Chin. Ann. Math. Ser. B
45(4), 2024, 637–660
DOI: 10.1007/s11401-024-0031-7

Chinese Annals of
Mathematics, Series B

Limits of Riemann Solutions for Isentropic MHD in a Variable Cross-Section Duct as Magnetic Field Vanishes*

Wancheng SHENG[1] Tao XIAO[2]

Abstract The stability for magnetic field to the solution of the Riemann problem for the polytropic fluid in a variable cross-section duct is discussed. By the vanishing magnetic field method, the stable solutions are determined by comparing the limit solutions with the solutions of the Riemann problem for the polytropic fluid in a duct obtained by the entropy rate admissibility criterion.

Keywords Variable duct, Magnetogasdynamics, Riemann problem, Stability, Vanishing magnetic field method
2000 MR Subject Classification 35L65, 35L50, 35L80, 76J20, 76N10

1 Introduction

The non-conservative hyperbolic system plays an important role in many areas, such as the laminar flow in compliant tubes (see [2]), the shallow water (see [12]) and the multiphase flows (see [16]). The main difficulties of the Riemann problem for it are the existence and the uniqueness of the solution. In a recent paper [19], the Riemann problem for the isentropic, inviscid, simple flow of ideal gas, subjected to transverse magnetic field, in a duct with cross-sectional area $a(x) > 0$ in magnetogasdynamics, was studied. It is governed by the hyperbolic system

$$\begin{cases}(a\rho)_t + (a\rho u)_x = 0,\\ (a\rho u)_t + \Big(a\Big(\rho u^2 + p + \dfrac{B^2}{2\mu}\Big)\Big)_x = \Big(p + \dfrac{B^2}{2\mu}\Big)a_x\\ a_t = 0\end{cases} \tag{1.1}$$

with the Riemann initial data

$$(u, \rho, a) = \begin{cases}(u_-, \rho_-, a_-), & x < 0,\\ (u_+, \rho_+, a_+), & x > 0,\end{cases} \tag{1.2}$$

where $a_+ > a_- > 0,\ \rho_- > 0,\ \rho_+ > 0, u_-$ and u_+ are arbitrary constants. Symbols ρ, p, u, B and μ are the specific density, the pressure, the velocity, the transverse magnetic field and the

Manuscript received September 8, 2021. Revised March 24, 2024.
[1]Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail: mathwcsheng@shu.edu.cn
[2]Department of Mathematics, Shanghai University, Shanghai 200444, China; Shanghai Aircraft Design and Research Institute, Shanghai 201210, China. E-mail: xiaotao@comac.cc taoxiao@shu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 12171305).

magnetic permeability, resp. (see [18]). The pressure function and the transverse magnetic field function are given by $p=\kappa\rho^\gamma$ and $B=k\rho$, resp., where $\gamma\in(1,2),\kappa,k$ are positive constants. The existence has been obtained for any given initial data. However, for some initial data, there exist multi solutions.

In this article, we will select a proper unique solution mainly by the vanishing magnetic field method, motivated by the vanishing viscosity method (see [4]) and the vanishing pressure method (see [5]). It will be verified that the unique solution happens to the one obtained in [19] by the entropy rate admissibility criterion (see [6]).

We call a solution of (1.1) to be stable in a vanishing magnetic field, provided that the limit of it, as $k\to 0$, equals to the solution of

$$\begin{cases}(a\rho)_t+(a\rho u)_x=0,\\(a\rho u)_t+(a(\rho u^2+p))_x=pa_x,\\a_t=0\end{cases}\tag{1.3}$$

with the initial data (1.2). System (1.3) describes a compressible polytropic fluid flow in a nozzle and was studied in [13, 20]. The nonisentropic case was investigated in [1, 8, 22]. Putting $a_x=0$, (1.1) can be written in conservation form as

$$\begin{cases}\rho_t+(\rho u)_x=0,\\(\rho u)_t+\Big(\rho u^2+p+\dfrac{B^2}{2\mu}\Big)_x=0,\end{cases}\tag{1.4}$$

which describes an unsteady one-dimensional isentropic flow in magnetogasdynamic. The system was studied in [17]. In [7, 15, 21], the authors were concerned with the nonisentropic cases.

This paper is organised as follows. In Section 2, the elementary waves and some properties of them are collected. In Section 3, we present all the solutions of (1.1) and (1.3), for any given initial data (1.2). In Section 4, the unique solution is determined by choosing the stable solution in a vanishing magnetic field, which satisfies the entropy rate admissibility criterion, as it will be seen. Summary is given in Section 5.

2 Elementary Waves

System (1.1) has three real eigenvalues

$$\lambda_1=u-\omega,\quad \lambda_2=0,\quad \lambda_3=u+\omega,$$

where $\omega(\rho)=\sqrt{\frac{\mathrm{d}f}{\mathrm{d}\rho}}$ and $f(\rho)=p+\frac{B^2}{2\mu}$. It is strictly hyperbolic in the following three regions

$$\mathrm{I}=\{(u,\rho,a)\mid u<-\omega\},\quad \mathrm{II}=\{(u,\rho,a)\mid |u|<\omega\},\quad \mathrm{III}=\{(u,\rho,a)\mid u>\omega\}.$$

The characteristic fields λ_1 and λ_3 are genuinely nonlinear, and the characteristic field of λ_2 is linearly degenerate. For convenience, we set $\Sigma=\{u=-\omega\}$, $\Pi=\{u=\omega\}$, $\mathrm{II}^-=\mathrm{II}\cap\{u<0\}$ and $\mathrm{II}^+=\mathrm{II}\cap\{u>0\}$. There exist three different elementary waves.

2.1 Rarefaction waves

Define the point (u, ρ, a) in (u, ρ, a)-space as U or $U(u, \rho, a)$. Centered rarefaction waves $R_1(U_0, U)$ and $R_3(U_0, U)$ (abb. $R_1(U_0)$ and $R_3(U_0)$, resp.) are

$$\begin{cases} R_1(U_0) : a = a_0, \quad u = u_0 - \displaystyle\int_{\rho_0}^{\rho} \frac{\omega}{\rho}\,\mathrm{d}\rho, \quad \rho < \rho_0, \\ R_3(U_0) : a = a_0, \quad u = u_0 + \displaystyle\int_{\rho_0}^{\rho} \frac{\omega}{\rho}\,\mathrm{d}\rho, \quad \rho > \rho_0 \end{cases} \tag{2.1}$$

for any given left hand state $U_0(u_0, \rho_0, a_0)$. $R_1(U_0)$ is convex and monotonic decreasing while $R_3(U_0)$ is concave and monotonic increasing.

2.2 Shock waves

Considering discontinuous solutions, we obtain the Rankine-Hugoniot jump condition of the system for any given smooth function $a = a(x)$,

$$\sigma[a\rho] = [a\rho u], \quad \sigma[a\rho u] = \Big[a\Big(\rho u^2 + p + \frac{B^2}{2\mu}\Big)\Big].$$

Here σ represents the speed of the discontinuity, $[a\rho] = a_r\rho_r - a_l\rho_l$. The component a remains invariant across shock waves $S_1(U_0, U)$ and $S_3(U_0, U)$ (abb. $S_1(U_0)$ and $S_3(U_0)$, resp.) satisfying

$$-\sigma\,[\rho] + [\rho u] = 0, \quad -\sigma\,[\rho u] + [g(u, \rho)] = 0, \quad [a] = 0, \tag{2.2}$$

where

$$g(u, \rho) = \rho u^2 + \kappa\rho^\gamma + \frac{k^2}{2\mu}\rho^2. \tag{2.3}$$

By Lax entropy conditions (see [9]), $S_i(U_0)$ can be expressed as

$$\begin{cases} S_1(U_0) : a = a_0, \quad u = u_0 - \sqrt{\dfrac{1}{\rho\rho_0}\,[f]\,[\rho]}, \quad \sigma = u_0 + \rho\dfrac{[u]}{[\rho]}, \quad \rho > \rho_0,\ u < u_0, \\ S_3(U_0) : a = a_0, \quad u = u_0 - \sqrt{\dfrac{1}{\rho\rho_0}\,[f]\,[\rho]}, \quad \sigma = u_0 + \rho\dfrac{[u]}{[\rho]}, \quad \rho < \rho_0,\ u < u_0. \end{cases} \tag{2.4}$$

$S_1(U_0)$ is convex and monotonic decreasing while $S_3(U_0)$ is concave and monotonic increasing. We obtain the following lemma by direct calculations to (2.4).

Lemma 2.1 *On the shock waves $S_1(U_0)$ (resp., $S_3(U_0)$), it holds that*

(i) $\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} < 0$ (*resp.,* $\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} > 0$);

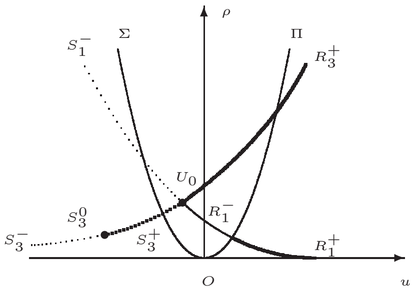
(ii) $\frac{\mathrm{d}u}{\mathrm{d}\rho} < -\frac{\omega}{\rho}$ (*resp.,* $\frac{\mathrm{d}u}{\mathrm{d}\rho} > \frac{\omega}{\rho}$);

(iii) *there exists a unique state $U \in \mathrm{II}^+$ (resp., $U \in \mathrm{I}$), denoted by $S_1^0(U_0)$ (resp., $S_3^0(U_0)$), such that $\sigma(U_0, U) = 0$ if and only if $U_0 \in \mathrm{III}$ (resp., $U_0 \in \mathrm{II}^-$).*

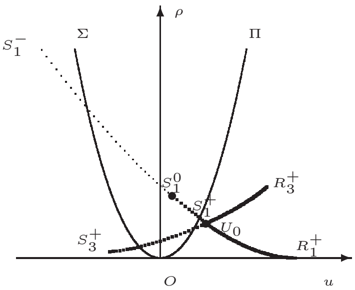
For any given left hand states U_0, we define

$$R_i^-(U_0) = R_i(U_0) \cap \{\lambda_i(U) \le 0\}, \quad S_i^-(U_0) = S_i(U_0) \cap \{\sigma(U_0, U) \le 0\},$$
$$R_i^+(U_0) = R_i(U_0) \cap \{\lambda_i(U) \ge 0\}, \quad S_i^+(U_0) = S_i(U_0) \cap \{\sigma(U_0, U) \ge 0\},$$
$$W_i^\pm(U_0) = R_i^\pm(U_0) \cup S_i^\pm(U_0), \quad W_i(U_0) = R_i(U_0) \cup S_i(U_0), \quad i = 1, 3.$$

Figure 1 shows the notations above for two cases, $U_0 \in \mathrm{II}^-$ and $U_0 \in \mathrm{III}$.



(a) $U_- \in \mathrm{II}^-$.



(b) $U_- \in \mathrm{III}$.

Figure 1 The solid (stressed solid) curves connected with U_0 are rarefaction waves R_i^- (R_i^+), while the dotted (stressed dotted) ones are shock waves S_i^- (S_i^+) for $i = 1, 3$. U_0 is omitted in the notations for convenience.

2.3 Stationary waves

For the case $[a] \neq 0$, following [10–11], the generalized Rankie-Hugoniot jump relations are

$$-\sigma(M - M_0) + \int_0^1 A(\phi(s; M_0, M)) \frac{\partial \phi}{\partial s}(s; M_0, M) \mathrm{d}s = 0 \tag{2.5}$$

for any given left-hand state M_0 and right-hand state M. Here $M(x,t) = (m_1, m_2, m_3)^{\mathrm{T}} = (a\rho, a\rho u, a)^{\mathrm{T}}$ and

$$A(M) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\mathrm{d}f}{\mathrm{d}\rho}\Big(\frac{m_1}{m_3}\Big) - \Big(\frac{m_2}{m_1}\Big)^2 & 2\frac{m_2}{m_1} & -\frac{m_1}{m_3}\frac{\mathrm{d}f}{\mathrm{d}\rho}\Big(\frac{m_1}{m_3}\Big) \\ 0 & 0 & 0 \end{pmatrix}.$$

$\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}} : [0,1] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a fixed Lipchitz continuous family of paths satisfying

$$\phi(0; M_0, M) = M_0, \quad \phi(1; M_0, M) = M$$

with some other properties, see also [12]. A direct calculation to (2.5) yields

$$\begin{cases} -\sigma[a\rho] + [a\rho u] = 0, \\ -\sigma[a\rho u] + \displaystyle\int_0^1 \Big(\Big(\frac{\partial \phi_1}{\partial s} - \frac{\phi_1}{\phi_3}\frac{\partial \phi_3}{\partial s}\Big) f'\Big(\frac{\phi_1}{\phi_3}\Big) - \Big(\frac{\phi_2}{\phi_1}\Big)^2 \frac{\partial \phi_1}{\partial s} + 2\frac{\phi_2}{\phi_1}\frac{\partial \phi_2}{\partial s}\Big) \mathrm{d}s = 0, \\ -\sigma[a] = 0. \end{cases} \tag{2.6}$$

The second equation of (2.6) leads to different relation if we define different paths ϕ_1 and ϕ_3. Especially, (2.6) is equivalent to (2.2) if we define ϕ_3 as a constant when $[a] = 0$. From the other two equations of (2.6), we have $\sigma = 0$ and $[a\rho u] = 0$. It motivates us to transform (2.6) to the form of the steady solution to (1.1). Under the definitions

$$\phi_1 = \begin{cases} a_0\rho_0, & 0 \le s \le s_1, \\ a_0\rho_0 + \dfrac{s - s_1}{s_2 - s_1}[a\rho], & s_1 < s < s_2, \\ a\rho, & s_2 \le s \le 1, \end{cases} \qquad \phi_3 = \begin{cases} a_0 + \dfrac{s}{s_1}(\widetilde{a} - a_0), & 0 \le s \le s_1, \\ \widetilde{a}, & s_1 < s < s_2, \\ \widetilde{a} + \dfrac{s - s_2}{1 - s_2}(a - \widetilde{a}), & s_2 \le s \le 1 \end{cases}$$

with arbitrary constants s_1, s_2 and the proper constant $\widetilde{a}$, we finally obtain the generalized Rankine-Hugoniot jump relations of stationary waves

$$W_2 : [a\rho u] = 0, \quad [h(u,\rho)] = 0, \quad \sigma = 0, \tag{2.7}$$

where

$$h(u,\rho) = \frac{u^2}{2} + \frac{\kappa\gamma}{\gamma - 1}\rho^{\gamma-1} + \frac{k^2}{\mu}\rho.$$

It is not reachable to obtain the explicit expressions of U in (2.7) as we have done in discussing rarefaction waves and shock waves for the given state $U_0(u_0, \rho_0, a_0)$. However, once we assume that $a > a_0$ and $\rho_0 u_0 \neq 0$, there exist two different solutions, denoted by $\overline{U}_0(\overline{u}_0, \overline{\rho}_0, a)$ and $\underline{U_0}(\underline{u_0}, \underline{\rho_0}, a)$ of (2.7). They satisfy

$$|u_0| > |\overline{u}_0|, \quad \rho_0 < \overline{\rho}_0, \quad |u_0| < |\underline{u_0}|, \quad \rho_0 > \underline{\rho_0}. \tag{2.8}$$

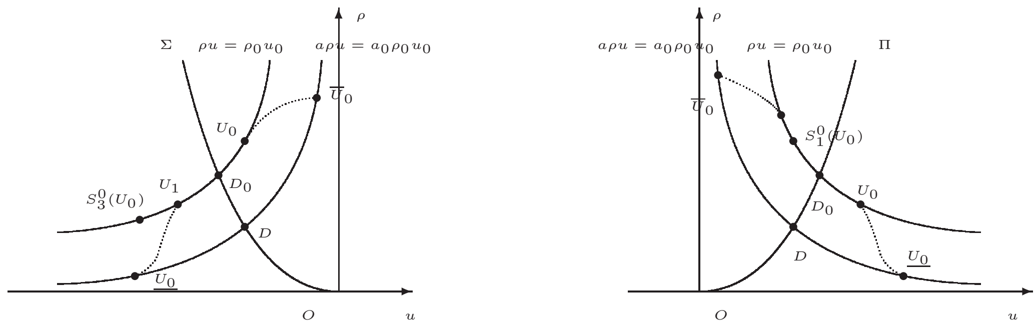
In fact, equation $[a\rho u] = 0$ implies that (u, ρ) lies on the curve $a\rho u = a_0\rho_0 u_0$. Along this curve, it holds that

$$\frac{\mathrm{d}h}{\mathrm{d}\rho} = \frac{\omega^2 - u^2}{\rho}.$$

Thus h reaches its minimum value at D, the intersection point of $a\rho u = a_0\rho_0 u_0$ with $\Pi \cup \Sigma$. We obtain

$$h(u_0, \rho_0) \ge h(D_0) > h(D),$$

where D_0 is the intersection point of $\rho u = \rho_0 u_0$ with $\Pi \cup \Sigma$. Therefore, two states solve (2.7). Inequalities in (2.8) can be achieved easily by comparing $h(u_0, \rho_0)$ with $h\big(u_0, \frac{a\rho u}{a_0 u_0}\big)$. In particular, when $a = a_0$, the two solutions $\overline{U}_0 = U_0$, $\underline{U_0} \in \mathrm{I} \cup \mathrm{III}$ if $U_0 \in \mathrm{II}$, while $\overline{U}_0 \in \mathrm{II}$, $\underline{U_0} = U_0$ if $U_0 \in \mathrm{I} \cup \mathrm{III}$. Figure 2 shows two cases, $U_0 \in \mathrm{III}$ and $U_0 \in \mathrm{II}^-$. The dotted curves reveal the behaviours of the solutions for (2.7) as a decreases to a_0. That can be obtained by differentiating (2.7) with respect to a, resp.. State U_1 will be defined later. The fact that there exists no stationary wave solution for (1.4) motivates us to suggest the stability stationary wave condition to remove the unreasonable solution.



(a) $U_- \in \mathrm{II}^-$. (b) $U_- \in \mathrm{III}$.

Figure 2 The stationary wave solutions $\overline{U}_0$ and $\underline{U_0}$.

Stability Stationary Wave Condition: The state $U(u,\rho,a)$ is called a stable stationary solution of (2.7), if u and ρ are continuous functions of a, and the two states U and U_0 satisfy the Rankine-Hugoniot jump condition (2.2) when $a = a_0$.

As a conclusion, we obtain the following lemma.

Lemma 2.2 *For any given* $U_0(u_0,\rho_0,a_0)$ *and* $a > a_0$, *the two solutions* $\overline{U}_0(\overline{u}_0,\overline{\rho}_0,a)$ *and* $\underline{U_0}(\underline{u_0},\underline{\rho_0},a)$ *of* (2.7) *satisfy* (2.8) *and*

(i) $\overline{U}_0 \in \mathrm{II}^{\pm}$ *is the unique stable stationary solution, if* $U_0 \in \mathrm{II}^{\pm}$;

(ii) $\underline{U_0} \in \mathrm{I}$ (*resp.,* III) *is the unique stable stationary solution, if* $U_0 \in \mathrm{I}$ (*resp.,* III);

(iii) *both* $\underline{U_0} \in \mathrm{I}$ (*resp.,* III) *and* $\overline{U}_0 \in \mathrm{II}$ *are the stable stationary solutions, if* $U_0 \in \Sigma$ (*resp.,* Π).

Proof We only give the proof for the case $U_0 \in \mathrm{II}^-$, see Figure 2(a). The others can be obtained by similar discussions. Define $U_z(u_z,\rho_z,a_0) = S_3^0(U_0)$ and $U_1(u_1,\rho_1,a_0) = \underline{U_0}$. It is clear that $\overline{U}_0$ is the stable stationary solution. Furthermore, following jump conditions (2.2) and (2.7), we have

$$g(u_z,\rho_z) = g(u_0,h_0), \quad h(u_1,\rho_1) = h(u_0,h_0), \quad \rho_z u_z = \rho_1 u_1 = \rho_0 u_0.$$

Along the curve $\rho u = \rho_0 u_0$, it holds that

$$\rho_1 \int_{\rho_1}^{\rho_z} h'(\rho)\mathrm{d}\rho > \int_{\rho_1}^{\rho_z} \rho h'(\rho)\mathrm{d}\rho = \int_{\rho_1}^{\rho_z} g'(\rho)\mathrm{d}\rho = \int_{\rho_1}^{\rho_0} g'(\rho)\mathrm{d}\rho > \rho_1 \int_{\rho_1}^{\rho_0} h'(\rho)\mathrm{d}\rho.$$

Thus we get

$$h(u_z,\rho_z) > h(u_0,h_0) = h(u_1,\rho_1),$$

which implies that $\underline{U_0}$ does not satisfy the stability stationary wave condition. We complete the proof.

For any given left hand state U_0 and right hand state U, $\overline{W}_2(U_0,U)$ denotes the stationary wave satisfying $U = \overline{U}_0$, while $\underline{W_2}(U_0,U)$ denotes the stationary wave satisfying $U = \underline{U_0}$.

It is clear that a changes only when the gas passes across the stationary wave. When there exists no confusion, symbols denote the projections of themselves on (u, ρ) plane, either. For example, U_- denotes both $U_-(u_-, \rho_-, a_-)$ and $U_-(u_-, \rho_-)$.

3 The Riemann Solutions for $k \geq 0$

For any given $k \geq 0$, Riemann problem (1.1) with (1.2) can be solved constructively by the two cases $U_- \in \Delta_\pm$ with

$$\Delta_+(\Delta_-) \triangleq \Big\{U(u,\rho)\Big| u + \int_0^\rho \frac{\omega(\rho)}{\rho} \mathrm{d}\rho > (\leq) 0\Big\},$$

which are separated by $R_1(U, O) : u = -\int_0^\rho \frac{\omega(\rho)}{\rho} \mathrm{d}\rho$, see Figure 3. The Riemann problem can be solved constructively as

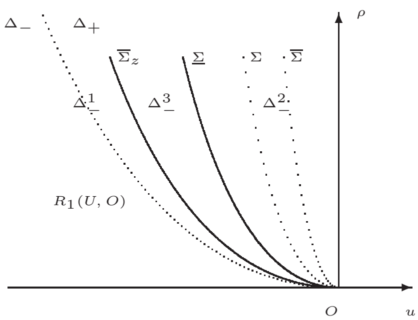
$$W_1^- \oplus W_3^- \oplus W_2 \oplus W_1^+ \oplus W_3^+, \tag{3.1}$$

if it contains a unique stable stationary wave, here "$\oplus$" means "followed by". Each elementary wave may not appear except W_2. A solution contains two stable stationary waves is constructed as

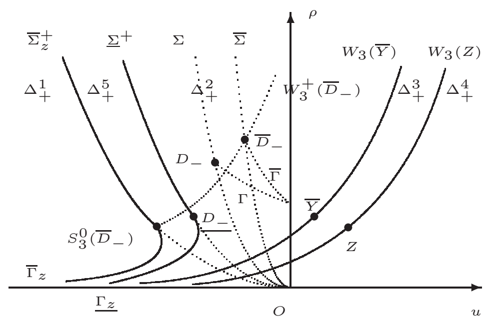
$$W_1^- \oplus W_3^- \oplus W_2 \oplus S_i \oplus W_2 \oplus W_1^+ \oplus W_3^+. \tag{3.2}$$

Here the cross-section areas a on both sides of the zero-speed shock wave S_i ($i = 1$ or 3) satisfy $a \in (a_-, a_+)$.

Case 1 $U_- \in \Delta_-$. The curves are defined by



(a) $U_- \in \Delta_-$.



(b) $U_- \in \Delta_+$.

Figure 3 Different regions separated by the solid curves.

$$\underline{\Sigma} = \{U \mid U = \underline{U_0}, U_0 \in \Sigma\}, \quad \overline{\Sigma} = \{U \mid U = \overline{U}_0, U_0 \in \Sigma\}, \quad \overline{\Sigma}_z = \{U \mid U = S_3^0(\overline{U}_0), U_0 \in \Sigma\}$$

with the cross-section area of U_0 being a_-, and the one of U being a_+. It can be proved that $\underline{\Sigma}$ is on the right of $\overline{\Sigma}_z$. Thus the solid curves $\overline{\Sigma}_z$ and $\underline{\Sigma}$ in Figure 1(a) separate the upper half (u, ρ) plane into three regions, Δ_-^1, Δ_-^2 and Δ_-^3 (including $\overline{\Sigma}_z$ and $\underline{\Sigma}$).

The states, connected by the coalescence waves $W_1^- \oplus W_3^-$ with U_-, are located on the left of Σ. Considering the Riemann solution is illustrated as (3.1) with $\underline{W_2}$, we require that U_+ is located on the left of $\underline{\Sigma}$ following Lemmas 2.1–2.2. U_+ must be located on the right of $\overline{\Sigma}_z$ if the solution is illustrated as (3.1) with $\overline{W}_2$. It is easy to prove that the states connected by the coalescence waves $W_1^- \oplus W_3^- \oplus \overline{W}_2 \oplus S_3 \oplus \underline{W_2}$ are located on $\Delta_-^3 \subset I$. Thus, the Riemann solutions of (1.1) with (1.2) are illustrated as follows.

Subcase 1.1 $U_- \in \Delta_-,\ U_+ \in \Delta_-^1$ (see Figure 4). The solution is unique and structured in

$$Q_-^1 : W_1^- \oplus W_3^- \oplus \underline{W_2}.$$

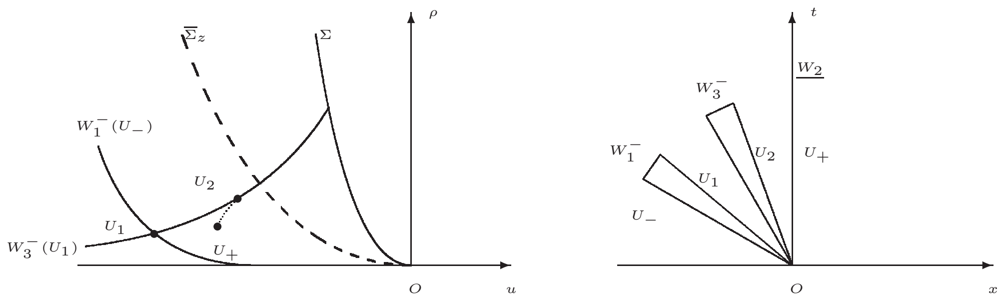


Figure 4 Subcase 1.1, Q_-^1 : $W_1^- \oplus W_3^- \oplus \underline{W_2}$.

Subcase 1.2 $U_- \in \Delta_-,\ U_+ \in \Delta_-^2$ (see Figure 5). The solution is unique and structured in

$$Q_-^2 : W_1^- \oplus R_3^- \oplus \overline{W}_2 \oplus W_3^+.$$

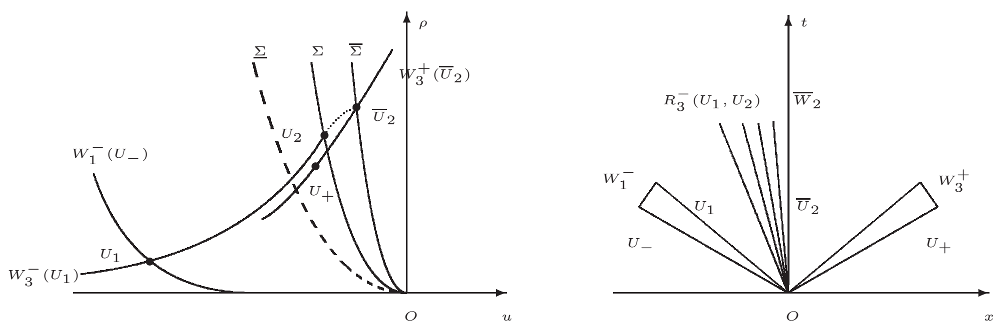


Figure 5 Subcase 1.2, Q_-^2 : $W_1^- \oplus R_3^- \oplus \overline{W}_2 \oplus W_3^+$.

Subcase 1.3 $U_- \in \Delta_-,\ U_+ \in \Delta_-^3$, the solution is not unique. Besides Q_-^1, Q_-^2, the other solution with two stable stationary waves can be constructed as

$$Q_-^3 : W_1^-(U_-, U_1) \oplus R_3^-(U_1, U_3) \oplus \overline{W}_2(U_3, U_4) \oplus S_3(U_4, U_2) \oplus \underline{W_2}(U_2, U_+).$$

This special solution and Q_+^5, which will be listed later, were not discussed in [13]. The states $U_3(u_3, \rho_3, a_-)$, $U_4(u_4, \rho_4, a)$ and $U_2(u_2, \rho_2, a)$ satisfy that

$$\sigma(U_4, U_2) = 0, \quad u_2, u_3, u_4 < 0, \ a \in [a_-, a_+].$$

$a = a_-$ holds if and only if $U_+ \in \underline{\Sigma}$, and then $Q_-^3 = Q_-^1$. $a = a_+$ holds if and only if $U_+ \in \overline{\Sigma}_z$, and then $Q_-^3 = Q_-^2$. When $U_+ \in \Delta_-^2 \backslash (\overline{\Sigma}_z \cup \underline{\Sigma})$, Q_-^3 is unstable (see [14]). Because it contains a standing shock wave $S_3(U_4, U_2)(\sigma(U_4, U_2) = 0)$, which occurs in contracting duct.

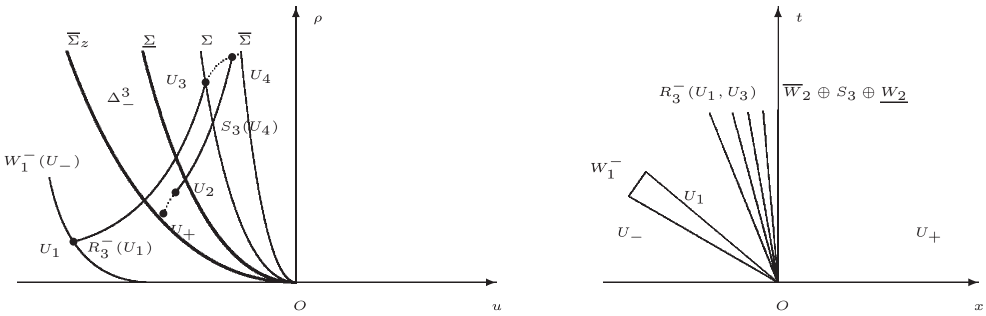


Figure 6 Subcase 1.3, Q_-^3 : $W_1^- \oplus R_3^- \oplus \overline{W}_2 \oplus S_3 \oplus \underline{W_2}$.

Case 2 $U_- \in \Delta_+$. The solid curves $\overline{\Sigma}_z^+$, $\overline{\Gamma}_z$, $\underline{\Sigma}^+$, $\underline{\Gamma_z}$, $W_3(\overline{Y})$ and $W_3(Z)$ in Figure 1(b) separate the upper half (u, ρ) plane into Δ_+^1, Δ_+^2 (including $W_3(\overline{Y})$), Δ_+^3, Δ_+^4 (including $W_3(Z)$) and Δ_+^5 (including the boundaries). Γ is the part of the curve $W_1(U_-)$, the ends of which are $D_- \in \Sigma$ and the one on ρ axis, resp.. Define

$$\overline{\Gamma} = \{U \mid U = \overline{U}_0, \ U_0 \in \Gamma\}, \quad \overline{\Gamma}_z = \{U \mid U = S_3^0(\overline{U}_0), U_0 \in \Gamma\},$$
$$\Gamma_z = \{\widetilde{U} \mid \widetilde{U} = S_3^0(U_0), \ U_0 \in \Gamma\}, \quad \underline{\Gamma_z} = \{U \mid U = \underline{U_0}, \ U_0 \in \Gamma_z\}$$

with the cross-section areas of $U_0, \widetilde{U}$ being a_- and the one of U being a_+. Obviously, $\underline{D_-}$ is an end of $\underline{\Gamma_z}$ and $S_3^0(\overline{D}_-)$ is an end of $\overline{\Gamma}_z$. For convenience, let

$$\overline{\Sigma}_z^+ = \overline{\Sigma}_z \cap \{\rho \geq \rho(S_3^0(\overline{D}_-))\}, \quad \underline{\Sigma}^+ = \underline{\Sigma} \cap \{\rho \geq \rho(\underline{D_-})\},$$

where $\rho(\underline{D_-})$ denotes the ρ coordinate at $\underline{D_-}$. It can be proved that $\overline{\Gamma}_z$ is at the left of $\underline{\Gamma_z}$, and $W_3(\overline{Y})$ is at the left of $W_3(Z)$. Here,

$$Y = \begin{cases} S_1^0(U_-), & \text{if } U_- \in \mathrm{III}, \\ D_+, & \text{otherwise}, \end{cases} \qquad Z = \begin{cases} S_1^0(\underline{U_-}), & \text{if } U_- \in \mathrm{III}, \\ S_1^0(\underline{D_+}), & \text{otherwise}, \end{cases}$$

where D_+ is the intersection point of Π with $W_1(U_-)$.

After a similar discussion as we have done in Case 1, the Riemann solutions are constructed as follows.

Subcase 2.1 $U_- \in \Delta_+,\ U_+ \in \Delta_+^1$ (see Figure 7). The solution is unique and structured in

$$Q_+^1 : W_1^- \oplus W_3^- \oplus \underline{W_2}.$$

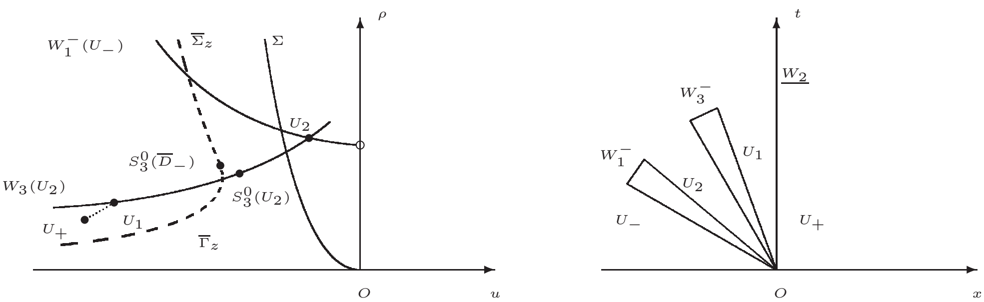


Figure 7 Subcase 2.1, $Q_+^1 : W_1^- \oplus W_3^- \oplus \underline{W_2}$.

Subcase 2.2 $U_- \in \Delta_+,\ U_+ \in \Delta_+^2$. The solution is unique and structured in

$$Q_+^2 : W_1^-(\oplus R_3^-) \oplus \overline{W}_2 \oplus W_3^+,$$

where R_3^- appears if and only if U_+ is located on the left of $W_3^+(\overline{D}_-)$ (see Figure 8).

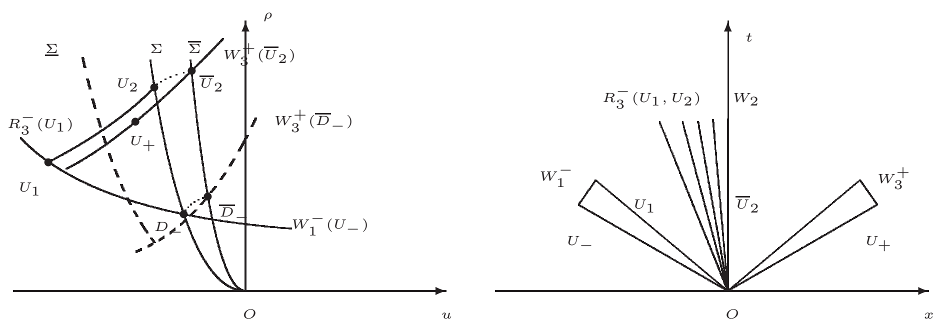


Figure 8 Subcase 2.2, $Q_+^2 : W_1^- \oplus R_3^- \oplus \overline{W}_2 \oplus W_3^+$.

Subcase 2.3 $U_- \in \Delta_+,\ U_+ \in \Delta_+^3$. The solution is unique and structured in

$$Q_+^3 : (R_1^- \oplus)\underline{W_2} \oplus S_1 \oplus \overline{W}_2 \oplus W_3^+,$$

where the gas velocities of both sides of the standing shock wave S_1 are positive. Therefore, S_1 occurs in a compacting duct, and R_1^- appears if and only if $u_- < \omega(\rho_-)$. Figure 9 shows the case $U_- \in$ Ⅲ.

Subcase 2.4 $U_- \in \Delta_+,\ U_+ \in \Delta_+^4$. The solution is unique and structured in

$$Q_+^4 : (R_1^- \oplus)\underline{W_2} \oplus W_1^+ \oplus W_3^+,$$

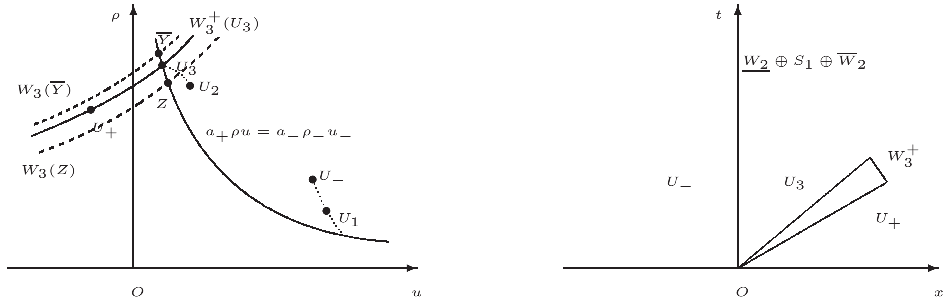


Figure 9 Subcase 2.3, $Q_+^3 : \underline{W_2} \oplus S_1 \oplus \overline{W}_2 \oplus W_3^+$.

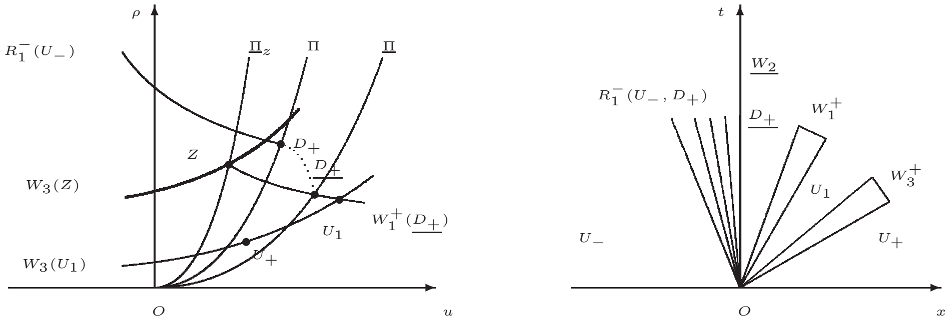


Figure 10 Subcase 2.4, $Q_+^4 : R_1^- \oplus \underline{W_2} \oplus W_1^+ \oplus W_3^+$.

where R_1^- appears if and only if $u_+ < \omega(\rho_+)$ (see Figure 10).

Subcase 2.5 $U_- \in \Delta_+, U_+ \in \Delta_+^5$. In this case, the solution loses uniqueness. Besides Q_+^1 and Q_+^2, the other solution can be structured in

$$Q_+^5 : W_1^-(\oplus R_3^-) \oplus \overline{W}_2 \oplus S_3 \oplus \underline{W_2},$$

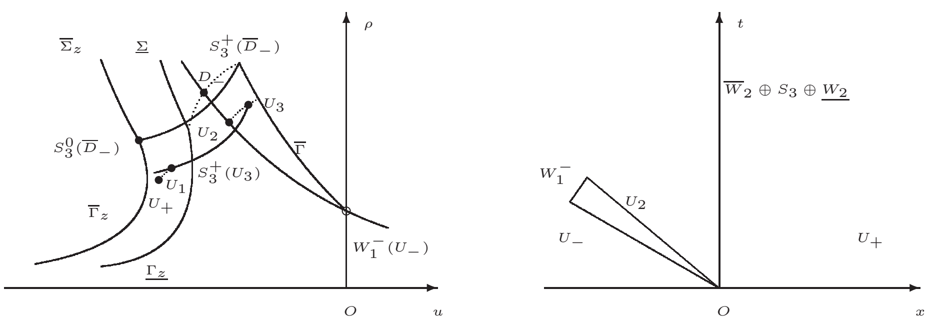


Figure 11 Subcase 2.5: $Q_+^5 : W_1^- \oplus \overline{W}_2 \oplus S_3 \oplus \underline{W_2}$.

where, R_3^- appears if and only if U_+ is located on the left of $W_3^+(\overline{D}_-)$. Figure 11 shows Q_+^5 without R_3^-. The states $U_1(u_1, \rho_1, a_-) \in \mathrm{I}$, $U_2(u_2, \rho_2, a) \in \mathrm{I\!I}^-$ and $U_3(u_3, \rho_3, a) \in \mathrm{I\!I}^-$ satisfy that

$$g(U_3) = g(U_1), \quad h(U_2) = h(U_3), \quad h(U_1) = h(U_+),$$
$$a_-\rho_2 u_2 = a\rho_3 u_3 = a\rho_1 u_1 = a_+\rho_+ u_+, \quad a \in [a_-, a_+].$$

$a = a_-$ holds if and only if $U_+ \in \overline{\Sigma}_z^+ \cup \overline{\Gamma}_z$, and then $Q_+^5 = Q_+^1$. $a = a_+$ holds if and only if $U_+ \in \underline{\Sigma}^+ \cup \underline{\Gamma_z}$, and then $Q_+^5 = Q_+^2$. Otherwise, similar to Q_-^3, the solution Q_+^5 will not be considered, either.

Under the stability stationary wave condition, the unique stable solution of (2.7) is guaranteed except $U_0 \in \Sigma \cup \Pi$ following Lemma 4.3. The existence of the Riemann problem has been obtained case by case. However, the solutions lose uniqueness in Subcases 1.3 and 2.5. Suggesting the entropy rate admissibility criterion, [19] obtained the unique admissible solutions.

Entropy Rate Admissibility Criterion A solution $U(x,t)$ of the Riemann problem (1.1) with (1.2) is admissible provided that $D_+H_U(\tau) < D_+H_{\widehat{U}}(\tau), \tau \in [0,t]$, where $\widehat{U}(x,t) \neq U(x,t)$ is any other Riemann solution of (1.1).

The total entropy rate of a solution $U(x,t)$ is defined as

$$D_+H_U(t) = \lim_{\Delta t \to 0^+} \frac{H_U(x_0, t+\Delta t) - H_U(x_0,t)}{\Delta t},$$

where the total entropy is

$$H_U(x_0,t) = \int_{-x_0+u_-t}^{x_0+u_+t} a\rho\eta(U)\mathrm{d}x$$

with $t \geq 0$. Here $x_0 > 0$ is large enough so that for any $\tau \in [0,t]$, $U(x,\tau) = U_+$ as $x \geq x_0 + u_+\tau$ and $U(x,\tau) = U_-$ as $x \leq -x_0 + u_-\tau$. The definition of the special entropy $\eta(U) = -u$ of (1.1) with the entropy flux $q(U) = -h(u,\rho)$ is motivated by the works [3, 6, 9]. Then direct calculations yield that $D_+H_{Q_-^2}$ is less than $D_+H_{Q_-^1}$ and $D_+H_{Q_-^3}$ in Subcase 1.3, while $D_+H_{Q_+^2}$ is less than $D_+H_{Q_+^1}$ and $D_+H_{Q_+^5}$ in Subcase 2.5. Hence the unique admissible solution of (1.1) with (1.2) is $Q_\pm^2$ in Subcases 1.3 and 2.5, respectively.

4 The Behaviour of the Solution as $k \to 0$

It has been declaimed that in Subcase 1.3 (resp., 2.5), both Q_-^1 and Q_-^2 (resp., Q_+^1 and Q_+^2) are the solutions of (1.1) for $k \geq 0$. By the entropy rate admissibility criterion, we can construct the solution uniquely for any given initial data (1.2). In this section, we firstly study the limit solutions of (1.1) with any initial data (1.2) as $k \to 0$. Secondly, we compare the limit solutions with the solutions of (1.3). We want to check whether the limit solution is the one selected by the entropy rate admissibility criterion. The variation of k leads to the changes of the solid curves in Figure 3. Thus, the structure of the solution may change if k vanishes for the fixed initial data. To study the limit solution of (1.1) with (1.2), we only need to concentrate on the case that U_+ is located on the solid curves when $k = 0$ in Figure 3. Then our goal is to investigate the behaviour of the solid curves as $k \to 0$. For simplify calculations, we replace $\frac{k^2}{\mu}$ with k. Then $\omega,\ f,\ g$ and h are rewritten as

$$\omega(\rho) = \sqrt{\kappa\gamma\rho^{\gamma-1} + k\rho}, \quad f(\rho) = \kappa\rho^{\gamma} + \tfrac{k}{2}\rho^2,$$
$$g(u,\rho) = \rho u^2 + \kappa\rho^{\gamma} + \tfrac{k}{2}\rho^2, \quad h(u,\rho) = \tfrac{u^2}{2} + \tfrac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1} + k\rho,$$

respectively, from now on. To be more exactly, $\omega(\rho)$ is the abbreviation of $\omega(\rho,k)$, etc. We define $\Sigma(0) = \{U(u,\rho) \mid u = -\omega(\rho,0) = -\sqrt{\kappa\gamma\rho^{\gamma-1}}\}$, etc.

4.1 The behaviour of the solution as $k \to 0$ when $U_- \in \Delta_-(0)$

We have the following lemma when we investigate the behaviour of $\overline{\Sigma}_z$ and $\underline{\Sigma}$ as $k \to 0$.

Lemma 4.1 *There exists a sufficient small constant $k_0 > 0$ such that, for any $k \in (0,k_0)$,* (i) $\underline{\Sigma}$ *is at the left of* $\underline{\Sigma}(0)$; (ii) $\overline{\Sigma}_z$ *is at the left of* $\overline{\Sigma}_z(0)$.

Proof (i) Assume that $U_0(0) = (u_0(0), \rho_0(0), a_-) \in \Sigma(0)$ is an arbitrary state. Define $U_0 = (u_0, \rho_0, a_-) \in \Sigma$, $\underline{U_0} = (\underline{u_0}, \underline{\rho_0}, a_+) \in \underline{\Sigma}$, $\overline{U}_0 = (\overline{u}_0, \overline{\rho}_0, a_+) \in \overline{\Sigma}$ and $\overline{U}_z = (\overline{u}_z, \overline{\rho}_z, a_+) \in \overline{\Sigma}_z$ satisfying that

$$u_0^2 = \omega_0^2 = \kappa\gamma\rho_0^{\gamma-1} + k\rho_0, \quad h(u_0,\rho_0) = h(\underline{u_0}, \underline{\rho_0}) = h(\overline{u}_0, \overline{\rho}_0), \tag{4.1}$$

$$g(\overline{u}_z, \overline{\rho}_z) = g(\overline{u}_0, \overline{\rho}_0), \tag{4.2}$$

$$a_-\rho_0(0)u_0(0) = a_-\rho_0 u_0 = a_+\underline{\rho_0 u_0} = a_+\overline{\rho}_0\overline{u}_0 = a_+\overline{\rho}_z\overline{u}_z \tag{4.3}$$

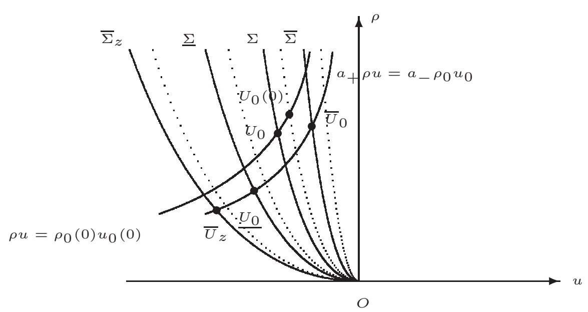


Figure 12 Curves $\overline{\Sigma}_z,\ \underline{\Sigma},\ \Sigma$ and $\overline{\Sigma}$, move to the dotted lines $\overline{\Sigma}_z(0),\ \underline{\Sigma}(0),\ \Sigma(0)$ and $\overline{\Sigma}(0)$, resp., from their left as $k \to 0$.

(see Figure 12). Lemma 2.2 implies that the following inequalities hold,

$$\overline{\rho}_z < \underline{\rho_0} < \rho_0 < \overline{\rho}_0, \quad \overline{u}_z^2 > \overline{\omega}_z^2, \quad \underline{u_0}^2 > \underline{\omega_0}^2, \quad \overline{u}_0^2 < \overline{\omega}_0^2,$$

where $\overline{\omega}_z = \omega(\overline{\rho}_z)$, etc.

Differentiating (4.3) with respect to k, resp., one gets

$$0 = \rho_0' u_0 + \rho_0 u_0' = \underline{\rho_0}'\underline{u_0} + \underline{\rho_0}\underline{u_0}' = \overline{\rho}_0'\overline{u}_0 + \overline{\rho}_0\overline{u}_0' = \overline{\rho}_z'\overline{u}_z + \overline{\rho}_z\overline{u}_z', \tag{4.4}$$

here and hereafter, $' = \frac{\mathrm{d}}{\mathrm{d}k}$. From (4.1) and (4.4), we get

$$\rho_0 = \frac{\omega_0^2 - u_0^2}{\rho_0}\rho_0' + \rho_0 = \frac{\underline{\omega_0}^2 - \underline{u_0}^2}{\underline{\rho_0}}\underline{\rho_0}' + \underline{\rho_0} = \frac{\overline{\omega}_0^2 - \overline{u}_0^2}{\overline{\rho}_0}\overline{\rho}_0' + \overline{\rho}_0, \tag{4.5}$$

which follows $\underline{\rho_0}' < 0$. Because the intersection point of $\underline{\Sigma}$ with $\rho u =$ const. is unique if k is fixed. We say $\underline{\Sigma}$ to be at the left of $\underline{\Sigma}(0)$.

(ii) Differentiating (4.2) with respect to k and from (4.5), we obtain

$$(\overline{\omega}_z^2 - \overline{u}_z^2)\overline{\rho}_z' = (\overline{\omega}_0^2 - \overline{u}_0^2)\overline{\rho}_0' + \frac{\overline{\rho}_0^2 - \overline{\rho}_z^2}{2} = \rho_0\overline{\rho}_0 - \overline{\rho}_0^2 + \frac{\overline{\rho}_0^2 - \overline{\rho}_z^2}{2} = \frac{2\rho_0\overline{\rho}_0 - \overline{\rho}_0^2 - \overline{\rho}_z^2}{2}.$$

We now prove that $2\rho_0 > \overline{\rho}_0 + \overline{\rho}_z$ holds at $k = 0$ for any given state $U_0(u_0, \rho_0, a_-)$ with $0 < -u_0 \le \omega_0$. For $a \ge a_-$, we define $\overline{U}_0(\overline{u}_0, \overline{\rho}_0, a)$ and $\overline{U}_z(\overline{u}_z, \overline{\rho}_z, a) = S_3^0(\overline{U}_0)$ satisfying

$$h_0 = \overline{h}_0, \quad \overline{g}_0 = \overline{g}_z, \quad a_-\rho_0 u_0 = a\overline{\rho}_0\overline{u}_0 = a\overline{\rho}_z\overline{u}_z. \tag{4.6}$$

It is clear that $\overline{U}_0$ and $\overline{U}_z$ are functions of a, from the stable stationary wave condition. The following inequalities hold when $a > a_-$,

$$\overline{\rho}_z < \rho_0 < \overline{\rho}_0, \quad \overline{u}_z < u_0 < \overline{u}_0 < 0, \quad \overline{u}_z^2 > \overline{\omega}_z^2, \quad u_0^2 < \omega_0^2, \quad \overline{u}_0^2 > \overline{\omega}_0^2.$$

We have $\overline{U}_0 = U_0$ and $\rho_0 \ge \overline{\rho}_z$, thus $2\rho_0 \ge \overline{\rho}_0 + \overline{\rho}_z$, if $a = a_-$. From (4.6), it holds that

$$\begin{aligned}
&\frac{\mathrm{d}(\overline{\rho}_0 + \overline{\rho}_z)}{\mathrm{d}a} = \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2 - \overline{u}_0^2)} + \frac{2\overline{\rho}_z\overline{u}_z^2 - \overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_z^2 - \overline{u}_z^2)} \\
&= \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2 - \overline{u}_0^2)(\overline{\omega}_z^2 - \overline{u}_z^2)}\Big(\kappa\gamma\overline{\rho}_z^{\gamma-1} - \frac{\overline{\rho}_0}{\overline{\rho}_z}\frac{\overline{f}_0 - f_z}{\overline{\rho}_0 - \overline{\rho}_z} + \Big(2\frac{\overline{\rho}_0}{\overline{\rho}_z} - 1\Big)\Big(\kappa\gamma\overline{\rho}_0^{\gamma-1} - \frac{\overline{\rho}_z}{\overline{\rho}_0}\frac{\overline{f}_0 - f_z}{\overline{\rho}_0 - \overline{\rho}_z}\Big)\Big) \\
&= \frac{\overline{\rho}_0\overline{u}_0^2\kappa\overline{\rho}_z^{\gamma-1}}{a(\overline{\omega}_0^2 - \overline{u}_0^2)(\overline{\omega}_z^2 - \overline{u}_z^2)}\Big(\gamma - \alpha\frac{\alpha^\gamma - 1}{\alpha - 1} + (2\alpha - 1)\Big(\gamma\alpha^{\gamma-1} - \frac{1}{\alpha}\frac{\alpha^\gamma - 1}{\alpha - 1}\Big)\Big)
\end{aligned}$$

for $k = 0$, where

$$\frac{\overline{f}_0 - f_z}{\overline{\rho}_0 - \overline{\rho}_z} = \frac{\kappa\overline{\rho}_0^\gamma - \kappa\overline{\rho}_z^\gamma}{\overline{\rho}_0 - \overline{\rho}_z} = \kappa\overline{\rho}_z^{\gamma-1}\frac{(\frac{\overline{\rho}_0}{\overline{\rho}_z})^\gamma - 1}{\frac{\overline{\rho}_0}{\overline{\rho}_z} - 1} = \kappa\overline{\rho}_z^{\gamma-1}\frac{\alpha^\gamma - 1}{\alpha - 1}$$

and $\alpha = \frac{\overline{\rho}_0}{\overline{\rho}_z} > 1$ for $a > a_-$. Therefore, $\frac{\mathrm{d}(\overline{\rho}_0 + \overline{\rho}_z)}{\mathrm{d}a}$ has the different sign with the auxiliary function

$$M_1(\alpha) = \gamma\alpha(\alpha - 1) - \alpha^2(\alpha^\gamma - 1) + (2\alpha - 1)(\gamma\alpha^\gamma(\alpha - 1) - \alpha^\gamma + 1).$$

It is easy to check that $M_1(\alpha) > M_1(1) = 0$ by $\frac{\mathrm{d}M_1}{\mathrm{d}\alpha}(\alpha) > 0$. In fact,

$$\begin{aligned}
&\frac{\mathrm{d}^3 M_1}{\mathrm{d}\alpha^3}(\alpha) = \gamma(\gamma + 1)\alpha^{\gamma-3}(\alpha^2(2\gamma - 1)(\gamma + 2) - \alpha(3\gamma + 2)(\gamma - 1) + (\gamma - 2)(\gamma - 1)) > 0, \\
&\frac{\mathrm{d}^2 M_1}{\mathrm{d}\alpha^2}(\alpha) = (\gamma + 1)(\alpha^\gamma(2\gamma - 1)(\gamma + 2) - \alpha^{\gamma-1}(3\gamma + 2)\gamma + \alpha^{\gamma-2}\gamma(\gamma - 1) + 2) > 0,
\end{aligned}$$

Then we obtain

$$\frac{\mathrm{d}M_1}{\mathrm{d}\alpha}(\alpha) > \frac{\mathrm{d}M_1}{\mathrm{d}\alpha}(1) = 0.$$

As a result, we have that $2\rho_0 > \overline{\rho}_0 + \overline{\rho}_z$ holds for any $a > a_-$. Thus

$$2\rho_0\overline{\rho}_0 - \overline{\rho}_0^2 - \overline{\rho}_z^2 > (\overline{\rho}_0 + \overline{\rho}_z)\overline{\rho}_0 - \overline{\rho}_0^2 - \overline{\rho}_z^2 = \overline{\rho}_z(\overline{\rho}_0 - \overline{\rho}_z) > 0,$$

which implies $\overline{\rho}_z'(0) < 0$. We complete the proof.

Lemma 4.1 points out that there exists a constant $k_0 > 0$ such that for any $k \in (0, k_0)$: (1) $U_+ \in \Delta_-^2(k)$, if $U_+ \in \underline{\Sigma}(0) \subset \Delta_-^3(0)$; (2) $U_+ \in \Delta_-^3(k)$, if $U_+ \in \overline{\Sigma}_z(0) \subset \Delta_-^3(0)$. Recall the conclusion that the Riemann solution losses uniqueness if and only if $U_- \in \Delta_-, U_+ \in \Delta_-^3$ or $U_- \in \Delta_+, U_+ \in \Delta_+^5$. We then obtain the following corollary.

Corollary 4.1 *When the solution losses uniqueness, we choose* $Q_\pm^1$*(resp.,* $Q_\pm^2$*) as the unique solution of* (1.1) *with* (1.2) *for any* $k \geq 0$*. There exists a* $k_0 > 0$ *such that for any* $k \in (0, k_0)$*, we have that*:

(i) $U_- \in \Delta_-$ *and* $U_+ \in \Delta_-^3$*, if* $U_- \in \Delta_-(0)\backslash R_1(U, O)|_{k=0}$ *and* $U_+ \in \overline{\Sigma}_z(0)$*. Then the unique solution of* (1.1) *is* Q_-^1*(resp.,* Q_-^2*), while the unique solution of* (1.3) *is* $Q_-^1(0)$*(resp.,* $Q_-^2(0)$*).*

(ii) $U_- \in \Delta_-$ *and* $U_+ \in \Delta_-^2$*, if* $U_- \in \Delta_-(0)\backslash R_1(U, O)|_{k=0}$ *and* $U_+ \in \underline{\Sigma}(0)$*. Then the unique solution of* (1.1) *is* Q_-^2*, while the unique solution of* (1.3) *is* $Q_-^1(0)$*(resp.,* $Q_-^2(0)$*).*

(iii) $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$ *is located on the left of* $W_3^+(\overline{D}_-)$*, if* $U_- \in R_1(U, O)|_{k=0}$ *and* $U_+ \in \overline{\Sigma}_z(0)$*. Then the unique solution of* (1.1) *is* Q_+^1*(resp.,* $Q_+^2 : W_1^- \oplus R_3^- \oplus W_2 \oplus W_3^+$*), while the unique solution of* (1.3) *is* $Q_-^1(0)$*(resp.,* $Q_-^2(0)$*).*

(iv) $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^2$ *is located on the left of* $W_3^+(\overline{D}_-)$*, if* $U_- \in R_1(U, O)|_{k=0}$ *and* $U_+ \in \underline{\Sigma}(0)$*. Then the unique solution of* (1.1) *is* $Q_+^2 : W_1^- \oplus R_3^- \oplus W_2 \oplus W_3^+$*, while the unique solution of* (1.3) *is* $Q_-^1(0)$*(resp.,* $Q_-^2(0)$*).*

4.2 The behaviour of the solution as $k \to 0$ when $U_- \in \Delta_+(0)$

We have the following lemma when we investigate the behaviour of $\overline{\Gamma}_z$ and $\underline{\Gamma_z}$ as $k \to 0$.

Lemma 4.2 *There exists a sufficient small constant* $k_0 > 0$ *such that, for any* $k \in (0, k_0)$*,* (i) $\underline{\Gamma_z}$ *is at the left of* $\underline{\Gamma_z}(0)$; (ii) $\overline{\Gamma}_z$ *is at the left of* $\overline{\Gamma}_z(0)$*.*

Proof (i) Assume that $U_0(0)(u_0(0), \rho(0), a_-) \in \Gamma(0)$ is an arbitrary state. Let $U_0(u_0, \rho_0, a_-) \in$ II be the intersection point of $\rho u = \rho_0(0)u_0(0)$ with Γ, $\underline{U_z}(\underline{u_z}, \underline{\rho_z}, a_+) \in$ I, $\overline{U}_0(\overline{u}_0, \overline{\rho}_0, a_+) \in$ I, and $\overline{U}_v(\overline{u}_v, \overline{\rho}_v, a_+) \in$ I satisfying that $\sigma(U_0, U_z) = 0$ *and* $\sigma(\overline{U}_0, \overline{U}_v) = 0$. More precisely, we have

$$a_-\rho_0(0)u_0(0) = a_-\rho_0 u_0 = a_-\rho_z u_z = a_+\underline{\rho_z}\,\underline{u_z} = a_+\overline{\rho}_0\overline{u}_0 = a_+\overline{\rho}_v\overline{u}_v, \tag{4.7}$$

$$g_0 = g_z, \quad h_z = \underline{h_z}, \quad h_0 = \overline{h}_0, \quad \overline{g}_0 = \overline{g}_v, \tag{4.8}$$

$$u_0^2 < \omega_0^2, \quad \underline{u_z}^2 > \underline{\omega_z}^2, \quad \overline{u}_0^2 < \overline{\omega}_0^2, \quad \overline{u}_v^2 > \overline{\omega}_z^2, \quad u_z^2 > \omega_z^2, \quad \overline{\rho}_v < \underline{\rho_z} < \rho_z < \rho_0 < \overline{\rho}_0 \tag{4.9}$$

(see Figure 13). Differentiating (4.8) with respect to k, resp., one obtains

$$(\omega_0^2-u_0^2)\rho_0'+\frac{\rho_0^2}{2}=(\omega_z^2-u_z^2)\rho_z'+\frac{\rho_z^2}{2},\quad \frac{\omega_z^2-u_z^2}{\rho_z}\rho_z'+\rho_z=\frac{\underline{\omega_z}^2-\underline{u_z}^2}{\underline{\rho_z}}\underline{\rho_z}'+\underline{\rho_z},\tag{4.10}$$

$$\frac{\omega_0^2-u_0^2}{\rho_0}\rho_0'+\rho_0=\frac{\overline{\omega}_0^2-\overline{u}_0^2}{\overline{\rho}_0}\overline{\rho}_0'+\overline{\rho}_0,\quad (\overline{\omega}_0^2-\overline{u}_0^2)\overline{\rho}_0'+\frac{\overline{\rho}_0^2}{2}=(\overline{\omega}_v^2-\overline{u}_v^2)\overline{\rho}_v'+\frac{\overline{\rho}_v^2}{2}.\tag{4.11}$$

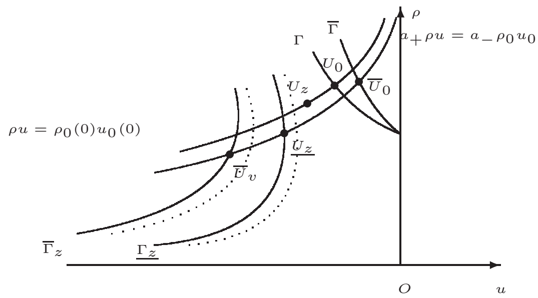


Figure 13 $\overline{\Gamma}_z$ and $\underline{\Gamma_z}$ move to the dotted lines $\overline{\Gamma}_z(0)$ and $\underline{\Gamma_z}(0)$, resp., from their left as $k\to 0$.

The sign of $\underline{\rho_z}'$ can be determined by ρ_0', which can be obtained by the following two cases. When $U_0\in R_1(U_-)$, it holds

$$u_0=u_--\int_{\rho_-}^{\rho_0}\frac{\omega}{\rho}\mathrm{d}\rho,\quad \rho_0<\rho_-.$$

Associating it with the first equation of (4.7), we have

$$u_0'=\frac{u_0}{\omega_0-u_0}\int_{\rho_-}^{\rho_0}\frac{1}{2\omega}\mathrm{d}\rho,\quad \rho_0'=-\frac{\rho_0}{\omega_0-u_0}\int_{\rho_-}^{\rho_0}\frac{1}{2\omega}\mathrm{d}\rho.\tag{4.12}$$

Therefore $\rho_0'>0$. When $U_0\in S_1(U_-)$, setting $\omega_\theta^2=\frac{f_0-f_-}{\rho_0-\rho_-}$, we achieve $\rho_-<\rho_0$ and

$$u_--u_0=\frac{(\rho_0-\rho_-)\omega_\theta}{\sqrt{\rho_-\rho_0}},\quad u_0'=\frac{f_0'\big(\frac{1}{\rho_-}-\frac{1}{\rho_0}\big)+(f_0-f_-)\frac{\rho_0'}{\rho_0^2}}{2(u_0-u_-)}\tag{4.13}$$

from (2.2). The speed of the shock wave can be rewritten as

$$\sigma=\frac{\rho_0u_0-\rho_-u_-}{\rho_0-\rho_-}=u_0+\rho_-\frac{u_0-u_-}{\rho_0-\rho_-}=u_0-\sqrt{\frac{\rho_-}{\rho_0}}\omega_\theta.$$

Applying (4.13), we have

$$(u_0^2-\sigma^2-\omega_0^2)\rho_0'=\frac{\rho_0^2}{2}.\tag{4.14}$$

From (4.10), determined by

$$\frac{\underline{\omega_z}^2-\underline{u_z}^2}{\underline{\rho_z}}\underline{\rho_z}'=\frac{\omega_z^2-u_z^2}{\rho_z}\rho_z'+\rho_z-\underline{\rho_z}=\frac{1}{2\rho_z}(2(\omega_0^2-u_0^2)\rho_0'+(\rho_0^2+\rho_z^2-2\underline{\rho_z}\rho_z)),$$

$\underline{\rho_z}' < 0$ holds when $U_0 \in R_1(U_-)$. Since $\rho_0 \geq \underline{\rho_z}$ and $\rho_0' > 0$, when $U_0 \in S_1(U_-)$, it holds that

$$\begin{aligned}\frac{\underline{\underline{\omega_z}}^2 - \underline{u_z}^2}{\underline{\rho_z}}\underline{\rho_z}' &= \frac{1}{2\rho_z}\Big(-\frac{\omega_0^2 - u_0^2}{\omega_0^2 - u_0^2 + \sigma^2}\rho_0^2 + \rho_0^2 + \rho_z^2 - 2\underline{\rho_z}\rho_z\Big) \\ &\geq \frac{1}{2\rho_z\omega_0^2}\big(-(\omega_0^2 - u_0^2)\rho_0^2 + \omega_0^2\rho_0^2 + \omega_0^2\rho_z^2 - 2\omega_0^2\underline{\rho_z}\rho_z\big) \\ &= \frac{1}{2\rho_z\omega_0^2}\big(u_0^2\rho_0^2 + \omega_0^2\rho_z^2 - 2\omega_0^2\underline{\rho_z}\rho_z\big) > \frac{1}{2\rho_z\omega_0^2}2(\rho_z - \underline{\rho_z})\rho_z\omega_0^2 > 0.\end{aligned}$$

In fact, it is clear that

$$\begin{aligned}\rho_0^2u_0^2 - \rho_z^2\omega_0^2 &= \frac{\kappa\rho_z}{\rho_0 - \rho_z}\big(\rho_0^{\gamma+1} - \rho_z^{\gamma}\rho_0 - \gamma\rho_0^{\gamma}\rho_z + \gamma\rho_0^{\gamma-1}\rho_z^2\big) \\ &= \frac{\kappa\rho_z^{\gamma+1}\rho_0}{\rho_0 - \rho_z}\Big(\Big(\frac{\rho_0}{\rho_z}\Big)^{\gamma} - 1 - \gamma\Big(\frac{\rho_0}{\rho_z}\Big)^{\gamma-1} + \gamma\Big(\frac{\rho_0}{\rho_z}\Big)^{\gamma-2}\Big) > 0\end{aligned}$$

for $\rho_0 > \rho_z$. Thus, we also have $\underline{\rho_z}' < 0$.

(ii) When $U_0 \in R_1^-(U_-)$, it is apparent that for $k = 0$,

$$(\overline{\omega}_v^2 - \overline{u}_v^2)\overline{\rho}_v' = (\overline{\omega}_0^2 - \overline{u}_0^2)\overline{\rho}_0' + \frac{\overline{\rho}_0^2}{2} - \frac{\overline{\rho}_v^2}{2} = \frac{\overline{\rho}_0(\omega_0^2 - u_0^2)}{\rho_0}\rho_0' + \rho_0\overline{\rho}_0 - \overline{\rho}_0^2 - \frac{\overline{\rho}_v^2}{2} > 0,$$

following (4.11)– (4.12) and the proof of Lemma 4.1. Therefore, $\overline{\rho}_v'(0) < 0$.

When $U_0 \in S_1^-(U_-)$, from (4.11) and (4.14), we have

$$\begin{aligned}(\overline{\omega}_v^2 - \overline{u}_v^2)\overline{\rho}_v' &= \frac{\overline{\rho}_0\rho_0}{2}\Big(2 - \frac{\omega_0^2 - u_0^2}{\omega_0^2 - u_0^2 + \sigma^2}\Big) + \frac{-\overline{\rho}_0^2 - \overline{\rho}_v^2}{2} \\ &\geq \frac{1}{2\omega_0^2}\big((\omega_0^2 + u_0^2)\rho_0\overline{\rho}_0 - \omega_0^2\overline{\rho}_0^2 - \omega_0^2\overline{\rho}_v^2\big).\end{aligned}$$

To declaim $\overline{\rho}_v'(0) < 0$, we are about to prove that

$$\rho_0 > \frac{\overline{\rho}_0^2 + \overline{\rho}_v^2}{\overline{\rho}_0}\frac{\omega_0^2}{\omega_0^2 + u_0^2}, \quad \text{if } \alpha \triangleq \frac{\overline{\rho}_0}{\overline{\rho}_v} \leq \alpha_0, \tag{4.15}$$

$$\rho_0 \geq \Big(\overline{\rho}_0 + \frac{\overline{\rho}_v}{\gamma}\Big)\frac{\omega_0^2}{\omega_0^2 + u_0^2}, \quad \text{if } \alpha > \alpha_0, \tag{4.16}$$

where $\alpha_0 > \gamma > 1$ is the root of the equation $\alpha_0^2 - 2\alpha_0 - 1 = 0$.

For any given state $U_0(u_0, \rho_0, a_-)$ with $u_0 < 0$, $u_0^2 < \omega_0^2$, we define $\overline{U}_0(\overline{u}_0, \overline{\rho}_0, a)$ and $\overline{U}_v(\overline{u}_v, \overline{\rho}_v, a) = S_3^0(\overline{U}_0)$ for $a \geq a_-$. It is clear that when $a = a_-$ and $k = 0$, it holds that

$$\rho_0 > \frac{\rho_0^2 + \overline{\rho}_v^2}{\rho_0}\frac{\omega_0^2}{\omega_0^2 + u_0^2},$$

which is equivalent to

$$\rho_0^2u_0^2 = \rho_0\overline{\rho}_v\frac{\kappa\rho_0^{\gamma} - \kappa\overline{\rho}_v^{\gamma}}{\rho_0 - \overline{\rho}_v} > \overline{\rho}_v^2\kappa\gamma\rho_0^{\gamma-1}.$$

Since

$$1 - \Big(\frac{\overline{\rho}_v}{\rho_0}\Big)^{\gamma} > \gamma\frac{\overline{\rho}_v}{\rho_0}\Big(1 - \frac{\overline{\rho}_v}{\rho_0}\Big)$$

holds when $\rho_0 > \overline{\rho}_v$, we have

$$\frac{\mathrm{d}}{\mathrm{d}a}\Big(\frac{\overline{\rho}_0^2+\overline{\rho}_v^2}{\overline{\rho}_0}\Big) = \frac{(2\overline{\rho}_0^2-\overline{\rho}_0^2-\overline{\rho}_v^2)\frac{\mathrm{d}\overline{\rho}_0}{\mathrm{d}a}+2\overline{\rho}_0\overline{\rho}_v\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a}}{\overline{\rho}_0^2} = \frac{(\overline{\rho}_0^2-\overline{\rho}_v^2)\frac{\mathrm{d}\overline{\rho}_0}{\mathrm{d}a}+2\overline{\rho}_0\overline{\rho}_v\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a}}{\overline{\rho}_0^2}$$

$$< \frac{(2\overline{\rho}_0\overline{\rho}_v-(\overline{\rho}_0^2-\overline{\rho}_v^2))}{\overline{\rho}_0^2}\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a} = \frac{-(\alpha^2-2\alpha-1)}{\overline{\rho}_0^2\overline{\rho}_v^2}\frac{\mathrm{d}\overline{\rho}_v}{\mathrm{d}a} < 0$$

for $1 \le \alpha < \alpha_0$. So far, (4.15) has been proved.

When $\alpha \ge \alpha_0$, for $a = a_-$ and $k = 0$, it holds that $\overline{U}_0 = U_0,\ U_z = \underline{U_z} = \overline{U}_v$. From

$$\rho_0 u_0^2 = \overline{\rho}_v\frac{\kappa\rho_0^\gamma-\kappa\overline{\rho}_v^\gamma}{\rho_0-\overline{\rho}_v} > \overline{\rho}_v\kappa\rho_0^{\gamma-1},$$

we have

$$\rho_0 \ge \Big(\overline{\rho}_0+\frac{\overline{\rho}_v}{\gamma}\Big)\frac{\omega_0^2}{\omega_0^2+u_0^2}.$$

Direct calculations lead to

$$\frac{\mathrm{d}}{\mathrm{d}a}\Big(\overline{\rho}_0+\frac{\overline{\rho}_v}{\gamma}\Big) = \frac{\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_0^2-\overline{u}_0^2)}+\frac{1}{\gamma}\frac{2\overline{\rho}_v\overline{u}_v^2-\overline{\rho}_0\overline{u}_0^2}{a(\overline{\omega}_v^2-\overline{u}_v^2)}$$

$$= \frac{\overline{\rho}_0\overline{u}_0^2\kappa\overline{\rho}_v^{\gamma-1}}{a\gamma(\overline{\omega}_0^2-\overline{u}_0^2)(\overline{\omega}_v^2-\overline{u}_v^2)}\Big(\gamma^2-\gamma\alpha\frac{\alpha^\gamma-1}{\alpha-1}+(2\alpha-1)\Big(\gamma\alpha^{\gamma-1}-\frac{1}{\alpha}\frac{\alpha^\gamma-1}{\alpha-1}\Big)\Big),$$

which has different sign with the auxiliary function

$$M_2(\alpha) = \gamma^2\alpha(\alpha-1)-\gamma\alpha^2(\alpha^\gamma-1)+(2\alpha-1)\gamma\alpha^\gamma(\alpha-1)-(2\alpha-1)(\alpha^\gamma-1)$$
$$= \gamma\alpha^{\gamma+2}-(3\gamma+2)\alpha^{\gamma+1}+(\gamma+1)\alpha^\gamma+(\gamma^2+\gamma)\alpha^2+(2-\gamma^2)\alpha-1.$$

For $\gamma \in (1,2)$, the third derivative

$$\frac{1}{\gamma+1}\frac{\mathrm{d}^3M_2}{\mathrm{d}\alpha^3}(\alpha) = \gamma\alpha^{\gamma-3}(\gamma(\gamma+2)\alpha^2-(\gamma-1)(3\gamma+2)\alpha+(\gamma-1)(\gamma-2))$$
$$> \gamma\alpha^{\gamma-3}(\gamma^2+2\gamma-3\gamma^2+\gamma+2+\gamma^2-3\gamma+2) = \gamma\alpha^{\gamma-3}(4-\gamma^2) > 0$$

holds for any $\alpha > 1$. By using the equation $\alpha_0^2-2\alpha_0-1=0$, we have

$$\frac{1}{\gamma+1}\frac{\mathrm{d}^2M_2}{\mathrm{d}\alpha^2}(\alpha) = (\gamma+2)\gamma\alpha^\gamma-(3\gamma+2)\gamma\alpha^{\gamma-1}+\gamma(\gamma-1)\alpha^{\gamma-2}+2\gamma$$
$$\ge \frac{1}{\gamma+1}\frac{\mathrm{d}^2M_2}{\mathrm{d}\alpha^2}(\alpha_0) > 0.$$

Thus, we get $\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha) > \frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0)$ and

$$\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0) = -(\gamma^2+\gamma+2)\alpha_0^\gamma+\gamma(2\gamma+2)\alpha_0^{\gamma-1}+2(\gamma+1)\gamma\alpha_0+2-\gamma^2+\gamma\alpha_0^{\gamma-1}.$$

It holds that

$$\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0) > \Big(2\gamma^2 + 2\gamma - \frac{5}{2}\gamma^2 - \frac{5}{2}\gamma - 5 + 2\gamma^2 + 2\gamma\Big)\alpha_0^{\gamma-1} > 0$$

if $\gamma^2 \geq 2$. Meanwhile

$$\frac{\mathrm{d}M_2}{\mathrm{d}\alpha}(\alpha_0) > \Big(2\gamma^2 + 3\gamma - \frac{5}{2}\gamma^2 - \frac{5}{2}\gamma - 5 + 3\gamma^2 + 3\gamma\Big)\alpha_0^{\gamma-1} = \frac{1}{2}(5\gamma^2 + 7\gamma - 10)\alpha_0^{\gamma-1} > 0$$

holds if $\gamma^2 < 2$. We finally achieve

$$\begin{aligned} M_2(\alpha) \geq M_2(\alpha_0) &= -(\gamma+2)\alpha_0^{\gamma+1} + (2\gamma+1)\alpha_0^{\gamma} + (\gamma^2 + 2\gamma + 2)\alpha_0 + (\gamma^2 + \gamma - 1) \\ &> \alpha_0^{\gamma}\Big(-\frac{1}{2}\gamma - 4\Big) + (\gamma^2 + 2\gamma + 2)\alpha_0 + (\gamma^2 + \gamma - 1) \triangleq M_3(\gamma) > 0. \end{aligned}$$

In fact, $M_3(\gamma)$ is concave in $(1, 2)$, $M_3(1) > 0$ and $M_3(2) = 0$. We complete the proof.

Lemmas 4.1–4.2 point out that there exists a constant $k_0 > 0$ such that for any $k \in (0, k_0)$: (1) $U_+ \in \Delta_+^2(k)$, if $U_+ \in \underline{\Sigma}^+(0) \subset \Delta_+^5(0)$; (2) $U_+ \in \Delta_+^5(k)$, if $U_+ \in \overline{\Sigma}_z^+(0) \subset \Delta_+^5(0)$. Recall the conclusion that the Riemann solution losses uniqueness if and only if $U_- \in \Delta_-, U_+ \in \Delta_-^3$ or $U_- \in \Delta_+, U_+ \in \Delta_+^5$. We then obtain the following corollary.

Corollary 4.2 *When the initial data satisfy* $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$*, we choose* Q_+^1*(resp.,* Q_+^2*) as the unique solution of* (1.1) *with* (1.2) *for any* $k \geq 0$*. Lemmas* 4.1–4.2 *imply that there exists a* $k_0 > 0$*, such that for any* $k \in (0, k_0)$*, we have that*:

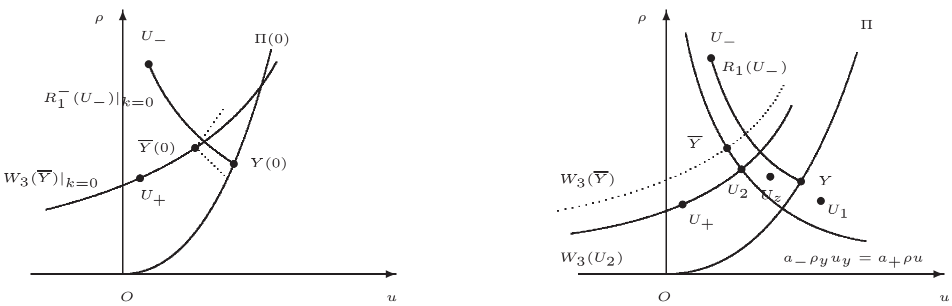
(i) $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$*, if* $U_- \in \Delta_+(0), U_+ \in \overline{\Sigma}_z^+(0) \cup \overline{\Gamma}_z(0)$*. Then the unique solution of* (1.1) *is* Q_+^1*(resp.,* Q_+^2*), while the unique solution of* (1.3) *is* $Q_+^1(0)$*(resp.,* $Q_+^2(0)$*).*

(ii) $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^2$*, if* $U_- \in \Delta_+(0), U_+ \in \underline{\Sigma}^+(0) \cup \underline{\Gamma_z}(0)$*. Then the unique solution of* (1.1) *is* Q_+^2*, while the unique solution of* (1.3) *is* $Q_+^1(0)$ *(resp.,* $Q_+^2(0)$*).*

To discuss the behaviour of $W_3(\overline{Y})$ as $k \to 0$, we consider the Riemann initial data satisfying

$$U_- \in \Delta_+(0) \backslash \text{Ш}(0), \quad U_+ \in W_3(\overline{Y})|_{k=0} \subset \Delta_+^2(0) \tag{4.17}$$

(see Figure 14(a)).



(a) $k = 0,\ Q_+^2(0): R_1^- \oplus \overline{W}_2 \oplus W_3^+$. (b) $k > 0,\ Q_+^3: R_1^- \oplus \underline{W_2} \oplus S_1 \oplus \overline{W}_2 \oplus W_3^+$.

Figure 14 The Riemann solution of (1.1) with (1.2) satisfying (4.17).

Thus, the solution of (1.3) is structured in

$$Q_+^2(0): R_1^-(U_-, Y(0)) \oplus \overline{W}_2(Y(0), \overline{Y}(0)) \oplus W_3^+(\overline{Y}(0), U_+).$$

More precisely, the following equations hold,

$$\begin{cases} u_y(0) = u_- - \displaystyle\int_{\rho_-}^{\rho_y} \frac{\sqrt{\kappa\rho^{\gamma-1}}}{\rho} \mathrm{d}\rho, \quad u_y(0) = \omega_y(0), \\ a_-\rho_y(0)u_y(0) = a_+\overline{\rho}_y(0)\overline{u}_y(0), \quad h_y(0) = h_1(0). \end{cases} \tag{4.18}$$

Here, we used the fact that $Y(0) = D_+(0) = (u_y(0), \rho_y(0), a_-)$ and $\overline{Y}(0) = (\overline{u}_y(0), \overline{\rho}_y(0), a_+)$. Whether U_+ is located on Δ_+^2 or not depends on both U_- and U_+ for $k > 0$. As we will see, the value of $\frac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y}|_{k=0}$ changes in

$$\Big(-\infty, -\frac{\overline{u}_y}{\overline{\rho}_y}\Big) \cup \Big(\frac{\overline{u}_y(\overline{\omega}_y^2 - u_y^2)}{\overline{\rho}_y(u_y^2 - \overline{u}_y^2)}, +\infty\Big).$$

By the definitions of Y and $\overline{Y}$, we have

$$u_y = u_- - \int_{\rho_-}^{\rho_y} \frac{\omega}{\rho} \mathrm{d}\rho = \omega_y, \quad a_-\rho_y u_y = a_+\overline{\rho}_y\overline{u}_y, \quad h_y = \overline{h}_y. \tag{4.19}$$

The equations follow that

$$\begin{cases} \rho_y' = \dfrac{\omega_y \int_{\rho_y}^{\rho_-} \frac{1}{\omega}\mathrm{d}\rho - \rho_y}{\kappa\gamma\rho_y^{\gamma-2}(\gamma+1) + 3k}, \\ u_y' = \dfrac{\kappa\gamma\rho_y^{\gamma-1}(\gamma+3) + 5k\rho_y}{\kappa\gamma\rho_y^{\gamma-1}(\gamma+1) + 3k\rho_y} \displaystyle\int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho - \frac{\omega_y}{\kappa\gamma\rho_y^{\gamma-2}(\gamma+1) + 3k}. \end{cases} \tag{4.20}$$

Direct calculations to (4.19)–(4.20) yield

$$\begin{cases} (\overline{\omega}_y^2 - \overline{u}_y^2)\overline{\rho}_y' = \overline{\rho}_y(\rho_y - \overline{\rho}_y) + \dfrac{\overline{\rho}_y}{u_y}(u_y^2 - \overline{u}_y^2) \displaystyle\int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho, \\ \dfrac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y} = \dfrac{\frac{\overline{u}_y(\overline{\omega}_y^2 - u_y^2)}{u_y} \int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho + \overline{u}_y(\overline{\rho}_y - \rho_y)}{\frac{\overline{\rho}_y}{u_y}(u_y^2 - \overline{u}_y^2) \int_{\rho_y}^{\rho_-} \frac{1}{2\omega}\mathrm{d}\rho + \overline{\rho}_y(\rho_y - \overline{\rho}_y)}, \end{cases} \tag{4.21}$$

where the value of $\frac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y}$ depends on U_- for the fixed Y. It is no doubt that for k being sufficient small, $W_3(\overline{Y})$ is always at the right of $W_3(\overline{Y})|_{k=0}$ if $\overline{\rho}_y' \leq 0$. However, when $\overline{\rho}_y' > 0$, $W_3(\overline{Y})$ (at least $S_3(\overline{Y})$) may be at the left of $W_3(\overline{Y})|_{k=0}$. Since the minimum value of $\frac{\mathrm{d}\overline{u}_y}{\mathrm{d}\overline{\rho}_y}$ obtained by (4.21), we have that

$$\frac{\overline{u}_y(\overline{\omega}_y^2 - u_y^2)}{\overline{\rho}_y(u_y^2 - \overline{u}_y^2)} < \frac{\overline{u}_y}{\overline{\rho}_y} < \frac{\overline{\omega}_y}{\overline{\rho}_y}.$$

Likewise, when the initial data satisfy that

$$U_- \in \Delta_+(0), \quad U_+ \in W_3(Z)|_{k=0} \subset \Delta_+^4(0),$$

the solution of (1.1) with (1.2) may change from Q_+^3 to $Q_+^4(0)$ as $k \to 0$.

4.3 The stability of the limit solution

Even though that U_+ located on either Δ_+^2 or Δ_+^3 can not be determined, when we discuss the limit solution of (1.1) with (1.2) satisfying (4.17), we have the following lemma.

Lemma 4.3 *As $k \to 0$, the limit solution of* (1.1) *equals to the solution of* (1.3), *provided that the initial data* (1.2) *satisfy the condition* (4.17).

Proof As an example, we now prove that the solution of (1.1) with (1.2),

$$Q_+^3 : R_1^-(U_-, Y) \oplus \underline{W_2}(Y, U_1) \oplus S_1(U_1, U_z) \oplus \overline{W}_2(U_z, U_2) \oplus W_3^+(U_2, U_+),$$

tends to the solution of (1.3) with (1.2),

$$Q_+^2(0) : R_1^-(U_-, Y(0)) \oplus \overline{W}_2(Y(0), \overline{Y}(0)) \oplus S_3^+(\overline{Y}(0), U_+)$$

as $k \to 0$. The situation is that $U_- \in \Delta_+(0)$, $U_+ \in S_3(\overline{Y})|_{k=0}$, and for some small k_0, $U_+ \in \Delta_+^3$ if $k \in (0, k_0)$. More precisely, we have (4.18) and

$$\begin{cases} u_y = u_- - \displaystyle\int_{\rho_-}^{\rho_y} \frac{\omega}{\rho} \mathrm{d}\rho = \omega_y, \quad u_+ = u_2 - \sqrt{\frac{\big(\kappa\rho_2^\gamma + \frac{k}{2}\rho_2^2 - \kappa\rho_+^\gamma - \frac{k}{2}\rho_+^2\big)(\rho_2 - \rho_+)}{\rho_2\rho_+}}, \\ a_-\rho_y u_y = a\rho_1 u_1 = a\rho_z u_z = a_+\rho_2 u_2, \quad h_y = h_1, \quad g_1 = g_z, \quad h_z = h_2, \\ u_+ = \overline{u}_y(0) - \sqrt{\dfrac{(\kappa\overline{\rho}_y^\gamma(0) - \kappa\rho_+^\gamma)(\overline{\rho}_y(0) - \rho_+)}{\overline{\rho}_y(0)\rho_+}}, \quad a \in (a_-, a_+), \end{cases} \tag{4.22}$$

(see Figure 14). Our problem reduces to proving that $\rho_y \to \rho_y(0)$, $u_y \to u_y(0)$, $\rho_2 \to \rho_2(0) = \overline{\rho}_y(0)$, $u_2 \to u_2(0) = \overline{u}_y(0)$, $a \to a_-$ as $k \to 0$. To this end, we now show that $\rho_y'(0)$, $u_y'(0)$, $\rho_2'(0)$ and $u_2'(0)$ are finite. Direct calculations to (4.22) yield that

$$a_+\rho_2' u_2 + a_+\rho_2 u_2' = a_-\rho_y' u_y + a_-\rho_y u_y' = -a_-\rho_y \int_{\rho_-}^{\rho_y} \frac{1}{2\omega} \mathrm{d}\rho,$$

$$0 = u_2' + \rho_2'\Big(\frac{\omega_2^2}{2(u_+ - u_2)}\frac{\rho_2 - \rho_+}{\rho_+\rho_2} + \frac{(u_+ - u_2)\rho_+}{2\rho_2(\rho_2 - \rho_+)}\Big) + \frac{\rho_2^2 - \rho_+^2}{4(u_+ - u_2)}\Big(\frac{1}{\rho_+} - \frac{1}{\rho_2}\Big)$$

where u_y' and ρ_y' given by (4.20) resp., are finite at $k = 0+$ for the given state $U_- \in \Delta_+$. Thus

$$\rho_2'\Big(\frac{\omega_2^2}{2(u_+ - u_2)}\frac{\rho_2 - \rho_+}{\rho_+} + \frac{(u_+ - u_2)\rho_+}{2\rho_2(\rho_2 - \rho_+)} - u_2\Big) = \frac{a_-\rho_y}{a_+}\int_{\rho_-}^{\rho_y} \frac{1}{2\omega}\mathrm{d}\rho - \frac{\rho_2^2 - \rho_+^2}{4(u_+ - u_2)}\frac{\rho_2 - \rho_+}{\rho_+},$$

which implies that $\rho_2'(0)$ is finite, and so is $u_2'(0)$. Because the coefficient of $\rho_2'(0)$ is not greater than $-\frac{\omega_2 + u_2}{\rho_2} < 0$. Hence we have $h_y \to h_y(0)$ and $h_2 \to h_2(0)$ as $k \to 0$. From $h_y(0) = \overline{h}_y(0) = h_2(0)$, $h_1 = h_y$, we notice that $h_1 \to h_y(0)$ and $h_z \to h_2(0)$ as $k \to 0$. Associating with $g_1 = g_z$, ones obtain $\rho_1 \to \rho_z$, $u_1 \to u_z$ and $a \to a_-$ as $k \to 0$. For the other cases, the lemma can be obtained similarly. We complete the proof.

A similar argument shows that the limit solution of (1.1) as $k \to 0$ equals to the solution for (1.3) with (1.2) satisfying neither $U_+ \in \underline{\Sigma}(0)$ when $U_- \in \Delta_-(0)$ nor $U_+ \in \underline{\Sigma}^+(0) \cup \underline{\Gamma_z}(0)$ when $U_- \in \Delta_+(0)$, by Corollarys 4.1– 4.2. We achieve the following two theorems.

Theorem 4.1 *The solution of* (1.1) *with* (1.2) *is stable in a vanishing magnetic field, provided that, the unique solution is defined as* Q_-^2 *(resp.,* Q_+^2*), when* $U_- \in \Delta_-$ *and* $U_+ \in \Delta_-^3$ *(resp.,* $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$*).*

Theorem 4.2 *The solution of* (1.1) *with* (1.2) *is unstable in a vanishing magnetic field, provided that, the unique solution is defined as* Q_-^1 *(resp.,* Q_+^1*), when* $U_- \in \Delta_-$ *and* $U_+ \in \Delta_-^3$ *(resp.,* $U_- \in \Delta_+$ *and* $U_+ \in \Delta_+^5$*).*

Proof When the initial data satisfy $U_- \in \Delta_-(0)$ and $U_+ \in \underline{\Sigma}(0)$, the solution of (1.3) is

$$Q_-^1(0) : W_1^-(U_-, U_2) \oplus R_3(U_2, D_2) \oplus \underline{W_2}(D_2, U_+),$$

where $D_2(u_{D_2}, \rho_{D_2}) \in \Sigma(0)$. By Corollary 4.2, we know that $U_+ \in \Delta_3$. Similar as we have done in Lemma 4.3, as $k \to 0$, the limit solution of (1.1) with (1.2) is

$$Q_-^2(0) : W_1^-(U_-, U_1) \oplus R_3(U_1, D_1) \oplus W_2(D_1, \overline{D}_1) \oplus S_1^+(\overline{D}_1, U_+),$$

where $D_1(u_{D_1}, \rho_{D_1}) \in \Sigma(0)$, $\overline{D}_1(u_{\overline{D}_1}, \rho_{\overline{D}_1}, a_+) \in \overline{\Sigma}(0)$. Figure 15 shows the two solutions in (u, ρ) plane, and Figure 16 shows them in (x, t) plane.

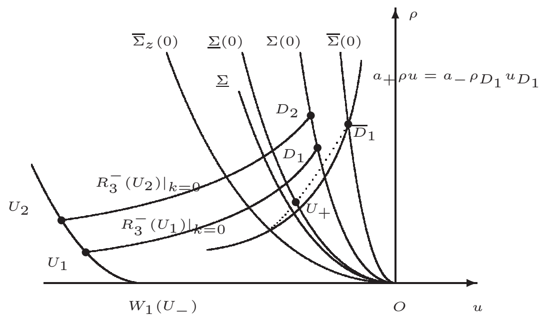
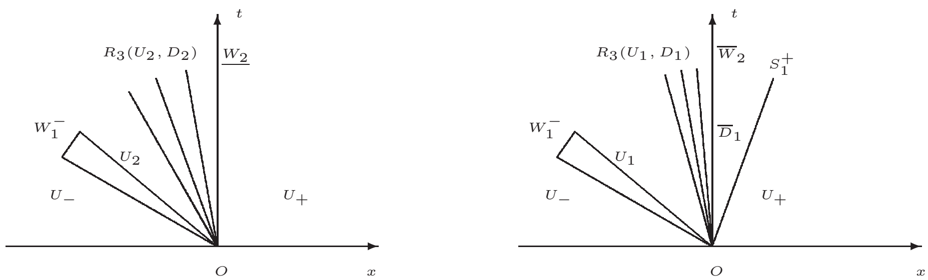


Figure 15 $Q_-^1(0)$ and $Q_-^2(0)$ in (u, ρ) plane. The dotted line is $S_3^+(\overline{D}_1)|_{k=0}$.



a. Solution $Q_-^1(0)$. b. The limit solution $Q_-^2(0)$.

Figure 16 $Q_-^1(0)$ and $Q_-^2(0)$ in (x, t) plane.

It holds that

$$a_+\rho_+u_+ = a_-\rho_{D_2}u_{D_2}, \quad h_+ = h_{D_2}, \quad \sigma(\overline{D}_1, U_+) = \frac{\rho_+u_+ - \rho_{\overline{D}_1}u_{\overline{D}_1}}{\rho_+ - \rho_{\overline{D}_1}} > 0.$$

In fact, by the definition of $\overline{\Sigma}_z$, the unique state $S_3^0(\overline{D}_1)$ is on the curve $\overline{\Sigma}_z(0)$, which is located on the left of $\underline{\Sigma}(0)$. Thus

$$a_-\rho_{D_2}u_{D_2} = a_+\rho_+u_+ < a_+\rho_{\overline{D}_1}u_{\overline{D}_1} = a_-\rho_{D_1}u_{D_1},$$

which implies that $\rho_{D_2} > \rho_{D_1}$ since both D_1 and D_2 are on the curve $\Sigma(0)$. It is clear that $\rho_2 > \rho_1$ and $\lambda_3(U_1) > \lambda_3(U_2)$. As far, we have proved that $Q_-^1(0)$ and $Q_-^2(0)$ are totally different. Likewise, the case that the initial data (1.2) satisfy

$$U_- \in \Delta_+(0), \quad U_+ \in \underline{\Sigma}^+(0) \cup \underline{\Gamma_z}(0),$$

shows the solution of (1.1) is unstable in a vanishing magnetic field. We complete the proof.

5 Summary

We have presented all the possible solutions of system (1.1) with arbitrary initial data (1.2). When the initial data satisfy the condition $U_- \in \Delta_-$, $U_+ \in \Delta_-^3$ or $U_- \in \Delta_+$, $U_+ \in \Delta_+^5$, the system has multi solutions. The conditions and the solutions change with the variation of k. Investigating the limits of the solutions as k vanishes, we obtain a unique stable solution, which satisfies the entropy rate admissibility criterion as well, to system (1.1) with any given initial data (1.2) in a vanishing magnetic field.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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