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Centered Waves for the Two-dimensional Pseudo-Steady van der Waals Gas Satisfied Maxwell's Law Around a Sharp Corner*

Shuangrong LI[1] Wancheng SHENG[2]

Abstract In this paper, the authors study the centered waves for the two-dimensional (2D for short) pseudo-steady supersonic flow with van der Waals gas satisfied Maxwell's law around a sharp corner. In view of the initial value of the specific volume and the properties of van der Waals gas, the centered waves at the sharp corner are constructed by classification. It is shown that the supersonic incoming flow turns the sharp corner by a centered simple wave or a centered simple wave with right-contact discontinuity or a composite wave (jump-fan, fan-jump or fan-jump-fan), or a combination of waves and constant state. Moreover, the critical angle of the sharp corner corresponding to the appearance of the vacuum phenomenon is obtained.

Keywords Two-dimensional Euler equations, Van der Waals gas, Centered simple wave, Composite wave
2000 MR Subject Classification 35L65, 35L50, 76J20, 76N10

1 Introduction

The 2D steady supersonic flow around a bend or sharp corner was studied by Courant and Friedrichs [7]. In this paper, we focus on the pseudo-steady flow with van der Waals gas satisfied Maxwell's law around a sharp corner. Based on this purpose, we pay attention to the 2D isentropic Euler equations

$$\begin{cases}\rho_t + (\rho u)_x + (\rho v)_y = 0,\\ (\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho uv)_y = 0,\\ (\rho v)_t + (\rho uv)_x + \left(\rho v^2 + p\right)_y = 0,\end{cases} \tag{1.1}$$

where (u, v), ρ, p are designated as the velocity, the density and the pressure respectively. Here, we take van der Waals gas equation of state

$$p = \frac{A}{(\tau - b)^{\delta+1}} - \frac{a}{\tau^2}, \tag{1.2}$$

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[1]Department of Mathematics, Shanghai University, Shanghai 200444, China; Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou 310023, China. E-mail: li_sr@shu.edu.cn
[2]Corresponding author. Department of Mathematics, Newtouch Center for Mathematics of Shanghai University, Shanghai University, Shanghai 200444, China. E-mail: mathwcsheng@shu.edu.cn
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in which $\tau = \frac{1}{\rho}$ is the specific volume, $A > 0$ is a constant depending on the entropy, a and b are two positive constants representing the attraction between the particles and the compressibility limit of the molecules in the gas respectively, $0 < \delta < 1$ is a constant. The equation (1.2) can be seen as the dusty gas for the case $a = 0$ and the polytropic gas for the case $a = b = 0$.

As Figure 1, the straight ground and the ramp form a sharp corner at the origin O. The incoming supersonic flow with a constant state $(u_0, 0, \rho_0)$ goes along the straight ground wall to point O and diffuses to vacuum. The problem is how the flow turns over the corner. Thus, we discuss the system (1.1) with the initial data

$$(u, v, \rho)(x, y, 0) = \begin{cases} (u_0, 0, \rho_0), & (x, y) \in \{x < 0, y > 0\}, \\ \text{vacuum}, & (x, y) \in \{x > 0, y \geq 0\} \cup \{y < 0, x > y \cot\theta\} \end{cases} \tag{1.3}$$

and the boundary data

$$\begin{cases} (\rho v)(x, 0, t) = 0, & (x, y) \in \{x < 0, y = 0\}, \quad t \geq 0, \\ (\rho v)(x, y, t) = (\rho u)(x, y, t)\tan\theta, & (x, y) \in \{y < 0, x = y\cot\theta\}, \ t \geq 0, \end{cases} \tag{1.4}$$

where u_0, ρ_0 are two constants, and $-\pi < \theta < 0$ is the angle of the positive half of the $x-$axis to the ramp.

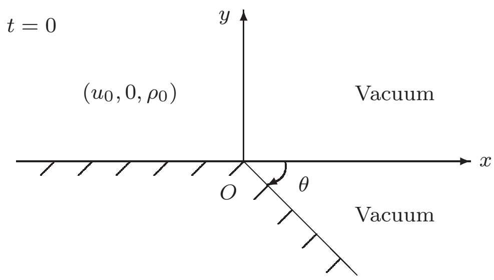


Figure 1 Initial boundary value conditions.

This is a kind of 2D initial boundary value Riemann problem. Two dimensional Riemann problem is significant and well-studied in analysis and applications. Up to now, there are some progresses for the system (1.1) (see [2, 3, 11–12, 17, 21, 24–29, 32, 36, 38, 41–43, 46, 51–52]) and its simplified models (see [5, 9–10, 15, 33, 35, 45, 47–49]).

Recently, many researches considered the expansion problem of a gas into vacuum. Sheng and You [43–44] investigated the problem (1.1) with (1.3)–(1.4) for the polytropic gases $p = \rho^\gamma$ at $1 < \gamma < \sqrt{2} + 1$. In their work, the centered simple wave is complete. After that, the same problem is studied by Sheng and Yao [42], but the centered simple wave is incomplete. Lai and Sheng [24] extended the range of γ to $1 < \gamma < 3$ and got the global solution when the incoming flow is sonic or subsonic. Chen et al. [4] discussed the problem with the Noble-Abel gas $p = \frac{A\rho^\gamma}{(1-a\rho)^\gamma}$, where A and a are two constants, and $\gamma \in (1, 3)$. Li and Sheng considered the expansion problem with the isothermal flow $p = \rho$ (see [30]) and van der Waals gas satisfied some conditions (see [31]).

For the van der Waals gas equation of state, Maxwell pointed out that $p'(\tau) > 0$ violates the law of thermodynamic equilibrium (see [8]). According to the law of thermal stability, $p'(\tau)$ must be non-positive. Thus, the model we focus on may be seen as a van der Waals type equation of state satisfied Maxwell's law. In brief, Maxwell's law introduces two constant specific volumes $b < \tau_1 < \tau_2$ which are given by the thermodynamic equilibrium and limit the mixture zone (see Figure 2). And in the region $[\tau_1, \tau_2]$, the pressure is constant p_m which is a mean pressure. As a consequence, the pressure is always nonincreasing and its derivative (thus the sound speed) vanishes in the mixture zone. The main difficulty is that $p(\tau)$ is discontinuous at τ_1 and τ_2, which leads to a local degenerate problem. Indeed, such a problem, usually called resonance, has been widely studied (see [6, 13, 18–20, 34–35, 39]). Moreover, Godlewski et al. [14] studied the Riemann problem for the isothermal p-system of phase transition. In their work, the equation of state is van der Waals gas satisfied Maxwell's law. Sheng and Wang [40] considered the Riemann problem and interaction of elementary waves to the Euler equations for van der Waals gas satisfied Maxwell's law. In their work, τ_1 and τ_2 are polished. In addition, more studies related to van der Waals gas are referred to [1, 16, 22–23, 37, 50]. The main result of this paper is as follows.

Theorem 1.1 (Main theorem) *When $u_0 > c_0$, $\tau_0 > b$ and $\theta \leq \min\{\theta_2, \theta_4, \theta_6, \theta_8\}$, the supersonic incoming flow turns the sharp corner O locally by a centered simple wave, a right-contact discontinuity, a composite wave (jump-fan, fan-jump or fan-jump-fan), or a combination of waves and constant state. Here, θ_2, θ_4, θ_6 and θ_8 are given by* (4.11), (5.7), (5.15) *and* (5.20), *respectively.*

This paper is organized as follows. In Section 2, we give some preliminaries on the equation of state satisfied Maxwell's law and 2D self-similar Euler equations. The centered waves at point O as $\tau_0 \in [\tau_2, +\infty)$, $\tau_0 \in [\tau_1, \tau_2)$ and $\tau_0 \in (b, \tau_1)$ are constructed in Sections 3–5, respectively.

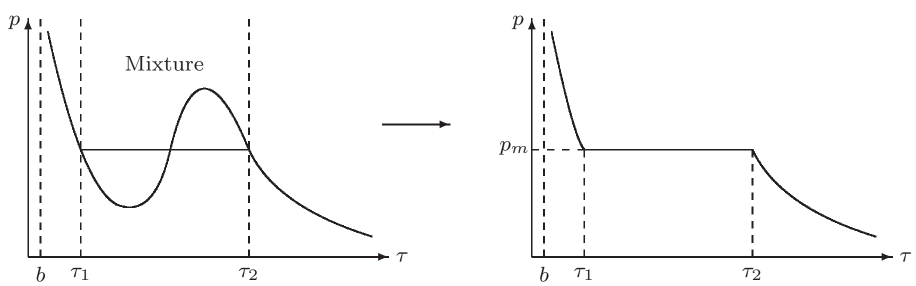


Figure 2 The equation of state for the van der Waals gas and Maxwell's law in (τ, p) plan.

2 Preliminaries

2.1 Equation of state

Throughout this paper, we make the following assumptions which satisfy the Maxwell's law (see Figure 2):

(A1) $p'(\tau) < 0$ and $p''(\tau) > 0$ for $\tau \in (b, \tau_1) \cup (\tau_2, +\infty)$.

(A2) $p(\tau) = p_m$ for $\tau \in [\tau_1, \tau_2]$.

(A3) $\lim\limits_{\tau \to b^+} p(\tau) = +\infty$ and $\lim\limits_{\tau \to +\infty} p(\tau) = 0$.

(A4) $\tau p''(\tau) + 2p'(\tau) > 0$ for $\tau \in (b, \tau_1) \cup (\tau_2, +\infty)$.

In addition, we use the following notations in this paper for convenience

$$\Lambda(\tau) = \frac{2p'(\tau)}{\tau p''(\tau) + 2p'(\tau)}, \quad c^2 = -\tau^2 p'(\tau), \quad \alpha_0 = \arcsin\frac{c_0}{u_0}. \tag{2.1}$$

2.2 2D self-similar Euler equations

By the self-similar transformation $\xi = \frac{x}{t}, \eta = \frac{y}{t}$, the system (1.1) can be rewritten as follows

$$\begin{cases} U \cdot \rho_\xi + \rho \cdot u_\xi + V \cdot \rho_\eta + \rho \cdot v_\eta = 0, \\ U \cdot u_\xi + V \cdot u_\eta + \tau \cdot p_\xi = 0, \\ U \cdot v_\xi + V \cdot v_\eta + \tau \cdot p_\eta = 0, \end{cases} \tag{2.2}$$

in which $(U, V) = (u - \xi, v - \eta)$ is called the pseudo-velocity.

The eigenvalues of system (2.2) are

$$\lambda_0 = \frac{V}{U}, \quad \lambda_\pm = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2}. \tag{2.3}$$

Obviously, system (2.2) is hyperbolic as $q^2 > c^2$, degenerate-hyperbolic as $q^2 = c^2$ and elliptic-hyperbolic as $q^2 < c^2$, where $q^2 = U^2 + V^2$.

If the flow is irrotational, namely, $u_y = v_x$, we get the pseudo-Bernoulli law by (2.2) as follows

$$\frac{U^2 + V^2}{2} + \int_{\tau_0}^{\tau} \tau p'(\tau)\ \mathrm{d}\tau + \varphi = \text{Const.}, \tag{2.4}$$

where the potential function φ satisfies $\varphi_\xi = U$ and $\varphi_\eta = V$.

We refer to α (β) as the characteristic angles of the C_+ (C_-) characteristic curves, i.e., $\tan\alpha = \lambda_+$ and $\tan\beta = \lambda_-$. Then we have

$$u - \xi = c\frac{\cos\sigma}{\sin\omega}, \quad v - \eta = c\frac{\sin\sigma}{\sin\omega}. \tag{2.5}$$

Here, $\sigma = \frac{\alpha+\beta}{2}$ and $\omega = \frac{\alpha-\beta}{2}$ are the pseudo-flow characteristic angle and the pseudo-Mach angle, respectively.

3 The Centered Waves for $\tau_0 \in [\tau_2, +\infty)$

In this section, we construct the centered waves at point O as $\tau_0 \in [\tau_2, +\infty)$. In addition, we define $c_2^2 = -\tau_2^2 \cdot \lim\limits_{\tau \to \tau_2^+} p'(\tau)$ in Section 3.

Theorem 3.1 *Assume that the supersonic incoming flow* $(u_0, 0, c_0)$ *satisfies* $\tau_0 \geq \tau_2$. *Then the incoming flow turns the sharp corner* O *locally by a* C_+ *type centered simple wave* R_1, *which satisfies*

$$R_1 : \begin{cases} u = \sqrt{B(\tau) + D_1} \cos\alpha + c \sin\alpha, \\ v = \sqrt{B(\tau) + D_1} \sin\alpha - c\cos\alpha, \\ c = \displaystyle\int_{\alpha_0}^{\alpha} \frac{1}{1 - \Lambda(\tau)} \cdot \sqrt{B(\tau) + D_1}\, \mathrm{d}\alpha + c_0, \end{cases} \tag{3.1}$$

in which

$$B(\tau) = -\frac{A}{\delta(\tau - b)^{(\delta+2)}} \left((\delta+1)(\delta+2)\tau^2 - 2(\delta+2)b\tau + 2b^2\right) + \frac{6a}{\tau} \tag{3.2}$$

and

$$D_1 = u_0^2 - c_0^2 + \frac{A}{\delta(\tau_0 - b)^{(\delta+2)}}((\delta+1)(\delta+2)\tau_0^2 - 2(\delta+2)b\tau_0 + 2b^2) - \frac{6a}{\tau_0}. \tag{3.3}$$

Moreover, we define

$$\theta_1 := \int_{\tau_0}^{+\infty} \frac{-\sqrt{-p'(\tau)} \cdot \sqrt{q^2 + \tau^2 p'(\tau)}}{q^2}\, \mathrm{d}\tau, \tag{3.4}$$

where $q^2 = u^2 + v^2 = B(\tau) + D_1 + c^2$. *Specifically, there are two cases*:

(i) *When* $\theta \leq \theta_1$, *the complete simple wave* R_1 *connects the constant state* $(u_0, 0, c_0)$ *and vacuum* (*see Figure* 3(b)).

(ii) *When* $\theta_1 < \theta < 0$, *the incomplete simple wave* R_1 *connects two constant states* $(u_0, 0, c_0)$ *and* (u_3, v_3, c_3), *where* (u_3, v_3, c_3) *is governed by* (3.1)–(3.3) *and* $\tan\theta = \frac{v(\alpha_3)}{u(\alpha_3)}$ (*see Figure* 3(c)).

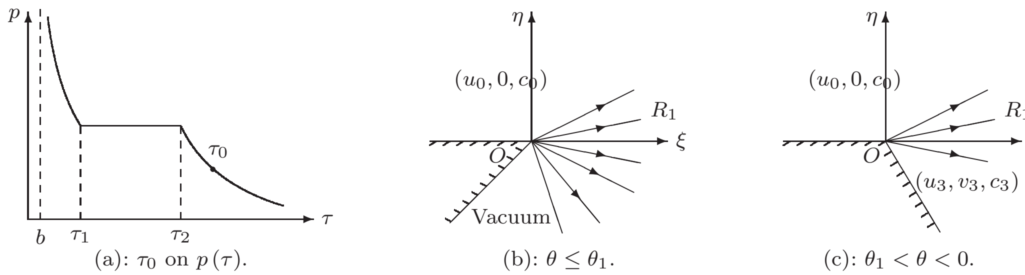


Figure 3 $\tau_0 \in [\tau_2, +\infty)$.

Proof For the proof of this theorem, Li and Sheng [31] gave a detailed explanation. In what follows, we explain briefly for completeness.

Near the point O, i.e., $\xi \to 0$, $\eta \to 0$, the centered simple wave (u, v, c) and φ satisfy

$$\begin{cases} \sin\alpha \cdot \dfrac{\mathrm{d}v}{\mathrm{d}\alpha} + \cos\alpha \cdot \dfrac{\mathrm{d}u}{\mathrm{d}\alpha} = 0, \\ \dfrac{1}{2}(u^2 + v^2) + \dfrac{A}{(\tau - b)^{\delta}}\Big(\dfrac{\delta+1}{\delta} + \dfrac{b}{\tau - b}\Big) - \dfrac{2a}{\tau} = \text{Const.} - \varphi(\alpha_0), \quad \dfrac{\mathrm{d}\varphi}{\mathrm{d}\alpha} = 0, \\ u \cdot \sin\alpha - v \cdot \cos\alpha = c. \end{cases} \tag{3.5}$$

We decompose pseudo-flow velocity (U, V) along the directions $(\cos\alpha, \sin\alpha)$ and $(\sin\alpha, -\cos\alpha)$ respectively. When $\xi, \eta \to 0$, we have

$$g = q\cos\omega = u\cos\alpha + v\sin\alpha, \quad c = q\sin\omega = u\sin\alpha - v\cos\alpha, \tag{3.6}$$

where g is a function of α and c is the sound speed. Then (3.6) yields

$$u = g\cos\alpha + c\sin\alpha, \quad v = g\sin\alpha - c\cos\alpha. \tag{3.7}$$

Taking the derivatives of (3.7) with respect to α and combining with (3.5), one has

$$g_\alpha = -c \tag{3.8}$$

and

$$(1 - \Lambda(\tau))\, c_{\alpha\alpha} - \frac{\Lambda'(\tau)\cdot\Lambda(\tau)}{\sqrt{-p'(\tau)}}\cdot c_\alpha^2 + c = 0. \tag{3.9}$$

Solving (3.9) with the initial data $g^2(\alpha_0) + c_0^2 = u_0^2$, we get (3.1)–(3.3).

Moreover, (3.5), $u = q\cdot\cos\sigma$ and $v = q\cdot\sin\sigma$ lead to

$$\sigma = \int_{\tau_0}^{\tau} \frac{-\sqrt{-p'(\tau)}\cdot\sqrt{q^2 + \tau^2 p'(\tau)}}{q^2}\,\mathrm{d}\tau + \sigma(\tau_0), \tag{3.10}$$

along the centered simple wave R_1, in which $q^2 = u^2 + v^2 = B(\tau) + D_1 + c^2$. Letting $\theta_1 := \lim\limits_{\tau\to+\infty}\sigma$, by simple calculations, we find that $\int_{\tau_0}^{+\infty} \frac{-\sqrt{-p'(\tau)}\cdot\sqrt{q^2+\tau^2p'(\tau)}}{q^2}\,\mathrm{d}\tau$ converges absolutely. That is to say, θ_1 is a bounded quantity. Then the proof is completed.

4 The Centered Waves for $\tau_0 \in [\tau_1, \tau_2)$

The centered waves at point O locally as $\tau_0 \in [\tau_1, \tau_2)$ and $\theta < \theta_2$ is constructed in this section, in which θ_2 is given by (4.11). Then there is a point $\tau_4 \in (\tau_2, +\infty)$ such that (see Figure 4)

$$p'(\tau_4) = \frac{p(\tau_4) - p(\tau_0)}{\tau_4 - \tau_0}. \tag{4.1}$$

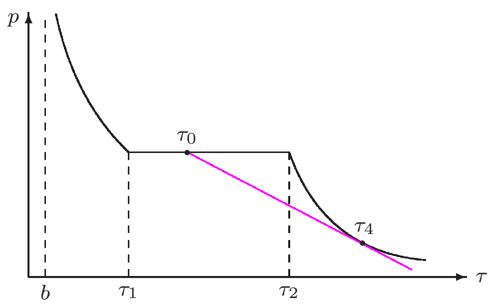


Figure 4 τ_0 on $p(\tau)$ and $\tau_0 \in [\tau_1, \tau_2)$.

Obviously, the incoming flow $(u_0, 0, c_0)$ is connected to (u_4, v_4, c_4) by a right-contact discontinuity J_1 and diffuses into the vacuum adjacent to the rarefaction wave.

Lemma 4.1 *Suppose that the incoming flow* $(u_0, 0, c_0)$ *is supersonic and satisfies* $\tau_1 \le \tau_0 < \tau_2$*. Then the system* (2.2) *exists a right-contact discontinuity* J_1 *whose wave front is* $(u_0, 0, c_0)$ *and wave back is* (u_4, v_4, c_4)*. Here,*

$$\begin{cases} u_4 = \dfrac{u_0^2 - \tau_0 (\tau_4 - \tau_0) p'(\tau_4)}{u_0}, \\ v_4 = \dfrac{(\tau_0 - \tau_4) \cdot \sqrt{-p'(\tau_4)(u_0^2 + \tau_0^2 p'(\tau_4))}}{u_0}, \\ \dfrac{d\eta}{d\xi}\Big|_{J_1} = \dfrac{\tau_0 \sqrt{-p'(\tau_4)}}{\sqrt{u_0^2 + \tau_0^2 p'(\tau_4)}}. \end{cases} \tag{4.2}$$

Proof Near the point O, the right-contact discontinuity J_1 satisfies the Rankine-Hugoniot condition

$$\frac{d\eta}{d\xi}\Big|_{J_1} [\rho u] = [\rho v], \quad \frac{d\eta}{d\xi}\Big|_{J_1} \left[\rho u^2 + p\right] = [\rho uv], \quad \frac{d\eta}{d\xi}\Big|_{J_1} [\rho uv] = \left[\rho v^2 + p\right], \tag{4.3}$$

which gives

$$\begin{cases} \dfrac{d\eta}{d\xi}\Big|_{J_1} \cdot (\rho_0 u_0 - \rho_4 u_4) = -\rho_4 v_4, \\ \dfrac{d\eta}{d\xi}\Big|_{J_1} \cdot (\rho_0 u_0^2 + p_0 - \rho_4 u_4^2 - p_4) = -\rho_4 u_4 v_4, \\ \dfrac{d\eta}{d\xi}\Big|_{J_1} \cdot (-\rho_4 u_4 v_4) = p_0 - \rho_4 v_4^2 - p_4. \end{cases} \tag{4.4}$$

By the last two equations of (4.4), one has

$$u_4 = \frac{u_0^2 - \tau_0 (\tau_4 - \tau_0) p'(\tau_4)}{u_0}. \tag{4.5}$$

Inserting (4.5) into the first equation of (4.4) yields

$$v_4 = -\frac{d\eta}{d\xi}\Big|_{J_1} \cdot \frac{\rho_0 u_0^2 - \rho_4 u_0^2 + \tau_0 \rho_4 (\tau_4 - \tau_0) p'(\tau_4)}{\rho_4 u_0}. \tag{4.6}$$

Insert (4.5) and (4.6) into the last equation of (4.4) and let $\frac{d\eta}{d\xi}\big|_{J_1} > 0$, which leads to

$$\frac{d\eta}{d\xi}\Big|_{J_1} = \frac{\tau_0 \sqrt{-p'(\tau_4)}}{\sqrt{u_0^2 + \tau_0^2 p'(\tau_4)}}. \tag{4.7}$$

Moreover, inserting (4.5) and (4.7) into the first equation of (4.4), we have

$$v_4 = \frac{(\tau_0 - \tau_4) \cdot \sqrt{-p'(\tau_4)(u_0^2 + \tau_0^2 p'(\tau_4))}}{u_0}. \tag{4.8}$$

Combining with (4.5) and (4.7)–(4.8), the proof is completed.

Remark 4.1 We take $\frac{d\eta}{d\xi}\big|_{J_1} > 0$ in proof of Lemma 4.1. In fact, according to the analysis, the wave back (u_4, v_4, c_4) of the right-contact discontinuity J_1 is the wave front of the adjacent rarefaction wave R_2. Thus, it needs to satisfy $\frac{d\eta}{d\xi}\big|_{J_1} = \tan\alpha(u_4, v_4, c_4) = \frac{u_4 v_4 + c_4\sqrt{u_4^2 + v_4^2 - c_4^2}}{u_4^2 - c_4^2}$. If we take $\frac{d\eta}{d\xi}\big|_{J_1} > 0$, that is obviously true. But if we take $\frac{d\eta}{d\xi}\big|_{J_1} < 0$, we do not get that.

Theorem 4.1 *If $\tau_1 \le \tau_0 < \tau_2$ and $\theta \le \theta_2$, then supersonic flow $(u_0, 0, c_0)$ turns the sharp corner O as follows:*

(i) *When $\theta = \theta_2$, the incoming flow turns the sharp corner O by a right-contact discontinuity J_1(see Figure 5(a)).*

(ii) *When $\theta \le \theta_3$, the composite wave made of J_1 and a complete simple wave R_2 connect the constant state $(u_0, 0, c_0)$ and vacuum (see Figure 5(b)). This type of composite wave is usually called jump-fan (JF).*

(iii) *When $\theta_3 < \theta < \theta_2$, the composite wave (JF) made of J_1 and an incomplete simple wave R_2 connect two constant states $(u_0, 0, c_0)$ and (u_5, v_5, c_5) (see Figure 5(c)).*

Here, J_1 is given by Lemma 4.1 and R_2 shows

$$R_2: \begin{cases} u = \sqrt{B(\tau) + D_2}\cos\alpha + c\sin\alpha, \\ v = \sqrt{B(\tau) + D_2}\sin\alpha - c\cos\alpha, \\ c = \displaystyle\int_{\alpha_4}^{\alpha} \frac{1}{1 - \Lambda(\tau)} \cdot \sqrt{B(\tau) + D_2}\, \mathrm{d}\alpha + c_4, \end{cases} \tag{4.9}$$

in which

$$D_2 = u_0^2 + \tau_0^2 p'(\tau_4) + \frac{A}{\delta(\tau_4 - b)^{(\delta+2)}}((\delta+1)(\delta+2)\tau_4^2 - 2(\delta+2)b\tau_4 + 2b^2) - \frac{6a}{\tau_4}. \tag{4.10}$$

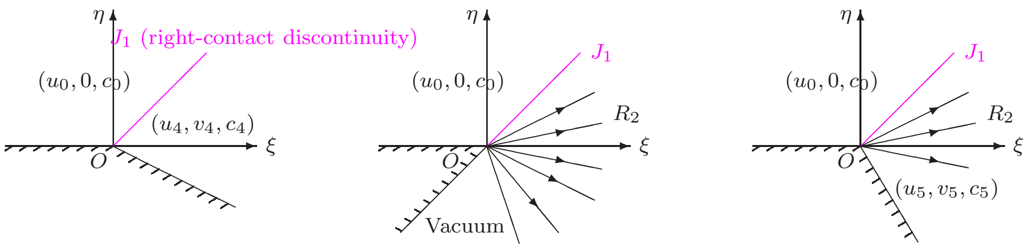
Moreover, we get

$$\theta_2 = \arcsin \frac{(\tau_0 - \tau_4)\cdot\sqrt{-p'(\tau_4)\left(u_0^2 + \tau_0^2 p'(\tau_4)\right)}}{u_0\sqrt{u_0^2 + (\tau_0 - \tau_4)^2 p'(\tau_4)}} \tag{4.11}$$

and

$$\theta_3 = \int_{\tau_4}^{+\infty} \frac{-\sqrt{-p'(\tau)}\cdot\sqrt{q^2 + \tau^2 p'(\tau)}}{q^2}\, \mathrm{d}\tau + \theta_2, \tag{4.12}$$

where $q^2 = B(\tau) + D_2 + c^2$, $B(\tau)$ and $p'(\tau_4)$ are given by (3.2) and (4.1), respectively, (u_5, v_5, c_5) is governed by (4.9)–(4.10) and $\tan\theta = \frac{v(\alpha_5)}{u(\alpha_5)}$.



(a): $\theta = \theta_2$. (b): $\theta \le \theta_3$. (c): $\theta_3 < \theta < \theta_2$.

Figure 5 $\theta \le \theta_2$ and $\tau_0 \in [\tau_1, \tau_2)$.

Proof The proof is similar to the proof of Theorem 3.1. For the centered simple wave R_2, we have

$$\begin{cases} u = g\cos\alpha + c\sin\alpha, \quad v = g\sin\alpha - c\cos\alpha, \quad g_\alpha = -c, \\ (1 - \Lambda(\tau))\, c_{\alpha\alpha} - \dfrac{\Lambda'(\tau)\cdot\Lambda(\tau)}{\sqrt{-p'(\tau)}}\cdot c_\alpha^2 + c = 0. \end{cases} \tag{4.13}$$

Combining with the initial data $g^2(\alpha_4) = u_4^2 + v_4^2 - c_4^2 = u_0^2 + \tau_0^2 \cdot p'(\tau_4)$, we get (4.9)–(4.10).

Moreover, along the centered simple wave R_2, one has

$$\sigma = \int_{\tau_4}^{\tau} \frac{-\sqrt{-p'(\tau)} \cdot \sqrt{q^2 + \tau^2 p'(\tau)}}{q^2} \, \mathrm{d}\tau + \sigma(\tau_4), \tag{4.14}$$

in which $q^2 = u^2 + v^2 = B(\tau) + D_2 + c^2$. By (4.2), we obtain $q^2(\tau_4) = u_0^2 + (\tau_0 - \tau_4)^2 \cdot p'(\tau_4)$. We define

$$\theta_2 := \sigma(\tau_4) = \arcsin \frac{v_4}{q(\tau_4)} = \arcsin \frac{(\tau_0 - \tau_4) \cdot \sqrt{-p'(\tau_4)\left(u_0^2 + \tau_0^2 p'(\tau_4)\right)}}{u_0 \sqrt{u_0^2 + (\tau_0 - \tau_4)^2 p'(\tau_4)}},$$

and obtain case (i) when $\theta = \theta_2$.

When $\theta < \theta_2$, incoming flow turns the sharp corner O by a composite wave made of a right-contact discontinuity J_1 and a simple wave R_2. We define

$$\theta_3 := \int_{\tau_4}^{+\infty} \frac{-\sqrt{-p'(\tau)} \cdot \sqrt{q^2 + \tau^2 p'(\tau)}}{q^2} \, \mathrm{d}\tau + \theta_2$$

and get case (ii) for $\theta \le \theta_3$ and case (iii) for $\theta_3 < \theta < \theta_2$.

Remark 4.2 If $\tau_0 = \tau_1$ and $p'(\tau_4) = \lim\limits_{\tau \to \tau_1^-} p'(\tau)$, then J_1 is a double-contact discontinuity.

5 The Centered Waves for $\tau_0 \in (b, \tau_1)$

In this section, we construct the centered waves at point O as $\tau_0 \in (b, \tau_1)$ and $\theta \le \min\{\theta_4, \theta_6, \theta_8\}$. Here, θ_4, θ_6, and θ_8 are given by (5.7), (5.15) and (5.20), respectively. In addition, we define $c_1^2 = -\tau_1^2 \cdot \lim\limits_{\tau \to \tau_1^-} p'(\tau)$ in Section 5.

Theorem 5.1 *When the supersonic flow* $(u_0, 0, c_0)$ *satisfies* $b < \tau_0 < \tau_1$*, the incoming flow turns the sharp corner* O *by a* C_+ *type centered simple wave* R_3 *at first, where,* R_3 *is*

$$R_3 : \begin{cases} u = \sqrt{B(\tau) + D_1} \cos\alpha + c \sin\alpha, \\ v = \sqrt{B(\tau) + D_1} \sin\alpha - c \cos\alpha, \\ c = \displaystyle\int_{\alpha_0}^{\alpha} \frac{1}{1 - \Lambda(\tau)} \cdot \sqrt{B(\tau) + D_1} \, \mathrm{d}\alpha + c_0, \end{cases} \tag{5.1}$$

in which $B(\tau)$ *and* D_1 *are given by* (3.2) *and* (3.3), *respectively.*

Proof Obviously, the incoming flow turns the sharp corner O by a C_+ type centered simple wave. It is similar to the calculation of Theorem 3.1, the expression of R_3 is (5.1).

In what follows, we make a line l through τ_1 such that the line l is tangent to $p = p(\tau)$ at point $\tau_6 > \tau_2$. Then, there are three cases as follows:

(1) The line l is also tangent to $p = p(\tau)$ at point τ_1 (see Figure 6(a)), i.e.,

$$p'(\tau_6) = \frac{p(\tau_6) - p(\tau_1)}{\tau_6 - \tau_1} = \lim_{\tau \to \tau_1^-} p'(\tau). \tag{5.2}$$

(2) The slope of line l is greater than the slope of $p = p(\tau)$ at $\tau \to \tau_1^-$ (see Figure 6(b)), i.e.,

$$p'(\tau_6) = \frac{p(\tau_6) - p(\tau_1)}{\tau_6 - \tau_1} > \lim_{\tau \to \tau_1^-} p'(\tau). \tag{5.3}$$

(3) The slope of line l is less than the slope of $p=p(\tau)$ at $\tau\to\tau_1^-$ (see Figure 6(c)), i.e.,

$$p'(\tau_6)=\frac{p(\tau_6)-p(\tau_1)}{\tau_6-\tau_1}<\lim_{\tau\to\tau_1^-}p'(\tau). \tag{5.4}$$

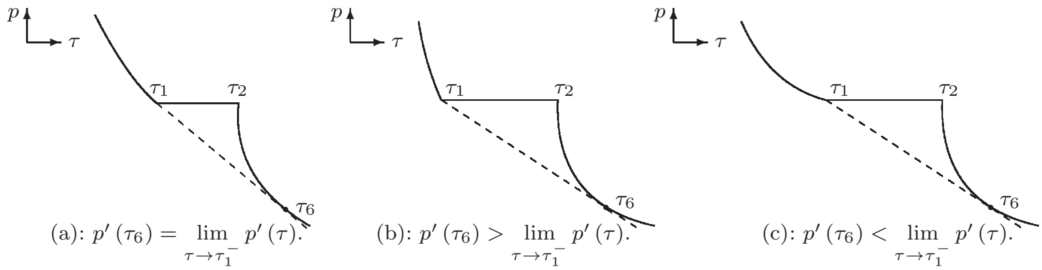


Figure 6 The relationship between l and $p=p(\tau)$.

5.1 $p'(\tau_6)=\lim_{\tau\to\tau_1^-}p'(\tau)$

Theorem 5.2 *If $b<\tau_0<\tau_1$ and $\theta\le\theta_4$, then supersonic flow $(u_0,0,c_0)$ turns the sharp corner O as follows:*

(i) As $\theta=\theta_4$, the incoming flow turns the sharp corner O by a composite wave consist of a simple wave R_3 and a double-contact discontinuity J_2 (see Figure 7(b)). This type of composite wave is usually called fan-jump (FJ).

(ii) As $\theta\le\theta_5$, the composite wave consist of R_3, J_2 and a complete simple wave R_4 connect the constant state $(u_0,0,c_0)$ and vacuum (see Figure 7(c)). This type of composite wave is usually called fan-jump-fan (FJF).

(iii) As $\theta_5<\theta<\theta_4$, the composite wave (FJF) consist of R_3, J_2 and an incomplete simple wave R_4 connect two constant states $(u_0,0,c_0)$ and (u_7,v_7,c_7) (see Figure 7(d)).

Here, both the wave back of R_3 and the wave front of J_2 are (u_1,v_1,c_1) which is governed by (5.1) and $\tau=\tau_1$. The wave back of J_2 is (u_6,v_6,c_6) satisfied (5.2) and

$$\begin{cases}\rho_6\left(\rho_1u_1\left(u_1-u_6\right)+p_1-p_6\right)^2=\rho_1v_1^2\left(\rho_1\rho_6\left(u_1-u_6\right)-\left(\rho_1-\rho_6\right)\left(p_1-p_6\right)\right),\\ v_6=\dfrac{\rho_1v_1\left(\rho_6u_6\left(u_1-u_6\right)+p_1-p_6\right)}{\rho_6\left(\rho_1u_1\left(u_1-u_6\right)+p_1-p_6\right)},\\ \left.\dfrac{\mathrm{d}\eta}{\mathrm{d}\xi}\right|_{J_2}=\dfrac{\rho_1v_1\left(u_1-u_6\right)}{\rho_1u_1\left(u_1-u_6\right)+p_1-p_6}.\end{cases} \tag{5.5}$$

Moreover, R_4 shows

$$R_4:\begin{cases}u=\sqrt{B(\tau)+D_3}\cos\alpha+c\sin\alpha,\\ v=\sqrt{B(\tau)+D_3}\sin\alpha-c\cos\alpha,\\ c=\displaystyle\int_{\alpha_6}^{\alpha}\frac{1}{1-\Lambda(\tau)}\cdot\sqrt{B(\tau)+D_3}\,\mathrm{d}\alpha+c_6,\end{cases} \tag{5.6}$$

in which $D_3=u_6^2+v_6^2-c_6^2-B(\tau_6)$ and $B(\tau)$ is given by (3.2). In addition, we have

$$\theta_4=\arcsin\frac{v_6}{\sqrt{u_6^2+v_6^2}} \tag{5.7}$$

and

$$\theta_5 = \int_{\tau_6}^{+\infty} \frac{-\sqrt{-p'(\tau)} \cdot \sqrt{q^2 + \tau^2 p'(\tau)}}{q^2} \, d\tau + \theta_4, \tag{5.8}$$

where $q^2 = B(\tau) + D_3 + c^2$, (u_7, v_7, c_7) *is given by* (5.6) *and* $\tan\theta = \frac{v(\alpha_7)}{u(\alpha_7)}$.

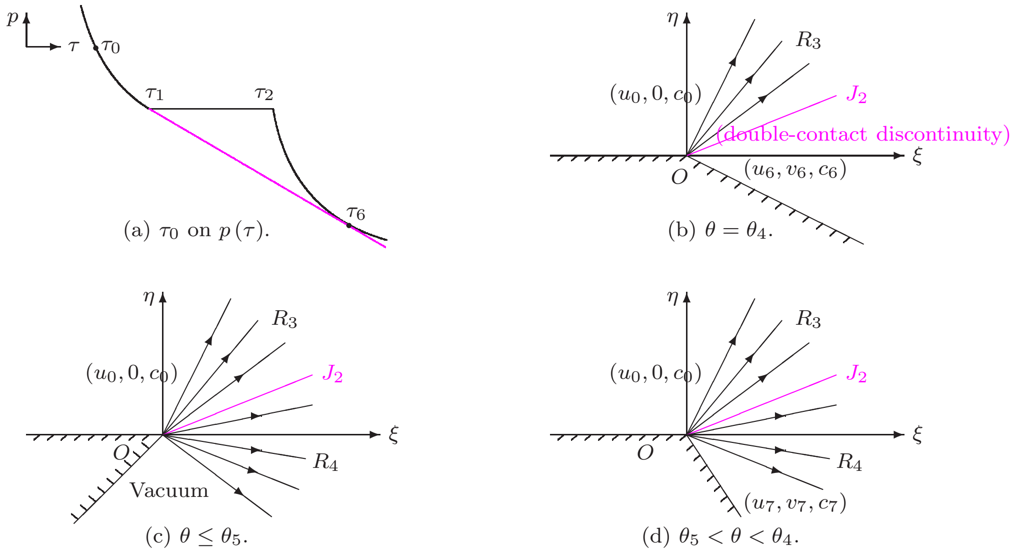


Figure 7 $\theta \le \theta_4$ and $\tau_0 \in (b, \tau_1)$ with $p'(\tau_6) = \lim\limits_{\tau \to \tau_1^-} p'(\tau)$.

Proof At first, we show that when $b < \tau_0 < \tau_1$ and $\theta \le \theta_4$, there exists a double-contact discontinuity J_2 whose wave front is (u_1, v_1, c_1) and wave back is (u_6, v_6, c_6). The double-contact discontinuity J_2 satisfies Rankine-Hugoniot condition

$$\begin{cases} \left.\dfrac{d\eta}{d\xi}\right|_{J_2} \cdot (\rho_1 u_1 - \rho_6 u_6) = \rho_1 v_1 - \rho_6 v_6, \\ \left.\dfrac{d\eta}{d\xi}\right|_{J_2} \cdot \left(\rho_1 u_1^2 + p_1 - \rho_6 u_6^2 - p_6\right) = \rho_1 u_1 v_1 - \rho_6 u_6 v_6, \\ \left.\dfrac{d\eta}{d\xi}\right|_{J_2} \cdot (\rho_1 u_1 v_1 - \rho_6 u_6 v_6) = \rho_1 v_1^2 + p_1 - \rho_6 v_6^2 - p_6. \end{cases} \tag{5.9}$$

It is easy to get (5.5). After a calculation similar to that for Theorem 3.1, we obtain the expression of R_4, i.e., (5.6). Then, like Theorem 4.1, we prove Theorem 5.2.

5.2 $p'(\tau_6) > \lim\limits_{\tau \to \tau_1^-} p'(\tau)$

Theorem 5.3 *When* $b < \tau_0 < \tau_1$ *and* $\theta \le \theta_4$, *the supersonic flow* $(u_0, 0, c_0)$ *turns the sharp corner* O *as follows*:

(i) *As* $\theta = \theta_4$, *incoming flow turns the sharp corner* O *by a simple wave* R_3, *a constant state* (u_1, v_1, c_1) *and a right-contact discontinuity* J_3 (*see Figure* 8(b)).

(ii) *As* $\theta \le \theta_5$, *the incoming flow turns the sharp corner* O *by* R_3, *a constant state* (u_1, v_1, c_1), *a composite wave* (JF) *consist of* J_3 *and a complete simple wave* R_5 *whose wave back is vacuum* (*see Figure* 8(c)).

(iii) *When $\theta_5 < \theta < \theta_4$, the incoming flow turns the sharp corner O by R_3, a constant state (u_1, v_1, c_1), a composite wave (JF) consist of J_3 and an incomplete simple wave R_5 whose wave back is (u_7, v_7, c_7) (see Figure* 8(d)*).*

Here, (u_1, v_1, c_1) is governed by (5.1) *and $\tau = \tau_1$, τ_6 is given by* (5.3)*. Both the wave back of J_3 and the wave front of R_5 are (u_6, v_6, c_6) satisfied*

$$\begin{cases} \rho_6\left(\rho_1 u_1\left(u_1-u_6\right)+p_1-p_6\right)^2=\rho_1 v_1^2\left(\rho_1 \rho_6\left(u_1-u_6\right)-\left(\rho_1-\rho_6\right)\left(p_1-p_6\right)\right), \\ v_6=\dfrac{\rho_1 v_1\left(\rho_6 u_6\left(u_1-u_6\right)+p_1-p_6\right)}{\rho_6\left(\rho_1 u_1\left(u_1-u_6\right)+p_1-p_6\right)}, \\ \left.\dfrac{\mathrm{d} \eta}{\mathrm{d} \xi}\right|_{J_3}=\dfrac{\rho_1 v_1\left(u_1-u_6\right)}{\rho_1 u_1\left(u_1-u_6\right)+p_1-p_6}. \end{cases} \tag{5.10}$$

Moreover, R_5 shows

$$R_5: \begin{cases} u=\sqrt{B(\tau)+D_3} \cos \alpha+c \sin \alpha, \\ v=\sqrt{B(\tau)+D_3} \sin \alpha-c \cos \alpha, \\ c=\displaystyle\int_{\alpha_6}^\alpha \frac{1}{1-\Lambda(\tau)} \cdot \sqrt{B(\tau)+D_3}\, \mathrm{d} \alpha+c_6, \end{cases} \tag{5.11}$$

in which $D_3 = u_6^2 + v_6^2 - c_6^2 - B(\tau_6)$ and $B(\tau)$ is given by (3.2)*. In addition, θ_4, θ_5 and (u_7, v_7, c_7) are governed by Theorem* 5.2*.*

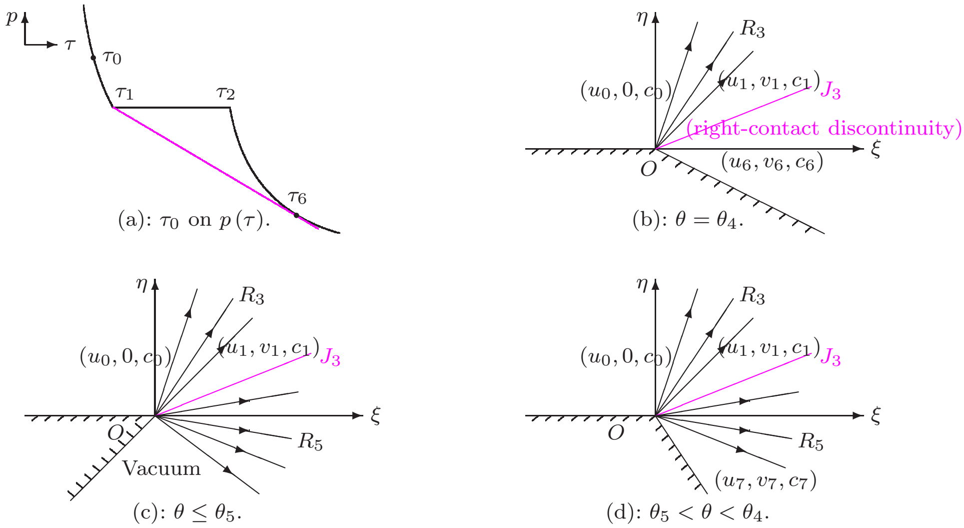


Figure 8 $\theta \le \theta_4$ and $\tau_0 \in (b, \tau_1)$ with $p'(\tau_6) > \lim\limits_{\tau\to\tau_1^-} p'(\tau)$.

Proof The proof is similar to the proof of Theorem 5.2.

5.3 $p'(\tau_6) < \lim\limits_{\tau\to\tau_1^-} p'(\tau)$

In view of (5.5), there is two points $\tau_1^* \in (b, \tau_1)$ and $\tau_2^* \in (\tau_6, +\infty)$ such that

$$p'\left(\tau_1^*\right)=\frac{p\left(\tau_1^*\right)-p\left(\tau_2^*\right)}{\tau_1^*-\tau_2^*}=p'\left(\tau_2^*\right). \tag{5.12}$$

Now, we discuss it in two cases $\tau_0 \in (b, \tau_1^*)$ and $\tau_0 \in [\tau_1^*, \tau_1)$.

5.3.1 $\tau_0 \in \left(b, \tau_1^*\right)$

Theorem 5.4 *When $b < \tau_0 < \tau_1^*$ and $\theta \le \theta_6$, then supersonic flow $(u_0, 0, c_0)$ turns the sharp corner O as follows:*

(i) *For $\theta = \theta_6$, the incoming flow turns the sharp corner O by a composite wave (FJ) made of a simple wave R_3 and a double-contact discontinuity J_4 (see Figure 9(b)).*

(ii) *For $\theta \le \theta_7$, the composite wave (FJF) made of R_3, J_4 and a complete simple wave R_6 connect the constant state $(u_0, 0, c_0)$ and vacuum (see Figure 9(c)).*

(iii) *For $\theta_7 < \theta < \theta_6$, the composite wave (FJF) made of R_3, J_4 and an incomplete simple wave R_6 connect two constant states $(u_0, 0, c_0)$ and (u_8, v_8, c_8) (see Figure 9(d)).*

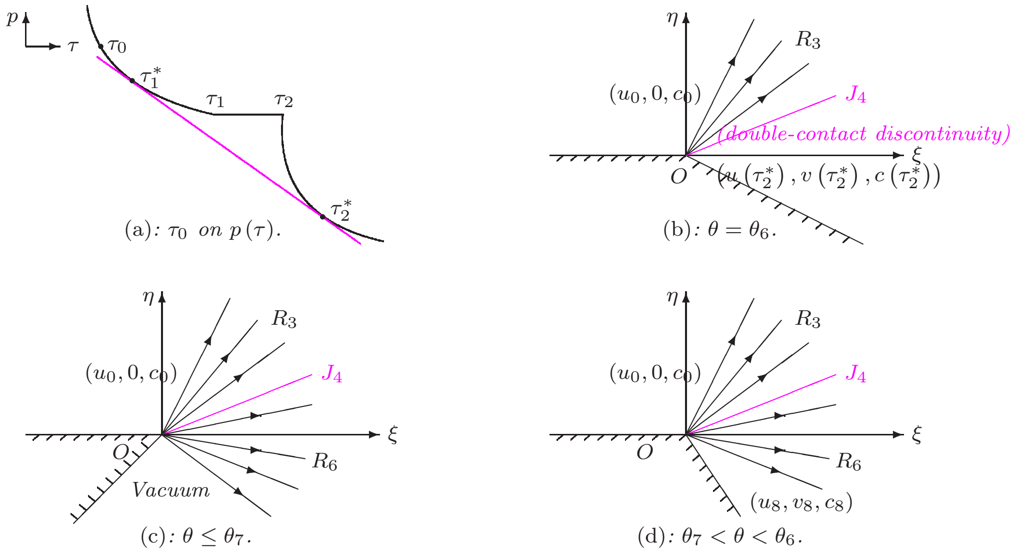


Figure 9 $\theta \le \theta_6$ and $\tau_0 \in (b, \tau_1^*)$ with $p'(\tau_6) < \lim\limits_{\tau \to \tau_1^-} p'(\tau)$.

Here, the wave back of R_3 and the wave front of J_4 are $\left(u\left(\tau_1^*\right), v\left(\tau_1^*\right), c\left(\tau_1^*\right)\right)$ governed by (5.1) and $\tau = \tau_1^*$. The wave back of J_4 is $\left(u\left(\tau_2^*\right), v\left(\tau_2^*\right), c\left(\tau_2^*\right)\right)$ which satisfies (5.12) and

$$\begin{cases} \quad \tau_1^*\left(\frac{1}{\tau_1^*}u\left(\tau_1^*\right)\cdot\left(u\left(\tau_1^*\right)-u\left(\tau_2^*\right)\right)+p\left(\tau_1^*\right)-p\left(\tau_2^*\right)\right)^2 \\ = \tau_2^* v^2\left(\tau_1^*\right)\left(\frac{1}{\tau_1^*\tau_2^*}\left(u\left(\tau_1^*\right)-u\left(\tau_2^*\right)\right)-\left(\frac{1}{\tau_1^*}-\frac{1}{\tau_2^*}\right)\left(p\left(\tau_1^*\right)-p\left(\tau_2^*\right)\right)\right), \\ v\left(\tau_2^*\right) = \dfrac{v\left(\tau_1^*\right)\left(u\left(\tau_2^*\right)\left(u\left(\tau_1^*\right)-u\left(\tau_2^*\right)\right)+\tau_2^*\left(p\left(\tau_1^*\right)-p\left(\tau_2^*\right)\right)\right)}{u\left(\tau_1^*\right)\cdot\left(u\left(\tau_1^*\right)-u\left(\tau_2^*\right)\right)+\tau_1^*\left(p\left(\tau_1^*\right)-p\left(\tau_2^*\right)\right)}, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi}\Big|_{J_4} = \dfrac{v\left(\tau_1^*\right)\cdot\left(u\left(\tau_1^*\right)-u\left(\tau_2^*\right)\right)}{u\left(\tau_1^*\right)\cdot\left(u\left(\tau_1^*\right)-u\left(\tau_2^*\right)\right)+\tau_1^*\left(p\left(\tau_1^*\right)-p\left(\tau_2^*\right)\right)}. \end{cases} \tag{5.13}$$

Moreover, R_6 is

$$R_6: \begin{cases} u = \sqrt{B(\tau)+D_4}\cos\alpha + c\sin\alpha, \\ v = \sqrt{B(\tau)+D_4}\sin\alpha - c\cos\alpha, \\ c = \displaystyle\int_{\alpha\left(\tau_2^*\right)}^{\alpha} \frac{1}{1-\Lambda(\tau)}\cdot\sqrt{B(\tau)+D_4}\,\mathrm{d}\alpha + c\left(\tau_2^*\right), \end{cases} \tag{5.14}$$

where $D_4 = u^2(\tau_2^*) + v^2(\tau_2^*) - c^2(\tau_2^*) - B(\tau_2^*)$ *and* $B(\tau)$ *is given by* (3.2). *In addition, we get*

$$\theta_6 = \arcsin \frac{v(\tau_2^*)}{\sqrt{u^2(\tau_2^*) + v^2(\tau_2^*)}} \tag{5.15}$$

and

$$\theta_7 = \int_{\tau_2^*}^{+\infty} \frac{-\sqrt{-p'(\tau)} \cdot \sqrt{q^2 + \tau^2 p'(\tau)}}{q^2} \, \mathrm{d}\tau + \theta_6, \tag{5.16}$$

in which $q^2 = B(\tau) + D_4 + c^2$, (u_8, v_8, c_8) *is given by* (5.14) *and* $\tan\theta = \frac{v(\alpha_8)}{u(\alpha_8)}$.

Proof The proof is similar to the proof of Theorem 5.2.

5.3.2 $\tau_0 \in [\tau_1^*, \tau_1)$

In this subsection, we have

$$p'(\tau_9) = \frac{p(\tau_9) - p(\tau_0)}{\tau_9 - \tau_0}. \tag{5.17}$$

Obviously, we have $\tau_9 = \tau_2^*$ as $\tau_0 = \tau_1^*$.

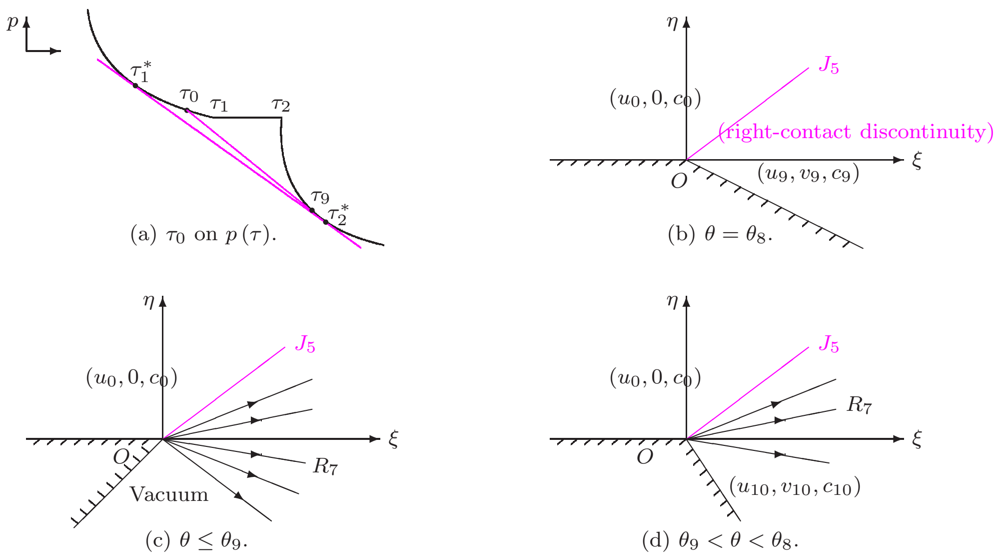


Figure 10 $\theta \le \theta_8$ and $\tau_0 \in [\tau_1^*, \tau_1)$ with $p'(\tau_6) < \lim\limits_{\tau \to \tau_1^-} p'(\tau)$.

Theorem 5.5 *If* $\tau_1^* \le \tau_0 < \tau_1$ *and* $\theta \le \theta_8$, *then supersonic flow* $(u_0, 0, c_0)$ *turns the sharp corner* O *as follows*:

(i) *For* $\theta = \theta_8$, *the incoming flow turns the sharp corner* O *by a right-contact discontinuity* J_5 *whose wave front is* $(u_0, 0, c_0)$ *and wave back is* (u_9, v_9, c_9) (*see Figure* 10(b)).

(ii) *For* $\theta \le \theta_9$, *the composite wave* (*JF*) *made of* J_5 *and a complete simple wave* R_7 *connect the constant state* $(u_0, 0, c_0)$ *and vacuum* (*see Figure* 10(c)).

(iii) *For $\theta_9 < \theta < \theta_8$, the composite wave (JF) made of J_5 and an incomplete simple wave R_7 connect two constant states $(u_0, 0, c_0)$ and (u_{10}, v_{10}, c_{10}) (see Figure 10(d)). Here, (u_9, v_9, c_9) and J_5 satisfy (5.17) and*

$$\begin{cases} u_9 = \dfrac{u_0^2 - \tau_0\left(\tau_9 - \tau_0\right)p'\left(\tau_9\right)}{u_0}, \\ v_9 = \dfrac{\left(\tau_0 - \tau_9\right)\cdot\sqrt{-p'\left(\tau_9\right)\left(u_0^2 + \tau_0^2 p'\left(\tau_9\right)\right)}}{u_0}, \\ \left.\dfrac{\mathrm{d}\eta}{\mathrm{d}\xi}\right|_{J_5} = \dfrac{\tau_0\sqrt{-p'\left(\tau_9\right)}}{\sqrt{u_0^2 + \tau_0^2 p'\left(\tau_9\right)}}. \end{cases} \tag{5.18}$$

Moreover, R_7 shows

$$R_7 : \begin{cases} u = \sqrt{B\left(\tau\right) + D_5}\cos\alpha + c\sin\alpha, \\ v = \sqrt{B\left(\tau\right) + D_5}\sin\alpha - c\cos\alpha, \\ c = \displaystyle\int_{\alpha_9}^{\alpha} \frac{1}{1 - \Lambda\left(\tau\right)}\cdot\sqrt{B\left(\tau\right) + D_5}\ \mathrm{d}\alpha + c_9, \end{cases} \tag{5.19}$$

in which $D_5 = u_9^2 + v_9^2 - c_9^2 - B\left(\tau_9\right)$ and $B\left(\tau\right)$ is given by (3.2). In addition, we get

$$\theta_8 = \arcsin\frac{\left(\tau_0 - \tau_9\right)\cdot\sqrt{-p'\left(\tau_9\right)\left(u_0^2 + \tau_0^2 p'\left(\tau_9\right)\right)}}{u_0\sqrt{u_0^2 + \left(\tau_0 - \tau_9\right)^2 p'\left(\tau_9\right)}} \tag{5.20}$$

and

$$\theta_9 = \int_{\tau_9}^{+\infty} \frac{-\sqrt{-p'\left(\tau\right)}\cdot\sqrt{q^2 + \tau^2 p'\left(\tau\right)}}{q^2}\ \mathrm{d}\tau + \theta_8, \tag{5.21}$$

where $q^2 = B\left(\tau\right) + D_5 + c^2$, (u_{10}, v_{10}, c_{10}) is given by (5.19) and $\tan\theta = \frac{v(\alpha_{10})}{u(\alpha_{10})}$. Specially, we have $(u_9, v_9, c_9) = \left(u\left(\tau_2^*\right), v\left(\tau_2^*\right), c\left(\tau_2^*\right)\right)$ as $\tau_0 = \tau_1^*$. And at this time, J_5 is a double-contact discontinuity.

Proof The proof is similar to the proofs of Lemma 4.1 and the Theorem 4.1.

To sum up, we have completed the discussions and get Theorem 1.1.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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