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Global Tangentially Analytical Solutions of the 3D Axially Symmetric Prandtl Equations*

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Abstract In this paper, the authors will prove the global existence of solutions to the three dimensional axially symmetric Prandtl boundary layer equations with small initial data, which lies in H^1 Sobolev space with respect to the normal variable and is analytical with respect to the tangential variables. The main novelty of this paper relies on careful constructions of a tangentially weighted analytic energy functional and a specially designed good unknown for the reformulated system. The result extends that of Paicu-Zhang in [Paicu, M. and Zhang, P., Global existence and the decay of solutions to the Prandtl system with small analytic data, *Arch. Ration. Mech. Anal.*, **241**(1), 2021, 403–446]. from the two dimensional case to the three dimensional axially symmetric case, but the method used here is a direct energy estimates rather than Fourier analysis techniques applied there.

Keywords Global existence, Tangentially analytical solutions, Axially symmetric, Prandtl equations
2000 MR Subject Classification 35Q35, 76D10

1 Introduction

The main purpose of this paper is to study the well-posedness of the initial-boundary value problem for the three dimensional axially symmetric Prandtl boundary layer equations in the domain $\{(t,x,y,z)\in\mathbb{R}^4:\ t>0,(x,y)\in\mathbb{R}^2,z>0\}$.

The general three dimensional Prandtl boundary layer equations read as follows,

$$\begin{cases}\partial_t\widetilde{u}+(\widetilde{u}\partial_x+\widetilde{v}\partial_y+\widetilde{w}\partial_z)\widetilde{u}+\partial_x p=\partial_z^2\widetilde{u},\\ \partial_t\widetilde{v}+(\widetilde{u}\partial_x+\widetilde{v}\partial_y+\widetilde{w}\partial_z)\widetilde{v}+\partial_y p=\partial_z^2\widetilde{v},\\ \partial_x\widetilde{u}+\partial_y\widetilde{v}+\partial_z\widetilde{w}=0,\\ (\widetilde{u},\widetilde{v},\widetilde{w})|_{z=0}=0,\qquad \lim\limits_{z\to+\infty}(\widetilde{u},\widetilde{v})=(U(t,x,y),V(t,x,y)),\end{cases}\tag{1.1}$$

where $(U(t,x,y),V(t,x,y))$ and $p(t,x,y)$ are respectively the tangential velocity fields and

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pressure of the Euler flow, satisfying

$$\begin{cases} \partial_t U + U\partial_x U + V\partial_y U + \partial_x p = 0, \\ \partial_t V + U\partial_x V + V\partial_y V + \partial_y p = 0. \end{cases} \tag{1.2}$$

Here we write $\widetilde{\boldsymbol{u}} = (\widetilde{u}, \widetilde{v}, \widetilde{w})$ and $\boldsymbol{U} = (U(t,x,y), V(t,x,y))$.

The Prandtl equations were proposed by Prandtl [26] in 1904 in order to explain the mismatch between the no slip boundary condition of the Navier-Stokes equations and the corresponding Euler equations when the vanishing viscosity limit $\nu \to 0$. Reader can see [23] and references therein for more introductions on the boundary layer theory and check [10] for some recent development on this topic.

Since Prandtl equations (1.1) have no tangential diffusion and the advection term will cause one order tangential derivative loss when we perform finite-order energy estimates, local in time well-posedness of the Prandtl equations in Sobolev spaces for general data without structure assumptions is still an open question.

For data in Sobolev spaces, under the monotonic assumption on the tangential velocity of the outflow, Oleinik and Samokhin [23] proved the local existence and uniqueness by using Crocco transform for the two dimensional Prandtl equations. Recently, in [2] (see also [22]), the second author of the present work and their collaborators introduced a nice change of variable such that the cancellation property of the bad term was discovered and the local well-posedness in Sobolev spaces was proved by direct weighted energy estimates. Ill-posedness in Sobolev spaces for the Prandtl equations around non-monotonic outflow can be found in E and Engquist [6], Gerard-Varet and Dormy [7], and Gerard-Varet and Nguyen [9]. For the three dimensional Prandtl equations, Liu, Wang and Yang [20] proved the local well-posedness of solutions in Sobolev spaces under some constraints on the flow structure in addition to the monotonic assumption. While this flow structure is violated, in [19], they showed the ill-posedness of the 3D Prandtl equations in Sobolev spaces, which indicates that the monotonicity condition on tangential velocity fields is not sufficient for the well-posedness of the three-dimensional Prandtl equations.

As for the long time behavior of the Prandtl equations in Sobolev spaces, Oleinik and Samokhin [23] showed global regular solutions existence when the tangential variable belongs to a finite interval with the amplitude being small. Xin and Zhang [29] proved the global existence of weak solutions under an additional favorable sign condition on the pressure p, and the regularity and uniqueness results are obtained in the recent paper [30]. The second author of the present paper and Zhang [31] proved that the lifespan of the solution is $\mathcal{O}(\ln \frac{1}{\varepsilon})$ if the initial datum is a small ε perturbation around the monotonic shear flow in Sobolev spaces. All the above results are discussed in the two-dimensional spaces.

For data in analytical spaces, Sammartino and Caflisch [27] established the local well-posedness in both tangential and normal variables by using the abstract Cauchy-Kowalewski theorem. The analyticity on the normal variable was removed in [21]. Later in [14], Kukavica and Vicol gave an energy-based proof of the local well-posedness result with data analytical only with respect to the tangential variable. The above results are both valid for the two and three dimensional Prandtl equations. To relax the analyticity condition is not easy. In the case where the data has a single non-degenerate critical point in the normal variable at each fixed tangential variable point, Gérard-Varet and Masmoudi [8] proved the local well-posedness of the two dimensional Prandtl equations in Gevrey class $\frac{7}{4}$ with respect to the tangential variable, which was extended to Gevrey class 2 in [15] for data that are small perturbations of a shear flow with a single non-degenerate critical point for the three-dimensional Prandtl equations. Note that this exponent 2 is optimal in view of the instability mechanism of [7]. Recently, Dietert and Gérard-varet [5] improved the well-posedness to Gevrey class 2 by removing the hypothesis on the number and order of the critical points for the two-dimensional Prandtl equations, which was extended to the three-dimensional case in [16].

For the long time existence of the Prandtl equations with analytical data, the first result appeared in Zhang and Zhang [32] where authors proved that the lifespan of the tangentially analytical solution is $\mathcal{O}(\varepsilon^{\frac{-4}{3}})$ if the datum is an ε size and the outflow is of size $\varepsilon^{\frac{5}{3}}$ for the two and three-dimensional Prandtl equations. Later, an almost global existence result was proved in [12] in two-dimensional case, where a good unknown combining the tangential component of the velocity and its derivative on the normal variable is introduced to extend the existence time. This result was extended to the three cases in [17]. Most recently, global existence of tangentially analytical solutions with small data was proved in [24] for the two dimensional Prandtl equations. This result was improved to the optimal Gevrey class 2 in [28]. As far as the authors know, there is not any results concerning on the global existence of tangentially analytical solutions for the three Prandtl equations.

The main purpose of this paper is to study the global existence of tangentially analytical solutions for the three-dimensional axially symmetric Prandtl equations. As far as the authors know, study on the axially symmetric flow has attracted more and more attention recently, such as pointwise blow-up criteria and Liouville type theorems for the axially symmetric Navier-Stokes equations in [3-4, 13, 25] and references therein. Most recently, Albritton, Brué and Colombo obtained the non-uniqueness of Leray solutions of the forced axially symmetric Navier-Stokes equations in [1]. The novelty of our present work lies in the followings: First, we will construct an energy functional which involves in a polynomial weight on the tangential variables. This carefully constructed energy is based on the special structure of the axially symmetric Prandtl equations and mainly set to overcome the order mismatch between the tangentially radial velocity u^r, and the normal velocity u^z, with respect to the distance to the symmetric axis r, when we use the divergence free condition to connect them each other. Second, the unknown acted on by the energy functional is specially designed, which is a combination of the tangentially radial velocity u^r, and its primitive one in the normal variable. This quantity has a sufficiently fast decay-in-time rate for our constructed weighted analytical energy, which ensures the positive lower bound of the analytical radius for any time. Its two dimensional originality can be traced to Paicu-Zhang [24].

2 Reformulation of the Problem and the Main Theorem

2.1 Reformulation of the equations

In the following, we give a derivation of the three dimensional axially symmetric Prandtl equations in cylindrical coordinates $(r,\, \theta,\, z)$, i.e., for $\boldsymbol{x} = (x,\, y,\, z) \in \mathbb{R}^3$,

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}.$$

A solution of (1.1) and (1.2) are said to be an axisymmetic solution, if and only if

$$\begin{aligned}
&\widetilde{\boldsymbol{u}} = \widetilde{u}^r(t,r,z)e_r + \widetilde{u}^\theta(t,r,z)e_\theta + \widetilde{u}^z(t,r,z)e_z,\\
&\boldsymbol{U} = U^r(t,r,z)e_r + U^\theta(t,r,z)e_\theta,\\
&p = p(t,r)
\end{aligned}$$

satisfy the system (1.1) and (1.2), separately, where the components of $\widetilde{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{U}}$ in cylindrical coordinates are independent of θ and the basis vectors e_r, e_θ, e_z are

$$e_r = \Big(\frac{x}{r}, \frac{y}{r}, 0\Big), \quad e_\theta = \Big(-\frac{y}{r}, \frac{x}{r}, 0\Big), \quad e_z = (0,0,1).$$

Then in cylindrical coordinates, systems (1.1) and (1.2) satisfy

$$\begin{cases}
\partial_t \widetilde{u}^r + (\widetilde{u}^r\partial_r + \widetilde{u}^z\partial_z)\widetilde{u}^r - \dfrac{(\widetilde{u}^\theta)^2}{r} + \partial_r p = \partial_z^2 \widetilde{u}^r,\\
\partial_t \widetilde{u}^\theta + (\widetilde{u}^r\partial_r + \widetilde{u}^z\partial_z)\widetilde{u}^\theta + \dfrac{\widetilde{u}^\theta\widetilde{u}^r}{r} = \partial_z^2 \widetilde{u}^\theta,\\
\dfrac{\partial_r(r\widetilde{u}^r)}{r} + \partial_z \widetilde{u}^z = 0,\\
(\widetilde{u}^r, \widetilde{u}^\theta, \widetilde{u}^z)\big|_{z=0} = 0, \quad \lim\limits_{z\to+\infty}(\widetilde{u}^r, \widetilde{u}^\theta) = (U^r, U^\theta)
\end{cases} \tag{2.1}$$

and

$$\begin{cases}
\partial_t U^r + U^r\partial_r U - \dfrac{U_\theta^2}{r} + \partial_r p = 0,\\
\partial_t U^\theta + U^r\partial_r U^\theta + \dfrac{U^r U^\theta}{r} = 0.
\end{cases}$$

Now we consider that the flow is swirl free, which means $u^\theta = U^\theta \equiv 0$. Also we consider the simple case of the outflow $U^r \equiv 0$, which indicates that $\partial_r p \equiv 0$. Then (2.1) is simplified to

$$\begin{cases}
\partial_t \widetilde{u}^r + (\widetilde{u}^r\partial_r + \widetilde{u}^z\partial_z)\widetilde{u}^r - \partial_z^2 \widetilde{u}^r = 0,\\
\dfrac{\partial_r(r\widetilde{u}^r)}{r} + \partial_z \widetilde{u}^z = 0,\\
(\widetilde{u}^r, \widetilde{u}^z)|_{z=0} = 0, \quad \lim\limits_{z\to+\infty} \widetilde{u}^r = 0.
\end{cases} \tag{2.2}$$

This simplified axially symmetric boundary layer equations (2.2) have appeared in [23, Chapter 4.1]. If the axially symmetric velocity $\widetilde{u} = \widetilde{u}^r(t,r,z)e_r + \widetilde{u}^\theta(t,r,z)e_\theta + \widetilde{u}^z(t,r,z)e_z$ is smooth and divergence free, we can deduce that

$$\widetilde{u}^r\big|_{r=0} = \widetilde{u}^\theta\big|_{r=0} \equiv 0.$$

See (reference [18]). Then there is not singularity for the quantity $\frac{\widetilde{u}^r}{r}$ at $r = 0$.

Set the new unknowns

$$(u^r, u^z) := \Big(\frac{\widetilde{u}^r}{r}, \widetilde{u}^z\Big),$$

which satisfy the following new formation of axially symmetric Prandtl boundary layer equations

$$\begin{cases} \partial_t u^r + (ru^r\partial_r + u^z\partial_z)u^r - \partial_z^2 u^r + (u^r)^2 = 0, \\ r\partial_r u^r + 2u^r + \partial_z u^z = 0, \\ (u^r, u^z)\big|_{z=0} = 0, \quad \lim\limits_{z\to+\infty} u^r = 0. \end{cases} \tag{2.3}$$

2.2 The linearly good unknown

We assume that u^r, u^z decay sufficiently fast as $z \to \infty$ and define

$$\phi(t,r,z) := -\int_z^{+\infty} u^r(t,r,\bar z)\mathrm{d}\bar z, \tag{2.4}$$

which also decays sufficiently fast at z infinity. By integrating $(2.3)_1$ on $[z, +\infty]$ with respect to z variable, we have

$$\begin{cases} \partial_t\phi - \partial_z^2\phi - u^r u^z + \displaystyle\int_z^\infty (u^r)^2 \mathrm{d}\bar z - 2\int_z^\infty \partial_z u^r u^z \mathrm{d}\bar z = 0, \\ \partial_z\phi\big|_{z=0} = 0, \quad \lim\limits_{z\to+\infty} \phi = 0, \\ \phi\big|_{t=0} = \phi_0 = \displaystyle\int_z^\infty u^r(0,r,\bar z)\mathrm{d}\bar z. \end{cases}$$

And (u^r, u^z) is obtained from ϕ as

$$u^r = \partial_z\phi, \quad u^z = -r\partial_r\phi - 2\phi.$$

Inspired by the good unknown in [24], we define

$$g := \partial_z\phi + \frac{z}{2\langle t\rangle}\phi = u^r + \frac{z}{2\langle t\rangle}\phi, \tag{2.5}$$

which satisfies

$$\begin{cases} \partial_t g + (ru^r\partial_r + u^z\partial_z)g - \partial_z^2 g + \dfrac{1}{\langle t\rangle}g + (u^r)^2 - \dfrac{1}{2\langle t\rangle}u^z\partial_z(z\phi) + \dfrac{z}{\langle t\rangle}u^r\phi \\ \quad + \dfrac{z}{2\langle t\rangle}\displaystyle\int_z^\infty (u^r)^2\mathrm{d}\bar z - \dfrac{z}{\langle t\rangle}\int_z^\infty \partial_z u^r u^z \mathrm{d}\bar z = 0, \\ g\big|_{z=0} = 0, \quad \lim\limits_{z\to+\infty} g = 0, \\ g\big|_{t=0} = g_0 = u^r(0,r,z) + \dfrac{z}{2}\phi_0(r,z). \end{cases} \tag{2.6}$$

The introduced g can control the velocity u^r and u^z nicely with a lower order time weight which leads to the possibility of closing our energy functional defined below for any $t > 0$.

2.3 Energy functional spaces and the main result

Set

$$\theta(t,z) := \exp\Big(\frac{z^2}{8\langle t\rangle}\Big).$$

For $\lambda \in \mathbb{R}$, set

$$\theta_\lambda(t,z) = \exp\Big(\frac{\lambda z^2}{8\langle t\rangle}\Big).$$

Then for any $\lambda, \mu \in \mathbb{R}$, $\theta_{\lambda+\mu} = \theta_\lambda \cdot \theta_\mu$.

Denote

$$M_n = \frac{(n+1)^4}{n!}, \quad \partial_h^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$$

and

$$\langle r\rangle = (r^2+1)^{\frac{1}{2}} = \sqrt{x^2+y^2+1}, \quad \langle t\rangle = (t+1), \quad (x,y) \in \mathbb{R}^2,\ t \geq 0.$$

For a positive time-dependent function $\tau := \tau(t)$, we introduce the Sobolev weighted semi-norms

$$\begin{aligned}
X_n &= X_n(g,\tau) = \sum_{|\alpha|=n} \|\theta\langle r\rangle^n \partial_h^\alpha g\|_{L^2}\tau^n M_n,\ n \in \mathbb{N};\\
D_n &= D_n(g,\tau) = \sum_{|\alpha|=n} \|\theta\langle r\rangle^n \partial_h^\alpha \partial_z g\|_{L^2}\tau^n M_n = X_n(\partial_z g, \tau), \quad n \in \mathbb{N};\\
Y_n &= Y_n(g,\tau) = \sum_{|\alpha|=n} \|\theta\langle r\rangle^n \partial_h^\alpha g\|_{L^2}\tau^{n-1} n M_n,\ n \in (\mathbb{N}/\{0\}).
\end{aligned} \tag{2.7}$$

We consider the following functional space that is real-analytic in $\boldsymbol{x}_h = (x,y)$ and lies in a weighted L^2 space with respect to z,

$$\mathcal{X}_\tau = \{\forall \alpha \in \mathbb{N}^2, \langle r\rangle^{|\alpha|}\partial_h^\alpha g(t,r,z) \in L^2(\mathbb{R}^3_+; \theta^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z) : \|g\|_{\mathcal{X}_\tau} < \infty\},$$

where

$$\|g\|_{\mathcal{X}_\tau} = \sum_{n \geqq 0} X_n(g,\tau).$$

Remark 2.1 In the first equation of (2.7), there is a weight $\langle r\rangle^n$ for the tangential nth order derivative, which is set to match and control the term $r\partial_r g$ appeared in (2.6).

We also define the semi-norm

$$\|g\|_{\mathcal{Y}_\tau} = \sum_{n \geq 1} Y_n(g,\tau),$$

which encodes the one-derivative gain in the analytic estimates. Note that for $\beta > 1$, we have

$$\|g\|_{\mathcal{Y}_\tau} \leq \tau^{-1}\|g\|_{\mathcal{X}_{\beta\tau}} \sup_{n \geq 1}\big(n\beta^{-n}\big) \leq C_\beta \tau^{-1}\|g\|_{\mathcal{X}_{\beta\tau}}.$$

The gain of a z derivative shall be encoded in the dissipative semi-norm

$$\|g\|_{\mathcal{D}_\tau} = \sum_{n \geqq 0} D_n(g,\tau) = \|\partial_z g\|_{\mathcal{X}_\tau}.$$

Having introduced the functional spaces in our paper and before presenting the main results, we give a definition of solution to the reformulated Prandtl equation (2.6).

Definition 2.1 (Classical in tangential variables and weak in normal variable) *For a fixed time $t > 0$, let $\mathcal{H}$ be the closure of the set of functions*

$$\big\{f(t,x,y,z) \in C_c^\infty(\mathbb{R}^2 \times [0,+\infty)) : \ f|_{z=0} = 0\big\}$$

under the space norm

$$\|f(t)\|_{\mathcal{H}}^2 := \sum_{|\alpha|\le 3} \int_{\mathbb{R}^3_+} |\partial_h^\alpha f(t,x,y,z)|^2 \exp\Big(\frac{z^2}{4\langle t\rangle}\Big)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

For $T > 0$, we say that a function g is classical in x, y and weak in z solution of (2.6) *if*

$$\|g(t)\|_{\mathcal{H}} \in L^\infty([0,T)) \quad \text{and} \quad \|\partial_z g(t)\|_{\mathcal{H}} \in L^2([0,T)),$$

and (2.6) *holds when tested by $C_c^\infty([0,T) \times \mathbb{R}^2 \times [0,+\infty))$.*

Theorem 2.1 *Let $g_0(r,z)$ be tangentially analytical with radius of analyticity being $\tau_0 > 0$. Then, for any $0 < \delta \le \frac{1}{4}$, there exists a ε_0, depending only on δ and τ_0, such that for any $\varepsilon \le \varepsilon_0$, if $\int_0^\infty u^r(0,r,z)\mathrm{d}z = 0$ and*

$$\|g_0\|_{X_{\tau_0}} \le \varepsilon,$$

then (2.6) *has a globally in-time solution g, which is tangentially analytical with the radius of analyticity $\tau(t) \ge \frac{1}{2}\tau_0$ and for any $t > 0$, it satisfies*

$$\begin{aligned}
&\langle t\rangle^{\frac{5}{4}-\delta}\|g(t)\|_{\mathcal{X}_{\tau(t)}} + \frac{\delta}{12}\int_0^t (\langle s\rangle^{\frac{1}{4}-\delta}\|g(s)\|_{\mathcal{X}_{\tau(s)}} + \langle s\rangle^{\frac{3}{4}-\delta}\|g(s)\|_{\mathcal{D}_{\tau(s)}})\mathrm{d}s \\
&+ C_0\int_0^t \frac{\langle s\rangle^{\frac{5}{4}-\delta}}{\tau^2(s)}(\|g(s)\|_{\mathcal{X}_{\tau(s)}} + \langle s\rangle^{\frac{1}{4}}\|g(s)\|_{\mathcal{D}_{\tau(s)}})\|g(s)\|_{\mathcal{Y}_{\tau(s)}}\mathrm{d}s \le \|g_0\|_{\mathcal{X}_{\tau_0}} \le \varepsilon_0. \qquad (2.8)
\end{aligned}$$

Remark 2.2 It follows from the estimates in Lemma (3.1) and Lemma (3.2) below that bounds on g, $\partial_z g$ in (2.8) in $\mathcal{X}_\tau$ imply similar estimates on u^r and u^z. So global existence and uniqueness of tangentially analytical solutions in Theorem 2.1 indicates global existence and uniqueness of tangentially analytical solutions for the original system (2.3) and (2.2). The proof of Theorem 2.1 mainly consists of a priori estimates (cf. Section 3) and the local well-posedness. Since the local existence and uniqueness of the tangentially analytical solutions has already shown in many references, e. g. [14, 32], here we only present the a priori estimate (2.8).

Remark 2.3 In the model (2.2), we only consider the case that the outflow $U^r \equiv 0$. Actually the proof can be also applied to the case that $U^r = r\varepsilon f(t)$, where $\varepsilon > 0$ is sufficiently small and $f(t)$ decays sufficiently fast as $t \to +\infty$. The computation will be more elaborated and complicated. For simplicity and convenience of presenting the main idea, we omit this extension and leave it to the interested reader.

Remark 2.4 Here we only consider the the axially symmetric Prandtl equation, and extensions of Theorem 2.1 to the axially symmetric MHD boundary layer system and in the tangential Gevrey spaces will be considered in our future work.

For a function $f(t,x,y,z)$ and $1\le p,q\le+\infty$, define

$$\|f(t)\|_{L^p_hL^q_z}:=\Big(\int_0^{+\infty}\Big(\int_{\mathbb{R}^2}|f(t,x,y,z)|^p\mathrm{d}x\mathrm{d}y\Big)^{\frac{q}{p}}\mathrm{d}z\Big)^{\frac{1}{q}}.$$

If $p=q$, we simply write it as $\|f\|_{L^p}$ and besides, if $p=q=2$, we will simply denote it as $\|f\|$. Throughout the paper, $C_{a,b,c,\cdots}$ denotes a positive constant depending on a, b, $c,\cdots$ which may be different from line to line. We also apply $A\lesssim_{a,b,c,\cdots}B$ to denote $A\le C_{a,b,c,\cdots}B$. For a two dimensional multi-index $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$, we write $\partial_h^\alpha=\partial_x^{\alpha_1}\partial_y^{\alpha_2}$ and $\partial_h^k=\{\partial_h^\alpha;\ |\alpha|=k\}$. For a norm $\|\cdots\|$, we use $\|(f,g,\cdots)\|$ to denote $\|f\|+\|g\|+\cdots$.

3 A Priori Estimates and Proof of the Main Theorem

First, we state a simple version of the local well-posedness result on the three dimensional Prandtl equations in tangentially analytical spaces (see [14, Theorem 3.1, Remark 3.3]).

Theorem 3.1 (see [14, Theorem 3.1] *with the outflow being zero in three dimensional spaces*) *Fix the constant $\nu>\frac{1}{2}$, denote $\langle z\rangle:=1+z$. For a function $f(t,x,y,z)$ and $\tau(t)>0$, define*

$$\|f(t)\|^2_{\widetilde{\mathcal{X}}_{\tau(t)}}:=\sum_{n\ge0}\sum_{|\alpha|=n}\|\langle z\rangle^\nu\partial^\alpha_{x,y}f(t,x,y,z)\|^2_{L^2(\mathbb{R}^3_+)}\tau^{2n}(t)M_n^2.$$

Then, for $\tau_0>0$, if the solution to (1.1) *with the outflow $\boldsymbol{U}$ being zero satisfies*

$$(\widetilde{u},\widetilde{v})|_{t=0}:=(\widetilde{u}_0,\widetilde{v}_0)\in\widetilde{\mathcal{X}}_{\tau_0},$$

then there exists a $T_*=T_*(\nu,\tau_0,\|(\widetilde{u}_0,\widetilde{v}_0)\|_{\widetilde{\mathcal{X}}_{\tau_0}})>0$, such that the three dimensional Prandtl equations (1.1) *have a unique real-analytical solution in $[0,T_*)$ satisfying for any $t\in[0,T_*)$, $\tau(t)>0$ and*

$$\|(\widetilde{u},\widetilde{v})(t)\|_{\widetilde{\mathcal{X}}_{\tau(t)}}<+\infty.$$

Based on the above local well-posedness result of the three dimensional Prandtl equations, The proof of Theorem 2.1 is simplified to continuity argument and the following a prior estimate, stated as Proposition 3.1.

Proposition 3.1 *For $T>0$, let g be the tangentially analytical solution of* (2.6) *and $g_0(r,z)$ be tangentially analytical with radius of analyticity being $\tau_0>0$. Then, for any $0<\delta\le\frac{1}{4}$, there exists a ε_0, depending only on δ and τ_0 such that for any $\varepsilon\le\varepsilon_0$, if $\int_0^\infty u^r(0,r,z)\mathrm{d}z=0$ and*

$$\|g_0\|_{\mathcal{X}_{\tau_0}}\le\varepsilon,$$

then for any $0<t<T$, the solution g satisfies

$$\begin{aligned}&\langle t\rangle^{\frac{5}{4}-\delta}\|g(t)\|_{\mathcal{X}_{\tau(t)}}+\frac{\delta}{12}\int_0^t(\langle s\rangle^{\frac{1}{4}-\delta}\|g(s)\|_{\mathcal{X}_{\tau(s)}}+\langle s\rangle^{\frac{3}{4}-\delta}\|g(s)\|_{\mathcal{D}_{\tau(s)}})\mathrm{d}s\\&+C_0\int_0^t\frac{\langle s\rangle^{\frac{5}{4}-\delta}}{\tau^2(s)}(\|g(s)\|_{\mathcal{X}_{\tau(s)}}+\langle s\rangle^{\frac{1}{4}}\|g(s)\|_{\mathcal{D}_{\tau(s)}})\|g(s)\|_{\mathcal{Y}_{\tau(s)}}\mathrm{d}s\le\|g_0\|_{X_{\tau_0}}\le\varepsilon_0,\end{aligned}$$

and the tangentially analytical radius $\tau(t)\ge\frac{1}{2}\tau_0$.

Before proving Proposition 3.1, we give two lemmas which concern on bounds of u^r, u^z, ϕ in terms of g.

3.1 Bounds of u^r, u^z, ϕ in terms of g

Lemma 3.1 *Let* (u^r, u^z) *be the solution of* (2.3), ϕ *and* g *be respectively the functions defined in* (2.4) *and* (2.5). *For any* $n \in \mathbb{N}$, $|\alpha| = n$ *and* $0 \le \lambda < 1$, *we have*

$$|\theta_\lambda \langle r\rangle^n \partial_h^\alpha \phi| \lesssim_\lambda \theta_{\lambda-1} \langle t\rangle^{\frac{1}{4}} \|\theta \langle r\rangle^n \partial_h^\alpha g\|_{L_z^2}, \tag{3.1}$$

$$|\theta_\lambda \langle r\rangle^n \partial_h^\alpha u^r| \lesssim_\lambda |\theta_\lambda \langle r\rangle^n \partial_h^\alpha g| + \frac{z}{\langle t\rangle^{\frac{3}{4}}} \theta_{\lambda-1} \|\theta \langle r\rangle^n \partial_h^\alpha g\|_{L_z^2} \tag{3.2}$$

and

$$\begin{aligned}|\theta_\lambda \langle r\rangle^n \partial_h^\alpha \partial_z u^r| &\lesssim_\lambda \frac{z}{\langle t\rangle} |\theta_\lambda \langle r\rangle^n \partial_h^\alpha g(z)| + |\theta_\lambda \langle r\rangle^n \partial_h^\alpha \partial_z g| \\ &\quad + \Big(\frac{1}{\langle t\rangle} + \frac{z^2}{\langle t\rangle^2}\Big) \theta_{\lambda-1} \langle t\rangle^{\frac{1}{4}} \|\theta \langle r\rangle^n \partial_h^\alpha g\|_{L_z^2}.\end{aligned} \tag{3.3}$$

Proof We only show the proof of that $n = 0$ since the case $n > 0$ follows the same line. From the second equation of (2.3), we have

$$r\partial_r \int_0^\infty u_r \mathrm{d}z + 2 \int_0^\infty u_r \mathrm{d}z = -\int_0^\infty \partial_z u_z \mathrm{d}z = u_z(t, r, 0) = 0,$$

which indicates that

$$r \int_0^\infty u^r \mathrm{d}z = 0.$$

Since when $r > 0$, the above equality implies that $\int_0^\infty u^r \mathrm{d}z = 0$ for $r > 0$, then continuity of u^r indicates that

$$\int_0^\infty u^r \mathrm{d}z \equiv 0.$$

By the definitions of ϕ and g in (2.4)–(2.5), we have

$$\begin{cases} \partial_z \phi + \dfrac{z}{2\langle t\rangle} \phi = g, \\ \phi\big|_{z=0} = 0. \end{cases} \tag{3.4}$$

Solving the ODE, we get

$$\phi(t, r, z) = \exp\Big(-\frac{z^2}{4\langle t\rangle}\Big) \int_0^z g(t, r, \bar{z}) \exp\Big(\frac{\bar{z}^2}{4\langle t\rangle}\Big) \mathrm{d}\bar{z}. \tag{3.5}$$

For any $0 \le \lambda < 1$, by multiplying the above equality with θ_λ, we have

$$\theta_\lambda \phi = \theta_{\lambda-1}(z) \int_0^z \theta(\bar{z}) g(\bar{z}) \exp\Big(\frac{1}{8\langle t\rangle}(\bar{z}^2 - z^2)\Big) \mathrm{d}\bar{z}. \tag{3.6}$$

Differentiating (3.5) on z gives that

$$u^r(t,r,z)=\partial_z\phi=-\frac{z}{2\langle t\rangle}\exp\Big(-\frac{z^2}{4\langle t\rangle}\Big)\int_0^z g(t,r,\bar z)\exp\Big(\frac{\bar z^2}{4\langle t\rangle}\Big)\mathrm{d}\bar z+g. \tag{3.7}$$

Multiplying (3.7) by θ_λ gives that

$$\theta_\lambda u^r=\theta_\lambda g-\frac{z}{2\langle t\rangle}\theta_{\lambda-1}(z)\int_0^z\theta(\bar z)g(\bar z)\exp\Big(\frac{1}{8\langle t\rangle}(\bar z^2-z^2)\Big)\mathrm{d}\bar z. \tag{3.8}$$

Differentiating (3.7) on z and multiplying the resulted equation by θ_λ give that

$$\begin{aligned}\theta_\lambda\partial_z u^r=&\,\theta_\lambda\partial_z g-\frac{z}{2\langle t\rangle}\theta_\lambda g\\&-\Big(\frac{1}{2\langle t\rangle}-\frac{z^2}{4\langle t\rangle^2}\Big)\theta_{\lambda-1}\int_0^z\theta(\bar z)g(\bar z)\exp\Big(\frac{1}{8\langle t\rangle}(\bar z^2-z^2)\Big)\mathrm{d}\bar z.\end{aligned} \tag{3.9}$$

Using the fact that for any $\beta\ge 0$,

$$\sup_{\zeta\geqq 0}\zeta^\beta e^{-\zeta^2}\le C_\beta,$$

we have

$$\Big|\Big(\frac{z}{\sqrt{\langle t\rangle}}\Big)^\beta\theta_{\lambda-1}\Big|\le C_{\lambda,\beta}.$$

Moreover, by considering $0\le\zeta\le 1$ and $\zeta>1$, it is not hard to check that

$$e^{-\zeta^2}\int_0^\zeta e^{\bar\zeta^2}d\bar\zeta\le\frac{2}{1+\zeta}.$$

Then a change of variable indicates that

$$\int_0^z\exp\Big(\frac{1}{4\langle t\rangle}(\bar z^2-z^2)\Big)\mathrm{d}\bar z\le\frac{C}{1+\zeta}\sqrt{\langle t\rangle}. \tag{3.10}$$

Here $\zeta=\frac{z}{\sqrt{\langle t\rangle}}$. In (3.6), by using Hölder inequality on z, we have

$$\begin{aligned}|\theta_\lambda\phi|\ge&\,\theta_{\lambda-1}(z)\|\theta g\|_{L^2_z}\Big(\int_0^z\exp\Big(\frac{1}{4\langle t\rangle}(\bar z^2-z^2)\Big)\mathrm{d}\bar z\Big)^{\frac12}\\\lesssim&\,\theta_{\lambda-1}\|\theta g\|_{L^2_z}\langle t\rangle^{\frac14}(1+\zeta)^{-\frac14}\\\lesssim&\,\theta_{\lambda-1}\|\theta g\|_{L^2_z}\langle t\rangle^{\frac14},\end{aligned} \tag{3.11}$$

which is (3.1) for $n=0$.

In (3.8), by using Hölder inequality and (3.10), we have

$$\begin{aligned}|\theta_\lambda u^r|\ge&\,|\theta_\lambda g|+\frac{z}{\langle t\rangle}\theta_{\lambda-1}\|\theta g\|_{L^2_z}\Big(\int_0^z\exp\Big(\frac{1}{4\langle t\rangle}(\bar z^2-z^2)\Big)\mathrm{d}\bar z\Big)^{\frac12}\\\lesssim&\,|\theta_\lambda g|+\frac{z}{\langle t\rangle^{\frac34}}\theta_{\lambda-1}\|\theta g\|_{L^2_z},\end{aligned}$$

which is (3.2) for $n=0$.

In (3.9), by using Hölder inequality and (3.10), we have

$$\begin{aligned}|\theta_\lambda\partial_z u^r| &\lesssim_\lambda \frac{z}{\langle t\rangle}|\theta_\lambda g| + |\theta_\lambda\partial_z g| \\ &\quad + \Big(\frac{1}{\langle t\rangle} + \frac{z^2}{\langle t\rangle^2}\Big)\theta_{\lambda-1}\langle t\rangle^{\frac14}\|\theta g\|_{L^2_z}(1+\zeta)^{-\frac12} \\ &\lesssim_\lambda \frac{z}{\langle t\rangle}|\theta_\lambda g| + |\theta_\lambda\partial_z g| + \Big(\frac{1}{\langle t\rangle} + \frac{z^2}{\langle t\rangle^2}\Big)\theta_{\lambda-1}\langle t\rangle^{\frac14}\|\theta g\|_{L^2_z},\end{aligned}$$

which is (3.3) for $n=0$.

By applying $\langle r\rangle^n\partial_h^\alpha$ to (3.6) and (3.8)–(3.9), the above derivation from (3.11) also stands by replacing ϕ, u^r, $\partial_z u^r$ and g by $\langle r\rangle^n\partial_h^\alpha\phi$, $\langle r\rangle^n\partial_h^\alpha u^r$, $\langle r\rangle^n\partial_h^\alpha\partial_z u^r$ and $\langle r\rangle^n\partial_h^\alpha g$, respectively.

Based on the rough estimates in Lemma 3.1, we have the following much more subtle integration controls of u^r, u^z and ϕ in terms of the weighted L^2 norm of g.

Lemma 3.2 (Bounds of u^r, u^z, ϕ in terms of g) *For any $n\in\mathbb{N}$, $|\alpha|=n$ and $0\le\lambda<1$, we have the following estimates*

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha\phi\|_{L^2_z} \lesssim_\lambda \langle t\rangle^{\frac12}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2_z}, \tag{3.12}$$

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L^2} \lesssim_\lambda \|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}, \tag{3.13}$$

$$\sum_{|\alpha|=n}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L^\infty_h L^2_z} \lesssim_\lambda (n+1)^2\sum_{|\alpha|=n}^{n+2}\|\theta\langle r\rangle^{|\alpha|}\partial_h^\alpha g\|_{L^2}, \tag{3.14}$$

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L^2_h L^\infty_z} \lesssim_\lambda \|\theta\langle r\rangle^n\partial_h^\alpha(g,\partial_z g)\|_{L^2}, \tag{3.15}$$

$$\sum_{|\alpha|=n}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L^\infty_h L^\infty_z} \lesssim_\lambda (n+1)^2\sum_{|\alpha|=n}^{n+2}\|\theta\langle r\rangle^{|\alpha|}\partial_h^\alpha(g,\partial_z g)\|_{L^2}, \tag{3.16}$$

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^z\|_{L^2_h L^\infty_z} \lesssim_\lambda \langle t\rangle^{\frac14}\|\theta\langle r\rangle^n\partial_h^\alpha(r\partial_r g, g)\|_{L^2}, \tag{3.17}$$

$$\sum_{|\alpha|=n}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^z\|_{L^\infty_h L^\infty_z} \lesssim_\lambda (n+1)^2\langle t\rangle^{\frac14}\sum_{|\alpha|=n}^{n+2}\|\theta\langle r\rangle^{|\alpha|}\partial_h^\alpha(r\partial_r g, g)\|_{L^2}, \tag{3.18}$$

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha\partial_z u^r\|_{L^2} \lesssim_\lambda \langle t\rangle^{-\frac12}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2} + \|\theta\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L^2}, \tag{3.19}$$

$$\begin{aligned}&\sum_{|\alpha|=n}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha\partial_z u^r\|_{L^\infty_h L^2_z} \\ &\lesssim_\lambda (n+1)^2\sum_{|\alpha|=n}^{n+2}(\langle t\rangle^{-\frac12}\|\theta\langle r\rangle^{|\alpha|}\partial_h^\alpha g\|_{L^2} + \|\theta\langle r\rangle^{|\alpha|}\partial_h^\alpha\partial_z g\|_{L^2}).\end{aligned} \tag{3.20}$$

Proof From (3.1), we have

$$\begin{aligned}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha\phi\|_{L^2_z} &\lesssim_\lambda \|\theta_{\lambda-1}\|_{L^2_z}\langle t\rangle^{\frac14}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2_z} \\ &\lesssim_\lambda \langle t\rangle^{\frac12}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2_z},\end{aligned}$$

where we have used the fact that when $\lambda-1<0$,

$$\|\theta_{\lambda-1}\|_{L^2_z} \lesssim_\lambda \langle t\rangle^{\frac14}.$$

Hence, we have obtained (3.12).

From (3.2), we have

$$\begin{aligned}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L^2} &\lesssim_\lambda \|\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\|_{L^2}+\|\frac{z}{\langle t\rangle^{\frac34}}\theta_{\lambda-1}\|_{L_z^2}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}\\ &\lesssim_\lambda \|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2},\end{aligned}\tag{3.21}$$

which is (3.13).

Using the two-dimensional Sobolev inequality

$$\|f\|_{L_h^\infty}\lesssim\|f\|_{L_h^2}+\|\partial_h^2 f\|_{L_h^2},$$

we have

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L_h^\infty L_z^2}\lesssim\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L_h^2L_z^2}+\|\theta_\lambda\partial_h^2\left[\langle r\rangle^n\partial_h^\alpha u^r\right]\|_{L_h^2L_z^2}.\tag{3.22}$$

It is easy to show that for $n\in\mathbb{N}/\{0\}$,

$$\left|\partial_h^2\left[\langle r\rangle^n\partial_h^\alpha u^r\right]\right|\lesssim\frac{(n+1)^2}{\langle r\rangle^2}\sum_{|\gamma|=0}^{2}|\langle r\rangle^{n+|\gamma|}\partial_h^{\alpha+\gamma}u^r|.\tag{3.23}$$

Inserting (3.23) into (3.22) and summing over $|\alpha|=n$, we have

$$\sum_{|\alpha|=n}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L_h^\infty L_z^2}\lesssim(n+1)^2\sum_{|\alpha|=n}^{n+2}\|\theta_\lambda\langle r\rangle^{|\alpha|}\partial_h^\alpha u^r\|_{L^2}.\tag{3.24}$$

Inserting (3.21) into (3.24), we obtain (3.14).

Also from (3.2), we have

$$\begin{aligned}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L_z^\infty} &\lesssim \|\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\|_{L_z^\infty}+\|\frac{z}{\langle t\rangle^{\frac34}}\theta_{\lambda-1}\|_{L_z^\infty}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}\\ &\lesssim_\lambda \|\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\|_{L_z^\infty}+\langle t\rangle^{-\frac14}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}.\end{aligned}\tag{3.25}$$

Using one-dimensional Sobolev embedding

$$\begin{aligned}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\|_{L_z^\infty} &\lesssim \|\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\|_{L_z^2}^{\frac12}\|\partial_z(\theta_\lambda\langle r\rangle^n\partial_h^\alpha g)\|_{L_z^2}^{\frac12}\\ &\lesssim \|\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\|_{L_z^2}^{\frac12}\left[\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L_z^2}+\left\|\frac{z}{\langle t\rangle}\theta_\lambda\langle r\rangle^n\partial_h^\alpha g\right\|_{L_z^2}\right]^{\frac12}\\ &\lesssim_\lambda \|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L_z^2}+\|\theta\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L_z^2}.\end{aligned}$$

Inserting the above inequality into (3.25), we can have

$$\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L_z^\infty}\lesssim_\lambda\|\theta\langle r\rangle^n\partial_h^\alpha(g,\partial_z g)\|_{L_z^2}.\tag{3.26}$$

The bound (3.15) follows from taking L^2 norms in x,y variables of the above inequality (3.26).

Similar to (3.24), we can have

$$\sum_{|\alpha|=n}\|\theta_\lambda\langle r\rangle^n\partial_h^\alpha u^r\|_{L_h^\infty L_z^\infty}\lesssim(n+1)^2\sum_{|\alpha|=n}^{n+2}\left\|\theta_\lambda\langle r\rangle^{|\alpha|}\partial_h^\alpha u^r\right\|_{L_h^2L_z^\infty}.\tag{3.27}$$

Integrating (3.26) on the tangential variables and inserting the resulted inequality into (3.27), we can get (3.16).

From the incompressible condition, i.e., the second equation of (2.3), we have

$$u^z(z) = -\int_z^\infty \partial_z u^z(\bar{z})\mathrm{d}\bar{z} = \int_z^\infty (r\partial_r u^r + 2u^r)(\bar{z})\mathrm{d}\bar{z},$$

then we can get

$$\begin{aligned}\|\theta_\lambda \langle r\rangle^n \partial_h^\alpha u^z\|_{L_z^\infty} &\le \|\theta_\lambda \langle r\rangle^n \partial_h^\alpha (r\partial_r u^r + 2u^r)\|_{L_z^1} \\ &\lesssim_\lambda \|\theta \langle r\rangle^n \partial_h^\alpha (r\partial_r u^r + 2u^r)\|_{L_z^2} \|\theta_{\lambda-1}\|_{L_z^2} \\ &\lesssim_\lambda \langle t\rangle^{\frac14} \|\theta \langle r\rangle^n \partial_h^\alpha (r\partial_r u^r + 2u^r)\|_{L_z^2}.\end{aligned} \tag{3.28}$$

From (3.2), we have

$$\begin{aligned}&\|\theta_\lambda \langle r\rangle^n \partial_h^\alpha (r\partial_r u^r + 2u^r)\|_{L_z^2} \\ &\lesssim_\lambda \|\theta_\lambda \langle r\rangle^n \partial_h^\alpha (r\partial_r + 2)g\|_{L_z^2} + \Big\|\frac{z}{\langle t\rangle^{\frac34}}\theta_{\lambda-1}\Big\|_{L_z^2} \|\theta\langle r\rangle^n \partial_h^\alpha (r\partial_r + 2)g\|_{L_z^2} \\ &\lesssim_\lambda \|\theta\langle r\rangle^n \partial_h^\alpha (r\partial_r + 2)g\|_{L_z^2}.\end{aligned}$$

Similar to (3.24), we can have

$$\sum_{|\alpha|=n} \|\theta_\lambda \langle r\rangle^n \partial_h^\alpha u^z\|_{L_h^\infty L_z^\infty} \lesssim_\lambda (n+1)^2 \sum_{|\alpha|=n}^{n+2} \|\theta_\lambda \langle r\rangle^{|\alpha|} \partial_h^\alpha u^z\|_{L_h^2 L_z^\infty}. \tag{3.29}$$

Inserting (3.17) into (3.29), we can get (3.18).

Inserting the above inequality into (3.28) and then integrating the resulted equation in the tangential variables imply that

$$\|\theta_\lambda \langle r\rangle^n \partial_h^\alpha u^z\|_{L_h^2 L_z^\infty} \lesssim_\lambda \langle t\rangle^{\frac14} \|\theta\langle r\rangle^n \partial_h^\alpha (r\partial_r + 2)g\|_{L^2},$$

which corresponds to (3.17).

Similar to (3.24), using the estimate (3.17), we can get (3.18).

From (3.3), we can get

$$\begin{aligned}\|\theta_\lambda \langle r\rangle^n \partial_h^\alpha \partial_z u^r\|_{L^2} &\lesssim_\lambda \|\frac{z}{\langle t\rangle}\theta_{\lambda-1}\|_{L_z^\infty} \|\theta\langle r\rangle^n \partial_h^\alpha g\|_{L^2} + \|\theta\langle r\rangle^n \partial_h^\alpha \partial_z g\|_{L^2} \\ &\quad + \Big\|\Big(\frac{1}{\langle t\rangle} + \frac{z^2}{\langle t\rangle^2}\Big)\theta_{\lambda-1}\Big\|_{L_z^2} \langle t\rangle^{\frac14} \|\theta\langle r\rangle^n \partial_h^\alpha g\|_{L^2} \\ &\lesssim_\lambda \langle t\rangle^{-\frac12} \|\theta\langle r\rangle^n \partial_h^\alpha g\|_{L^2} + \|\theta\langle r\rangle^n \partial_h^\alpha \partial_z g\|_{L^2},\end{aligned}$$

which is (3.19).

Then almost in the same as (3.24), we can get

$$\begin{aligned}\sum_{|\alpha|=n} \|\theta_\lambda \langle r\rangle^n \partial_h^n \partial_z u^r\|_{L_h^\infty L_z^2} &\lesssim_\lambda (n+1)^2 \sum_{|\alpha|=n}^{n+2} \|\theta_\lambda \langle r\rangle^{|\alpha|} \partial_h^\alpha \partial_z u^r\|_{L_h^2 L_z^2} \\ &\lesssim_\lambda (n+1)^2 \sum_{|\alpha|=n}^{n+2} (\langle t\rangle^{-\frac12} \|\theta\langle r\rangle^{|\alpha|} \partial_h^\alpha g\|_{L^2} + \|\theta\langle r\rangle^{|\alpha|} \partial_h^\alpha \partial_z g\|_{L^2}),\end{aligned}$$

which is (3.20).

3.2 Weighted energy estimates for the good unknown g

Now we perform the weighted energy estimates for the good unknown g. Rewrite the first equation of (2.6) as

$$\begin{aligned}\partial_t g-\partial_z^2 g+\frac{1}{\langle t\rangle}g&=-(ru^r\partial_r+u^z\partial_z)g-(u^r)^2+\frac{1}{2\langle t\rangle}u^z\partial_z(z\phi)-\frac{z}{\langle t\rangle}u^r\phi\\&\quad-\frac{z}{2\langle t\rangle}\int_z^\infty(u^r)^2\mathrm{d}\bar z+\frac{z}{\langle t\rangle}\int_z^\infty\partial_z u^r u^z\mathrm{d}\bar z.\end{aligned}\tag{3.30}$$

Let $n\ge 0$ and $|\alpha|=n$. Applying $\langle r\rangle^n\partial_h^\alpha$ to (3.30) and multiplying the resulted equation with $\theta^2\langle r\rangle^n\partial_h^\alpha g$, and then integrating over $\mathbb{R}^3_+$, we give

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{dt}}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}^2+\|\theta\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L^2}^2+\frac{3}{4\langle t\rangle}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}^2\\
&=-\int\theta\langle r\rangle^n\partial_h^\alpha(u^r r\partial_r g)\theta\langle r\rangle^n\partial_h^\alpha g-\int\theta\langle r\rangle^n\partial_h^\alpha(u^z\partial_z g)\theta\langle r\rangle^n\partial_h^\alpha g\\
&\quad-\int\theta\langle r\rangle^n\partial_h^\alpha(u^r)^2\theta\langle r\rangle^n\partial_h^\alpha g+\frac{1}{2\langle t\rangle}\int\theta\langle r\rangle^n\partial_h^\alpha(u^z\partial_z(z\phi))\theta\langle r\rangle^n\partial_h^\alpha g\\
&\quad-\frac{1}{\langle t\rangle}\int z\theta\langle r\rangle^n\partial_h^\alpha(u^r\phi)\theta\langle r\rangle^n\partial_h^\alpha g-\frac{1}{2\langle t\rangle}\int z\theta\int_z^\infty\langle r\rangle^n\partial_h^\alpha(u^r)^2\mathrm{d}\bar z\theta\langle r\rangle^n\partial_h^\alpha g\\
&\quad+\frac{1}{\langle t\rangle}\int z\theta\int_z^\infty\langle r\rangle^n\partial_h^\alpha(\partial_z u^r u^z)\mathrm{d}\bar z\theta\langle r\rangle^n\partial_h^\alpha g\\
&:=\sum_{j=1}^7 I_j^\alpha.\end{aligned}$$

Here for a function $f(t,x,y,z)$, we have denoted $\int_{\mathbb{R}^3_+}f(t,x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ simply by $\int f$ if no confusion is caused.

Dividing the above equality by $\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}$ and multiplying the resulted equation by $\tau^n(t)M_n$, then by summing for $|\alpha|=n$, we can get that for $n\ge 0$,

$$\frac{\mathrm{d}}{\mathrm{dt}}X_n+\sum_{|\alpha|=n}\frac{\|\theta\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L^2}^2}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}+\frac{3}{4\langle t\rangle}X_n=\dot\tau(t)Y_n+\sum_{|\alpha|=n}\frac{\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\sum_{j=1}^7 I_j^\alpha,\tag{3.31}$$

where when $n=0$, we set $Y_0=0$.

Here we present a lemma to characterize the quantitative relation between $\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}^2$ and $\|\theta\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L^2}^2$.

Lemma 3.3 *Let g be a smooth enough function in x,y variables and belong to H^1 in z variable, which decays to zero sufficiently fast as $z\to+\infty$. Then we have*

$$\frac{1}{2\langle t\rangle}\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}^2\le\|\theta\langle r\rangle^n\partial_h^\alpha\partial_z g\|_{L^2}^2.\tag{3.32}$$

Inequality (3.32) is a special case of Treves inequality that can be found in [11]. Proof of Lemma 3.3 can be found in [24, Lemma 3.1] (see also [12, Lemma 3.3]). Here, we omit the details.

Using (3.32), from (3.31) we can obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}X_n+\frac{1}{\sqrt{2\langle t\rangle}}D_n+\frac{3}{4\langle t\rangle}X_n\le\dot\tau(t)Y_n+\sum_{|\alpha|=n}\frac{\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\sum_{j=1}^7 I_j^\alpha.\tag{3.33}$$

3.3 Proof of Proposition 3.1 and the main theorem

First, we state a proposition concerning on the estimates of the nonlinear terms in (3.33).

Proposition 3.2 (Estimates of the nonlinear terms) *For the nonlinear terms in* (3.33), *we have the following estimate*

$$\begin{aligned}&\sum_{n\geqq 0}\sum_{|\alpha|=n}\frac{\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\sum_{j=1}^{7}I_j^\alpha\\ &\le C\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{Y}_\tau}+C\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{X}_\tau}.\end{aligned}$$

We postpone the proof of Proposition 3.2 till Section 4 and continue to prove the a priori estimate in Proposition 3.1.

Proof of Proposition 3.1 From (3.33), by summing on $n\ge 0$, we get for a uniform constant C_0,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|g\|_{\mathcal{X}_\tau}+\frac{1}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}+\frac{3}{4\langle t\rangle}\|g\|_{\mathcal{X}_\tau}\\ &\le(\dot\tau+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau}))\|g\|_{\mathcal{Y}_\tau}\\ &\quad+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{X}_\tau}.\end{aligned}\tag{3.34}$$

By using (3.32), for any small $\delta_1>0$, we have

$$\begin{aligned}\frac{1}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}&=\frac{\delta_1}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}+\frac{(1-\delta_1)}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}\\ &\ge\frac{\delta_1}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}+\frac{(1-\delta_1)}{2\langle t\rangle}\|g\|_{\mathcal{X}_\tau}\\ &\ge\frac{\delta_1}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}+\frac{\delta_1}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{1-3\delta_1}{2\langle t\rangle}\|g\|_{\mathcal{X}_\tau}.\end{aligned}$$

Inserting the above inequality into (3.34), we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|g\|_{\mathcal{X}_\tau}+\frac{\frac54-\frac32\delta_1}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\Big(\frac{\delta_1}{\langle t\rangle}\|g\|_{X_\tau}+\frac{\delta_1}{\sqrt{2\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}\Big)\\ &\le(\dot\tau+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau}))\|g\|_{\mathcal{Y}_\tau}+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{X}_\tau}.\end{aligned}$$

For $\delta\in\big(0,\frac14\big]$, by choosing $\delta_1=\frac{\delta}{3}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|g\|_{\mathcal{X}_\tau}+\frac{\frac54-\frac12\delta}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{\delta}{6}(\frac{1}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{1}{\sqrt{\langle t\rangle}}\|g\|_{\mathcal{D}_\tau})\\ &\le(\dot\tau+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau}))\|g\|_{\mathcal{Y}_\tau}\\ &\quad+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{X}_\tau}.\end{aligned}\tag{3.35}$$

Now, we assume the a prior assumption that for any $t>0$,

$$\langle t\rangle^{\frac54-\delta}\|g\|_{\mathcal{X}_\tau}\le 2\varepsilon_0,\quad \tau(t)\ge\frac14\tau_0.\tag{3.36}$$

Using this a priori assumption (3.36) and by choosing suitable $\tau(t)$ and sufficiently small ε_0, depending on τ_0 and δ, we will show that

$$\langle t\rangle^{\frac{5}{4}-\delta}\|g\|_{\mathcal{X}_\tau}\le\varepsilon_0,\quad \tau(t)\ge\frac{1}{2}\tau_0. \tag{3.37}$$

Then continuity argument insures that (3.37) stands for any $t>0$.

First, inserting (3.36) into (3.35), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|g\|_{\mathcal{X}_\tau}+\frac{\frac{5}{4}-\frac{1}{2}\delta}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{\delta}{6}\Big(\frac{1}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{1}{\sqrt{\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}\Big)\\&\le(\dot\tau+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau}))\|g\|_{\mathcal{Y}_\tau}+\frac{32\varepsilon_0C_0}{\tau_0^2\langle t\rangle^{\frac54-\delta}}(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau}).\end{aligned}$$

By choosing ε_0 such that $\frac{32\varepsilon_0C_0}{\tau_0^2}<\frac{\delta}{12}$, then we can have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|g\|_{\mathcal{X}_\tau}+\frac{\frac{5}{4}-\delta}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{\delta}{12}\Big(\frac{1}{\langle t\rangle}\|g\|_{\mathcal{X}_\tau}+\frac{1}{\sqrt{\langle t\rangle}}\|g\|_{\mathcal{D}_\tau}\Big)\\&\le(\dot\tau+C_0\tau^{-2}(t)(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau}))\|g\|_{\mathcal{Y}_\tau}.\end{aligned} \tag{3.38}$$

We choose $\tau(t)$ such that

$$\dot\tau+\frac{2C_0}{\tau^2(t)}(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})=0. \tag{3.39}$$

Then (3.38) indicates that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}(\langle t\rangle^{\frac54-\delta}\|g\|_{\mathcal{X}_\tau})+\frac{\delta}{12}(\langle t\rangle^{\frac14-\delta}\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac34-\delta}\|g\|_{\mathcal{D}_\tau})\\&+\frac{C_0\langle t\rangle^{\frac54-\delta}}{\tau^2(t)}(\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{Y}_\tau}\le0.\end{aligned} \tag{3.40}$$

Integrating (3.40), we can have

$$\begin{aligned}&\langle t\rangle^{\frac54-\delta}\|g\|_{\mathcal{X}_\tau}+\frac{\delta}{12}\int_0^t(\langle s\rangle^{\frac14-\delta}\|g\|_{\mathcal{X}_\tau}+\langle s\rangle^{\frac34-\delta}\|g\|_{\mathcal{D}_\tau})\mathrm{d}s\\&+C_0\int_0^t\frac{\langle s\rangle^{\frac54-\delta}}{\tau^2(s)}(\|g\|_{\mathcal{X}_\tau}+\langle s\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{Y}_\tau}\mathrm{d}s\le\|g_0\|_{\mathcal{X}_{\tau_0}}\le\varepsilon_0,\end{aligned} \tag{3.41}$$

which implies that

$$\int_0^t(\langle s\rangle^{\frac14-\delta}\|g\|_{\mathcal{X}_\tau}+\langle s\rangle^{\frac34-\delta}\|g\|_{\mathcal{D}_\tau})\mathrm{d}s\le\frac{12}{\delta}\varepsilon_0.$$

Then from (3.39), we see that

$$\begin{aligned}\tau^3(t)&=\tau_0^3-6C_0\int_0^t(\|g\|_{\mathcal{X}_\tau}+\langle s\rangle^{\frac14}\|g\|_{D_\tau})\mathrm{d}s\\&\ge\tau_0^3-\frac{72C_0\varepsilon_0}{\delta}\ge\Big(\frac12\tau_0\Big)^3,\end{aligned}$$

by choosing small ε_0. Then by choosing small ε_0, depending on τ_0 and δ, we obtain (3.37) and (3.41), which finishes the proof of Proposition 3.1.

End Proof of Theorem 2.1 Combining the local existence and uniqueness of the tangentially analytical solutions in Theorem 3.1 and continuity argument, we can obtain the validity of Theorem 2.1.

4 Technical Estimates of the Nonlinear Terms

In this section, we give the technical estimates for the nonlinear terms on the righthand of (3.33). When summing over $n \geq 0$, we can get the following tangentially analytical estimates for the nonlinear terms.

Lemma 4.1 (Estimates of the nonlinear terms separately) *We have the following estimates for the the nonlinear terms on the righthand of* (3.33).

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_1^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}(\|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{Y}_\tau}, \tag{4.1}$$

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_2^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}\langle t\rangle^{\frac14}(\|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{Y}_\tau})\|g\|_{\mathcal{D}_\tau}, \tag{4.2}$$

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_3^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} + \sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_6^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}(\|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{X}_\tau}, \tag{4.3}$$

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_4^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}\langle t\rangle^{-\frac14}(|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{Y}_\tau})\|g\|_{\mathcal{X}_\tau}, \tag{4.4}$$

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_5^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}(\|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{D}_\tau})\|g\|_{\mathcal{X}_\tau}, \tag{4.5}$$

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_7^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}(\langle t\rangle^{-\frac14}\|g\|_{\mathcal{X}_\tau}+\langle t\rangle^{\frac14}\|g\|_{\mathcal{D}_\tau})(\|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{Y}_\tau}). \tag{4.6}$$

Proof Before the proof, we give the following simple claim.

Claim For any $k \in \mathbb{N}$, $1 \leq p, q \leq +\infty$,

$$\sum_{|\alpha|=k}\|\theta\langle r\rangle^k\partial_h^\alpha(r\partial_r g)\|_{L_h^pL_z^q} \lesssim \sum_{|\alpha|=k+1}\|\theta\langle r\rangle^{k+1}\partial_h^\alpha g\|_{L_h^pL_z^q} + k\sum_{|\alpha|=k}\|\theta\langle r\rangle^k\partial_h^\alpha g\|_{L_h^pL_z^q}. \tag{4.7}$$

Proof of the Claim Without loss of generality, we assume $k \geq 1$, since the claim is obvious for $k = 0$. We write $r\partial_r = x\partial_x + y\partial_y := x_h\partial_h$. Then using Leibniz formula, we have

$$\begin{aligned}\big|\langle r\rangle^k\partial_h^\alpha(r\partial_r g)\big| &= \big|\langle r\rangle^k\partial_h^\alpha(x_h\partial_h g)\big| \\ &= \Big|\langle r\rangle^k x_h\partial_h^\alpha\partial_h g + \sum_{\beta\leq\alpha,|\beta|=1}\langle r\rangle^k\binom{\alpha}{\beta}\partial_h^{\alpha-\beta}\partial_h g\partial_h^\beta x_h\Big| \\ &\leq \langle r\rangle^{k+1}|\partial_h^\alpha\partial_h g| + 2k\langle r\rangle^k|\partial_h^\alpha g|.\end{aligned} \tag{4.8}$$

Then from (4.8), we can easily obtain (4.7).

In later calculations, for multi-indices α, β with $\beta \leq \alpha$, we will frequently use

$$\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}, \quad \sum_{|\alpha|=n}\sum_{|\beta|=k,\beta\leq\alpha}a_\beta b_{\alpha-\beta} = \Big(\sum_{|\beta|=k}a_\beta\Big)\Big(\sum_{|\gamma|=n-k}b_\gamma\Big) \tag{4.9}$$

for all sequences $\{a_\beta\}$ and $\{b_\gamma\}$.

Now we are ready to prove Lemma 4.1.

Estimate for term I_1 For the term I_1, by using (4.9), we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_1^n|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\
\le{}& \tau^n(t)M_n\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}\Big(\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma u^r\|_{L^2_hL^\infty_z}\Big)\Big(\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^\infty_hL^2_z}\Big)\\
&+\tau^n(t)M_n\sum_{k=[\frac{n}{2}]+1}^{n}\binom{n}{k}\Big(\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma u^r\|_{L^\infty}\Big)\Big(\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^2}\Big).
\end{aligned}$$

Then by using (3.15)–(3.16), and noting that $M_n\binom{n}{k}=\frac{(n+1)^4}{(n-k)!k!}$, we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_1^n|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\
\lesssim{}&\sum_{k=0}^{[\frac{n}{2}]}(X_{n-k}+D_{n-k})\frac{\tau^k}{k!}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^\infty_hL^2_z}\\
&+\tau^{-2}\sum_{k=[\frac{n}{2}]+1}^{n}\sum_{i=0}^{2}(X_{n-k+i}+D_{n-k+i})\frac{\tau^k(k+1)^4}{k!}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^2}.
\end{aligned}\tag{4.10}$$

Then by the same Sobolev embedding estimate as that in (3.24) and using (4.7), we can get

$$\begin{aligned}
&\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^\infty_hL^2_z}\\
\lesssim{}&(k+1)^2\sum_{|\beta|=k}^{k+2}\|\theta\langle r\rangle^{|\beta|}\partial_h^\beta(r\partial_r g)\|_{L^2}\\
\lesssim{}&(k+1)^2\sum_{|\beta|=k+1}^{k+3}\|\theta\langle r\rangle^{|\beta|}\partial_h^\beta g\|_{L^2}+(k+1)^2|\beta|\sum_{|\beta|=k}^{k+2}\|\theta\langle r\rangle^{|\beta|}\partial_h^\beta g\|_{L^2}.
\end{aligned}$$

Then it is not hard to check that

$$\frac{\tau^k}{k!}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^\infty_hL^2_z}\lesssim\tau^{-2}\sum_{i=0}^{3}Y_{k+i},\tag{4.11}$$

where, when $k=i=0$, we have set $Y_0=0$.

Also by using (4.7), we can obtain

$$\frac{\tau^k(k+1)^4}{k!}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta(r\partial_r g)\|_{L^2}\lesssim Y_k+Y_{k+1},\tag{4.12}$$

where we used that $\tau\le\tau_0$ since later we will chosen $\tau(t)$ to be a decreased function of t .

Inserting (4.11)–(4.12) into (4.10), we can get

$$\sum_{|\alpha|=n}\frac{|I_1^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\le\tau^{-2}\sum_{k=0}^{n}\sum_{i=0}^{2}(X_{n-k+i}+D_{n-k+i})\sum_{i=0}^{3}Y_{k+i}.\tag{4.13}$$

Then by using the following inequality

$$\sum_{n\geq 0}\sum_{k=0}^{n} a_{n-k}b_k \leq \sum_{k\geq 0} a_k \sum_{j\geq 0} b_j, \tag{4.14}$$

we can get from (4.13),

$$\sum_{n\geq 0}\sum_{|\alpha|=n}\frac{|I_1^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \tau^{-2}\sum_{k\geq 0}\Big(X_k+D_k\Big)\sum_{k\geq 0}Y_k = \tau^{-2}\Big(\|g\|_{\mathcal{X}_\tau}+\|g\|_{\mathcal{D}_\tau}\Big)\|g\|_{\mathcal{Y}_\tau},$$

which is (4.1) for term I_1.

Estimate for term I_2 Now we come to estimate term I_2. By using (4.9), we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_2^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \\
&\leq \tau^n(t)M_n\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{k}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma u^z\|_{L_h^2L_z^\infty}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta\partial_z g\|_{L_h^\infty L_z^2} \\
&\quad + \tau^n(t)M_n\sum_{k=[\frac{n}{2}]+1}^{n}\binom{n}{k}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma u^z\|_{L^\infty}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta\partial_z g\|_{L^2}.
\end{aligned} \tag{4.15}$$

Then by using (3.17)–(3.18), and noting that $M_n\binom{n}{k} = \frac{(n+1)^4}{(n-k)!k!}$, we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_2^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \\
&\lesssim \langle t\rangle^{\frac14}\tau^n(t)\Big\{\sum_{k=0}^{[\frac{n}{2}]}\frac{(n-k+1)^4}{(n-k)!k!}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma(r\partial_r g,g)\|_{L^2}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta\partial_z g\|_{L_h^\infty L_z^2} \\
&\quad + \sum_{k=[\frac{n}{2}]+1}^{n}\frac{(k+1)^4(n-k+1)^2}{(n-k)!k!}\sum_{|\gamma|=n-k}^{n-k+2}\|\langle r\rangle^{|\gamma|}\partial_h^\gamma(r\partial_r g,g)\|_{L^2}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta\partial_z g\|_{L^2}\Big\}.
\end{aligned}$$

By using Sobolev embedding, we have

$$\frac{1}{k!}\|\theta\langle r\rangle^k\partial_h^\beta\partial_z g\|_{L_h^\infty L_z^2} \lesssim \frac{(k+1)^2}{k!}\sum_{|\beta|=k}^{k+2}\|\theta\langle r\rangle^{|\beta|}\partial_h^\beta\partial_z g\|_{L^2} \lesssim \tau^{-2}\sum_{i=0}^{2}D_{k+i}.$$

Combining the above two inequalities, we obtain

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_2^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \\
&\lesssim \langle t\rangle^{\frac14}\tau^{-2}\sum_{k=0}^{[\frac{n}{2}]}\frac{\tau^{n-k}(n-k+1)^4}{(n-k)!}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma(r\partial_r g,g)\|_{L^2}\sum_{i=0}^{2}D_{k+i} \\
&\quad + \langle t\rangle^{\frac14}\sum_{k=[\frac{n}{2}]+1}^{n}\frac{\tau^{n-k}(n-k+1)^2}{(n-k)!}\sum_{|\gamma|=n-k}^{n-k+2}\|\langle r\rangle^{|\gamma|}\partial_h^\gamma(r\partial_r g,g)\|_{L^2}D_k.
\end{aligned} \tag{4.16}$$

We have that

$$\frac{\tau^{n-k}(n-k+1)^4}{(n-k)!}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma(r\partial_r g,g)\|_{L^2}\lesssim X_{n-k}+Y_{n-k+1}+Y_{n-k}, \tag{4.17}$$

$$\frac{\tau^{n-k}(n-k+1)^2}{(n-k)!}\sum_{|\gamma|=n-k}^{n-k+2}\|\langle r\rangle^{|\gamma|}\partial_h^\gamma(r\partial_r g,g)\|_{L^2}\leq X_{n-k}+\tau^{-2}\sum_{i=0}^{3}Y_{n-k+i}. \tag{4.18}$$

Inserting the above two inequalities into (4.16), we can obtain

$$\sum_{|\alpha|=n}\frac{|I_2^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\lesssim\langle t\rangle^{\frac14}\tau^{-2}\sum_{k=0}^{n}\Big(X_{n-k}+\sum_{i=0}^{3}Y_{n-k+i}\Big)\sum_{i=0}^{2}D_{k+i}. \tag{4.19}$$

Summing (4.19) over $n\geq 0$ and using (4.14), we can obtain (4.2).

Estimate for term I_3 Now we come to estimate term I_3. By using (4.9), we have

$$\begin{aligned}&\sum_{|\alpha|=n}\frac{|I_3^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\ &\leq\tau^n(t)M_n\sum_{k=0}^{[\frac n2]}\binom{n}{k}\sum_{|\gamma|=n-k}\|\theta_{\frac12}\langle r\rangle^{n-k}\partial_h^\gamma u^r\|_{L_h^2L_z^\infty}\sum_{|\beta|=k}\|\theta_{\frac12}\langle r\rangle^k\partial_h^\beta u^r\|_{L_h^\infty L_z^2}\\ &\quad+\tau^n(t)M_n\sum_{k=[\frac n2]+1}^{n}\binom{n}{k}\sum_{|\gamma|=n-k}\|\theta_{\frac12}\langle r\rangle^{n-k}\partial_h^\gamma u^r\|_{L^\infty}\sum_{|\beta|=k}\|\theta_{\frac12}\langle r\rangle^k\partial_h^\beta u^r\|_{L^2}.\end{aligned} \tag{4.20}$$

Then by using (3.13)–(3.16), and noting that $M_n\binom{n}{k}=\frac{(n+1)^4}{(n-k)!k!}$, we have

$$\begin{aligned}&\sum_{|\alpha|=n}\frac{|I_3^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\ &\lesssim\tau^{-2}\sum_{k=0}^{[\frac n2]}(X_{n-k}+D_{n-k})\sum_{i=0}^{2}X_{k+i}+\tau^{-2}\sum_{i=[\frac n2]+1}^{n}\sum_{i=0}^{2}(X_{n-k+i}+D_{n-k+i})X_k\\ &\lesssim\tau^{-2}\sum_{k=0}^{n}\sum_{i=0}^{2}(X_{n-k+i}+D_{n-k+i})\sum_{i=0}^{2}X_{k+i}.\end{aligned} \tag{4.21}$$

Summing (4.21) over $n\geq 0$ and using (4.14), we can obtain (4.3) for term I_3.

Estimate for term I_4 For the terms I_4^n, from the first equation of (3.4), we first have

$$\partial_z(z\phi)=\Big(1-\frac{z^2}{2\langle t\rangle}\Big)\phi+zg.$$

Then from (3.12), we have, for $|\alpha|=k$,

$$\begin{aligned}&\|\theta_\lambda\langle r\rangle^k\partial_h^\alpha\partial_z(z\phi)\|_{L_z^2}\\ \leq&\|\theta_\lambda\langle r\rangle^k\partial_h^\alpha\phi\|_{L_z^2}+\Big\|\theta_\lambda\frac{z^2}{\langle t\rangle}\langle r\rangle^k\partial_h^\alpha\phi\Big\|_{L_z^2}+\|\theta_\lambda z\langle r\rangle^k\partial_h^\alpha g\|_{L_z^2}\\ \leq&\sqrt{\langle t\rangle}\|\theta_\lambda\langle r\rangle^k\partial_h^\alpha g\|_{L_z^2}+\|\theta_{\frac{1+\lambda}{2}\alpha}\langle r\rangle^k\partial_h^\alpha\phi\|_{L_z^2}+\sqrt{\langle t\rangle}\|\theta_{\frac{1+\lambda}{2}}\langle r\rangle^k\partial_h^\alpha g\|_{L_z^2}\end{aligned}$$

$$\leq\sqrt{\langle t\rangle}\|\theta\langle r\rangle^k\partial_h^\alpha g\|_{L_z^2}. \tag{4.22}$$

Now we come to estimate term I_4. By using (4.9) and (4.22), we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_4^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\
&\leq \langle t\rangle^{-\frac12}\tau^n(t)M_n\sum_{|\alpha|=n}\sum_{\substack{\beta\leq\alpha\\|\beta|\leq[\frac n2]}}\binom{\alpha}{\beta}\|\theta_{\frac12}\langle r\rangle^{n-|\beta|}\partial^{\alpha-\beta}u^z\|_{L_h^2L_z^\infty}\|\theta\langle r\rangle^{|\beta|}\partial_h^\beta g\|_{L_h^\infty L_z^2}\\
&\quad+\langle t\rangle^{-\frac12}\tau^n(t)M_n\sum_{|\alpha|=n}\sum_{\substack{\beta\leq\alpha\\|\beta|>[\frac n2]}}\binom{\alpha}{\beta}\|\theta_{\frac12}\langle r\rangle^{n-|\beta|}\partial^{\alpha-\beta}u^z\|_{L^\infty}\|\theta\langle r\rangle^{|\beta|}\partial_h^\beta g\|_{L^2}.
\end{aligned}$$

Then almost in the same estimate as that in (4.15) by replacing $\partial_z g$ with g indicates a similar estimate as (4.19) as follows:

$$\sum_{|\alpha|=n}\frac{|I_4^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\lesssim\langle t\rangle^{-\frac14}\tau^{-2}\sum_{k=0}^n\Big(X_{n-k}+\sum_{i=0}^3Y_{n-k+i}\Big)\sum_{i=0}^2X_{k+i}. \tag{4.23}$$

Summing (4.23) over $n\geq0$ and using (4.14), we can obtain (4.4).

Estimate for term I_5 It is easy to see that, from (3.12),

$$\|\theta_\lambda\langle r\rangle^k\partial_h^\alpha(z\phi)\|_{L_z^2}\lesssim_\lambda\sqrt{\langle t\rangle}\|\theta_{\frac{1+\lambda}{2}}\langle r\rangle^k\partial_h^\alpha\phi\|_{L_z^2}\lesssim_\lambda\langle t\rangle\|\theta\langle r\rangle^k\partial_h^\alpha g\|_{L_z^2}. \tag{4.24}$$

By using (4.9) and (4.24), we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_5^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\
&\lesssim\tau^n(t)M_n\sum_{k=0}^{[\frac n2]}\binom{n}{k}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma u^r\|_{L_h^2L_z^\infty}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta g\|_{L_h^\infty L_z^2}\\
&\quad+\tau^n(t)M_n\sum_{k=[\frac n2]+1}^{n}\binom{n}{k}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma u^r\|_{L^\infty}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta g\|_{L^2}.
\end{aligned}$$

Then by using (3.15)–(3.16), and noting that $M_n\binom{n}{k}=\frac{(n+1)^4}{(n-k)!k!}$, we have

$$\begin{aligned}
&\sum_{|\alpha|=n}\frac{|I_5^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\
&\lesssim\sum_{k=0}^{[\frac n2]}(X_{n-k}+D_{n-k})\frac{\tau^k}{k!}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta g\|_{L_h^\infty L_z^2}\\
&\quad+\tau^{-2}\sum_{k=[\frac n2]+1}^{n}\sum_{i=0}^2(X_{n-k+i}+D_{n-k+i})X_k.
\end{aligned} \tag{4.25}$$

By using Sobolev embedding, it is easy to check that

$$\frac{\tau^k}{k!}\sum_{|\beta|=k}\|\theta\langle r\rangle^k\partial_h^\beta g\|_{L_h^\infty L_z^2}\lesssim\tau^{-2}\sum_{i=0}^2X_{k+i}.$$

Inserting the above inequality into (4.25), we can obtain

$$\sum_{|\alpha|=n}\frac{|I_5^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\lesssim\langle t\rangle^{-\frac{1}{2}}\tau^{-2}\sum_{k=0}^{n}\sum_{i=0}^{2}(X_{n-k+i}+D_{n-k+i})\sum_{i=0}^{2}X_{k+i}. \tag{4.26}$$

Summing (4.26) over $n\geq 0$ and using (4.14), we can obtain (4.5).

Estimate for term I_6 First, we have

$$\begin{aligned}\frac{|I_6^\alpha|}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}&\leq\frac{1}{\langle t\rangle}\|z\theta(z)\int_z^\infty\langle r\rangle^n\partial_h^\alpha(u^r)^2(\bar z)\mathrm{d}\bar z\|_{L^2}\\&=\frac{1}{\langle t\rangle}\|z\theta_{-\frac12}(z)\theta_{\frac32}(z)\int_z^\infty\langle r\rangle^n\partial_h^\alpha(u^r)^2(\bar z)\mathrm{d}\bar z\|_{L^2}\\&\leq\frac{1}{\langle t\rangle}\|z\theta_{-\frac12}(z)\|_{L_h^\infty L_z^2}\|\theta_{\frac32}(z)\int_z^\infty\langle r\rangle^n\partial_h^\alpha(u^r)^2(\bar z)\mathrm{d}\bar z\|_{L_h^2L_z^\infty}\\&\lesssim\langle t\rangle^{-\frac14}\|\theta_{\frac32}(z)\int_z^\infty\langle r\rangle^n\partial_h^\alpha(u^r)^2(\bar z)\mathrm{d}\bar z\|_{L_h^2L_z^\infty},\end{aligned}$$

while

$$\begin{aligned}&\Big\|\theta_{\frac32}(z)\int_z^\infty\langle r\rangle^n\partial_h^\alpha(u^r)^2(\bar z)\mathrm{d}\bar z\Big\|_{L_z^\infty}\\&\leq\sup_{z\geq0}\Big\{\theta_{\frac32}(z)\Big(\int_z^\infty\theta_{-\frac72}(\bar z)\mathrm{d}\bar z\Big)^{\frac12}\Big\}\|\theta_{\frac74}(z)\langle r\rangle^n\partial_h^\alpha(u^r)^2\|_{L_z^2}\\&\leq\langle t\rangle^{\frac14}\|\theta_{\frac74}(z)\langle r\rangle^n\partial_h^\alpha(u^r)^2\|_{L_z^2}.\end{aligned}$$

Then

$$\sum_{|\alpha|=n}\frac{|I_6^\alpha|\tau(t)M_n}{\|\theta(\langle r\rangle\partial_r)^n g\|_{L^2}}\leq\tau(t)M_n\sum_{|\alpha|=n}\|\theta_{\frac74}(z)\langle r\rangle^n\partial_h^\alpha(u^r)^2\|_{L^2}. \tag{4.27}$$

The rest is the same as I_3^α in (4.20) by replacing $\frac12$ with $\frac78$ which indicates (4.3) for term I_6^α.

Estimate for term I_7 Repeating the proof for (4.27), we can get

$$\sum_{|\alpha|=n}\frac{|I_7^\alpha|\tau(t)M_n}{\|\theta(\langle r\rangle\partial_r)^n g\|_{L^2}}\leq\tau(t)M_n\sum_{|\alpha|=n}\|\theta_{\frac74}(z)\langle r\rangle^n\partial_h^\alpha(u^z\partial_z u^r)\|_{L^2}.$$

By using (4.9), we have

$$\begin{aligned}&\sum_{|\alpha|=n}\frac{|I_7^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}}\\&\leq\tau^n(t)M_n\sum_{k=0}^{[\frac n2]}\binom{n}{k}\sum_{|\gamma|=n-k}\|\theta_{\frac78}\langle r\rangle^{n-k}\partial_h^\gamma u^z\|_{L_h^2L_z^\infty}\sum_{|\beta|=k}\|\theta_{\frac78}\langle r\rangle^k\partial_h^\beta\partial_z u^r\|_{L_h^\infty L_z^2}\\&\quad+\tau^n(t)M_n\sum_{k=[\frac n2]+1}^{n}\binom{n}{k}\sum_{|\gamma|=n-k}\|\theta_{\frac78}\langle r\rangle^{n-k}\partial_h^\gamma u^z\|_{L^\infty}\sum_{|\beta|=k}\|\theta_{\frac78}\langle r\rangle^k\partial_h^\beta\partial_z u^r\|_{L^2}.\end{aligned}$$

Then by using (3.17)–(3.20), and noting that $M_n\binom{n}{k} = \frac{(n+1)^4}{(n-k)!k!}$, we have

$$\begin{aligned}
&\sum_{|\alpha|=n} \frac{|I_7^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \\
&\lesssim \langle t\rangle^{\frac14}\tau^{-2}\sum_{k=0}^{[\frac n2]}\frac{(n-k+1)^4\tau^{n-k}}{(n-k)!}\sum_{|\gamma|=n-k}\|\langle r\rangle^{n-k}\partial_h^\gamma(r\partial_r g, g)\|_{L^2}\sum_{i=0}^{2}(\langle t\rangle^{-\frac12}X_{k+i}+D_{k+i}) \\
&\quad + \langle t\rangle^{\frac14}\sum_{k=[\frac n2]+1}^{n}\frac{(n-k+1)^2\tau^{n-k}}{(n-k)!}\sum_{|\gamma|=n-k}^{n-k+2}\|\langle r\rangle^{|\gamma|}\partial_h^\gamma(r\partial_r g, g)\|_{L^2}(\langle t\rangle^{-\frac12}X_k+D_k).
\end{aligned}$$

Then using (4.17)–(4.18), we obtain

$$\sum_{|\alpha|=n} \frac{|I_7^\alpha|\tau^n(t)M_n}{\|\theta\langle r\rangle^n\partial_h^\alpha g\|_{L^2}} \lesssim \langle t\rangle^{\frac14}\tau^{-2}\sum_{k=0}^{n}\Big(X_{n-k}+\sum_{i=0}^{3}Y_{n-k+i}\Big)\sum_{i=0}^{2}(\langle t\rangle^{-\frac12}X_{k+i}+D_{k+i}). \tag{4.28}$$

Summing (4.28) over $n \geq 0$ and using (4.14), we can obtain (4.6).

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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