_0^t \sigma(s,X_s)\mathrm{d}B_s, \quad t\in[0,T].\end{aligned}\tag{5.3}$$

Define

$$\widetilde{B}_t := B_t+\int_0^t h(s,X_s,u(s,X_s),v(s,X_s))\mathrm{d}s.$$

From (5.3), we have

$$X_t = x+\int_0^t b(s,X_s)\mathrm{d}s+\int_0^t \sigma(s,X_s)\mathrm{d}\widetilde{B}_s, \quad t\in[0,T].\tag{5.4}$$

Since u is bounded, from Girsanov theorem we know that $\{\widetilde{B}_t\}_{t\in[0,T]}$ is a Brownian motion under the equivalent probability measure $\widetilde{\mathbb{P}}$, where

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := \mathcal{E}\Big(-\int_0^\cdot [h\big(s,X_s,u(s,X_s),v(s,X_s)\big)]^{\mathrm{T}}\mathrm{d}B_s\Big)_T.\tag{5.5}$$

So from the uniqueness of solutions to SDE (5.4), we know that X and $\widetilde{X}$ have the same law.

On the other hand, from (5.2), for $i=1,\cdots,d$, we have

$$\begin{aligned}u^i(t,\widetilde{X}_t) =&g^i(\widetilde{X}_T)+\int_t^T [f^i(s,\widetilde{X}_s,u(s,\widetilde{X}_s),v(s,\widetilde{X}_s))+v^i(s,\widetilde{X}_s)h(s,\widetilde{X}_s,u(s,\widetilde{X}_s),v(s,\widetilde{X}_s))]\mathrm{d}s\\ &-\int_t^T v^i(s,\widetilde{X}_s)\mathrm{d}B_s\\ =&g^i(\widetilde{X}_T)+\int_t^T [f^i(s,\widetilde{X}_s,u(s,\widetilde{X}_s),v(s,\widetilde{X}_s))+v^i(s,\widetilde{X}_s)h(s,\widetilde{X}_s,u(s,\widetilde{X}_s),v(s,\widetilde{X}_s))]\mathrm{d}s\\ &+\int_t^T v^i(s,\widetilde{X}_s)\sigma^{-1}(s,\widetilde{X}_s)b(s,\widetilde{X}_s)\mathrm{d}s-\int_t^T v^i(s,\widetilde{X}_s)\sigma^{-1}(s,\widetilde{X}_s)\mathrm{d}\widetilde{X}_s.\end{aligned}\tag{5.6}$$

Since X has the same law as $\widetilde{X}$, for $i=1,\cdots,d$, we have

$$\begin{aligned}
u^i(t,X_t) =& g^i(X_T)+\int_t^T [f^i(s,X_s,u(s,X_s),v(s,X_s))+v^i(s,X_s)h(s,X_s,u(s,X_s),v(s,X_s))]\mathrm{d}s\\
&+\int_t^T v^i(s,X_s)\sigma^{-1}(s,X_s)b(s,X_s)\mathrm{d}s-\int_t^T v^i(s,X_s)\sigma^{-1}(s,X_s)\mathrm{d}X_s\\
=& g^i(X_T)+\int_t^T f^i(s,X_s,u(s,X_s),v(s,X_s))\mathrm{d}s\\
&+\int_t^T v^i(s,X_s)\sigma^{-1}(s,X_s)[\sigma(s,X_s)h(s,X_s,u(s,X_s),v(s,X_s))]\mathrm{d}s\\
&+\int_t^T v^i(s,X_s)\sigma^{-1}(s,X_s)b(s,X_s)\mathrm{d}s-\int_t^T v^i(s,X_s)\sigma^{-1}(s,X_s)\mathrm{d}X_s\\
=& g^i(X_T)+\int_t^T f^i(s,X_s,u(s,X_s),v(s,X_s))\mathrm{d}s-\int_t^T v^i(s,X_s)\mathrm{d}B_s.
\end{aligned} \tag{5.7}$$

Combining (5.3) and (5.7), we know that $\{(X_t,u(t,X_t),v(t,X_t)),0\le t\le T\}\in\mathbb{S}^2(\mathbb{R}^m)\times\mathbb{S}^\infty(\mathbb{R}^d)\times\mathcal{M}(\mathbb{R}^{d\times m})$ is a solution to FBSDE (5.1).

Remark 5.1 Similar assertions can be inferred from Theorem 4.1 for $d=1$. In addition, as stated in Remark 3.1, (F1)–(F2) can be replaced with (F3) in assumption (C1).

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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