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Essentially Commuting Dual Truncated Toeplitz Operators*

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Abstract In this paper, the authors completely characterize when two dual truncated Toeplitz operators ar e essentially commuting and when the semicommutator of two dual truncated Toeplitz operators is compact. Their main idea is to study dual truncated Toeplitz operators via Hankel operators, Toeplitz operators and function algebras.

Keywords Hardy space, Dual truncated Toeplitz operator, Essentially commuting
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1 Introduction

Let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D}$ be its boundary. Let L^2 denote the Lebesgue space of square integrable functions on the unit circle $\partial\mathbb{D}$. The Hardy space H^2 is the closed subspace of L^2, which is spanned by the space of analytic polynomials. Thus there is an orthogonal projection P from L^2 onto H^2. For φ in L^∞, the space of essentially bounded measurable functions on $\partial\mathbb{D}$, the Toeplitz operator T_φ and the Hankel operator H_φ with symbol φ on H^2 are defined by

$$T_\varphi f = P(\varphi f)$$

and

$$H_\varphi f = (I-P)(\varphi f)$$

for $f \in H^2$, respectively. Moreover, the dual Toeplitz operator S_φ on $(H^2)^\perp$ is defined by

$$S_\varphi h = (I-P)(\varphi h), \quad h \in (H^2)^\perp.$$

For more information on the topics of Toeplitz and Hankel operators we refer to [8, 26].

Let T_z^* be the adjoint of the forward shift operator T_z. Suppose that u is a nonconstant inner function. The invariant subspace for T_z^*,

$$K_u^2 = H^2 \ominus uH^2$$

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is called the model space (see [10]). Let P_u be the orthogonal projection from L^2 onto K_u^2. For $\varphi \in L^2$, the dual truncated Toeplitz operator D_φ with symbol φ on the orthogonal complement of K_u^2 is densely defined by

$$D_\varphi f = (I - P_u)(\varphi f)$$

on the subspace $(K_u^2)^\perp \cap L^\infty$ of $(K_u^2)^\perp = L^2 \ominus K_u^2$. Noting that $L^2 = H^2 \oplus \overline{zH^2}$ and $K_u^2 = H^2 \ominus uH^2$, we obtain

$$(K_u^2)^\perp = uH^2 \oplus \overline{zH^2},$$

and moreover,

$$P_u = P - M_u P M_{\overline{u}}$$

and

$$I - P_u = M_u P M_{\overline{u}} + (I - P),$$

where M_u is the multiplication operator on H^2 with symbol u.

Toeplitz operators and Hankel operators have played an especially important role in function theory and operator theory. There are many fascinating problems about those two classes of operators. The essentially commuting problem of two bounded linear operators arises from studying Fredholm theory of operators on a Hilbert space. The answer to the commuting problem for two Topelitz operators on the Hardy space was obtained by Brown and Halmos [4] in 1964, which states that two Toeplitz operators are commuting if and only if either both symbols of these operators are analytic, or both symbols of these operators are co-analytic, or a nontrivial linear combination of their symbols is constant. Axler and Čučkvoić obtained the analogous result for Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk (see [2]). Using some techniques in multiple complex-variable functions, Ding, Sun and Zheng [7] established a necessary and sufficient condition for two Toeplitz operators to be commuting on the Hardy space over the bidisk.

The problem of when the commutator or semicommutator of two operators is compact on function spaces has been investigated by many people. The beautiful Axler-Chang-Sarason-Volberg theorem (see [1, 23]) states that the semicommutator $T_f T_g - T_{fg}$ of two Hardy Toeplitz operators T_f and T_g is compact if and only if either $\overline{f}$ or g is in H^∞ on each support set (which will be introduced in the next section). An elementary characterization for the compactness of the semicommutator of two Hardy Toeplitz operators in terms of Hankel operators was obtained by Zheng [24]. The compactness for the semicommutator of two Toeplitz operators on other analytic function spaces was studied in [13, 16, 25].

In 1999, Gorkin and Zheng [12] completely characterized the compact commutator $T_f T_g - T_g T_f$ of two Toeplitz operators on the Hardy space in terms of Douglas algebras or support sets. More precisely, The characterization in [12] can be stated as follows: two Toeplitz operators are essentially commuting if and only if either the restrictions of their symbols on each support set S are in $H^\infty|_S$, or the restrictions of the conjugations of their symbols on each S belong to $H^\infty|_S$, or a nontrivial linear combination of the restrictions of their symbols on each support set S is constant. The essentially commuting problem for Toeplitz operators with bounded harmonic symbols on the Bergman space was solved by Stroethoff [20] in 1993.

Dual Toeplitz operators on the orthogonal complement of the Bergman space were studied in [22]. Dual truncated Toeplitz operator is a new class of operators on the orthogonal complement of the model space, which was first introduced in [6]. In [5], asymmetric dual truncated Toeplitz operators acting between the orthogonal complements of two (eventually different) model spaces were introduced. Although these operators differ in many ways from Toeplitz operators on the Hardy space, they do have some of the same interesting properties, see [6] and [18] for more information. In the present paper, we focus on the following problems:

Problem 1.1 When is the commutator $[D_f, D_g] = D_f D_g - D_g D_f$ of two dual truncated Toeplitz operators D_f and D_g with f and g in L^∞ compact?

Problem 1.2 When is the semicommutator $[D_f, D_g) = D_f D_g - D_{fg}$ of two dual truncated Toeplitz operators D_f and D_g with f and g in L^∞ compact?

In order to study the dual truncated Toeplitz operators, we use the useful matrix representation for the dual truncated Toeplitz operator to establish a connection between the Toeplitz operator, Hankel operator and dual truncated Toeplitz operator. Then the above essentially commuting (semicommuting) problem can be reduced to the study of the compactness of products of Toeplitz, Hankel and dual Toeplitz operators. The difficult part in this paper is characterizing the compactness of the sum of the four products of Toeplitz, Hankel and dual Toeplitz operators. Our main idea here is to study dual truncated Toeplitz operators via the characterization for the essentially commuting Hankel and Toeplitz operators (see [14]) and function algebras. The first main result in this paper is the following theorem.

Theorem 1.1 *Let u be a nonconstant inner function and $f, g \in L^\infty$. The commutator $[D_f, D_g]$ is compact if and only if for each support set S, one of the following holds*:

(1) $f|_S$, $g|_S$, $((u-\lambda)\overline{f})|_S$ *and* $((u-\lambda)\overline{g})|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(2) $\overline{f}|_S$, $\overline{g}|_S$, $((u-\lambda)f)|_S$ *and* $((u-\lambda)g)|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(3) *there exist constants* a, b, *not both zero, such that* $(af+bg)|_S$ *is a constant.*

The above theorem is analogous to the characterization when two Toeplitz operators are essentially commuting on the Hardy space (see [12, Theorem 0.8]).

The second main result of our paper is the following characterization on the compactness of the semicommutator of two dual truncated Toeplitz operators.

Theorem 1.2 *Let u be a nonconstant inner function and $f, g \in L^\infty$. The semicommutator $[D_f, D_g)$ is compact if and only if for each support set S, one of the following holds*:

(1) $f|_S$, $g|_S$, $((u-\lambda)\overline{f})|_S$, $((u-\lambda)\overline{g})|_S$ *and* $\big((u-\lambda)\overline{fg}\big)|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(2) $\overline{f}|_S$, $\overline{g}|_S$, $((u-\lambda)f)|_S$, $((u-\lambda)g)|_S$ *and* $((u-\lambda)fg)|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(3) *either* $f|_S$ *or* $g|_S$ *is a constant.*

Theorem 1.2 is analogous to the characterization for the compactness of the semicommutator of two Hardy Toeplitz operators (see [1, 23]).

As the proof of Theorem 1.1 is long, it is divided into the necessary part in Section 3 and the sufficient part in Section 4. We will present the details for the proof of the necessary part and the sufficient part of Theorem 1.2 in Sections 5 and 6, respectively.

2 Notations and Preliminaries

In this section, we introduce some notations and include some important lemmas. Let us begin with the following matrix representation for the dual truncated Toeplitz operator on the space $(K_u^2)^\perp$, see [19, Lemma 2] for the details.

Lemma 2.1 *Suppose that $\varphi \in L^\infty$. The dual truncated Toeplitz operator D_φ on $(K_u^2)^\perp$ is unitarily equivalent to the following (2×2) operator matrix*

$$\begin{pmatrix} T_\varphi & H^*_{u\overline{\varphi}} \\ H_{u\varphi} & S_\varphi \end{pmatrix}$$

on the space $L^2 = H^2 \oplus \overline{zH^2}$. Moreover, the unitary operator here is given by

$$U = \begin{pmatrix} M_u & 0 \\ 0 & I \end{pmatrix}.$$

In view of the matrix representation in the above lemma, the essentially commuting problem for two dual truncated Toeplitz operators can be easily transformed into the compactness of the following four classical operators.

Lemma 2.2 *Suppose that u is a nonconstant inner function and $f, g \in L^\infty$. Then the commutator $D_f D_g - D_g D_f$ is compact if and only if*

$$T_f T_g + H^*_{u\overline{f}} H_{ug} - T_g T_f - H^*_{u\overline{g}} H_{uf},$$
$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f,$$
$$H_{uf} T_g + S_f H_{ug} - H_{ug} T_f - S_g H_{uf}$$

and

$$H_{uf} H^*_{u\overline{g}} + S_f S_g - H_{ug} H^*_{u\overline{f}} - S_g S_f$$

are compact.

Proof Let

$$T_1 = T_f T_g + H^*_{u\overline{f}} H_{ug} - T_g T_f - H^*_{u\overline{g}} H_{uf},$$
$$T_2 = T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f,$$
$$T_3 = H_{uf} T_g + S_f H_{ug} - H_{ug} T_f - S_g H_{uf}$$

and

$$T_4 = H_{uf} H^*_{u\overline{g}} + S_f S_g - H_{ug} H^*_{u\overline{f}} - S_g S_f.$$

Then we have by Lemma 2.1 that

$$\begin{aligned} & U^*(D_f D_g - D_g D_f)U \\ = & \begin{pmatrix} T_f & H^*_{u\overline{f}} \\ H_{uf} & S_f \end{pmatrix} \begin{pmatrix} T_g & H^*_{u\overline{g}} \\ H_{ug} & S_g \end{pmatrix} - \begin{pmatrix} T_g & H^*_{u\overline{g}} \\ H_{ug} & S_g \end{pmatrix} \begin{pmatrix} T_f & H^*_{u\overline{f}} \\ H_{uf} & S_f \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

Denote the above operator matrix by T. According to the fact that T is compact if and only if T_1, T_2, T_3 and T_4 are all compact, we finish the proof of this lemma.

Using the same method as that in the proof of Lemma 2.2, we obtain a similar conclusion for the compactness of the semicommutator $[D_f, D_g)$.

Lemma 2.3 *Suppose that u is a nonconstant inner function and $f,\ g \in L^\infty$. Then the semicommutator $D_f D_g - D_{fg}$ is compact if and only if*

$$T_f T_g + H^*_{u\overline{f}} H_{ug} - T_{fg},$$

$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - H^*_{u\overline{fg}},$$

$$H_{uf} T_g + S_f H_{ug} - H_{ufg}$$

and

$$H_{uf} H^*_{u\overline{g}} + S_f S_g - S_{fg}$$

are compact.

To study the compactness of products of Hankel and Toeplitz operators on the Hardy space, the following operator V is very useful. Define the operator $V:\ L^2 \to L^2$ by

$$Vf(z) = \overline{zf(z)}, \quad f \in L^2,\ z \in \partial\mathbb{D}.$$

It is easy to check that V is anti-unitary and moreover,

$$V = V^{-1} = V^*$$

on L^2. For a general anti-linear operator V, V^* is the anti-linear operator defined via the property

$$\overline{\langle Vf, g\rangle} = \langle f, V^* g\rangle$$

for f and g in L^2.

We will show in the next lemma that the operator V and the Hardy projection P satisfy the following equation.

Lemma 2.4 *For $f \in L^2$, then*

$$VP(f) = (I - P)V(f).$$

Proof For any f in L^2, we write $f = f_+ + f_-$, where $f_+ = Pf$ and $f_- = (I - P)f$. Then we have

$$\begin{aligned} VP(f)(w) &= Vf_+(w) \\ &= \overline{w}\overline{f_+(w)} \\ &= \overline{w}\overline{f_+(w)} + (I - P)(\overline{w}\overline{f_-(w)}) \end{aligned}$$

$$= (I-P)(\overline{w}\overline{f_+(w)} + \overline{w}\overline{f_-(w)})$$
$$= (I-P)(\overline{w}\overline{f(w)})$$
$$= (I-P)V(f)(w)$$

for each $w \in \partial\mathbb{D}$, to complete the proof.

Remark 2.1 Observe that Lemma 2.4 easily leads to the following two relations:

$$VH_\varphi = H_\varphi^* V \quad \text{and} \quad S_\varphi V = VT_{\overline{\varphi}},$$

which will be used repeatedly later on.

For x and y in L^2, we use $x \otimes y$ to denote the following rank-one operator: For $f \in L^2$,

$$(x \otimes y)(f) = \langle f, y\rangle x.$$

It is well-known that the operator norm of the above rank-one operator is given by $\|x \otimes y\| = \|x\|_2 \cdot \|y\|_2$. The following two lemmas about the Toeplitz and Hankel operators on H^2 established in [24, Lemmas 1–2] are useful tools to study the compactness of the product of Hankel operators and compact operators in the Toeplitz algebra.

Lemma 2.5 *Let f and g be in L^2 and $z \in \mathbb{D}$. Then*

$$H_f^* H_g - T_{\phi_z}^* H_f^* H_g T_{\phi_z} = V[(H_f k_z) \otimes (H_g k_z)]V^*.$$

Here

$$k_z(\mathrm{e}^{\mathrm{i}\theta}) = \frac{\sqrt{1-|z|^2}}{1-\overline{z}\mathrm{e}^{\mathrm{i}\theta}}$$

is the normalized reproducing kernel for the Hardy space, and ϕ_z denotes the Möbius map

$$\phi_z(w) = \frac{z-w}{1-\overline{z}w}, \quad z, w \in \mathbb{D}.$$

Lemma 2.6 *Let K be a compact operator on H^2. Then we have*

$$\lim_{|z|\to 1^-} \|K - T_{\phi_z}^* K T_{\phi_z}\| = 0.$$

As in [11], a Douglas algebra is, by definition, a closed subalgebra of L^∞ which contains H^∞. As Douglas algebras play a prominent role in various problems on Toeplitz and Hankel operators, we need to review some important properties of them. Observe that H^∞ is a commutative Banach algebra, we can identify the maximal ideal space $\mathcal{M}(H^\infty)$ as the set of multiplicative linear functionals on H^∞. Endowed with the weak star topology it inherits as a subset of the dual space of H^∞, $\mathcal{M}(H^\infty)$ is a compact Hausdorff space. Identifying a point in the open unit disk $\mathbb{D}$ with the functional of evaluation at this point, we may regard the disk $\mathbb{D}$ as a subset of $\mathcal{M}(H^\infty)$. Using the Gelfand transform we regard every function in H^∞ as a continuous function on $\mathcal{M}(H^\infty)$. The deepest result concerning $\mathcal{M}(H^\infty)$ is the famous corona theorem of Carleson, stating that $\mathbb{D}$ is dense in $\mathcal{M}(H^\infty)$ under the weak star topology (for details, see [9, 11]).

It is a consequence of the Gleason-Whitney theorem that the maximal ideal space of a Douglas algebra B is a naturally imbedded in $\mathcal{M}(H^\infty)$. Thus we may identify the maximal ideal space $\mathcal{M}(H^\infty + C)$ of the Sarason algebra $H^\infty + C$ with a subset of $\mathcal{M}(H^\infty)$, where C is the algebra of continuous functions on $\partial\mathbb{D}$. A subset of $\mathcal{M}(L^\infty)$ will be a support set if it is the (closed) support of the representing measure for a functional in $\mathcal{M}(H^\infty + C)$, see [11, 17] for more details. Let m be in $\mathcal{M}(H^\infty + C)$ and let $\mathrm{d}\mu_m$ denote the unique representing measure for m with support S_m, i.e.,

(1) for all f and g in H^∞,

$$m(fg) = \int_{S_m} fg \, \mathrm{d}\mu_m = \Big(\int_{S_m} f\mathrm{d}\mu_m\Big)\Big(\int_{S_m} g\mathrm{d}\mu_m\Big);$$

(2) if $h \geq 0$ a.e. in $L^1(\mathrm{d}\mu_m)$ such that

$$\int_{S_m} fh \, \mathrm{d}\mu_m = \int_{S_m} f\mathrm{d}\mu_m$$

for all $f \in H^\infty$, then we have $h = 1$ a.e. $\mathrm{d}\mu_m$.

Suppose that $m \in \mathcal{M}(H^\infty + C)$ and $z \mapsto \xi_z$ is a mapping from the unit disk $\mathbb{D}$ into some topological space X. Let η be in X. We use the notation

$$\lim_{z\to m} \xi_z = \eta$$

to denote that for each open set $\mathcal{U}(\eta) \subset X$ containing η, there exists an open subset $\mathcal{O}(m)$ of $\mathcal{M}(H^\infty + C)$ containing m such that $\xi_z \in \mathcal{U}$ for all $z \in \mathcal{O}(m) \cap \mathbb{D}$.

For a function F on the disk $\mathbb{D}$ and m in $\mathcal{M}(H^\infty + C)$, we say

$$\lim_{z\to m} F(z) = 0$$

if for every net $\{z_\alpha\} \subset \mathbb{D}$ converging to m,

$$\lim_{z_\alpha\to m} F(z_\alpha) = 0.$$

We shall emphasize here that we deal with nets rather than sequences since the the topology of $\mathcal{M}(H^\infty + C)$ is not metrizable.

With the above notations and concepts about H^2 theory on a support set, we quote the following lemma obtained in [12, Lemmas 2.5–2.6].

Lemma 2.7 *Let f be in L^∞ and $m \in \mathcal{M}(H^\infty + C)$. Denote the support set for m by S_m. Then the following three conditions are equivalent*:

(1) $f|_{S_m} \in H^\infty|_{S_m}$;

(2) $\lim\limits_{z\to m} \|H_f k_z\|_2 = 0$;

(3) $\underline{\lim}_{z\to m} \|H_f k_z\|_2 = 0$.

3 The Necessary Part of Theorem 1.1

In this section, we assume that $D_fD_g - D_gD_f$ is a compact operator. Recall that the four operators in Lemma 2.2 are compact. Now we are going to derive the necessary condition for the compactness of these four operators in terms of the boundary properties of the symbols f and g.

In the following proposition, we establish a necessary condition for the compactness of the first operator $T_fT_g + H^*_{u\overline{f}}H_{ug} - T_gT_f - H^*_{u\overline{g}}H_{uf}$ given in Lemma 2.2.

Proposition 3.1 *Let u be a nonconstant inner function, $f, g \in L^\infty$ and $m \in \mathcal{M}(H^\infty + C)$. Suppose that the operator*

$$T_fT_g + H^*_{u\overline{f}}H_{ug} - T_gT_f - H^*_{u\overline{g}}H_{uf}$$

is compact. Then for the support set S_m of m, one of following conditions holds:

(1) *Both $f|_{S_m}$ and $g|_{S_m}$ are in $H^\infty|_{S_m}$;*

(2) *both $\overline{f}|_{S_m}$ and $\overline{g}|_{S_m}$ are in $H^\infty|_{S_m}$;*

(3) *there exist constants a, b, not both zero, such that $(af + bg)|_{S_m}$ is a constant.*

Proof Suppose that

$$T_fT_g + H^*_{u\overline{f}}H_{ug} = T_gT_f + H^*_{u\overline{g}}H_{uf} + K,$$

where K is compact. Since $T_fT_g - T_gT_f = H^*_{\overline{g}}H_f - H^*_{\overline{f}}H_g$, we have

$$H^*_{\overline{g}}H_f - H^*_{\overline{f}}H_g = H^*_{u\overline{g}}H_{uf} - H^*_{u\overline{f}}H_{ug} + K.$$

By Lemmas 2.5–2.6, we have

$$K - T^*_{\phi_z}KT_{\phi_z} = V(H_{\overline{g}}k_z \otimes H_fk_z - H_{\overline{f}}k_z \otimes H_gk_z - H_{u\overline{g}}k_z \otimes H_{uf}k_z + H_{u\overline{f}}k_z \otimes H_{ug}k_z)V^*$$

and

$$H_{\overline{g}}k_z \otimes H_fk_z - H_{\overline{f}}k_z \otimes H_gk_z = H_{u\overline{g}}k_z \otimes H_{uf}k_z - H_{u\overline{f}}k_z \otimes H_{ug}k_z + \varepsilon(z), \tag{3.1}$$

where the operator $\varepsilon(z)$ satisfies $\lim\limits_{z\to m} \|\varepsilon(z)\| = 0$.

In the following, we still use the same notation $\varepsilon(z)$ to denote the various terms such that

$$\|\varepsilon(z)\| \to 0, \quad z \to m$$

for simplicity.

For $m \in \mathcal{M}(H^\infty + C)$, we use $[f|_{S_m}]$ denote to the coset $\{f|_{S_m} + h|_{S_m} : h|_{S_m} \in H^\infty|_{S_m}\}$. As $(L^\infty|_{S_m})/(H^\infty|_{S_m})$ is a Banach space, we consider the following three cases:

(1) $\dim(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\}) = 0$;

(2) $\dim(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\}) = 1$;

(3) $\dim(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\}) = 2$.

Case 1 If $\dim(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\}) = 0$, then $[f|_{S_m}] = [g|_{S_m}] = 0$, which implies that $f|_{S_m}, g|_{S_m} \in H^\infty|_{S_m}$.

Case 2 If $\dim(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\}) = 1$, we may assume that $[g|_{S_m}] \neq 0$. Then there is a constant c such that $[f|_{S_m}] = c[g|_{S_m}]$. By Lemma 2.7, now (3.1) can be rewritten as follows:

$$\begin{aligned} &H_{\overline{g}}k_z \otimes H_{cg}k_z - H_{\overline{f}}k_z \otimes H_gk_z \\ &= H_{u\overline{g}}k_z \otimes H_{cug}k_z - H_{u\overline{f}}k_z \otimes H_{ug}k_z + \varepsilon(z) \end{aligned}$$

to obtain

$$H_{\overline{cg}-\overline{f}}k_z \otimes H_gk_z = H_{u(\overline{cg}-\overline{f})}k_z \otimes H_{ug}k_z + \varepsilon(z), \tag{3.2}$$

where $\varepsilon(z)$ satisfies $\|\varepsilon(z)\| \to 0$ as $z \to m$.

To derive the desired conclusions, we are going to discuss two cases. First, if

$$\varliminf_{z\to m} \|H_{\overline{cg}-\overline{f}}k_z\|_2 = 0,$$

then $(\overline{cg}-\overline{f})|_{S_m} \in H^\infty|_{S_m}$. Since $[(f-cg)|_{S_m}] = 0$ and that the support set is a set of antisymmetry for $H^\infty + C$ (see [11] or [17]), we obtain that $(f-cg)|_{S_m}$ must be a constant.

Now we need to analyse the case of

$$\varliminf_{z\to m} \|H_{\overline{cg}-\overline{f}}k_z\|_2 > 0.$$

By (3.2), we have

$$\langle H_{\overline{cg}-\overline{f}}k_z, H_{\overline{cg}-\overline{f}}k_z\rangle H_g k_z = \langle H_{u(\overline{cg}-\overline{f})}k_z, H_{\overline{cg}-\overline{f}}k_z\rangle H_{ug}k_z + \varepsilon(z).$$

Thus there exists a constant $a(z)$ depending on z such that

$$H_g k_z = a(z)H_{ug}k_z + \varepsilon(z),$$

where $a(z)$ satisfies that

$$|a(z)| = \left|\frac{\langle H_{u(\overline{cg}-\overline{f})}k_z, H_{\overline{cg}-\overline{f}}k_z\rangle}{\|H_{\overline{cg}-\overline{f}}k_z\|_2^2}\right| \le 1$$

for all $z \in \mathcal{O}(m)\cap\mathbb{D}$, so $|a(z)|$ is bounded for $z \in \mathcal{O}(m)\cap\mathbb{D}$. For $m \in \mathcal{M}(H^\infty + C)$, $\mathcal{O}(m)$ denotes a neighborhood of it in $\mathcal{M}(H^\infty)$.

By the boundedness of $a(z)$ and by the corona theorem, there exists a net $\{z_\beta\}$ and a constant $a \in \mathbb{C}$ such that

$$\lim_\beta z_\beta = m \quad \text{and} \quad \lim_\beta a(z_\beta) = a.$$

Therefore, by the the equivalence between conditions (2)–(3) in Lemma 2.7, we obtain

$$H_g k_z = aH_{ug}k_z + \varepsilon(z).$$

Hence we have that

$$\lim_{z\to m} \|H_{(1-au)g}k_z\|_2 = 0.$$

Making a change of variables yields

$$\lim_{z\to m} \|(I-P)[(1-au\circ\phi_z)(g\circ\phi_z)]\|_2 = 0.$$

Since $|a| \le 1$ and that u is not a constant on S_m, we have by [15, Lemma 1] that $(1-au)$ is an outer function on the support set S_m. Therefore, for any $\varepsilon > 0$ there exists a function $p \in H^\infty$ such that

$$\int_{S_m} |p(1-au)-1|^2 \mathrm{d}\mu_m < \varepsilon.$$

For such $\varepsilon > 0$, there also exists a neighborhood $\mathcal{O}(m)$ of m such that

$$\Big|\int_{S_m}|p(1-au)-1|^2\mathrm{d}\mu_m-\int_{S_m}|p(1-au)-1|^2\cdot|k_z|^2\frac{\mathrm{d}\theta}{2\pi}\Big|<\varepsilon$$

for $z\in\mathcal{O}(m)\cap\mathbb{D}$. Changing of variable gives

$$\int_{S_m}|p\circ\phi_z(1-au\circ\phi_z)-1|^2\frac{\mathrm{d}\theta}{2\pi}<2\varepsilon.$$

Applying the Hölder inequality, we obtain that

$$\begin{aligned}&\|(I-P)\{(g\circ\phi_z)\cdot[p\circ\phi_z(1-au\circ\phi_z)-1]\}\|_{\frac{4}{3}}\\ &\leq C_1\|g\circ\phi_z\|_4\cdot\|p\circ\phi_z(1-au\circ\phi_z)-1\|_2\leq C_1\|g\|_\infty\varepsilon^{\frac{1}{2}}\end{aligned}$$

for some constant $C_1>0$. Combining the above inequality with the identity

$$(I-P)\{(g\circ\phi_z)(p\circ\phi_z)(1-au\circ\phi_z)\}=S_{p\circ\phi_z}H_{g\circ\phi_z}(1-au\circ\phi_z)$$

gives us

$$\begin{aligned}&\|(I-P)(g\circ\phi_z)\|_{\frac{4}{3}}\\ &\leq C_1\|g\|_\infty\varepsilon^{\frac{1}{2}}+\|(I-P)\{(g\circ\phi_z)(p\circ\phi_z)(1-au\circ\phi_z)\}\|_{\frac{4}{3}}\\ &\leq C_1\|g\|_\infty\varepsilon^{\frac{1}{2}}+\|p\|_\infty\cdot\|(I-P)[(1-au\circ\phi_z)(g\circ\phi_z)]\|_2.\end{aligned}$$

Recalling that

$$\lim_{z\to m}\|(I-P)[(1-au\circ\phi_z)(g\circ\phi_z)]\|_2=0,$$

we get

$$\lim_{z\to m}\|(I-P)(g\circ\phi_z)\|_{\frac{4}{3}}\leq C_1\|g\|_\infty\varepsilon^{\frac{1}{2}}.$$

Note that the projection P is bounded on L^4, there exists an absolute constant $C>0$ such that

$$\|(I-P)(g\circ\phi_z)\|_4\leq C\|g\|_\infty.$$

In addition, since

$$\|(I-P)(g\circ\phi_z)\|_2^2\leq\|(I-P)(g\circ\phi_z)\|_{\frac{4}{3}}\cdot\|(I-P)(g\circ\phi_z)\|_4,$$

it follows that

$$\lim_{z\to m}\|H_gk_z\|_2=\lim_{z\to m}\|(I-P)(g\circ\phi_z)\|_2=0.$$

Thus we conclude by Lemma 2.7 that $g|_{S_m}\in H^\infty|_{S_m}$, which contradicts our assumption.

Case 3 Suppose that $\dim(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\})=2$. In this case, we need to further consider the dimension of $\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}$.

Subcase 3(i) If $\dim(\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\})=0$, then we have $\overline{f}|_{S_m},\ \overline{g}|_{S_m}\in H^\infty|_{S_m}$.

Subcase 3(ii) Suppose that $\dim(\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\})=1$. Without loss of generality, we may assume that $[\overline{g}|_{S_m}]\neq 0$ and $[\overline{f}|_{S_m}]=d[\overline{g}|_{S_m}]$ for some constant d. Then $(\overline{f}-d\overline{g})|_{S_m}\in H^\infty|_{S_m}$, we have by Lemma 2.7 that

$$H_{\overline{f}}k_z=dH_{\overline{g}}k_z+\varepsilon(z)\quad\text{and}\quad H_{u\overline{f}}k_z=dH_{u\overline{g}}k_z+\varepsilon(z),$$

where the second equation follows from that $H_{u\varphi} = S_u H_\varphi$ for all $\varphi \in L^\infty$. Thus we can rewrite (3.1) as follows

$$H_{\overline{g}} k_z \otimes H_{f-\overline{d}g} k_z = H_{u\overline{g}} k_z \otimes H_{u(f-\overline{d}g)} k_z + \varepsilon(z).$$

Using the same arguments as the one in Case 2, we conclude that $(f - \overline{d}g)|_{S_m} \in H^\infty|_{S_m}$. So we have that $(f - \overline{d}g)|_{S_m}$ is a constant, as desired.

Subcase 3(iii) Finally, we consider the case that dim $(\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}) = 2$. In this subcase, $\varliminf_{z\to m} \|H_f k_z\|_2$, $\varliminf_{z\to m} \|H_g k_z\|_2$, $\varliminf_{z\to m} \|H_{\overline{f}} k_z\|_2$ and $\varliminf_{z\to m} \|H_{\overline{g}} k_z\|_2$ are all positive. By (3.1), we have

$$\begin{aligned} &\langle H_{u\overline{g}} k_z, H_{\overline{g}} k_z\rangle H_f k_z - \langle H_{u\overline{g}} k_z, H_{\overline{f}} k_z\rangle H_g k_z \\ &= \langle H_{u\overline{g}} k_z, H_{u\overline{g}} k_z\rangle H_{uf} k_z - \langle H_{u\overline{g}} k_z, H_{u\overline{f}} k_z\rangle H_{ug} k_z + \varepsilon(z) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\langle H_{u\overline{f}} k_z, H_{\overline{g}} k_z\rangle H_f k_z - \langle H_{u\overline{f}} k_z, H_{\overline{f}} k_z\rangle H_g k_z \\ &= \langle H_{u\overline{f}} k_z, H_{u\overline{g}} k_z\rangle H_{uf} k_z - \langle H_{u\overline{f}} k_z, H_{u\overline{f}} k_z\rangle H_{ug} k_z + \varepsilon(z). \end{aligned} \tag{3.4}$$

In order to complete the discussion of Subcase 3(iii), the following claim is required.

Claim 3.1 $\varliminf_{z\to m} (\|H_{u\overline{f}} k_z\|_2^2 \cdot \|H_{u\overline{g}} k_z\|_2^2 - |\langle H_{u\overline{f}} k_z, H_{u\overline{g}} k_z\rangle|^2) = \delta > 0$ for some δ.

As the proof of the above claim is long, let us assume that the Claim 3.1 holds for the moment and we will give its proof later.

Based on Claim 3.1, we have by (3.3)–(3.4) that there are $a_{11}(z)$, $a_{12}(z)$, $a_{21}(z)$ and $a_{22}(z)$ such that

$$\begin{cases} H_{uf} k_z = a_{11}(z) H_f k_z + a_{12}(z) H_g k_z + \varepsilon(z), \\ H_{ug} k_z = a_{21}(z) H_f k_z + a_{22}(z) H_g k_z + \varepsilon(z), \end{cases} \tag{3.5}$$

where $z \in \mathcal{O}(m) \cap \mathbb{D}$. Furthermore, observe that the functions $\{a_{ij}(z)\}_{i,j=1}^2$ are all bounded for $z \in \mathcal{O}(m) \cap \mathbb{D}$.

Applying the same technique as the one used in Case 2, we conclude that there exist constants $\{a_{11}, a_{12}, a_{21}, a_{22}\}$ which are independent of z such that for $z \in \mathcal{O}(m) \cap \mathbb{D}$:

$$\begin{cases} H_{uf} k_z = a_{11} H_f k_z + a_{12} H_g k_z + \varepsilon(z), \\ H_{ug} k_z = a_{21} H_f k_z + a_{22} H_g k_z + \varepsilon(z). \end{cases} \tag{3.6}$$

Without loss of generality, we may assume that the coefficient matrix of (3.6) has the following form:

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where the above two matrices are the Jordan canonical forms for (a_{ij}). In fact, there is an invertible matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

such that

$$B\begin{pmatrix} H_{uf}k_z \\ H_{ug}k_z \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B\begin{pmatrix} H_f k_z \\ H_g k_z \end{pmatrix} + \varepsilon(z)$$

or

$$B\begin{pmatrix} H_{uf}k_z \\ H_{ug}k_z \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} B\begin{pmatrix} H_f k_z \\ H_g k_z \end{pmatrix} + \varepsilon(z).$$

This gives that

$$\begin{pmatrix} H_{u(b_{11}f+b_{12}g)}k_z \\ H_{u(b_{21}f+b_{22}g)}k_z \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} H_{(b_{11}f+b_{12}g)}k_z \\ H_{(b_{21}f+b_{22}g)}k_z \end{pmatrix} + \varepsilon(z)$$

or

$$\begin{pmatrix} H_{u(b_{11}f+b_{12}g)}k_z \\ H_{u(b_{21}f+b_{22}g)}k_z \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} H_{(b_{11}f+b_{12}g)}k_z \\ H_{(b_{21}f+b_{22}g)}k_z \end{pmatrix} + \varepsilon(z).$$

Now define $F = b_{11}f + b_{12}g$ and $G = b_{21}f + b_{22}g$. Then we have that $\overline{f}|_{S_m},\ \overline{g}|_{S_m} \in H^\infty|_{S_m}$ if and only if $\overline{F}|_{S_m},\ \overline{G}|_{S_m} \in H^\infty|_{S_m}$, since the matrix (b_{ij}) is invertible.

If the above coefficient matrix for (3.6) is

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

then we have

$$\begin{cases} H_{uf}k_z = \lambda_1 H_f k_z + \varepsilon(z), \\ H_{ug}k_z = \lambda_2 H_g k_z + \varepsilon(z). \end{cases}$$

Solving the above system gives

$$|\lambda_1| = \frac{|\langle S_u H_f k_z, H_f k_z\rangle|}{\|H_f k_z\|_2^2} + \varepsilon(z)$$

and

$$|\lambda_2| = \frac{|\langle S_u H_g k_z, H_g k_z\rangle|}{\|H_g k_z\|_2^2} + \varepsilon(z)$$

for all $z \in \mathcal{O}(m) \cap \mathbb{D}$. Since u is an inner function, we conclude that $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$. Thus we have

$$H_{\overline{g}}k_z \otimes H_f k_z - H_{\overline{f}}k_z \otimes H_g k_z = H_{u\overline{g}}k_z \otimes \lambda_1 H_f k_z - H_{u\overline{f}}k_z \otimes \lambda_2 H_g k_z + \varepsilon(z)$$

to obtain

$$H_{(1-\overline{\lambda}_1 u)\overline{g}}k_z \otimes H_f k_z = H_{(1-\overline{\lambda}_2 u)\overline{f}}k_z \otimes H_g k_z + \varepsilon(z)$$

for $z \in \mathcal{O}(m) \cap \mathbb{D}$. This gives that

$$\begin{cases} \langle H_f k_z, H_f k_z\rangle H_{(1-\overline{\lambda}_1 u)\overline{g}}k_z = \langle H_f k_z, H_g k_z\rangle H_{(1-\overline{\lambda}_2 u)\overline{f}}k_z + \varepsilon(z), \\ \langle H_g k_z, H_f k_z\rangle H_{(1-\overline{\lambda}_1 u)\overline{g}}k_z = \langle H_g k_z, H_g k_z\rangle H_{(1-\overline{\lambda}_2 u)\overline{f}}k_z + \varepsilon(z), \end{cases} \tag{3.7}$$

where $z \in \mathcal{O}(m) \cap \mathbb{D}$.

Since $[f|_{S_m}]$ and $[g|_{S_m}]$ are linearly independent, we first show that

$$\varliminf_{z\to m} (\|H_f k_z\|_2^2 \cdot \|H_g k_z\|_2^2 - |\langle H_f k_z, H_g k_z\rangle|^2) = \mu > 0. \tag{3.8}$$

Otherwise, there is a net $\{z_\beta\} \subset \mathbb{D}$ such that

$$\lim_{z_\beta\to m} (\|H_f k_{z_\beta}\|_2^2 \cdot \|H_g k_{z_\beta}\|_2^2 - |\langle H_f k_{z_\beta}, H_g k_{z_\beta}\rangle|^2) = 0.$$

For $z \in \mathcal{O}(m) \cap \mathbb{D}$, we let

$$\lambda_z = \frac{\langle H_f k_z, H_g k_z\rangle}{\|H_g k_z\|_2^2}.$$

Clearly, λ_z is bounded for $z \in \mathcal{O}(m) \cap \mathbb{D}$, since $\varliminf\limits_{z\to m} \|H_g k_z\|_2 > 0$. Then

$$\|H_f k_z - \lambda_z H_g k_z\|_2^2 = \frac{\|H_f k_z\|_2^2 \cdot \|H_g k_z\|_2^2 - |\langle H_f k_z, H_g k_z\rangle|^2}{\|H_g k_z\|_2^2}$$

for each z in the neighborhood $\mathcal{O}(m) \cap \mathbb{D}$. On the other hand, we can choose a subnet $\{z_{\beta,\gamma}\}$ of $\{z_\beta\}$ such that $\lim\limits_{z_{\beta,\gamma}\to m} \lambda_{z_{\beta,\gamma}} = \lambda$ for some λ, and we also have

$$\lim_{z_{\beta,\gamma}\to m} \|H_f k_{z_{\beta,\gamma}} - \lambda H_g k_{z_{\beta,\gamma}}\|_2 = 0.$$

Now Lemma 2.7 gives

$$\lim_{z\to m} \|H_f k_z - \lambda H_g k_z\|_2 = 0$$

to obtain that $(f - \lambda g)|_{S_m} \in H^\infty|_{S_m}$, which is impossible since our assumption is

$$\dim\,(\mathrm{span}\{[f|_{S_m}],\ [g|_{S_m}]\}) = 2.$$

The contradiction implies that $\mu > 0$.

By (3.7), we have

$$(\|H_f k_z\|_2^2 \cdot \|H_g k_z\|_2^2 - |\langle H_f k_z, H_g k_z\rangle|^2) H_{(1-\overline{\lambda}_1 u)\overline{g}} k_z = \varepsilon(z)$$

and

$$(\|H_f k_z\|_2^2 \cdot \|H_g k_z\|_2^2 - |\langle H_f k_z, H_g k_z\rangle|^2) H_{(1-\overline{\lambda}_2 u)\overline{f}} k_z + \varepsilon(z) = 0.$$

Thus we conclude by (3.8) that

$$\lim_{z\to m} \|H_{(1-\overline{\lambda}_1 u)\overline{g}} k_z\|_2 = 0$$

and

$$\lim_{z\to m} \|H_{(1-\overline{\lambda}_2 u)\overline{f}} k_z\|_2 = 0.$$

Repeating the arguments in the last two paragraphs of Case 2, we have $\overline{f}|_{S_m},\ \overline{g}|_{S_m} \in H^\infty|_{S_m}$, which is a contradiction.

In order to finish the proof, it remains to consider the case that the coefficient matrix of (3.6) is

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

In this case, we have that for $z \in \mathcal{O}(m) \cap \mathbb{D}$:

$$\begin{cases} H_{uf}k_z = \lambda_1 H_f k_z + H_g k_z + \varepsilon(z), \\ H_{ug}k_z = \lambda_1 H_g k_z + \varepsilon(z). \end{cases}$$

Using the same arguments as the above, we also have $|\lambda_1| \le 1$ and

$$\begin{aligned} & H_{\overline{g}}k_z \otimes H_f k_z - H_{\overline{f}}k_z \otimes H_g k_z \\ &= H_{u\overline{g}}k_z \otimes (\lambda_1 H_f k_z + H_g k_z) - H_{u\overline{f}}k_z \otimes \lambda_1 H_g k_z + \varepsilon(z), \end{aligned}$$

which is equivalent to

$$H_{(1-\overline{\lambda}_1 u)\overline{g}}k_z \otimes H_f k_z = H_{u\overline{g}+(1-\overline{\lambda}_1 u)\overline{f}}k_z \otimes H_g k_z + \varepsilon(z).$$

Since $[f|_{S_m}]$ and $[g|_{S_m}]$ are linearly independent, we deduce from the (2×2) determinant argument that

$$\begin{cases} ((1-\overline{\lambda}_1 u)\overline{g})|_{S_m} \in H^\infty|_{S_m}, \\ ((1-\overline{\lambda}_1 u)\overline{f} + (u\overline{g}))|_{S_m} \in H^\infty|_{S_m}. \end{cases} \tag{3.9}$$

Then by the condition $((1-\overline{\lambda}_1 u)\overline{g})|_{S_m} \in H^\infty|_{S_m}$ and the last two two paragraphs of Case 2, we have

$$\overline{g}|_{S_m} \in H^\infty|_{S_m}.$$

This contradicts our assumption that $\dim\,(\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}) = 2$.

To complete the whole proof of Proposition 3.1, we need to show that the following result holds under the assumption that $\underline{\lim}_{z\to m} \|H_f k_z\|_2$, $\underline{\lim}_{z\to m} \|H_g k_z\|_2$, $\underline{\lim}_{z\to m} \|H_{\overline{f}} k_z\|_2$ and $\underline{\lim}_{z\to m} \|H_{\overline{g}} k_z\|_2$ are all positive:

$$\underline{\lim}_{z\to m} (\|H_{u\overline{f}}k_z\|_2^2 \cdot \|H_{u\overline{g}}k_z\|_2^2 - |\langle H_{u\overline{f}}k_z, H_{u\overline{g}}k_z\rangle|^2) = \delta > 0.$$

Proof of Claim 3.1 By the Cauchy-Schwarz inequality, we have

$$\underline{\lim}_{z\to m} (\|H_{u\overline{f}}k_z\|_2^2 \cdot \|H_{u\overline{g}}k_z\|_2^2 - |\langle H_{u\overline{f}}k_z, H_{u\overline{g}}k_z\rangle|^2) \ge 0.$$

If the above conclusion does not hold, we can find a net $\{z_\alpha\} \subset \mathbb{D}$ such that $z_\alpha \to m$ and

$$\lim_{z_\alpha\to m} (\|H_{u\overline{f}}k_{z_\alpha}\|_2^2 \cdot \|H_{u\overline{g}}k_{z_\alpha}\|_2^2 - |\langle H_{u\overline{f}}k_{z_\alpha}, H_{u\overline{g}}k_{z_\alpha}\rangle|^2) = 0.$$

We first show that $\underline{\lim}_{z\to m} \|H_{u\overline{g}}k_z\|_2 > 0$. If this was not the case, then Lemma 2.7 gives

$$\lim_{z\to m} \|H_{u\overline{g}}k_z\|_2 = 0.$$

Thus we can rewrite (3.1) as follows

$$H_{\overline{f}}k_z \otimes H_g k_z - H_{\overline{g}}k_z \otimes H_f k_z$$

$$= H_{u\overline{f}}k_z \otimes H_{ug}k_z + \varepsilon(z). \tag{3.10}$$

This implies that

$$\begin{aligned} &\langle H_{\overline{g}}k_z, H_{\overline{g}}k_z\rangle H_f k_z - \langle H_{\overline{g}}k_z, H_{\overline{f}}k_z\rangle H_g k_z \\ &= -\langle H_{\overline{g}}k_z, H_{u\overline{f}}k_z\rangle H_{ug}k_z + \varepsilon(z) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\langle H_{\overline{f}}k_z, H_{\overline{g}}k_z\rangle H_f k_z - \langle H_{\overline{f}}k_z, H_{\overline{f}}k_z\rangle H_g k_z \\ &= -\langle H_{\overline{f}}k_z, H_{u\overline{f}}k_z\rangle H_{ug}k_z + \varepsilon(z). \end{aligned} \tag{3.12}$$

Using the method as the one in the proof of (3.8), we obtain

$$\varliminf_{z\to m} (\|H_{\overline{f}}k_z\|_2^2 \cdot \|H_{\overline{g}}k_z\|_2^2 - |\langle H_{\overline{f}}k_z, H_{\overline{g}}k_z\rangle|^2) > 0,$$

since $[\overline{f}|_{S_m}]$ and $[\overline{g}|_{S_m}]$ are also linearly independent. Therefore, we have by (3.11)–(3.12) that there exists $b(z)$ such that

$$H_g k_z = b(z)H_{ug}k_z + \varepsilon(z)$$

for all $z \in \mathcal{O}(m) \cap \mathbb{D}$. Moreover, $b(z)$ is bounded for $z \in \mathcal{O}(m) \cap \mathbb{D}$. Thus we can choose a net $\{z_\zeta\}$ such that $\lim_\zeta z_\zeta = m$ and $\lim_\zeta b(z_\zeta) = b$. Using Lemma 2.7 again, we obtain that

$$H_g k_z = bH_{ug}k_z + \varepsilon(z) \tag{3.13}$$

for all $z \in \mathcal{O}(m) \cap \mathbb{D}$. As $\varliminf_{z\to m} \|H_g k_z\|_2 > 0$ and

$$\|H_g k_z\|_2 = \|bH_{ug}k_z + \varepsilon(z)\|_2 \le |b| \cdot \|H_g k_z\|_2 + \|\varepsilon(z)\|_2,$$

we conclude that $|b| \ge 1$.

Using (3.10), we have

$$H_{\overline{g}}k_z \otimes H_f k_z = H_{(\overline{b}-u)\overline{f}}k_z \otimes H_{ug}k_z + \varepsilon(z)$$

and

$$H_f k_z = c(z)H_{ug}k_z + \varepsilon(z),$$

where

$$c(z) = \frac{\langle H_{\overline{g}}k_z, H_{(\overline{b}-u)\overline{f}}k_z\rangle}{\|H_{\overline{g}}k_z\|_2^2}$$

is bounded for $z \in \mathcal{O}(m) \cap \mathbb{D}$. So there is a constant c (which is independent of z) such that

$$H_f k_z = cH_{ug}k_z + \varepsilon(z).$$

As we have shown

$$H_g k_z = bH_{ug}k_z + \varepsilon(z)$$

for all $z \in \mathcal{O}(m) \cap \mathbb{D}$, it follows that

$$H_f k_z = \frac{c}{b}H_g k_z + \varepsilon(z).$$

This implies $(f-\frac{c}{b}g)|_{S_m}\in H^\infty|_{S_m}$. But this contradicts our assumption that

$$\dim\,(\text{span}\{[f|_{S_m}],\ [g|_{S_m}]\})=2.$$

So we have $\varliminf\limits_{z\to m}\|H_{u\overline{g}}k_z\|_2>0$.

Recall that our assumption is

$$\varliminf_{z\to m}\,(\|H_{u\overline{f}}k_z\|_2^2\cdot\|H_{u\overline{g}}k_z\|_2^2-|\langle H_{u\overline{f}}k_z,H_{u\overline{g}}k_z\rangle|^2)=0.$$

Using the same method as the one in the proof of (3.8), there exists a constant λ' such that

$$\lim_{z\to m}\|H_{u\overline{f}}k_z-\lambda' H_{u\overline{g}}k_z\|_2=0.$$

Combining the above limit with (3.1) gives that

$$H_{\overline{g}}k_z\otimes H_fk_z-H_{\overline{f}}k_z\otimes H_gk_z=H_{u\overline{g}}k_z\otimes H_{u(f-\overline{\lambda'}g)}k_z+\varepsilon(z).$$

Rewrite the above formula as the following

$$H_{\overline{g}}k_z\otimes H_{(f-\overline{\lambda'}g)}k_z-H_{(\overline{f}-\lambda'\overline{g})}k_z\otimes H_gk_z=H_{u\overline{g}}k_z\otimes H_{u(f-\overline{\lambda'}g)}k_z+\varepsilon(z). \tag{3.14}$$

Since

$$\dim\,(\text{span}\{[f|_{S_m}],\ [g|_{S_m}]\})=\dim\,(\text{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\})=2,$$

we have

$$\dim\,(\text{span}\{[(f-\overline{\lambda'}g)|_{S_m}],\ [g|_{S_m}]\})=\dim\,(\text{span}\{[(\overline{f}-\lambda'\overline{g})|_{S_m}],\ [\overline{g}|_{S_m}]\})=2.$$

Comparing (3.14) with (3.10) and then repeating the same arguments as used in (3.13), we have

$$H_{(f-\overline{\lambda'}g)}k_z=b'H_{u(f-\overline{\lambda'}g)}k_z+\varepsilon(z)$$

and

$$H_gk_z=c'H_{u(f-\overline{\lambda'}g)}k_z+\varepsilon(z),$$

where $b',\ c'$ are independent of z and moreover, $|b'|\geq 1$ and $c'\neq 0$, since $[g|_{S_m}]\neq 0$. Thus we have

$$\lim_{z\to m}\|H_{(g-\frac{c'}{b'}(f-\overline{\lambda'}g))}k_z\|_2=0.$$

This yields that

$$\left(g-\frac{c'}{b'}(f-\overline{\lambda'}g)\right)\Big|_{S_m}\in H^\infty|_{S_m}.$$

But it is a contradiction, since $\dim\,(\text{span}\{[f|_{S_m}],\ [g|_{S_m}]\})=2$. This completes the proof of Claim 3.1 and hence the proof of Proposition 3.1.

Combining the preceding proposition with the two relations in Remark 2.1, we obtain the following proposition which gives a necessary condition for the compactness of the fourth operator $H_{uf}H^*_{u\overline{g}}+S_fS_g-H_{ug}H^*_{u\overline{f}}-S_gS_f$ in Lemma 2.2.

Proposition 3.2 *Let u be a nonconstant inner function, $f, g \in L^\infty$ and $m \in \mathcal{M}(H^\infty + C)$. Suppose that the operator*

$$H_{uf}H^*_{u\overline{g}} + S_f S_g - H_{ug}H^*_{u\overline{f}} - S_g S_f$$

is compact. Then for the support set S_m of m, one of the following conditions holds:

(1) *Both $f|_{S_m}$ and $g|_{S_m}$ are in $H^\infty|_{S_m}$;*

(2) *both $\overline{f}|_{S_m}$ and $\overline{g}|_{S_m}$ are in $H^\infty|_{S_m}$;*

(3) *there exist constants a, b, not both zero, such that $(af + bg)|_{S_m}$ is a constant.*

Next, we will obtain a necessary condition for the compactness of the second operator $T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f$ in Lemma 2.2. To do so, we need the following two lemmas.

Lemma 3.1 *Let f and g be in L^2. Then*

$$H_f T_g T_{\phi_z} - S_{\phi_z} H_f T_g = H_f k_z \otimes T_{\overline{g\phi_z}} k_z = -(H_f k_z) \otimes (V H_g k_z)$$

for all $z \in \mathbb{D}$.

Proof Using the identity (see [24, Page 480])

$$I = k_z \otimes k_z + T_{\phi_z} T_{\overline{\phi_z}},$$

we obtain

$$\begin{aligned} H_f T_g T_{\phi_z} &= H_f (k_z \otimes k_z + T_{\phi_z} T_{\overline{\phi_z}}) T_g T_{\phi_z} \\ &= (H_f k_z \otimes k_z) T_{g\phi_z} + H_f T_{\phi_z} T_{\overline{\phi_z} g \phi_z} \\ &= (H_f k_z) \otimes (T_{\overline{g}\overline{\phi_z}} k_z) + H_f T_{\phi_z} T_g. \end{aligned}$$

Using Identity (4.6) of [21], we have

$$S_{\phi_z} H_f = H_f T_{\phi_z}.$$

It follows that

$$H_f T_g T_{\phi_z} - S_{\phi_z} H_f T_g = (H_f k_z) \otimes (T_{\overline{g}\overline{\phi_z}} k_z).$$

To obtain the last equality, we recall that $V^2 = I$ and observe that

$$\begin{aligned} V T_{\overline{g\phi_z}} k_z &= V P(\overline{g\phi_z} k_z) \\ &= (I - P) V(\overline{g\phi_z} k_z) \\ &= (I - P)(\overline{w} g(w) \phi_z(w) \overline{k_z(w)}) \\ &= (I - P)\Big(\overline{w} g(w) \frac{z - w}{1 - \overline{z}w} \frac{\sqrt{1 - |z|^2}}{1 - z\overline{w}}\Big) \\ &= -(I - P)\Big(g(w) \frac{\sqrt{1 - |z|^2}}{1 - \overline{z}w}\Big) \\ &= -H_g k_z, \end{aligned}$$

which gives the desired result.

Lemma 3.2 *Let $K : H^2 \to \overline{zH^2}$ be a compact operator. Then*

$$\lim_{|z|\to 1^-} \|S_{\phi_z} K - K T_{\phi_z}\| = 0.$$

Proof Since each compact operator can be approximated by finite rank operator in norm, we need only to consider the case that K is a rank-one operator.

Suppose that $K = f \otimes g$, where $f \in \overline{zH^2}$ and $g \in H^2$. Then

$$\begin{aligned} S_{\phi_z} K - K T_{\phi_z} &= S_{\phi_z}(f \otimes g) - (f \otimes g) T_{\phi_z} \\ &= (S_{\phi_z} f) \otimes g - f \otimes (T^*_{\phi_z} g). \end{aligned}$$

For every w on $\partial\mathbb{D}$, letting $|z| \to 1^-$, we have

$$z - \phi_z(w) = \frac{1 - |z|^2}{1 - \overline{z}w} w \to 0.$$

So we have by the dominated convergence theorem that

$$\|zf - \phi_z f\|_2 \to 0 \quad \text{and} \quad \|\overline{z}g - \overline{\phi_z} g\|_2 \to 0$$

as $|z| \to 1^-$. It follows that $\|\xi f - \phi_z f\|_2 \to 0$ and $\|\overline{\xi} g - \overline{\phi_z} g\|_2 \to 0$ if $z \to \xi \in \partial\mathbb{D}$.

Using the assumption that $f \in \overline{zH^2}$ and $g \in H^2$, we obtain

$$\|\xi f - S_{\phi_z} f\|_2 = \|\xi f - (I - P)(\phi_z f)\|_2 \to 0$$

and

$$\|\overline{\xi} g - T^*_{\phi_z} g\|_2 = \|\overline{\xi} g - P(\overline{\phi_z} g)\|_2 \to 0$$

as $z \to \xi$. Then we obtain that

$$\begin{aligned} &\|(S_{\phi_z} f) \otimes g - f \otimes (T^*_{\phi_z} g)\| \\ &= \|(S_{\phi_z} f) \otimes g - \xi f \otimes g + f \otimes \overline{\xi} g - f \otimes T^*_{\phi_z} g\| \\ &\le \|(S_{\phi_z} f) \otimes g - \xi f \otimes g\| + \|f \otimes (\overline{\xi} g) - f \otimes (T^*_{\phi_z} g)\| \\ &= \|(S_{\phi_z} f - \xi f) \otimes g\| + \|f \otimes (\overline{\xi} g - T^*_{\phi_z} g)\| \\ &= \|S_{\phi_z} f - \xi f\|_2 \cdot \|g\|_2 + \|f\|_2 \cdot \|\overline{\xi} g - T^*_{\phi_z} g\|_2 \end{aligned}$$

to get

$$\lim_{|z|\to 1^-} \|S_{\phi_z} f \otimes g - f \otimes T^*_{\phi_z} g\| = 0,$$

which completes the proof.

Remark 3.1 The Carleson-Corona theorem (see [11]) tells us that the conclusions of Lemmas 2.6 and 3.2 are equivalent to the condition that for each $m \in \mathcal{M}(H^\infty + C)$,

$$\lim_{z\to m} \|K - T^*_{\phi_z} K T_{\phi_z}\| = 0 \quad \text{and} \quad \lim_{z\to m} \|S_{\phi_z} K - K T_{\phi_z}\| = 0$$

for z in the unit disk $\mathbb{D}$.

Combining Lemmas 3.1–3.2, we obtain the following necessary condition for the compactness of the operator $T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f$.

Proposition 3.3 *Suppose that u is a nonconstant inner function and $f, g \in L^\infty$. Let $m \in \mathcal{M}(H^\infty + C)$ and S_m be its support set. Suppose that the operator*

$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f$$

is compact and $f|_{S_m},\ g|_{S_m} \in H^\infty|_{S_m}$. Then either

(1) *$((u-\lambda)\overline{f})|_{S_m}$ and $((u-\lambda)\overline{g})|_{S_m}$ are in $H^\infty|_{S_m}$ for some constant λ; or*

(2) *there exist constants a, b, not both zero, such that $(af+bg)|_{S_m}$ is a constant.*

Proof Suppose that

$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f = K \tag{3.15}$$

for some compact operator K. Taking adjoint of (3.15), we have

$$H_{u\overline{g}} T_{\overline{f}} + S_{\overline{g}} H_{u\overline{f}} - H_{u\overline{f}} T_{\overline{g}} - S_{\overline{f}} H_{u\overline{g}} = K^*.$$

By identity (4.5) of [21]

$$H_{\varphi\psi} = H_\varphi T_\psi + S_\varphi H_\psi = H_\psi T_\varphi + S_\psi H_\varphi$$

for any $\varphi,\ \psi \in L^\infty$, we also have

$$H_{u\overline{g}} T_{\overline{f}} - H_{\overline{g}} T_{u\overline{f}} - H_{u\overline{f}} T_{\overline{g}} + H_{\overline{f}} T_{u\overline{g}} = K^*.$$

From Lemma 3.1, we have

$$\begin{aligned} K^* T_{\phi_z} - S_{\phi_z} K^* = {} & H_{u\overline{g}} k_z \otimes T_{f\overline{\phi_z}} k_z - H_{\overline{g}} k_z \otimes T_{\overline{u}f\overline{\phi_z}} k_z \\ & - H_{u\overline{f}} k_z \otimes T_{g\overline{\phi_z}} k_z + H_{\overline{f}} k_z \otimes T_{\overline{u}g\overline{\phi_z}} k_z. \end{aligned}$$

By Lemma 3.2, the norm of the left hand side in the above equality tends to 0 as $z \to m$. Thus we obtain

$$\begin{aligned} & H_{u\overline{f}} k_z \otimes T_{g\overline{\phi_z}} k_z - H_{u\overline{g}} k_z \otimes T_{f\overline{\phi_z}} k_z \\ & = H_{\overline{f}} k_z \otimes T_{\overline{u}g\overline{\phi_z}} k_z - H_{\overline{g}} k_z \otimes T_{\overline{u}f\overline{\phi_z}} k_z + \varepsilon(z). \end{aligned} \tag{3.16}$$

By Lemma 3.1 and (3.16), we have

$$\begin{aligned} & H_{u\overline{g}} k_z \otimes V H_{\overline{f}} k_z - H_{u\overline{f}} k_z \otimes V H_{\overline{g}} k_z \\ & = H_{\overline{g}} k_z \otimes V H_{u\overline{f}} k_z - H_{\overline{f}} k_z \otimes V H_{u\overline{g}} k_z + \varepsilon(z). \end{aligned} \tag{3.17}$$

For $[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}] \in (L^\infty|_{S_m})/(H^\infty|_{S_m})$, the dimension of $\operatorname{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}$ should be 0, 1, or 2. Let us analyse these three cases in the following.

Case 1 If $\dim(\operatorname{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}) = 0$, then $[\overline{f}|_{S_m}] = [\overline{g}|_{S_m}] = 0$, which implies that $\overline{f}|_{S_m},\ \overline{g}|_{S_m} \in H^\infty|_{S_m}$. This gives that $f|_{S_m}$ and $g|_{S_m}$ are constants, and $(f+g)|_{S_m}$ is also a constant.

Case 2 If $\dim(\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}) = 1$, we assume that $[\overline{g}|_{S_m}] \neq 0$. Then there is a constant λ such that $[(\overline{f} + \lambda\overline{g})|_{S_m}] = 0$, i.e.,

$$(\overline{f} + \lambda\overline{g})|_{S_m} \in H^\infty|_{S_m}.$$

On the other hand, since $f|_{S_m},\ g|_{S_m} \in H^\infty|_{S_m}$, we get that $(f + \overline{\lambda}g)|_{S_m}$ is a constant.

Case 3 If $\dim(\mathrm{span}\{[\overline{f}|_{S_m}],\ [\overline{g}|_{S_m}]\}) = 2$, Lemma 2.7 gives that

$$\underline{\lim}_{z\to m} \|H_{\overline{f}}k_z\|_2 \geq d_1 > 0 \quad \text{and} \quad \underline{\lim}_{z\to m} \|H_{\overline{g}}k_z\|_2 \geq d_2 > 0$$

for some constants d_1 and d_2. By (3.17), we have

$$\begin{aligned}&\langle VH_{\overline{f}}k_z, VH_{\overline{f}}k_z\rangle H_{u\overline{g}}k_z - \langle VH_{\overline{f}}k_z, VH_{\overline{g}}k_z\rangle H_{u\overline{f}}k_z\\ &= \langle VH_{\overline{f}}k_z, VH_{u\overline{f}}k_z\rangle H_{\overline{g}}k_z - \langle VH_{\overline{f}}k_z, VH_{u\overline{g}}k_z\rangle H_{\overline{f}}k_z + \varepsilon(z)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\langle VH_{\overline{g}}k_z, VH_{\overline{f}}k_z\rangle H_{u\overline{g}}k_z - \langle VH_{\overline{g}}k_z, VH_{\overline{g}}k_z\rangle H_{u\overline{f}}k_z\\ &= \langle VH_{\overline{g}}k_z, VH_{u\overline{f}}k_z\rangle H_{\overline{g}}k_z - \langle VH_{\overline{g}}k_z, VH_{u\overline{g}}k_z\rangle H_{\overline{f}}k_z + \varepsilon(z).\end{aligned} \tag{3.19}$$

Since V is anti-unitary, we also have

$$\begin{aligned}&\|H_{\overline{f}}k_z\|_2^2 H_{u\overline{g}}k_z - \langle H_{\overline{g}}k_z, H_{\overline{f}}k_z\rangle H_{u\overline{f}}k_z\\ &= \langle H_{u\overline{f}}k_z, H_{\overline{f}}k_z\rangle H_{\overline{g}}k_z - \langle H_{u\overline{g}}k_z, H_{\overline{f}}k_z\rangle H_{\overline{f}}k_z + \varepsilon(z)\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}&\langle H_{\overline{f}}k_z, H_{\overline{g}}k_z\rangle H_{u\overline{g}}k_z - \|H_{\overline{g}}k_z\|_2^2 H_{u\overline{f}}k_z\\ &= \langle H_{u\overline{f}}k_z, H_{\overline{g}}k_z\rangle H_{\overline{g}}k_z - \langle H_{u\overline{g}}k_z, H_{\overline{g}}k_z\rangle H_{\overline{f}}k_z + \varepsilon(z).\end{aligned} \tag{3.21}$$

Using the same arguments as the one in the proof of Claim 3.1, we conclude that

$$\underline{\lim}_{z\to m} (\|H_{\overline{f}}k_z\|_2^2 \cdot \|H_{\overline{g}}k_z\|_2^2 - |\langle H_{\overline{f}}k_z, H_{\overline{g}}k_z\rangle|^2) = \rho$$

for some constant $\rho > 0$. By (3.20)–(3.21), we can find $\{a_{ij}(z)\}_{i,j=1}^2$ such that

$$\begin{pmatrix} H_{u\overline{f}}k_z \\ H_{u\overline{g}}k_z \end{pmatrix} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \begin{pmatrix} H_{\overline{f}}k_z \\ H_{\overline{g}}k_z \end{pmatrix} + \varepsilon(z)$$

for $z \in \mathcal{O}(m) \cap \mathbb{D}$, where $\{a_{ij}(z)\}_{i,j=1}^2$ are bounded for z in $\mathcal{O}(m) \cap \mathbb{D}$. By Lemma 2.7, there are constants $\{a_{ij}\}_{i,j=1}^2$ (independent of z) such that

$$\begin{pmatrix} H_{u\overline{f}}k_z \\ H_{u\overline{g}}k_z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} H_{\overline{f}}k_z \\ H_{\overline{g}}k_z \end{pmatrix} + \varepsilon(z)$$

for $z \in \mathcal{O}(m) \cap \mathbb{D}$, to obtain

$$\begin{cases} H_{u\overline{f}}k_z = a_{11}H_{\overline{f}}k_z + a_{12}H_{\overline{g}}k_z + \varepsilon(z), \\ H_{u\overline{g}}k_z = a_{21}H_{\overline{f}}k_z + a_{22}H_{\overline{g}}k_z + \varepsilon(z). \end{cases} \tag{3.22}$$

Combining (3.17) and (3.22), we have

$$\begin{aligned}&H_{\overline{g}}k_z\otimes V(a_{22}H_{\overline{f}}k_z-a_{12}H_{\overline{g}}k_z)-H_{\overline{f}}k_z\otimes V(-a_{21}H_{\overline{f}}k_z+a_{11}H_{\overline{g}}k_z)\\ =\ &H_{\overline{g}}k_z\otimes VH_{u\overline{f}}k_z-H_{\overline{f}}k_z\otimes VH_{u\overline{g}}k_z+\varepsilon(z).\end{aligned}$$

Since $\left[\overline{f}|_{S_m}\right]$ and $[\overline{g}|_{S_m}]$ are linearly independent, we obtain

$$\begin{cases}H_{u\overline{f}}k_z=a_{22}H_{\overline{f}}k_z-a_{12}H_{\overline{g}}k_z+\varepsilon(z),\\ H_{u\overline{g}}k_z=-a_{21}H_{\overline{f}}k_z+a_{11}H_{\overline{g}}k_z+\varepsilon(z),\end{cases}\tag{3.23}$$

where $z\in\mathcal{O}(m)\cap\mathbb{D}$. (3.22)–(3.23) imply that $a_{11}=a_{22}$ and $a_{12}=a_{21}=0$. Thus there is a constant λ such that

$$\begin{cases}H_{u\overline{f}}k_z=\lambda H_{\overline{f}}k_z+\varepsilon(z),\\ H_{u\overline{g}}k_z=\lambda H_{\overline{g}}k_z+\varepsilon(z).\end{cases}\tag{3.24}$$

Therefore,

$$\lim_{z\to m}\|H_{(u-\lambda)\overline{f}}k_z\|_2=\|H_{(u-\lambda)\overline{g}}k_z\|_2=0,$$

which implies that $((u-\lambda)\overline{f})|_{S_m}$, $((u-\lambda)\overline{g})|_{S_m}\in H^\infty|_{S_m}$, to complete the proof of Proposition 3.3.

Proposition 3.3 yields the following necessary condition for the compactness of the third operator $H_{uf}T_g+S_fH_{ug}-H_{ug}T_f-S_gH_{uf}$ given in Lemma 2.2.

Proposition 3.4 *Let u be a nonconstant inner function, $f,g\in L^\infty$ and $m\in\mathcal{M}(H^\infty+C)$. Suppose that $\overline{f}|_{S_m}$, $\overline{g}|_{S_m}\in H^\infty|_{S_m}$ and the operator*

$$H_{uf}T_g+S_fH_{ug}-H_{ug}T_f-S_gH_{uf}$$

is compact. Then either

(1) $((u-\lambda)f)|_{S_m}$ *and* $((u-\lambda)g)|_{S_m}$ *are in* $H^\infty|_{S_m}$ *for some constant* λ; *or*

(2) *there exist constants* a, b, *not both zero, such that* $(af+bg)|_{S_m}$ *is a constant.*

Combining Propositions 3.1–3.4, we obtain the following necessary condition for the compactness of the commutators of D_f and D_g.

Theorem 3.1 *Let u be a nonconstant inner function, $f,g\in L^\infty$ and $m\in\mathcal{M}(H^\infty+C)$. If $[D_f,D_g]$ is compact, then for the support set S_m of m, one of the following holds:*

(1) $f|_{S_m}$, $g|_{S_m}$, $((u-\lambda)\overline{f})|_{S_m}$ *and* $((u-\lambda)\overline{g})|_{S_m}$ *are in* $H^\infty|_{S_m}$ *for some constant* λ;

(2) $\overline{f}|_{S_m}$, $\overline{g}|_{S_m}$, $((u-\lambda)f)|_{S_m}$ *and* $((u-\lambda)g)|_{S_m}$ *are in* $H^\infty|_{S_m}$ *for some constant* λ;

(3) *there exist constants* a, b, *not both zero, such that* $(af+bg)|_{S_m}$ *is a constant.*

4 The Sufficient Part of Theorem 1.1

In this section, we will complete the proof of the sufficient part of Theorem 1.1. To do so, we need two lemmas.

Lemma 4.1 *Let* $f, g \in L^\infty$ *and*

$$\begin{aligned} F_z = &H_{\overline{g}}k_z \otimes VH_{u\overline{f}}k_z - H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z \\ &- H_{\overline{f}}k_z \otimes VH_{u\overline{g}}k_z + H_{u\overline{f}}k_z \otimes VH_{\overline{g}}k_z, \end{aligned}$$

where $z \in \mathbb{D}$. *For each support set* S, *suppose that* f *and* g *satisfy one of the following conditions*:

(1) $f|_S$, $g|_S$, $((u-\lambda)\overline{f})|_S$ *and* $((u-\lambda)\overline{g})\,|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(2) $\overline{f}|_S$, $\overline{g}|_S$, $((u-\lambda)f)\,|_S$ *and* $((u-\lambda)g)\,|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(3) *there exist constants* a, b, *not both zero, such that* $(af+bg)|_S$ *is a constant.*

Then we have

$$\lim_{|z|\to 1^-} \|F_z\| = 0.$$

Proof For each $m \in \mathcal{M}(H^\infty + C)$, let S_m be the support set of m. By the Carleson-Corona theorem, we need only to show

$$\lim_{z\to m} \|F_z\| = 0.$$

If f and g satisfy Condition (2), then we have by Lemma 2.7 that

$$\lim_{z\to m} \|H_{\overline{f}}k_z\|_2 = 0 \quad \text{and} \quad \lim_{z\to m} \|H_{\overline{g}}k_z\|_2 = 0.$$

It follows that

$$\lim_{z\to m} \|F_z\| = 0.$$

Assume that Condition (1) holds for f and g, i.e.,

$$f|_{S_m},\ g|_{S_m},\ ((u-\lambda)\overline{f})|_{S_m} \quad \text{and} \quad ((u-\lambda)\overline{g})|_{S_m} \in H^\infty|_{S_m}.$$

According to Lemma 2.7, we have

$$\lim_{z\to m} \|H_f k_z\|_2 = \lim_{z\to m} \|H_g k_z\|_2 = 0,$$

and moreover,

$$\lim_{z\to m} \|H_{(u-\lambda)\overline{f}} k_z\|_2 = \lim_{z\to m} \|H_{(u-\lambda)\overline{g}} k_z\|_2 = 0.$$

Since

$$\begin{aligned} F_z &= H_{\overline{g}}k_z \otimes VH_{[(u-\lambda)\overline{f}+\lambda\overline{f}]}k_z - H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z \\ &\quad - H_{\overline{f}}k_z \otimes VH_{[(u-\lambda)\overline{g}+\lambda\overline{g}]}k_z + H_{u\overline{f}}k_z \otimes VH_{\overline{g}}k_z \\ &= H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z + \lambda H_{\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z \\ &\quad - H_{\overline{f}}k_z \otimes VH_{(u-\lambda)\overline{g}}k_z - \lambda H_{\overline{f}}k_z \otimes VH_{\overline{g}}k_z + H_{u\overline{f}}k_z \otimes VH_{\overline{g}}k_z \\ &= H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z + H_{(\lambda-u)\overline{g}}k_z \otimes VH_{\overline{f}}k_z \\ &\quad - H_{\overline{f}}k_z \otimes VH_{(u-\lambda)\overline{g}}k_z - H_{(\lambda-u)\overline{f}}k_z \otimes VH_{\overline{g}}k_z, \end{aligned}$$

we have

$$\|F_z\| \le \|H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z\| + \|H_{(\lambda-u)\overline{g}}k_z \otimes VH_{\overline{f}}k_z\|$$

$$
\begin{aligned}
&+\|H_{\overline{f}}k_z\otimes VH_{(u-\lambda)\overline{g}}k_z\|+\|H_{(\lambda-u)\overline{f}}k_z\otimes VH_{\overline{g}}k_z\|\\
&=\|H_{\overline{g}}k_z\|_2\cdot\|VH_{(u-\lambda)\overline{f}}k_z\|_2+\|H_{(\lambda-u)\overline{g}}k_z\|_2\cdot\|VH_{\overline{f}}k_z\|_2\\
&\quad+\|H_{\overline{f}}k_z\|_2\cdot\|VH_{(u-\lambda)\overline{g}}k_z\|_2+\|H_{(\lambda-u)\overline{f}}k_z\|_2\cdot\|VH_{\overline{g}}k_z\|_2\\
&=\|H_{\overline{g}}k_z\|_2\cdot\|H_{(u-\lambda)\overline{f}}k_z\|_2+\|H_{(\lambda-u)\overline{g}}k_z\|_2\cdot\|H_{\overline{f}}k_z\|_2\\
&\quad+\|H_{\overline{f}}k_z\|_2\cdot\|H_{(u-\lambda)\overline{g}}k_z\|_2+\|H_{(\lambda-u)\overline{f}}k_z\|_2\cdot\|H_{\overline{g}}k_z\|_2.
\end{aligned}
$$

This gives us that

$$\lim_{z\to m}\|F_z\|=0.$$

To finish our proof, we suppose that f and g satisfy Condition (3). Without loss of generality, we may assume that $(f-ag)|_{S_m}=c$ for some constant c. Then we get that

$$(f-ag)|_{S_m},\quad(\overline{f}-\overline{ag})|_{S_m}\in H^\infty|_{S_m}$$

and

$$(u(f-ag))|_{S_m},\quad(u(\overline{f}-\overline{ag}))|_{S_m}\in H^\infty|_{S_m}.$$

Noting that

$$
\begin{aligned}
F_z&=H_{\overline{g}}k_z\otimes VH_{[u(\overline{f}-\overline{ag})+u\overline{ag}]}k_z-H_{u\overline{g}}k_z\otimes VH_{(\overline{f}-\overline{ag}+\overline{ag})}k_z\\
&\quad-H_{\overline{f}}k_z\otimes VH_{u\overline{g}}k_z+H_{u\overline{f}}k_z\otimes VH_{\overline{g}}k_z\\
&=H_{\overline{g}}k_z\otimes VH_{u(\overline{f}-\overline{ag})}k_z+H_{\overline{ag}}k_z\otimes VH_{u\overline{g}}k_z\\
&\quad-H_{u\overline{g}}k_z\otimes VH_{(\overline{f}-\overline{ag})}k_z-H_{\overline{a}u\overline{g}}k_z\otimes VH_{\overline{g}}k_z\\
&\quad-H_{\overline{f}}k_z\otimes VH_{u\overline{g}}k_z+H_{u\overline{f}}k_z\otimes VH_{\overline{g}}k_z\\
&=H_{\overline{g}}k_z\otimes VH_{u(\overline{f}-\overline{ag})}k_z+H_{(\overline{ag}-\overline{f})}k_z\otimes VH_{u\overline{g}}k_z\\
&\quad-H_{u\overline{g}}k_z\otimes VH_{(\overline{f}-\overline{ag})}k_z-H_{u(\overline{ag}-\overline{f})}k_z\otimes VH_{\overline{g}}k_z,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\|F_z\|&\le\|H_{\overline{g}}k_z\otimes VH_{u(\overline{f}-\overline{ag})}k_z\|+\|H_{(\overline{ag}-\overline{f})}k_z\otimes VH_{u\overline{g}}k_z\|\\
&\quad+\|H_{u\overline{g}}k_z\otimes VH_{(\overline{f}-\overline{ag})}k_z\|+\|H_{u(\overline{ag}-\overline{f})}k_z\otimes VH_{\overline{g}}k_z\|\\
&=2(\|H_{\overline{g}}k_z\|_2\cdot\|VH_{u(\overline{f}-\overline{ag})}k_z\|_2+\|H_{u\overline{g}}k_z\|_2\cdot\|VH_{(\overline{f}-\overline{ag})}k_z\|_2)\\
&=2(\|H_{\overline{g}}k_z\|_2\cdot\|H_{u(\overline{f}-\overline{ag})}k_z\|_2+\|H_{u\overline{g}}k_z\|_2\cdot\|H_{(\overline{f}-\overline{ag})}k_z\|_2).
\end{aligned}
$$

By Lemma 2.7 again, now we conclude that

$$\lim_{z\to m}\|F_z\|=0$$

to complete the proof of Lemma 4.1.

The following lemma will be needed in the proof of Theorem 1.1, which was established in [14, Lemma 17].

Lemma 4.2 *Suppose that φ and ψ are in L^∞. Let $m \in \mathcal{M}(H^\infty + C)$. If*

$$\lim_{z\to m} \|H_\varphi k_z\|_2 = 0,$$

then we have

$$\lim_{z\to m} \|H_\varphi T_\psi k_z\|_2 = 0.$$

Now we are ready to complete the proof of Theorem 1.1.

Proof of the Sufficient Part of Theorem 1.1 Let m be in $\mathcal{M}(H^\infty + C)$. We suppose that one of Conditions (1), (2) and (3) in Theorem 1.1 holds on the support set S_m. By Lemma 2.2, we need to show that

$$T_f T_g + H^*_{u\overline{f}} H_{ug} - T_g T_f - H^*_{u\overline{g}} H_{uf},$$

$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f,$$

$$H_{uf} T_g + S_f H_{ug} - H_{ug} T_f - S_g H_{uf}$$

and

$$H_{uf} H^*_{u\overline{g}} + S_f S_g - H_{ug} H^*_{u\overline{f}} - S_g S_f$$

are compact.

Letting

$$\begin{aligned} K_1 &= T_f T_g + H^*_{u\overline{f}} H_{ug} - T_g T_f - H^*_{u\overline{g}} H_{uf} \\ &= (T_{fg} - H^*_{\overline{f}} H_g) + H^*_{u\overline{f}} H_{ug} - (T_{fg} - H^*_{\overline{g}} H_f) - H^*_{u\overline{g}} H_{uf} \\ &= (H^*_{\overline{g}} H_f - H^*_{\overline{f}} H_g) - (H^*_{u\overline{g}} H_{uf} - H^*_{u\overline{f}} H_{ug}), \end{aligned}$$

we are going to show that K_1 is compact first.

In order to show that K_1 is compact, we first check that each condition of Theorem 1.1 can imply (3.1). Indeed, if Condition (1) holds, then we have

$$f|_{S_m}, \quad g|_{S_m} \in H^\infty|_{S_m}.$$

Using Lemma 2.7 and identity (4.5) of [21]

$$H_{uf} = S_u H_f, \quad H_{ug} = S_u H_g,$$

we have

$$\lim_{z\to m} \|H_f k_z\|_2 = \lim_{z\to m} \|H_g k_z\|_2 = 0$$

and

$$\lim_{z\to m} \|H_{uf} k_z\|_2 = \lim_{z\to m} \|H_{ug} k_z\|_2 = 0,$$

which implies that

$$\lim_{z\to m} \|H_{\overline{g}} k_z \otimes H_f k_z - H_{\overline{f}} k_z \otimes H_g k_z\| = 0$$

and

$$\lim_{z\to m} \|H_{u\overline{g}} k_z \otimes H_{uf} k_z - H_{u\overline{f}} k_z \otimes H_{ug} k_z\| = 0.$$

Similarly, if $\overline{f}|_{S_m},\ \overline{g}|_{S_m} \in H^\infty|_{S_m}$, then we also have

$$\lim_{z\to m} \|H_{\overline{g}}k_z \otimes H_f k_z - H_{\overline{f}}k_z \otimes H_g k_z\| = 0$$

and

$$\lim_{z\to m} \|H_{u\overline{g}}k_z \otimes H_{uf} k_z - H_{u\overline{f}}k_z \otimes H_{ug} k_z\| = 0.$$

Thus Condition (1) or (2) in Theorem 1.1 can imply (3.1).

If Condition (3) holds, we have

$$(af + bg)|_{S_m} = c$$

for some constants $a,\ b,\ c$ with $|a| + |b| \neq 0$. Without loss of generality, we may assume that

$$(f - dg)|_{S_m} = e$$

for some constants d and e. Then we have

$$\begin{aligned}
&H_{\overline{g}}k_z \otimes H_f k_z - H_{\overline{f}}k_z \otimes H_g k_z \\
&= H_{\overline{g}}k_z \otimes H_{(f-dg+dg)} k_z - H_{\overline{f}}k_z \otimes H_g k_z \\
&= H_{\overline{g}}k_z \otimes H_{(f-dg)} k_z + \overline{d} H_{\overline{g}}k_z \otimes H_g k_z - H_{\overline{f}}k_z \otimes H_g k_z \\
&= H_{\overline{g}}k_z \otimes H_{(f-dg)} k_z + H_{(\overline{dg}-\overline{f})}k_z \otimes H_g k_z
\end{aligned}$$

and

$$\begin{aligned}
&H_{u\overline{g}}k_z \otimes H_{uf} k_z - H_{u\overline{f}}k_z \otimes H_{ug} k_z \\
&= H_{u\overline{g}}k_z \otimes H_{u(f-dg+dg)} k_z - H_{u\overline{f}}k_z \otimes H_{ug} k_z \\
&= H_{u\overline{g}}k_z \otimes H_{u(f-dg)} k_z + \overline{d} H_{u\overline{g}}k_z \otimes H_{ug} k_z - H_{u\overline{f}}k_z \otimes H_{ug} k_z \\
&= H_{u\overline{g}}k_z \otimes H_{u(f-dg)} k_z + H_{u(\overline{dg}-\overline{f})}k_z \otimes H_{ug} k_z.
\end{aligned}$$

Since $(f - dg)|_{S_m}$ is a constant, we conclude that $(u(f - dg))|_{S_m}$ and $(u(\overline{dg} - \overline{f}))|_{S_m}$ both belong to $H^\infty|_{S_m}$. Using Lemma 2.7 again, we get that

$$\lim_{z\to m} \|H_{\overline{g}}k_z \otimes H_f k_z - H_{\overline{f}}k_z \otimes H_g k_z\| = 0$$

and

$$\lim_{z\to m} \|H_{u\overline{g}}k_z \otimes H_{uf} k_z - H_{u\overline{f}}k_z \otimes H_{ug} k_z\| = 0,$$

which implies that the equation in (3.1) holds, as desired.

By the definition of K_1 and Lemma 2.5, we have

$$\begin{aligned}
&K_1 - T^*_{\phi_z} K_1 T_{\phi_z} \\
&= [(H^*_{\overline{g}} H_f - H^*_{\overline{f}} H_g) - (H^*_{u\overline{g}} H_{uf} - H^*_{u\overline{f}} H_{ug})] \\
&\quad - T^*_{\phi_z} [(H^*_{\overline{g}} H_f - H^*_{\overline{f}} H_g) - (H^*_{u\overline{g}} H_{uf} - H^*_{u\overline{f}} H_{ug})] T_{\phi_z} \\
&= V[(H_{\overline{g}}k_z \otimes H_f k_z - H_{\overline{f}}k_z \otimes H_g k_z)]V^* \\
&\quad - V[(H_{u\overline{g}}k_z \otimes H_{uf} k_z - H_{u\overline{f}}k_z \otimes H_{ug} k_z)]V^*.
\end{aligned}$$

It follows that

$$\lim_{|z|\to 1^-} \|K_1 - T^*_{\phi_z} K_1 T_{\phi_z}\| = 0. \tag{4.1}$$

On the other hand, since

$$H^*_{u\overline{f}} H_{ug} = T_{fg} - T_{\overline{u}f} T_{ug}$$

and

$$H^*_{u\overline{g}} H_{uf} = T_{fg} - T_{\overline{u}g} T_{uf},$$

we have

$$K_1 = (T_f T_g - T_g T_f) + (T_{\overline{u}g} T_{uf} - T_{\overline{u}f} T_{ug}),$$

which is a finite sum of finite products of Toeplitz operators. According to [15, Theorem 12], we obtain by (4.1) that K_1 is equal to a compact perturbation of a Toeplitz operator, i.e.,

$$K_1 = T_h + K$$

for some $h \in L^\infty$ and some compact operator K. Thus $K = K_1 - T_h$ belongs to the Toeplitz algebra $\mathscr{T}_{L^\infty}$. We conclude by [3, Corollary 6] that $h = 0$ a.e., which implies that $K_1 = K$ is compact.

To show the fourth operator $H_{uf} H^*_{u\overline{g}} + S_f S_g - H_{ug} H^*_{u\overline{f}} - S_g S_f$ is compact, we recall that

$$V H_\varphi = H^*_\varphi V \quad \text{and} \quad S_\varphi V = V T_{\overline{\varphi}}.$$

Then

$$\begin{aligned} & V(H_{uf} H^*_{u\overline{g}} + S_f S_g - H_{ug} H^*_{u\overline{f}} - S_g S_f)V \\ &= H^*_{uf} V V H_{u\overline{g}} + T_{\overline{f}} V V T_{\overline{g}} - H^*_{ug} V V H_{u\overline{f}} - T_{\overline{g}} V V T_{\overline{f}} \\ &= H^*_{uf} H_{u\overline{g}} + T_{\overline{f}} T_{\overline{g}} - H^*_{ug} H_{u\overline{f}} - T_{\overline{g}} T_{\overline{f}}, \end{aligned} \tag{4.2}$$

where the second equality follows from $V^2 = I$. Using the same method as the above, we can show similarly that

$$H^*_{uf} H_{u\overline{g}} + T_{\overline{f}} T_{\overline{g}} - H^*_{ug} H_{u\overline{f}} - T_{\overline{g}} T_{\overline{f}}$$

is compact. Furthermore, (4.2) gives us that

$$H_{uf} H^*_{u\overline{g}} + S_f S_g - H_{ug} H^*_{u\overline{f}} - S_g S_f$$

is also compact.

Now we turn to the proof of the compactness of the second operator

$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f.$$

Denoting the above operator by

$$K_2 = T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - T_g H^*_{u\overline{f}} - H^*_{u\overline{g}} S_f,$$

we need only to consider the compactness of $K_2 K_2^*$. From identity (4.5) in [21], we have

$$H_{\varphi\psi} = H_\varphi T_\psi + S_\varphi H_\psi = H_\psi T_\varphi + S_\psi H_\varphi$$

for any $\varphi,\ \psi \in L^\infty$, to obtain

$$K_2 = T_f H^*_{u\overline{g}} - T_{\overline{u}f} H^*_{\overline{g}} - T_g H^*_{u\overline{f}} + T_{\overline{u}g} H^*_{\overline{f}}$$

and

$$K_2^* = H_{u\overline{g}} T_{\overline{f}} - H_{\overline{g}} T_{u\overline{f}} - H_{u\overline{f}} T_{\overline{g}} + H_{\overline{f}} T_{u\overline{g}}.$$

Observe that the operator $K_2K_2^*$ is in the Toeplitz algebra $\mathscr{T}_{L^\infty}$ and the symbol map maps $K_2K_2^*$ to 0. By [15, Theorem 12] again, we need only to prove that

$$\lim_{z\to m} \|K_2K_2^* - T^*_{\phi_z} K_2K_2^* T_{\phi_z}\| = 0.$$

By Lemma 3.1 and $VT_{\varphi\overline{\phi_z}} k_z = -H_{\overline{\varphi}} k_z$ for all φ in L^∞,

$$K_2^* T_{\phi_z} = S_{\phi_z} K_2^* - F_z,$$

where F_z is introduced in Lemma 4.1 and $\lim\limits_{|z|\to 1^-} \|F_z\| = 0$. Thus we have

$$\begin{aligned}
&T^*_{\phi_z} K_2K_2^* T_{\phi_z}\\
&= (K_2^* T_{\phi_z})^* K_2^* T_{\phi_z}\\
&= (S_{\phi_z} K_2^* - F_z)^* (S_{\phi_z} K_2^* - F_z)\\
&= (K_2 S^*_{\phi_z} - F_z^*)(S_{\phi_z} K_2^* - F_z)\\
&= K_2 S^*_{\phi_z} S_{\phi_z} K_2^* - K_2 S^*_{\phi_z} F_z - F_z^* S_{\phi_z} K_2^* + F_z^* F_z\\
&= K_2 (I - Vk_z \otimes Vk_z) K_2^* - K_2 S^*_{\phi_z} F_z - F_z^* S_{\phi_z} K_2^* + F_z^* F_z\\
&= K_2K_2^* - (K_2Vk_z) \otimes (K_2Vk_z) - K_2 S^*_{\phi_z} F_z - F_z^* S_{\phi_z} K_2^* + F_z^* F_z.
\end{aligned}$$

It follows that

$$K_2K_2^* - T^*_{\phi_z} K_2K_2^* T_{\phi_z} = (K_2Vk_z) \otimes (K_2Vk_z) + K_2 S^*_{\phi_z} F_z + F_z^* S_{\phi_z} K_2^* - F_z^* F_z.$$

Therefore, in order to show that

$$\lim_{|z|\to 1^-} \|K_2K_2^* - T^*_{\phi} K_2K_2^* T_{\phi}\| = 0,$$

it is sufficient to show

$$\lim_{|z|\to 1^-} \|K_2Vk_z\|_2 = 0 \tag{4.3}$$

as $\lim\limits_{|z|\to 1^-} \|F_z\| = 0$. For this purpose, we will check that each condition of Theorem 1.1 can imply (4.3).

Recall that

$$T_\varphi V = VS_{\overline{\varphi}} \quad \text{and} \quad VH_\varphi = H^*_\varphi V$$

for all $\varphi \in L^\infty$, we get

$$K_2Vk_z = V(S_{\overline{f}} H_{u\overline{g}} k_z - S_{u\overline{f}} H_{\overline{g}} k_z + S_{u\overline{g}} H_{\overline{f}} k_z - S_{\overline{g}} H_{u\overline{f}} k_z). \tag{4.4}$$

If f and g satisfy Condition (2) in Theorem 1.1, we have by Lemma 2.7 that

$$\lim_{z\to m}\|H_{\overline{f}}k_z\|_2=\lim_{z\to m}\|H_{\overline{g}}k_z\|_2=0$$

and

$$\lim_{z\to m}\|H_{u\overline{f}}k_z\|_2=\lim_{z\to m}\|H_{u\overline{g}}k_z\|_2=0.$$

This gives that

$$\lim_{z\to m}\|K_2Vk_z\|_2=0.$$

Assume that Condition (1) holds, i.e.,

$$f|_{S_m},\quad g|_{S_m},\quad ((u-\lambda)\overline{f})|_{S_m}\text{ and }((u-\lambda)\overline{g})|_{S_m}\in H^\infty|_{S_m}.$$

It follows that

$$\lim_{z\to m}\|H_fk_z\|_2=\lim_{z\to m}\|H_gk_z\|_2=0$$

and

$$\lim_{z\to m}\|H_{(u-\lambda)\overline{f}}k_z\|_2=\lim_{z\to m}\|H_{(u-\lambda)\overline{g}}k_z\|_2=0.$$

Computing K_2Vk_z directly, we obtain

$$\begin{aligned}K_2Vk_z&=V(S_{\overline{f}}H_{u\overline{g}}k_z-S_{u\overline{f}}H_{\overline{g}}k_z+S_{u\overline{g}}H_{\overline{f}}k_z-S_{\overline{g}}H_{u\overline{f}}k_z)\\&=V\{S_{\overline{f}}H_{[(u-\lambda)\overline{g}+\lambda\overline{g}]}k_z-S_{u\overline{f}}H_{\overline{g}}k_z+S_{u\overline{g}}H_{\overline{f}}k_z-S_{\overline{g}}H_{[(u-\lambda)\overline{f}+\lambda\overline{f}]}k_z\}\\&=V[S_{\overline{f}}H_{(u-\lambda)\overline{g}}k_z-S_{(u-\lambda)\overline{f}}H_{\overline{g}}k_z+S_{(u-\lambda)\overline{g}}H_{\overline{f}}k_z-S_{\overline{g}}H_{(u-\lambda)\overline{f}}k_z]\\&=V[S_{\overline{f}}H_{(u-\lambda)\overline{g}}k_z-S_{\overline{g}}H_{(u-\lambda)\overline{f}}k_z]+V[S_{(u-\lambda)\overline{g}}H_{\overline{f}}k_z-S_{(u-\lambda)\overline{f}}H_{\overline{g}}k_z].\end{aligned}$$

Noting that

$$\begin{aligned}&S_{(u-\lambda)\overline{g}}H_{\overline{f}}k_z-S_{(u-\lambda)\overline{f}}H_{\overline{g}}k_z\\&=(I-P)[(u-\lambda)\overline{g}(I-P)(\overline{f}k_z)]-(I-P)[(u-\lambda)\overline{f}(I-P)(\overline{g}k_z)]\\&=(I-P)[(u-\lambda)\overline{g}\overline{f}k_z-(u-\lambda)\overline{g}P(\overline{f}k_z)-(u-\lambda)\overline{f}\overline{g}k_z+(u-\lambda)\overline{f}P(\overline{g}k_z)]\\&=(I-P)[(u-\lambda)\overline{f}P(\overline{g}k_z)-(u-\lambda)\overline{g}P(\overline{f}k_z)]\\&=H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z-H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z,\end{aligned}$$

we have

$$\begin{aligned}K_2Vk_z=V&[S_{\overline{f}}H_{(u-\lambda)\overline{g}}k_z-S_{\overline{g}}H_{(u-\lambda)\overline{f}}k_z]\\&+V[H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z-H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z]\end{aligned}$$

and

$$\begin{aligned}\|K_2Vk_z\|_2\le\|f\|_\infty\cdot\|H_{(u-\lambda)\overline{g}}k_z\|_2&+\|g\|_\infty\cdot\|H_{(u-\lambda)\overline{f}}k_z\|_2\\&+\|H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z\|_2+\|H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z\|_2.\end{aligned}$$

Since

$$((u-\lambda)\overline{g})|_{S_m}\quad\text{and}\quad((u-\lambda)\overline{f})|_{S_m}\in H^\infty|_{S_m},$$

we conclude by Lemma 4.2 that $\|K_2 V k_z\|_2 \to 0$ as $z \to m$.

Finally, we suppose that Condition (3) holds. Without loss of generality, we assume that

$$(f - \alpha g)|_{S_m} = \beta$$

for some constants α and β. Then we have

$$(f - \alpha g)|_{S_m} \quad \text{and} \quad (\overline{f} - \overline{\alpha g})|_{S_m}$$

are in $H^\infty|_{S_m}$. Observe that

$$\begin{aligned} & K_2 V k_z \\ &= V(S_{\overline{f}} H_{u\overline{g}} k_z - S_{u\overline{f}} H_{\overline{g}} k_z + S_{u\overline{g}} H_{\overline{f}} k_z - S_{\overline{g}} H_{u\overline{f}} k_z) \\ &= V[S_{\overline{f}} H_{u\overline{g}} k_z - S_{u\overline{f}} H_{\overline{g}} k_z + S_{u\overline{g}} H_{(\overline{f}-\overline{\alpha g}+\overline{\alpha g})} k_z - S_{\overline{g}} H_{u(\overline{f}-\overline{\alpha g}+\overline{\alpha g})} k_z] \\ &= V[S_{u\overline{g}} H_{(\overline{f}-\overline{\alpha g})} k_z - S_{\overline{g}} H_{u(\overline{f}-\overline{\alpha g})} k_z] + V[S_{(\overline{f}-\overline{\alpha g})} H_{u\overline{g}} k_z - S_{u(\overline{f}-\overline{\alpha g})} H_{\overline{g}} k_z]. \end{aligned}$$

Similarly, we calculate that

$$\begin{aligned} & S_{(\overline{f}-\overline{\alpha g})} H_{u\overline{g}} k_z - S_{u(\overline{f}-\overline{\alpha g})} H_{\overline{g}} k_z \\ &= (I-P)[(\overline{f}-\overline{\alpha g})(I-P)(u\overline{g}k_z)] - (I-P)[u(\overline{f}-\overline{\alpha g})(I-P)(\overline{g}k_z)] \\ &= (I-P)[(\overline{f}-\overline{\alpha g})u\overline{g}k_z - (\overline{f}-\overline{\alpha g})P(u\overline{g}k_z) - u(\overline{f}-\overline{\alpha g})\overline{g}k_z + u(\overline{f}-\overline{xg})P(\overline{g}k_z)] \\ &= (I-P)[u(\overline{f}-\overline{\alpha g})P(\overline{g}k_z) - (\overline{f}-\overline{\alpha g})P(u\overline{g}k_z)] \\ &= H_{u(\overline{f}-\overline{\alpha g})} T_{\overline{g}} k_z - H_{(\overline{f}-\overline{\alpha g})} T_{u\overline{g}} k_z. \end{aligned}$$

It follows that

$$\begin{aligned} \|K_2 V k_z\|_2 &= \|V[S_{u\overline{g}} H_{(\overline{f}-\overline{\alpha g})} k_z - S_{\overline{g}} H_{u(\overline{f}-\overline{\alpha g})} k_z] + V[S_{(\overline{f}-\overline{\alpha g})} H_{u\overline{g}} k_z - S_{u(\overline{f}-\overline{\alpha g})} H_{\overline{g}} k_z]\|_2 \\ &\le \|S_{u\overline{g}} H_{(\overline{f}-\overline{\alpha g})} k_z - S_{\overline{g}} H_{u(\overline{f}-\overline{\alpha g})} k_z\|_2 + \|S_{(\overline{f}-\overline{\alpha g})} H_{u\overline{g}} k_z - S_{u(\overline{f}-\overline{\alpha g})} H_{\overline{g}} k_z\|_2 \\ &\le \|g\|_\infty \cdot \|H_{(\overline{f}-\overline{\alpha g})} k_z\|_2 + \|g\|_\infty \cdot \|H_{u(\overline{f}-\overline{\alpha g})} k_z\|_2 \\ &\quad + \|H_{u(\overline{f}-\overline{\alpha g})} T_{\overline{g}} k_z - H_{(\overline{f}-\overline{\alpha g})} T_{u\overline{g}} k_z\|_2 \\ &\le \|g\|_\infty \cdot \|H_{(\overline{f}-\overline{\alpha g})} k_z\|_2 + \|g\|_\infty \cdot \|H_{u(\overline{f}-\overline{\alpha g})} k_z\|_2 \\ &\quad + \|H_{u(\overline{f}-\overline{\alpha g})} T_{\overline{g}} k_z\|_2 + \|H_{(\overline{f}-\overline{\alpha g})} T_{u\overline{g}} k_z\|_2. \end{aligned}$$

Using the conditions that

$$(\overline{f} - \overline{\alpha g})|_{S_m} \quad \text{and} \quad (u(\overline{f} - \overline{\alpha g}))|_{S_m}$$

are in $H^\infty|_{S_m}$, we again conclude by Lemma 4.2 that $\|K_2 V k_z\|_2 \to 0$ as $z \to m$.

To summarize, each condition in Theorem 1.1 implies

$$\lim_{|z| \to 1^-} \|K_2 K_2^* - T_\phi^* K_2 K_2^* T_\phi\| = 0,$$

which gives that K_2 is compact.

In order to complete the proof, it remains to show that the third operator

$$K_3 = H_{uf} T_g + S_f H_{ug} - H_{ug} T_f - S_g H_{uf}$$

is compact. Rewrite K_3 as follows:

$$\begin{aligned}K_3 &= H_{uf}T_g + S_f H_{ug} - H_{ug}T_f - S_g H_{uf}\\ &= H_{uf}T_g + H_{fug} - H_f T_{ug} - H_{ug}T_f - H_{guf} + H_g T_{uf}\\ &= H_{uf}T_g - H_f T_{ug} - H_{ug}T_f + H_g T_{uf}.\end{aligned}$$

Observe that

$$K_3^* = T_{\overline{g}}H_{uf}^* - T_{\overline{ug}}H_f^* - T_{\overline{f}}H_{ug}^* + T_{\overline{uf}}H_g^*$$

has the same form as K_2. Using the same arguments as in the proof of the compactness of K_2, we conclude that K_3^* is also compact, which implies that K_3 is compact.

Finally, as the necessity part of Theorem 1.1 was contained in Theorem 3.1, thus we finish the proof of Theorem 1.1.

5 The Necessary Part of Theorem 1.2

Section 5 is devoted to the proof of the necessary part of Theorem 1.2. Let us begin with the following necessary condition for the compactness of the first operator given in Lemma 2.3.

Proposition 5.1 *Let u be a nonconstant inner function, $f,\ g \in L^\infty$ and $m \in \mathcal{M}(H^\infty + C)$. Suppose that*

$$T_f T_g + H_{u\overline{f}}^* H_{ug} - T_{fg}$$

is compact. Then for the support set S_m of m, one of the following holds:

(1) $\overline{f}|_{S_m}$ *is in* $H^\infty|_{S_m}$;

(2) $g|_{S_m}$ *is in* $H^\infty|_{S_m}$.

Proof Suppose that

$$K = T_f T_g + H_{u\overline{f}}^* H_{ug} - T_{fg}$$

is compact. Clearly, K can be rewritten as

$$K = H_{u\overline{f}}^* H_{ug} - H_{\overline{f}}^* H_g.$$

By Lemmas 2.5–2.6, we have

$$\lim_{z\to m}\|K - T_{\phi_z}^* K T_{\phi_z}\| = \lim_{z\to m}\|V[H_{u\overline{f}}k_z \otimes H_{ug}k_z - H_{\overline{f}}k_z \otimes H_g k_z]V^*\| = 0,$$

which gives

$$\lim_{z\to m}\|H_{u\overline{f}}k_z \otimes H_{ug}k_z - H_{\overline{f}}k_z \otimes H_g k_z\| = 0. \tag{5.1}$$

For $[\overline{f}|_{S_m}] \in (L^\infty|_{S_m})/(H^\infty|_{S_m})$, let us consider the following two cases.

Case 1 If $[\overline{f}|_{S_m}] = 0$, then $\overline{f}|_{S_m} \in H^\infty|_{S_m}$, as desired.

Case 2 Suppose that $[\overline{f}|_{S_m}] \neq 0$. Then we have by Lemma 2.7 that

$$\underline{\lim}_{z\to m}\|H_{\overline{f}}k_z\|_2 > 0.$$

On the other hand, (5.1) gives that

$$\lim_{z\to m}\left\|\frac{\langle H_{\overline{f}}k_z, H_{u\overline{f}}k_z\rangle}{\|H_{\overline{f}}k_z\|_2^2}H_{ug}k_z - H_g k_z\right\|_2 = 0.$$

Note that $\frac{\langle H_{\overline{f}}k_z, H_{u\overline{f}}k_z\rangle}{\|H_{\overline{f}}k_z\|_2^2}$ is bounded for z in some small neighborhood $\mathcal{O}(m)\cap\mathbb{D}$ of m. Using the Bolzano-Weierstrass theorem, we can find a subnet $\{z_\alpha\}\subset\mathbb{D}$ such that

$$\lim_{z_\alpha\to m}\frac{\langle H_{\overline{f}}k_{z_\alpha}, H_{u\overline{f}}k_{z_\alpha}\rangle}{\|H_{\overline{f}}k_{z_\alpha}\|_2^2} = a$$

for some constant a with $|a|\le 1$. Furthermore, we have

$$\lim_{z_\alpha\to m}\|aH_{ug}k_{z_\alpha} - H_g k_{z_\alpha}\|_2 = 0.$$

Thus we conclude by Lemma 2.7 that

$$\lim_{z\to m}\|aH_{ug}k_z - H_g k_z\|_2 = 0$$

to get

$$\lim_{z\to m}\|H_{(1-au)g}k_z\|_2 = 0.$$

According to the last two paragraphs in the proof of Case 2 of Proposition 3.1, we obtain

$$\lim_{z\to m}\|H_g k_z\|_2 = 0,$$

which implies that $g|_{S_m}\in H^\infty|_{S_m}$. This completes the proof.

The next proposition again follows directly from the following equalities in Remark 2.1:

$$VT_\varphi = S_{\overline{\varphi}}V,\quad VH_\varphi = H_\varphi^* V\quad\text{and}\quad V^2 = I.$$

Proposition 5.2 *Let u be a nonconstant inner function, $f,\ g\in L^\infty$ and $m\in\mathcal{M}(H^\infty+C)$. Assume that*

$$H_{uf}H_{u\overline{g}}^* + S_f S_g - S_{fg}$$

is compact. Then for the support set S_m of m, one of the following holds:

(1) *$f|_{S_m}$ is in $H^\infty|_{S_m}$;*

(2) *$\overline{g}|_{S_m}$ is in $H^\infty|_{S_m}$.*

Combining Propositions 5.1–5.2, we obtain a necessary condition for the compactness of $[D_f, D_g)$.

Proposition 5.3 *Let u be a nonconstant inner function, $f,\ g\in L^\infty$ and $m\in\mathcal{M}(H^\infty+C)$. Suppose that the semicommutator $[D_f, D_g)$ is compact. Then for the support set S_m of m, one of following conditions holds:*

(1) *$f|_{S_m}$ and $g|_{S_m}$ are in $H^\infty|_{S_m}$;*

(2) *$\overline{f}|_{S_m}$ and $\overline{g}|_{S_m}$ are in $H^\infty|_{S_m}$;*

(3) *either $f|_{S_m}$ or $g|_{S_m}$ is a constant.*

We establish a necessary condition for the compactness of the operator $T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - H^*_{u\overline{fg}}$ in the following proposition.

Proposition 5.4 *Let u be a nonconstant inner function, $f,\ g \in L^\infty$ and $m \in \mathcal{M}(H^\infty + C)$. Suppose that*

$$T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - H^*_{u\overline{fg}}$$

is compact and $f|_{S_m}$, $g|_{S_m}$ are in $H^\infty|_{S_m}$. Then for the support set S_m of m, one of the following holds:

(1) *$((u-\lambda)\overline{f})|_{S_m}$, $((u-\lambda)\overline{g})|_{S_m}$ and $((u-\lambda)\overline{fg})|_{S_m}$ are in $H^\infty|_{S_m}$ for some constant λ;*

(2) *either $f|_{S_m}$ or $g|_{S_m}$ is constant.*

Proof Let K denote the compact operator given above, then

$$K^* = H_{u\overline{g}} T_{\overline{f}} + S_{\overline{g}} H_{u\overline{f}} - H_{u\overline{fg}}$$

is also compact. Using identity (4.5) of [21], we obtain

$$H_{u\overline{fg}} = H_{\overline{g}} T_{u\overline{f}} + S_{\overline{g}} H_{u\overline{f}}$$

to get

$$K^* = H_{u\overline{g}} T_{\overline{f}} - H_{\overline{g}} T_{u\overline{f}}.$$

By Lemmas 3.1–3.2, we obtain that

$$\lim_{z\to m} \|K^* T_{\phi_z} - S_{\phi_z} K^*\| = \lim_{z\to m} \|H_{u\overline{g}} k_z \otimes V H_{\overline{f}} k_z - H_{\overline{g}} k_z \otimes V H_{u\overline{f}} k_z\| = 0. \tag{5.2}$$

Before going further, we need to consider the following two cases.

Case 1 If $[\overline{f}|_{S_m}] = 0$, then $\overline{f}|_{S_m} \in H^\infty|_{S_m}$. Since $f|_{S_m}$ is also in $H^\infty|_{S_m}$, we conclude that $f|_{S_m}$ is a constant.

Case 2 If $[\overline{f}|_{S_m}] \neq 0$, then we have by Lemma 2.7 that

$$\underline{\lim}_{z\to m} \|H_{\overline{f}} k_z\|_2 > 0.$$

By (5.2), we have

$$\lim_{z\to m} \Big\| H_{u\overline{g}} k_z - \frac{\langle V H_{\overline{f}} k_z, V H_{u\overline{f}} k_z\rangle}{\|V H_{\overline{f}} k_z\|_2^2} H_{\overline{g}} k_z \Big\|_2 = 0.$$

Since V is anti-unitary, $\frac{\langle V H_{\overline{f}} k_z, V H_{u\overline{f}} k_z\rangle}{\|V H_{\overline{f}} k_z\|_2^2}$ is bounded for $z \in \mathcal{O}(m) \cap \mathbb{D}$. Using the Bolzano-Weierstrass theorem again, there is a subnet $\{z_\alpha\} \subset \mathbb{D}$ such that

$$\lim_{z_\alpha\to m} \frac{\langle V H_{\overline{f}} k_{z_\alpha}, V H_{u\overline{f}} k_{z_\alpha}\rangle}{\|V H_{\overline{f}} k_{z_\alpha}\|_2^2} = \lambda$$

for some constant λ, to obtain

$$\lim_{z_\alpha\to m} \|H_{u\overline{g}} k_{z_\alpha} - \lambda H_{\overline{g}} k_{z_\alpha}\|_2 = 0.$$

Now Lemma 2.7 gives us that

$$\lim_{z\to m} \|H_{(u-\lambda)\overline{g}} k_z\|_2 = 0,$$

which implies that $((u-\lambda)\overline{g})|_{S_m} \in H^\infty|_{S_m}$.

Furthermore, since

$$\begin{aligned}
&\|H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z\| \\
&= \|H_{\overline{g}}k_z \otimes VH_{u\overline{f}}k_z - H_{\lambda\overline{g}}k_z \otimes VH_{\overline{f}}k_z + H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z\| \\
&= \|H_{(u-\lambda)\overline{g}}k_z \otimes VH_{\overline{f}}k_z - (H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{\overline{g}}k_z \otimes VH_{u\overline{f}}k_z)\| \\
&\leq \|H_{(u-\lambda)\overline{g}}k_z\|_2 \cdot \|VH_{\overline{f}}k_z\|_2 + \|H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{\overline{g}}k_z \otimes VH_{u\overline{f}}k_z\|,
\end{aligned}$$

we conclude that

$$\lim_{z\to m} \|H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z\| = \lim_{z\to m} \|H_{\overline{g}}k_z\|_2 \cdot \|H_{(u-\lambda)\overline{f}}k_z\|_2 = 0.$$

As u is inner and $f,\ g \in L^\infty$, we obtain that

$$\lim_{z\to m} \|H_{\overline{g}}k_z\|_2 = 0 \quad \text{or} \quad \lim_{z\to m} \|H_{(u-\lambda)\overline{f}}k_z\|_2 = 0.$$

It follows from Lemma 2.7 that $\overline{g}|_{S_m}$ or $((u-\lambda)\overline{f})|_{S_m}$ is in $H^\infty|_{S_m}$.

In order to complete the proof of this proposition, we need to consider the following two subcases for $[\overline{g}|_{S_m}]$.

Subcase 2(i) If $\overline{g}|_{S_m} \in H^\infty|_{S_m}$, then we have by $g|_{S_m} \in H^\infty|_{S_m}$ that $g|_{S_m}$ is a constant.

Subcase 2(ii) If $\overline{g}|_{S_m}$ is not in $H^\infty|_{S_m}$, then we have $((u-\lambda)\overline{f})|_{S_m} \in H^\infty|_{S_m}$ and

$$\lim_{z\to m} \|H_{(u-\lambda)\overline{f}}k_z\|_2 = 0.$$

Since K^* is compact, we have

$$\lim_{z\to m} \|K^* k_z\|_2 = 0.$$

Moreover, we have by Lemma 4.2 that

$$\lim_{z\to m} \|H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z\|_2 = 0.$$

Noting that

$$\begin{aligned}
K^* k_z &= H_{u\overline{g}}T_{\overline{f}}k_z - H_{\overline{g}}T_{u\overline{f}}k_z \\
&= H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z + H_{\lambda\overline{g}}T_{\overline{f}}k_z - H_{\overline{g}}T_{u\overline{f}}k_z \\
&= H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z - H_{\overline{g}}T_{(u-\lambda)\overline{f}}k_z \\
&= H_{(u-\lambda)\overline{g}}T_{\overline{f}}k_z - H_{(u-\lambda)\overline{fg}}k_z + S_{\overline{g}}H_{(u-\lambda)\overline{f}}k_z,
\end{aligned}$$

we have $\|H_{(u-\lambda)\overline{fg}}k_z\|_2 \to 0$ as $z \to m$. Thus $((u-\lambda)\overline{fg})|_{S_m}$ is also in $H^\infty|_{S_m}$, to complete the proof of Proposition 5.4.

In view of Proposition 5.4, we obtain the following proposition which gives a necessary condition for the compactness of the operator $H_{uf}T_g + S_f H_{ug} - H_{ufg}$.

Proposition 5.5 *Let u be a nonconstant inner function, $f,\ g \in L^\infty$ and $m \in \mathcal{M}(H^\infty + C)$. Suppose that*

$$H_{uf}T_g + S_f H_{ug} - H_{ufg}$$

is compact and $\overline{f}|_{S_m}$, $\overline{g}|_{S_m}$ *are in* $H^\infty|_{S_m}$. *Then for the support set* S_m *of* m, *one of the following holds*:

(1) $((u-\lambda)f)\,|_{S_m}$, $((u-\lambda)g)\,|_{S_m}$ *and* $((u-\lambda)fg)\,|_{S_m}$ *are in* $H^\infty|_{S_m}$ *for some constant* λ;

(2) *either* $f|_{S_m}$ *or* $g|_{S_m}$ *is a constant.*

Combining Propositions 5.3–5.5, now we summarize the necessary condition for the compactness of the semicommutator $[D_f, D_g)$ in the following theorem.

Theorem 5.1 *Let* u *be a nonconstant inner function,* $f, g \in L^\infty$ *and* $m \in \mathcal{M}(H^\infty + C)$. *Suppose that the semicommutator* $[D_f, D_g)$ *is compact. Then for each support set* S_m *of* m, *one of the following conditions holds*:

(1) $f|_{S_m}$, $g|_{S_m}$, $((u-\lambda)\overline{f})|_{S_m}$, $((u-\lambda)\overline{g})|_{S_m}$ *and* $((u-\lambda)\overline{fg})|_{S_m}$ *are in* $H^\infty|_{S_m}$ *for some constant* λ;

(2) $\overline{f}|_{S_m}$, $\overline{g}|_{S_m}$, $((u-\lambda)f)|_{S_m}$, $((u-\lambda)g)|_{S_m}$ *and* $((u-\lambda)fg)|_{S_m}$ *are in* $H^\infty|_{S_m}$ *for some constant* λ;

(3) *either* $f|_{S_m}$ *or* $g|_{S_m}$ *is a constant.*

6 The Sufficient Part of Theorem 1.2

In the final section, we will present the proof of the sufficient part of Theorem 1.2. To do this, we need the following lemma analogous to Lemma 4.1.

Lemma 6.1 *Let* f, g *be in* L^∞ *and*

$$L_z = H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{\overline{g}}k_z \otimes VH_{u\overline{f}}k_z,$$

where $z \in \mathbb{D}$. *For each support set* S, *suppose that* f *and* g *satisfy one of following conditions*:

(1) $f|_S$, $g|_S$, $((u-\lambda)\overline{f})|_S$, $((u-\lambda)\overline{g})|_S$ *and* $((u-\lambda)\overline{fg})|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(2) $\overline{f}|_S$, $\overline{g}|_S$, $((u-\lambda)f)|_S$, $((u-\lambda)g)|_S$ *and* $((u-\lambda)fg)|_S$ *are in* $H^\infty|_S$ *for some constant* λ;

(3) *either* $f|_S$ *or* $g|_S$ *is constant.*

Then we have

$$\lim_{|z|\to 1^-} \|L_z\| = 0. \tag{6.1}$$

Proof For any m in $\mathcal{M}(H^\infty + C)$, let S_m be the corresponding support set. If Condition (2) or (3) holds, we have by Lemma 2.7 that

$$\lim_{z\to m} \|H_{\overline{f}}k_z\|_2 = \lim_{z\to m} \|H_{u\overline{f}}k_z\|_2 = 0$$

or

$$\lim_{z\to m} \|H_{\overline{g}}k_z\|_2 = \lim_{z\to m} \|H_{u\overline{g}}k_z\|_2 = 0.$$

It follows that $\lim\limits_{|z|\to m} \|L_z\| = 0$.

To finish this proof, we need to show that Condition (1) can imply (6.1). By Lemma 2.7, we have

$$\lim_{z\to m} \|H_f k_z\|_2 = \lim_{z\to m} \|H_g k_z\|_2 = 0,$$

$$\lim_{z\to m}\|H_{(u-\lambda)\overline{f}}k_z\|_2 = \lim_{z\to m}\|H_{(u-\lambda)\overline{g}}k_z\|_2 = 0$$

and

$$\lim_{z\to m}\|H_{(u-\lambda)\overline{fg}}k_z\|_2 = 0.$$

Since

$$\begin{aligned}\|L_z\| &= \|H_{u\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{\overline{g}}k_z \otimes VH_{u\overline{f}}k_z\|\\ &= \|H_{(u-\lambda)\overline{g}}k_z \otimes VH_{\overline{f}}k_z - H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z\|\\ &\le \|H_{(u-\lambda)\overline{g}}k_z \otimes VH_{\overline{f}}k_z\| + \|H_{\overline{g}}k_z \otimes VH_{(u-\lambda)\overline{f}}k_z\|\\ &= \|H_{(u-\lambda)\overline{g}}k_z\|_2 \cdot \|H_{\overline{f}}k_z\|_2 + \|H_{\overline{g}}k_z\|_2 \cdot \|H_{(u-\lambda)\overline{f}}k_z\|_2,\end{aligned}$$

we obtain $\|L_z\| \to 0$ as $z \to m$. This completes the proof.

We are now in position to prove the sufficiency for Theorem 1.2.

Proof of the Sufficient Part of Theorem 1.2 For any $m \in \mathcal{M}(H^\infty + C)$, let S_m be the support set of m. Suppose that one of Conditions (1), (2) and (3) in Theorem 1.2 holds. According to Lemma 2.3, we need to show that

$$\widetilde{K_1} = T_fT_g + H^*_{u\overline{f}}H_{ug} - T_{fg},$$

$$\widetilde{K_2} = T_fH^*_{u\overline{g}} + H^*_{u\overline{f}}S_g - H^*_{u\overline{fg}},$$

$$\widetilde{K_3} = H_{uf}T_g + S_fH_{ug} - H_{ufg}$$

and

$$\widetilde{K_4} = H_{uf}H^*_{u\overline{g}} + S_fS_g - S_{fg}$$

are compact operators.

As $T_{fg} - T_fT_g = H^*_{\overline{f}}H_g$, we get

$$\widetilde{K_1} = H^*_{u\overline{f}}H_{ug} - H^*_{\overline{f}}H_g.$$

By Lemma 2.5, we have

$$\widetilde{K_1} - T^*_{\phi_z}\widetilde{K_1}T_{\phi_z} = V[H_{u\overline{f}}k_z \otimes H_{ug}k_z - H_{\overline{f}}k_z \otimes H_gk_z]V^*. \tag{6.2}$$

Next we will show that each condition in Theorem 1.2 can imply that

$$\lim_{z\to m}\|\widetilde{K_1} - T^*_{\phi_z}\widetilde{K_1}T_{\phi_z}\| = 0. \tag{6.3}$$

If Condition (3) holds, then we have by Lemma 2.7 that

$$\lim_{z\to m}\|H_fk_z\|_2 = \lim_{z\to m}\|H_gk_z\|_2 = 0$$

and

$$\lim_{z\to m}\|H_{uf}k_z\|_2 = \lim_{z\to m}\|H_{ug}k_z\|_2 = 0.$$

Observing that

$$\|H_{u\overline{f}}k_z \otimes H_{ug}k_z - H_{\overline{f}}k_z \otimes H_gk_z\| \le \|H_{u\overline{f}}k_z\|_2 \cdot \|H_{ug}k_z\|_2 + \|H_{\overline{f}}k_z\|_2 \cdot \|H_gk_z\|_2,$$

we obtain

$$\lim_{z\to m}\|\widetilde{K_1}-T_{\phi_z}^*\widetilde{K_1}T_{\phi_z}\|=0.$$

Assume that Condition (1) holds. From the proof of the sufficient part of Theorem 1.1, we get that

$$\lim_{z\to m}\|H_f k_z\|_2=\lim_{z\to m}\|H_g k_z\|_2=0,$$
$$\lim_{z\to m}\|H_{uf} k_z\|_2=\lim_{z\to m}\|H_{ug} k_z\|_2=0,$$
$$\lim_{z\to m}\|H_{(u-\lambda)\overline{f}} k_z\|_2=\lim_{z\to m}\|H_{(u-\lambda)\overline{g}} k_z\|_2=0$$

and

$$\lim_{z\to m}\|H_{(u-\lambda)\overline{fg}} k_z\|_2=0.$$

Since

$$\|H_{u\overline{f}}k_z\otimes H_{ug}k_z-H_{\overline{f}}k_z\otimes H_g k_z\|\le\|H_{u\overline{f}}k_z\|_2\cdot\|H_{ug}k_z\|_2+\|H_{\overline{f}}k_z\|_2\cdot\|H_g k_z\|_2,$$

we conclude that

$$\lim_{z\to m}\|\widetilde{K_1}-T_{\phi_z}^*\widetilde{K_1}T_{\phi_z}\|=0.$$

Using the same techniques as above, we can show that Condition (2) implies

$$\lim_{z\to m}\|\widetilde{K_1}-T_{\phi_z}^*\widetilde{K_1}T_{\phi_z}\|=0.$$

Therefore, each condition of Theorem 1.2 implies that

$$\lim_{|z|\to 1^-}\|\widetilde{K_1}-T_{\phi_z}^*\widetilde{K_1}T_{\phi_z}\|=0.$$

On the other hand, noting

$$H_{u\overline{f}}^*H_{ug}=T_{fg}-T_{\overline{u}f}T_{ug},$$

it follows that

$$\widetilde{K_1}=T_fT_g+H_{u\overline{f}}^*H_{ug}-T_{fg}=(T_{fg}-T_{\overline{u}f}T_{ug})-(T_{fg}-T_fT_g),$$

which is a finite sum of finite products of Toeplitz operators. Using the same method as in the proof of the sufficient part of Theorem 1.1, we conclude by (6.3) that $\widetilde{K_1}$ is compact.

Using

$$VT_\varphi=S_{\overline{\varphi}}V,\quad VH_\varphi=H_\varphi^*V\quad\text{and}\quad V^2=I$$

again, we have

$$\begin{aligned}V\widetilde{K_4}V&=V(H_{uf}H_{u\overline{g}}^*+S_fS_g-S_{fg})V\\&=H_{uf}^*V^2H_{u\overline{g}}+T_{\overline{f}}V^2T_{\overline{g}}-T_{\overline{fg}}V^2\\&=H_{uf}^*H_{u\overline{g}}+T_{\overline{f}}T_{\overline{g}}-T_{\overline{fg}}.\end{aligned}$$

Using the same arguments as above, we conclude that

$$H_{uf}^*H_{u\overline{g}}+T_{\overline{f}}T_{\overline{g}}-T_{\overline{fg}}$$

is compact, which gives us that $\widetilde{K_4}$ is also compact.

To show the compactness of $\widetilde{K_2}$, we will show that $\widetilde{K_2}\widetilde{K_2}^*$ is compact as before. Recall that

$$\widetilde{K_2} = T_f H^*_{u\overline{g}} + H^*_{u\overline{f}} S_g - H^*_{u\overline{fg}}.$$

Using identity (4.5) in [21] again, we have

$$H_{u\overline{fg}} = S_{\overline{g}} H_{u\overline{f}} + H_{\overline{g}} T_{u\overline{f}}.$$

Thus we get

$$\widetilde{K_2}^* = H_{u\overline{g}} T_{\overline{f}} - H_{\overline{g}} T_{u\overline{f}}$$

and

$$\widetilde{K_2}\widetilde{K_2}^* = (T_f H^*_{u\overline{g}} - T_{\overline{u}f} H^*_{\overline{g}})(H_{u\overline{g}} T_{\overline{f}} - H_{\overline{g}} T_{u\overline{f}}).$$

Note that $\widetilde{K_2}\widetilde{K_2}^*$ is a finite sum of finite products of Toeplitz operators and the symbol map maps this operator to zero. Applying [15, Lemma 12] and [3, Corollary 6] again, it suffices to show that

$$\lim_{z\to m} \left\| \widetilde{K_2}\widetilde{K_2}^* - T^*_{\phi_z} \widetilde{K_2}\widetilde{K_2}^* T_{\phi_z} \right\| = 0.$$

By Lemma 3.1, we have

$$\widetilde{K_2}^* T_{\phi_z} = S_{\phi_z} \widetilde{K_2}^* - L_z,$$

where L_z is defined in Lemma 6.1. Thus we have

$$\begin{aligned}
&T^*_{\phi_z} \widetilde{K_2}\widetilde{K_2}^* T_{\phi_z}\\
&= (\widetilde{K_2}^* T_{\phi_z})^* \widetilde{K_2}^* T_{\phi_z}\\
&= (S_{\phi_z}\widetilde{K_2}^* - L_z)^* (S_{\phi_z}\widetilde{K_2}^* - L_z)\\
&= (\widetilde{K_2} S^*_{\phi_z} - L^*_z)(S_{\phi_z}\widetilde{K_2}^* - L_z)\\
&= \widetilde{K_2} S^*_{\phi_z} S_{\phi_z} \widetilde{K_2}^* - \widetilde{K_2} S^*_{\phi_z} L_z - L^*_z S_{\phi_z}\widetilde{K_2}^* + L^*_z L_z\\
&= \widetilde{K_2}(I - Vk_z \otimes Vk_z)\widetilde{K_2}^* - \widetilde{K_2} S^*_{\phi_z} L_z - L^*_z S_{\phi_z}\widetilde{K_2}^* + L^*_z L_z\\
&= \widetilde{K_2}\widetilde{K_2}^* - \widetilde{K_2} Vk_z \otimes \widetilde{K_2} Vk_z - \widetilde{K_2} S^*_{\phi_z} L_z - L^*_z S_{\phi_z}\widetilde{K_2}^* + L^*_z L_z.
\end{aligned}$$

Lemma 6.1 gives us that $\|\widetilde{K_2} S^*_{\phi_z} L_z\|$, $\|L^*_z S_{\phi_z}\widetilde{K_2}^*\|$ and $\|L^*_z L_z\|$ all converge to 0 as $z \to m$. Thus, we need to show that $\|\widetilde{K_2} Vk_z\|_2 \to 0$ as $z \to m$. In fact,

$$\begin{aligned}
\widetilde{K_2} Vk_z &= (H_{u\overline{g}} T_{\overline{f}} - H_{\overline{g}} T_{u\overline{f}})^* Vk_z\\
&= T_f H^*_{u\overline{g}} Vk_z - T_{\overline{u}f} H^*_{\overline{g}} Vk_z\\
&= V(S_{\overline{f}} H_{u\overline{g}} k_z - S_{u\overline{f}} H_{\overline{g}} k_z)\\
&= V\{(I-P)[\overline{f}(I-P)(u\overline{g}k_z)] - (I-P)[u\overline{f}(I-P)(\overline{g}k_z)]\}\\
&= V(I-P)[u\overline{f}P(\overline{g}k_z) - \overline{f}P(u\overline{g}k_z)]\\
&= VH_{u\overline{f}} T_{\overline{g}} k_z - VH_{\overline{f}} T_{u\overline{g}} k_z,
\end{aligned}$$

where the third equality follows from that

$$VT_\varphi = S_{\overline{\varphi}}V, \quad VH_\varphi = H_\varphi^* V \quad \text{and} \quad V^2 = I.$$

If Condition (2) of Theorem 1.2 holds, then we have

$$\lim_{z\to m} \|H_{\overline{f}}k_z\|_2 = \lim_{z\to m} \|H_{u\overline{f}}k_z\|_2 = 0.$$

It follows from Lemma 4.2 that

$$\lim_{z\to m} \|\widetilde{K_2}Vk_z\|_2 = \lim_{z\to m} \|H_{u\overline{f}}T_{\overline{g}}k_z - H_{\overline{f}}T_{u\overline{g}}k_z\|_2 = 0.$$

If Condition (3) holds, then $\overline{f}|_{S_m}$ or $\overline{g}|_{S_m}$ is also a constant. This yields

$$\lim_{z\to m} \|H_{\overline{f}}k_z\|_2 = \lim_{z\to m} \|H_{u\overline{f}}k_z\|_2 = 0$$

or

$$\lim_{z\to m} \|H_{\overline{g}}k_z\|_2 = \lim_{z\to m} \|H_{u\overline{g}}k_z\|_2 = 0.$$

By Lemma 4.2 again, we have

$$\lim_{z\to m} \|\widetilde{K_2}Vk_z\|_2 = \lim_{z\to m} \|H_{u\overline{f}}T_{\overline{g}}k_z - H_{\overline{f}}T_{u\overline{g}}k_z\|_2 = 0$$

or

$$\lim_{z\to m} \|\widetilde{K_2}Vk_z\|_2 = \lim_{z\to m} \|S_{\overline{f}}H_{u\overline{g}}k_z - S_{u\overline{f}}H_{\overline{g}}k_z\|_2 = 0.$$

Finally, we assume that Condition (1) holds. From Lemma 2.7, we get

$$\lim_{z\to m} \|H_{(u-\lambda)\overline{f}}k_z\|_2 = \lim_{z\to m} \|H_{(u-\lambda)\overline{g}}k_z\|_2 = \lim_{z\to m} \|H_{(u-\lambda)\overline{fg}}k_z\|_2 = 0.$$

Noting that

$$\begin{aligned}
&\|\widetilde{K_2}Vk_z\|_2 \\
&= \|H_{u\overline{f}}T_{\overline{g}}k_z - H_{\overline{f}}T_{u\overline{g}}k_z\|_2 \\
&= \|H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z - H_{\overline{f}}T_{(u-\lambda)\overline{g}}k_z\|_2 \\
&= \|H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z - [H_{(u-\lambda)\overline{fg}} - S_{\overline{f}}H_{(u-\lambda)\overline{g}}]k_z\|_2 \\
&= \|H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z - H_{(u-\lambda)\overline{fg}}k_z + S_{\overline{f}}H_{(u-\lambda)\overline{g}}k_z\|_2 \\
&\le \|H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z\|_2 + \|H_{(u-\lambda)\overline{fg}}k_z\|_2 + \|S_{\overline{f}}H_{(u-\lambda)\overline{g}}k_z\|_2 \\
&\le \|H_{(u-\lambda)\overline{f}}T_{\overline{g}}k_z\|_2 + \|H_{(u-\lambda)\overline{fg}}k_z\|_2 + \|f\|_\infty \cdot \|H_{(u-\lambda)\overline{g}}k_z\|_2,
\end{aligned}$$

we conclude by Lemma 4.2 that $\lim_{z\to m} \|\widetilde{K_2}Vk_z\|_2 = 0$. Moreover, since

$$\begin{aligned}
\|\widetilde{K_2}\widetilde{K_2}^* - T_{\phi_z}^*\widetilde{K_2}\widetilde{K_2}^*T_{\phi_z}\| &= \|\widetilde{K_2}Vk_z \otimes \widetilde{K_2}Vk_z + \widetilde{K_2}S_{\phi_z}^*L_z + L_z^*S_{\phi_z}\widetilde{K_2}^* - L_z^*L_z\| \\
&\le \|\widetilde{K_2}Vk_z \otimes \widetilde{K_2}Vk_z\| + \|\widetilde{K_2}S_{\phi_z}^*L_z\| + \|L_z^*S_{\phi_z}\widetilde{K_2}^*\| + \|L_z^*L_z\| \\
&= \|\widetilde{K_2}Vk_z\|_2^2 + \|\widetilde{K_2}S_{\phi_z}^*L_z\| + \|L_z^*S_{\phi_z}\widetilde{K_2}^*\| + \|L_z\|^2,
\end{aligned}$$

we have

$$\lim_{z\to m} \|\widetilde{K_2}\widetilde{K_2}^* - T_{\phi_z}\widetilde{K_2}\widetilde{K_2}^* T_{\phi_z}\| = 0.$$

Using the same idea as in the proof of the compactness of $\widetilde{K_1}$, we conclude that $\widetilde{K_2}\widetilde{K_2}^*$ is compact, so $\widetilde{K_2}$ is also compact.

In order to finish the proof, we observe that

$$\begin{aligned} V\widetilde{K_3}V &= VH_{uf}T_gV + VS_fH_{ug}V - VH_{ufg}V \\ &= H_{uf}^*S_{\overline{g}} + T_{\overline{f}}H_{ug}^* - H_{ufg}^*. \end{aligned}$$

Similarly we can show that $\widetilde{K_3}$ is compact, to complete the proof of Theorem 1.2.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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