Chin. Ann. Math. Ser. B
45(5), 2024, 661–676
DOI: 10.1007/s11401-024-0032-6

Chinese Annals of Mathematics, Series B

Stochastic Maximum Principle for Square-Integrable Optimal Control of Linear Stochastic Systems*

Shanjian TANG[1] Xueqi WANG[2]

Abstract The authors give a stochastic maximum principle for square-integrable optimal control of linear stochastic systems. The control domain is not necessarily convex and the cost functional can have a quadratic growth. In particular, they give a stochastic maximum principle for the linear quadratic optimal control problem.

Keywords Stochastic maximum principle, Optimal control, Linear stochastic system, Square integrability
2000 MR Subject Classification 60H10, 93E20

1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with filtration $(\mathscr{F}_t)_{t\geq 0}$, and $\{W_t := (W_t^1, \cdots, W_t^d)^*, 0 \leq t \leq T\}$ be a d-dimensional standard Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$. We use the asterisk to represent the transpose of a vector or matrix. We assume that $\mathscr{F}_t = \sigma(W_s : 0 \leq s \leq t)$, and T is a fixed terminal time.

We consider the following linear stochastic system

$$X_t = x + \int_0^t (A_s X_s + B_s u_s + \alpha_s)\,\mathrm{d}s + \int_0^t \sum_{j=1}^d (C_s^j X_s + D_s^j u_s + \beta_s^j)\,\mathrm{d}W_s^j, \quad t \in [0, T] \tag{1.1}$$

and the quadratic cost functional

$$J(u) = \mathbb{E}[M(X_T)] + \mathbb{E}\Big[\int_0^T l(t, X_t, u_t)\,\mathrm{d}s\Big]. \tag{1.2}$$

We define

$$\mathcal{L}^p_{\mathscr{F}}(0, T) := \Big\{u : \|u\|^p_{\mathcal{L}^p_{\mathscr{F}}(0,T)} := \mathbb{E}\Big[\int_0^T |u_t|^p\,\mathrm{d}t\Big] < \infty\Big\}.$$

Letting $U \subset \mathbb{R}^m$, our admissible control set is

$$\mathcal{U}_{ad} = \{u \in \mathcal{L}^2_{\mathscr{F}}(0, T) : u_t \in U \text{ a.e.a.s.}\}.$$

Manuscript received October 7, 2023. Revised April 2, 2024.
[1]Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China. E-mail: sjtang@fudan.edu.cn
[2]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: xueqiwang18@fudan.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 12031009).

The optimal control problem is to find a $u \in \mathcal{U}_{ad}$ which minimizes the cost functional $J(u)$ over $u \in \mathcal{U}_{ad}$. Note that the range of control U is allowed to be non-convex here.

The linear quadratic (LQ for short) optimal control problem is a classic case in stochastic control problems. However, most existing results (see [4]) assume that U is convex, and see among others [4, Theorem 3.2, p. 427], [2, Theorem 4.1, p. 30] and [3, Theorem 1.2, Chapter VI, p. 232].

When U is not convex, the method of convex variations fails to give the stochastic maximum principle for optimal stochastic controls.

Besides, when the control does not enter into the diffusion term, we also have the stochastic maximum principle (see [1, 2, 8, 11]) to solve LQ problems (see [2, Theorem 2.1, p. 19]). In this case, the variational calculus is quite analogous to the deterministic case.

When both the control range U is non-convex and the diffusion depends on the control, we can only appeal to the general stochastic maximum principle of Peng [16]. However, Peng [16, p. 967] assumes that admissible controls satisfy the following higher integrability:

$$\sup_{t\in[0,T]} \mathbb{E}|u_t|^m < \infty, \quad \forall m = 1, 2, \cdots, \tag{1.3}$$

which seems necessary in his second order Taylor's expansions of both the system and the cost functional at the optimal pair. For more details about the history of LQ problems, we refer the reader to Yong and Zhou's book [19, Chapter 6].

In particular, Ji and Xue [10, Theorem 4.4, p. 501] give a stochastic maximum principle for optimal control of one-dimensional linear stochastic controlled system subject to a quadratic cost functional and a particular non-convex range U of admissible control values. More precisely, they specify U as follows

$$U := C \cap \{0, 1\}^k$$

for a convex set C and an integer k. Their proof heavily relied on their particular control domain U and cost functional, and seems difficult to be generalized to our more general context. Note that our control domain U can be very general, and it can be any measurable subset in $\mathbb{R}^m$.

In this paper, we use the combined techniques of truncation and approximation to get a stochastic maximum principle for square-integrable optimal stochastic control. Firstly, we obtain the variational inequalities for such admissible controls u that $u - \overline{u} \in \mathcal{L}^4_{\mathscr{F}}(0,T)$, where the L^4-integrability is used to estimate the fourth-order moment of the first variation of the state $|\delta_1 X_t|^4$ and then the cost variation $J(u^\epsilon) - J(\overline{u})$. Here $\delta_1 X_t$ is the first variation of $X_t^\rho - \overline{X}_t$. For a given admissible control $u \in \mathcal{U}_{ad}$, we have the variational inequalities for a sequence of truncated (and thus L^4-integrable) admissible controls

$$u_t^k := \begin{cases} u_t, & \text{if } |u_t - \overline{u}_t| \le k, \\ \overline{u}_t, & \text{if } |u_t - \overline{u}_t| > k. \end{cases}$$

Since u^k converges to u in $\mathcal{L}^2_{\mathscr{F}}(0,T)$, we have the variational inequality (by passing to the limit in those variational inequalities for the preceding sequence of truncated admissible controls) for the preceding admissible control $u \in \mathcal{L}^2_{\mathscr{F}}(0,T)$ and thus the desired stochastic maximum principle. Finally, we illustrate our main result with the classical typical example: Optimal control of linear stochastic systems with a quadratic cost functional.

2 The Main Result

We give our assumptions.

Assumption 2.1 The coefficients of linear system (1.1) satisfy: $A : [0,T] \times \Omega \to \mathbb{R}^{n\times n}$, $B : [0,T] \times \Omega \to \mathbb{R}^{n\times m}$, $C^j : [0,T] \times \Omega \to \mathbb{R}^{n\times n}$, $D^j : [0,T] \times \Omega \to \mathbb{R}^{n\times m}$, $\alpha : [0,T] \times \Omega \to \mathbb{R}^n$, $\beta^j : [0,T] \times \Omega \to \mathbb{R}^n$ are $\mathscr{F}_t$-adapted processes, and A, B, C^j, D^j are bounded for almost everywhere $t \in [0,T]$ and almost surely $\omega \in \Omega$.

The terminal cost function $M(x)$ and running cost function $l(t,x,u)$ satisfy the following conditions.

Assumption 2.2 For $(x,u) \in \mathbb{R}^n \times U$, $l(\cdot,x,u)$ is an $\mathscr{F}_t$-adapted process, and $M(x)$ is $\mathscr{F}_T$-measurable variable. The functions $l(t,\omega,x,u)$, $M(\omega,x)$ are twice differential with respect to variable x. $l(t,\omega,x,u)$, $M(\omega,x)$, $l_x(t,\omega,x,u)$, $M_x(\omega,x)$, $l_{xx}(t,\omega,x,u)$, $M_{xx}(\omega,x)$ are continuous with respect to (x,u). $l(t,\omega,x,u)$ and $M(\omega,x)$ have a quadratic growth with respect to (x,u). Both $l_x(t,\omega,x,u)$ and $M_x(\omega,x)$ have linear growth with respect to (x,u). $l_{xx}(t,\omega,x,u)$ and $M_{xx}(\omega,x)$ are bounded. That is, there exists a constant C such that

$$
\begin{aligned}
&|l(t,\omega,x,u)| \le C(1+|x|^2+|u|^2), \quad |M(\omega,x)| \le C(1+|x|^2),\\
&|l_x(t,\omega,x,u)| \le C(1+|x|+|u|), \quad |M_x(\omega,x)| \le C(1+|x|),\\
&|l_{xx}(t,\omega,x,u)| \le C, \quad |M_{xx}(\omega,x)| \le C.
\end{aligned}
$$

For $t \in [0,T]$, $x \in \mathbb{R}^n$, $u \in U$, $p \in \mathbb{R}^n$, $q = (q^1,\cdots,q^d) \in (\mathbb{R}^n)^d$ and $\omega \in \Omega$, the Hamiltonian is

$$
H(t,\omega,x,u,p,q) = \langle p, A_t x + B_t u + \alpha_t\rangle + \sum_{j=1}^{d} \langle q^j, C_t^j x + D_t^j u + \beta_t^j\rangle + l(t,\omega,x,u). \tag{2.1}
$$

Then we have the following stochastic maximum principle.

Theorem 2.1 *Let Assumptions* 2.1–2.2 *hold. Let* $(\overline{X},\overline{u})$ *be an optimal pair for system* (1.1) *which minimizes cost functional* (1.2). *Let the two pairs of stochastic processes*

$$
(p;q^1,\cdots,q^d) \in \mathcal{L}^2_{\mathscr{F}}(0,T;R^n \times R^{n\times d})
$$

and

$$
(P;Q^1,\cdots,Q^d) \in \mathcal{L}^2_{\mathscr{F}}(0,T;R^{n\times n} \times (R^{n\times n})^d)
$$

solve the first- and second-order adjoint equations

$$\begin{cases} \mathrm{d}p_t = -\Big\{A_t^* p_t + \sum_{j=1}^d C_t^{j,*} q_t^j + l_x^*(t, \overline{X}_t, \overline{u}_t)\Big\}\,\mathrm{d}t + \sum_{j=1}^d q_t^j \,\mathrm{d}W_t^j, \quad t \in [0,T), \\ p_T = M_x^*(\overline{X}_T) \end{cases} \tag{2.2}$$

and

$$\begin{cases} \mathrm{d}P_t = -\Big\{A_t^* P_t + P_t A_t + \sum_{j=1}^d (C_t^{j,*} P_t C_t^j + C_t^{j,*} Q_t^j + Q_t^j C_t^j) + l_{xx}(t, \overline{X}_t, \overline{u}_t)\Big\}\,\mathrm{d}t \\ \qquad + \sum_{j=1}^d Q_t^j \,\mathrm{d}W_t^j, \quad t \in [0,T), \\ P_T = M_{xx}(\overline{X}_T). \end{cases} \tag{2.3}$$

Then, we have the maximum condition

$$\begin{aligned} \min_{u\in U} \Big\{ &\langle p_t, B_t(u - \overline{u}_t)\rangle + \sum_{j=1}^d \langle q_t^j, D_t^j(u - \overline{u}_t)\rangle + l(t, \overline{X}_t, u) - l(t, \overline{X}_t, \overline{u}_t) \\ &+ \frac{1}{2}\sum_{j=1}^d (u - \overline{u}_t)^* D_t^{j,*} P_t D_t^j (u - \overline{u}_t) \Big\} = 0, \quad a.e.a.s. \end{aligned} \tag{2.4}$$

3 Proof of Theorem 2.1

We have the following priori estimate on the solution of a stochastic differential equation (see [17, Lemma 7.1], [7, Basic theorem, pp. 756–757]).

Lemma 3.1 *Assume that the vector functions* $f : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ *satisfy the following two conditions*:

(i) *For each* $x \in \mathbb{R}^n$, $f(\cdot, x)$ *and* $g(\cdot, x)$ *are* $\{\mathscr{F}_t, 0 \le t \le T\}$*-adapted processes. Moreover,*

$$\int_0^T |f(t,0)|\,\mathrm{d}t < \infty, \quad \int_0^T |g(t,0)|^2\,\mathrm{d}t < \infty, \quad a.s.$$

(ii) *Lipschitz continuity: There exist two positive functions* α_1 *and* α_2 *such that they are* $\{\mathscr{F}_t, 0 \le t \le T\}$*-adapted. Moreover,*

$$\int_0^T \alpha_1(t)\,\mathrm{d}t < \infty, \quad \int_0^T |\alpha_2(t)|^2\,\mathrm{d}t < \infty, \quad a.s.$$

For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} |f(t,x) - f(t,y)| &\le \alpha_1(t)|x - y|, \\ |g(t,x) - g(t,y)| &\le \alpha_2(t)|x - y|. \end{aligned}$$

Then, the stochastic differential equation

$$\mathrm{d}x_t = f(t, x_t)\,\mathrm{d}t + g(t, x_t)\,\mathrm{d}W_t, \quad 0 \le t \le T,\ x(0) = h \tag{3.1}$$

has a unique strong solution. Moreover, if for $p \geq 1$ *the following conditions hold*

$$\mathbb{E}\|f(\cdot,0)\|^p_{L^1(0,T;\mathbb{R}^n)} := \mathbb{E}\Big[\Big(\int_0^T |f(s,0)|\,\mathrm{d}s\Big)^p\Big] < \infty,$$
$$\mathbb{E}\|g(\cdot,0)\|^p_{L^2(0,T;\mathbb{R}^{n\times d})} := \mathbb{E}\Big[\Big(\int_0^T |g(s,0)|^2\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big] < \infty,$$

then the solutions of (3.1) *satisfy the following*

$$\mathbb{E}\Big[\max_{t\in[0,T]} |x_t|^p\Big] \leq C_{p,T}(|h|^p + \mathbb{E}\|f(\cdot,0)\|^p_{L^1(0,T;\mathbb{R}^n)} + \mathbb{E}\|g(\cdot,0)\|^p_{L^2(0,T;\mathbb{R}^{n\times d})}).$$

We use $f(x) \lesssim g(x)$ to mean $f(x) \leq Cg(x)$ for a positive constant C.

Since the control range U is not necessarily convex, we use the spike variation. We pick up a $u \in \mathcal{U}_{ad}$ which satisfies

$$\|u-\overline{u}\|^4_{\mathcal{L}^4_{\mathscr{F}}(0,T)} := \mathbb{E}\Big[\int_0^T |u_t - \overline{u}_t|^4\,\mathrm{d}t\Big] < \infty.$$

Then according to Liapunov's range theorem of a vector-valued measure (see [13–14]), there exists I_ρ such that

$$\int_{I_\rho} G^*(t)\,\mathrm{d}t = \rho \int_{[0,T]} G^*(t)\,\mathrm{d}t, \tag{3.2}$$

where

$$G(t) := \Big(1, \mathbb{E}|u_t - \overline{u}_t|^4, \mathbb{E}\Big[\delta H(t;u_t) + \frac{1}{2}\sum_{j=1}^d (u_t - \overline{u}_t)^* D_t^{j,*} P_t D_t^j (u_t - \overline{u}_t)\Big]\Big)$$

and

$$\delta H(t;u_t) = H(t,\overline{X}_t,u_t,p_t,q_t) - H(t,\overline{X}_t,\overline{u}_t,p_t,q_t).$$

We define the spike variation u^ρ of $\overline{u}$ as follows:

$$u_t^\rho = \begin{cases} u_t, & t \in I_\rho, \\ \overline{u}_t, & t \notin I_\rho. \end{cases} \tag{3.3}$$

We denote by X^ρ the solution of (1.1) corresponding to the admissible control u^ρ. Let $\delta_1 X$ and $\delta_2 X$ be respectively the unique solutions of the following stochastic differential equations:

$$\delta_1 X_t = \int_0^t A_s \delta_1 X_s\,\mathrm{d}s + \int_0^t \sum_{j=1}^d (C_s^j \delta_1 X_s + D_s^j(u_s^\rho - \overline{u}_s))\,\mathrm{d}W_s^j \tag{3.4}$$

and

$$\delta_2 X_t = \int_0^t (A_s \delta_2 X_s + B_s(u_s^\rho - \overline{u}_s))\,\mathrm{d}s + \int_0^t \sum_{j=1}^d C_s^j \delta_2 X_s\,\mathrm{d}W_s^j. \tag{3.5}$$

Since

$$X_t^\rho - \overline{X}_t = \int_0^t [A_s(X_s^\rho - \overline{X}_s) + B_s(u_s^\rho - \overline{u}_s)]\,\mathrm{d}s + \int_0^t \sum_{j=1}^d [C_s^j(X_s^\rho - \overline{X}_s) + D_s^j(u_s^\rho - \overline{u}_s)]\,\mathrm{d}W_s^j, \tag{3.6}$$

using the existence and uniqueness theorem for stochastic differential equations (see [9, 15]), we have

$$X_t^\rho - \overline{X}_t = \delta_1 X_t + \delta_2 X_t. \tag{3.7}$$

Then we have the following estimates.

Lemma 3.2 *Let Assumption* 2.1 *hold. For* $u \in \mathcal{L}^2_{\mathscr{F}}(0,T)$ *such that*

$$\|u - \overline{u}\|^4_{\mathcal{L}^4_{\mathscr{F}}(0,T)} := \mathbb{E}\Big[\int_0^T |u_t - \overline{u}_t|^4\,\mathrm{d}t\Big] < \infty,$$

we have

$$\mathbb{E}\Big[\max_{t\in[0,T]} |X_t^\rho - \overline{X}_t|^4\Big] = (1 + \|u - \overline{u}\|^4_{\mathcal{L}^4_{\mathscr{F}}(0,T)})O(\rho^2), \tag{3.8}$$

$$\mathbb{E}\Big[\max_{t\in[0,T]} |\delta_1 X_t|^4\Big] = (1 + \|u - \overline{u}\|^4_{\mathcal{L}^4_{\mathscr{F}}(0,T)})O(\rho^2), \tag{3.9}$$

$$\mathbb{E}\Big[\max_{t\in[0,T]} |X_t^\rho - \overline{X}_t - \delta_1 X_t|^2\Big] = (1 + \|u - \overline{u}\|^4_{\mathcal{L}^4_{\mathscr{F}}(0,T)})O(\rho^2), \tag{3.10}$$

$$\mathbb{E}\Big[\max_{t\in[0,T]} |\delta_2 X_t|^2\Big] = (1 + \|u - \overline{u}\|^4_{\mathcal{L}^4_{\mathscr{F}}(0,T)})O(\rho^2). \tag{3.11}$$

Proof (i) According to (3.6) and Lemma 3.1, we have that for $p_1 \geq 2$,

$$\mathbb{E}\Big[\max_{t\in[0,T]} \big|X_t^\rho - \overline{X}_t\big|^{p_1}\Big] \leq C_{p_1,T}\,(I_{1,p_1} + I_{2,p_1}),$$

where

$$I_{1,p_1} := \mathbb{E}\Big[\Big(\int_0^T |B_s(u_s^\rho - \overline{u}_s)|\,\mathrm{d}s\Big)^{p_1}\Big], \quad I_{2,p_1} := \mathbb{E}\Big[\Big(\int_0^T \sum_{j=1}^d |D_s^j(u_s^\rho - \overline{u}_s)|^2\,\mathrm{d}s\Big)^{\frac{p_1}{2}}\Big].$$

Since B and D are bounded, from the definition of u^ρ in (3.3), we have

$$I_{1,p_1} \leq C_{B,p_1}\mathbb{E}\Big[\Big(\int_{I_\rho} |u_s - \overline{u}_s|\,\mathrm{d}s\Big)^{p_1}\Big], \quad I_{2,p_1} \leq C_{D,p_1}\mathbb{E}\Big[\Big(\int_{I_\rho} |u_s - \overline{u}_s|^2\,\mathrm{d}s\Big)^{\frac{p_1}{2}}\Big],$$

where C_{B,p_1} is a constant depending on the upper bound of $|B|$ and subscript p_1, and C_{D,p_1} has the same meaning. Using Hölder's inequality and Fubini lemma, we have

$$I_{1,p_1} \lesssim \mathbb{E}\Big[\Big(\int_{I_\rho} 1\,\mathrm{d}s\Big)^{p_1-1}\int_{I_\rho} |u_s - \overline{u}_s|^{p_1}\,\mathrm{d}s\Big] \lesssim |I_\rho|^{p_1-1}\mathbb{E}\Big[\int_{I_\rho} |u_s - \overline{u}_s|^{p_1}\,\mathrm{d}s\Big]$$

$$\lesssim |I_\rho|^{p_1-1}\int_{I_\rho}\mathbb{E}[|u_s-\overline{u}_s|^{p_1}]\,\mathrm{d}s \quad \text{for } p_1\geq 1$$

and

$$\begin{aligned} I_{2,p_1} &\lesssim \mathbb{E}\Big[\Big(\int_{I_\rho}1\,\mathrm{d}s\Big)^{\frac{p_1}{2}-1}\int_{I_\rho}|u_s-\overline{u}_s|^{p_1}\,\mathrm{d}s\Big] \\ &\lesssim |I_\rho|^{\frac{p_1}{2}-1}\mathbb{E}\Big[\int_{I_\rho}|u_s-\overline{u}_s|^{p_1}\,\mathrm{d}s\Big] \\ &\lesssim |I_\rho|^{\frac{p_1}{2}-1}\int_{I_\rho}\mathbb{E}[|u_s-\overline{u}_s|^{p_1}]\,\mathrm{d}s \quad \text{for } p_1\geq 2. \end{aligned}$$

Taking $p_1=4$ and using the definition of I_ρ in (3.2), we have

$$\mathbb{E}\Big[\max_{t\in[0,T]}|X_t^\rho-\overline{X}_t|^4\Big]\leq C_{B,D,T}\,\rho\int_{I_\rho}\mathbb{E}[|u_s-\overline{u}_s|^4]\,\mathrm{d}s\lesssim \rho^2\int_0^T\mathbb{E}|u_s-\overline{u}_s|^4\,\mathrm{d}s.$$

So the estimate (3.8) holds.

(ii) According to (3.4) and Lemma 3.1, we have

$$\mathbb{E}\Big[\max_{t\in[0,T]}|\delta_1X_t|^{p_2}\Big]\leq C_{p_2,T}\mathbb{E}\Big[\Big(\int_0^T\sum_{j=1}^d|D_s^j(u_s^\rho-\overline{u}_s)|^2\,\mathrm{d}s\Big)^{\frac{p_2}{2}}\Big],\quad p_2\geq 2.$$

We use the definition of u^ρ in (3.3), the fact that D is bounded, and Hölder's inequality to have

$$\begin{aligned} I_{2,p_2} &:= \mathbb{E}\Big[\Big(\int_0^T\sum_{j=1}^d|D_s^j(u_s^\rho-\overline{u}_s)|^2\,\mathrm{d}s\Big)^{\frac{p_2}{2}}\Big]=\mathbb{E}\Big[\Big(\int_{I_\rho}\sum_{j=1}^d|D_s^j(u_s-\overline{u}_s)|^2\,\mathrm{d}s\Big)^{\frac{p_2}{2}}\Big] \\ &\leq C_{D,p_2}\mathbb{E}\Big[\Big(\int_{I_\rho}|u_s-\overline{u}_s|^2\,\mathrm{d}s\Big)^{\frac{p_2}{2}}\Big]\lesssim |I_\rho|^{\frac{p_2}{2}-1}\mathbb{E}\Big[\int_{I_\rho}|u_s-\overline{u}_s|^{p_2}\,\mathrm{d}s\Big],\quad p_2\geq 2. \end{aligned}$$

Taking $p_2=4$ and using the definition of I_ρ in (3.2), we can deduce (3.9).

(iii) Using (3.4) and (3.6), we deduce the stochastic differential equation for $X_t^\rho-\overline{X}_t-\delta_1X_t$. From Lemma 3.1, we have

$$\mathbb{E}\Big[\max_{t\in[0,T]}|X_t^\rho-\overline{X}_t-\delta_1X_t|^{p_3}\Big]\leq C_{p_3,T}\mathbb{E}\Big[\Big(\int_0^T|B_s(u_s^\rho-\overline{u}_s)|\,\mathrm{d}s\Big)^{p_3}\Big],\quad p_3\geq 2.$$

Then from the definition of u^ρ in (3.3) and Hölder's inequality, since B is bounded, we have for $p_3\geq 2$,

$$\begin{aligned} I_{1,p_3} &:= \mathbb{E}\Big[\Big(\int_0^T|B_s(u_s^\rho-\overline{u}_s)|\,\mathrm{d}s\Big)^{p_3}\Big]\leq C_{B,p_3}\mathbb{E}\Big[\Big(\int_{I_\rho}|u_s-\overline{u}_s|\,\mathrm{d}s\Big)^{p_3}\Big] \\ &\lesssim |I_\rho|^{p_3-1}\mathbb{E}\Big[\int_{I_\rho}|u_s-\overline{u}_s|^{p_3}\,\mathrm{d}s\Big]=|I_\rho|^{p_3-1}\int_{I_\rho}\mathbb{E}|u_s-\overline{u}_s|^{p_3}\,\mathrm{d}s. \end{aligned}$$

Taking $p_3=2$, we have

$$\mathbb{E}\Big[\max_{t\in[0,T]}|X_t^\rho-\overline{X}_t-\delta_1X_t|^2\Big]\lesssim \rho\int_{I_\rho}\mathbb{E}|u_s-\overline{u}_s|^2\,\mathrm{d}s.$$

Using Hölder's inequality, we have

$$\int_{I_\rho} \mathbb{E}|u_s - \overline{u}_s|^2 \,\mathrm{d}s \le \int_{I_\rho} \{\mathbb{E}|u_s - \overline{u}_s|^4\}^{\frac{1}{2}} \,\mathrm{d}s$$
$$\le \int_{I_\rho} (1 + \mathbb{E}|u_s - \overline{u}_s|^4)^{\frac{1}{2}} \,\mathrm{d}s \le \int_{I_\rho} (1 + \mathbb{E}|u_s - \overline{u}_s|^4) \,\mathrm{d}s.$$

Then, according to the definition of I_ρ in (3.2), we have

$$\mathbb{E}\Big[\max_{t\in[0,T]} |X_t^\rho - \overline{X}_t - \delta_1 X_t|^2\Big] \lesssim \rho^2 \int_0^T (1 + \mathbb{E}|u_s - \overline{u}_s|^4) \,\mathrm{d}s.$$

So the estimate (3.10) holds.

(iv) According to (3.7), we have

$$\delta_2 X_t = X_t^\rho - \overline{X}_t - \delta_1 X_t.$$

So (3.10) deduces (3.11). The proof is complete.

According to (3.7) and Taylor's expansion, we have

$$\begin{aligned}
&J(u^\rho) - J(\overline{u}) \\
&= \mathbb{E}[M_x(\overline{X}_T)(\delta_1 X_T + \delta_2 X_T)] + \mathbb{E}\Big[\frac{1}{2} M_{xx}(\overline{X}_T)(\delta_1 X_T)^2\Big] \\
&\quad + \mathbb{E}\Big[(\widetilde{M}_{xx}(T) - \frac{1}{2} M_{xx}(\overline{X}_T))(\delta_1 X_T)^2 + \widetilde{M}_{xx}(T)((X_T^\rho - \overline{X}_T)^2 - (\delta_1 X_T)^2)\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T l_x(s, \overline{X}_s, \overline{u}_s)(\delta_1 X_s + \delta_2 X_s) + (l_x(s, \overline{X}_s, u_s^\rho) - l_x(s, \overline{X}_s, \overline{u}_s))(X_s^\rho - \overline{X}_s) \,\mathrm{d}s\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \frac{1}{2} l_{xx}(s, \overline{X}_s, \overline{u}_s)(\delta_1 X_s)^2 + \Big(\widetilde{l}_{xx}(s) - \frac{1}{2} l_{xx}(s, \overline{X}_s, \overline{u}_s)\Big)(X_s^\rho - \overline{X}_s)^2 \,\mathrm{d}s\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \frac{1}{2} l_{xx}(s, \overline{X}_s, \overline{u}_s)((X_s^\rho - \overline{X}_s)^2 - (\delta_1 X_s)^2) \,\mathrm{d}s\Big] + \mathbb{E}\Big[\int_0^T \delta l(s; u_s^\rho) \,\mathrm{d}s\Big],
\end{aligned}$$

where $\widetilde{M}_{xx}(T)$ and $\widetilde{l}_{xx}(t)$ are defined as

$$\widetilde{M}_{xx}(T) = \int_0^1 \int_0^1 \lambda M_{xx}(\overline{X}_T + \lambda\theta(X_T^\rho - \overline{X}_T)) \,\mathrm{d}\lambda \,\mathrm{d}\theta,$$
$$\widetilde{l}_{xx}(s) = \int_0^1 \int_0^1 \lambda l_{xx}(s, \overline{X}_s + \lambda\theta(X_s^\rho - \overline{X}_s), u_s^\rho) \,\mathrm{d}\lambda \,\mathrm{d}\theta,$$

respectively, and $\delta l(s; u_s^\rho)$ is defined as

$$\delta l(s; u_s^\rho) := l(s, \overline{X}_s, u_s^\rho) - l(s, \overline{X}_s, \overline{u}_s).$$

Using Hölder's inequality, we have

$$\mathbb{E}\Big[(\widetilde{M}_{xx}(T) - \frac{1}{2} M_{xx}(\overline{X}_T))(\delta_1 X_T)^2\Big]$$
$$\le \Big\{\mathbb{E}\Big[|\widetilde{M}_{xx}(T) - \frac{1}{2} M_{xx}(\overline{X}_T)|^2\Big]\Big\}^{\frac{1}{2}} \{\mathbb{E}[|\delta_1 X_T|^4]\}^{\frac{1}{2}}.$$

Since $M_{xx}(\cdot)$ is bounded, we use (3.8) and dominated convergence theorem to have

$$\lim_{\rho\to 0}\mathbb{E}\Big[\Big|\widetilde{M}_{xx}(T)-\frac{1}{2}M_{xx}(\overline{X}_T)\Big|^2\Big]=0.$$

Then we use Lemma 3.2 to deduce

$$\mathbb{E}\Big[(\widetilde{M}_{xx}(T)-\frac{1}{2}M_{xx}(\overline{X}_T))(\delta_1 X_T)^2\Big]=o(\rho).$$

Since $M_{xx}(\cdot)$ is bounded and (3.7) holds, we have

$$\begin{aligned}&\mathbb{E}[\widetilde{M}_{xx}(T)((X_T^\rho-\overline{X}_T)^2-(\delta_1 X_T)^2)]\\&\le C_M\mathbb{E}[|\delta_2 X_T||X_T^\rho-\overline{X}_T+\delta_1 X_T|]\\&\lesssim\{\mathbb{E}[|\delta_2 X_T|^2]\}^{\frac{1}{2}}\{\mathbb{E}[|X_T^\rho-\overline{X}_T+\delta_1 X_T|^2]\}^{\frac{1}{2}}.\end{aligned}$$

The C_M is a constant depending on the upper bound of the function $|M_{xx}|$. Besides, we have

$$\begin{aligned}\mathbb{E}[|X_T^\rho-\overline{X}_T+\delta_1 X_T|^2]&\le 2\mathbb{E}[|X_T^\rho-\overline{X}_T|^2]+2\mathbb{E}[|\delta_1 X_T|^2]\\&\le 2\{\mathbb{E}[|X_T^\rho-\overline{X}_T|^4]\}^{\frac{1}{2}}+2\{\mathbb{E}[|\delta_1 X_T|^4]\}^{\frac{1}{2}}.\end{aligned}$$

Then we use Lemma 3.2 to have

$$\{\mathbb{E}[|\delta_2 X_T|^2]\}^{\frac{1}{2}}\{\{\mathbb{E}[|X_T^\rho-\overline{X}_T|^4]\}^{\frac{1}{2}}+\{\mathbb{E}[|\delta_1 X_T|^4]\}^{\frac{1}{2}}\}^{\frac{1}{2}}=O(\rho^{\frac{3}{2}})=o(\rho).$$

Similarly, we have

$$\mathbb{E}\Big[\int_0^T\frac{1}{2}l_{xx}(s,\overline{X}_s,\overline{u}_s)((X_s^\rho-\overline{X}_s)^2-(\delta_1 X_s)^2)\,\mathrm{d}s\Big]=o(\rho).$$

In addition, since

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T\Big(\widetilde{l}_{xx}(s)-\frac{1}{2}l_{xx}(s,\overline{X}_s,\overline{u}_s)\Big)(X_s^\rho-\overline{X}_s)^2\,\mathrm{d}s\Big]\\&\le\Big\{\mathbb{E}\Big[\int_0^T\Big|\widetilde{l}_{xx}(s)-\frac{1}{2}l_{xx}(s,\overline{X}_s,\overline{u}_s)\Big|^2\mathrm{d}s\Big]\Big\}^{\frac{1}{2}}\Big\{\mathbb{E}\Big[\int_0^T|X_s^\rho-\overline{X}_s|^4\,\mathrm{d}s\Big]\Big\}^{\frac{1}{2}},\end{aligned}$$

we have

$$\mathbb{E}\Big[\int_0^T\Big(\widetilde{l}_{xx}(s)-\frac{1}{2}l_{xx}(s,\overline{X}_s,\overline{u}_s)\Big)(X_s^\rho-\overline{X}_s)^2\,\mathrm{d}s\Big]=o(\rho).$$

Through out above detailed computation, we have

$$\begin{aligned}&J(u^\rho)-J(\overline{u})\\&=\mathbb{E}\Big[M_x(X_T)(\delta_1 X_T+\delta_2 X_T)+\frac{1}{2}M_{xx}(X_T)(\delta_1 X_T)^2\Big]\\&\quad+\mathbb{E}\Big[\int_0^T l_x(s)(\delta_1 X_s+\delta_2 X_s)+\frac{1}{2}l_{xx}(s)(\delta_1 X_s)^2+\delta l(s;u_s^\rho)\,\mathrm{d}s\Big]+o(\rho).\end{aligned}$$

Using adjoint processes (p,q) and (P,Q) as the unique solutions of BSDEs (2.2)–(2.3), we have

$$J(u^\rho)-J(\overline{u})$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\int_0^T \delta H(t;u_t^\rho) + \frac{1}{2}\sum_{j=1}^d (u_t^\rho - \overline{u}_t)^* D_t^{j,*} P_t D_t^j (u_t^\rho - \overline{u}_t)\,\mathrm{d}t\Big] + o(\rho) \\
&= \mathbb{E}\Big[\int_{I_\rho} \delta H(t;u_t) + \frac{1}{2}\sum_{j=1}^d (u_t - \overline{u}_t)^* D_t^{j,*} P_t D_t^j (u_t - \overline{u}_t)\,\mathrm{d}t\Big] + o(\rho) \\
&= \rho\mathbb{E}\Big[\int_0^T \delta H(t;u_t) + \frac{1}{2}\sum_{j=1}^d (u_t - \overline{u}_t)^* D_t^{j,*} P_t D_t^j (u_t - \overline{u}_t)\,\mathrm{d}t\Big] + o(\rho) \\
&\geq 0,
\end{aligned}
$$

where

$$
\delta H(t;u) = \langle p_t, B_t(u - \overline{u}_t)\rangle + \sum_{j=1}^d \langle q_t^j, D_t^j(u - \overline{u}_t)\rangle + \delta l(t;u).
$$

Taking $\rho \to 0$, then we have the following lemma.

Lemma 3.3 *For any admissible control u such that*

$$
\mathbb{E}\Big[\int_0^T |u_t - \overline{u}_t|^4\,\mathrm{d}t\Big] < \infty, \tag{3.12}
$$

the following condition holds

$$
\mathbb{E}\Big[\int_0^T \delta H(t;u_t) + \frac{1}{2}\sum_{j=1}^d (u_t - \overline{u}_t)^* D_t^{j,*} P_t D_t^j (u_t - \overline{u}_t)\,\mathrm{d}t\Big] \geq 0. \tag{3.13}
$$

Next, we demonstrate that for any $u \in \mathcal{U}_{ad}$ (i.e., $u \in \mathcal{L}^2_{\mathscr{F}}(0,T)$), (3.13) still holds. Note that $u - \overline{u} \in \mathcal{L}^2_{\mathscr{F}}(0,T)$, we define

$$
(u - \overline{u})_t^k = \begin{cases} u_t - \overline{u}_t, & \text{if } |u_t - \overline{u}_t| \leq k, \\ 0, & \text{if } |u_t - \overline{u}_t| > k. \end{cases}
$$

Then $\{(u - \overline{u})^k\}_{k=1}^\infty \subset \mathcal{L}^4_{\mathscr{F}}(0,T)$ and this sequence satisfies:

(i) $(u - \overline{u})^k$ converges to $u - \overline{u}$ strongly in $\mathcal{L}^2_{\mathscr{F}}(0,T)$,

(ii) $|(u - \overline{u})_t^k| \leq |u_t - \overline{u}_t|$ a.e.a.s.

Set

$$
u_t^k = (u - \overline{u})_t^k + \overline{u}_t, \tag{3.14}
$$

so we have

$$
u_t^k = \begin{cases} u_t, & \text{if } |u_t - \overline{u}_t| \leq k, \\ \overline{u}_t, & \text{if } |u_t - \overline{u}_t| > k. \end{cases}
$$

Thus $u^k \in \mathcal{U}_{ad}$ and u^k satisfies (3.12)–(3.13). Then we have the following lemma.

Lemma 3.4 *For u^k which is defined in* (3.14), *we have*

$$
\lim_{k\to\infty} \mathbb{E}\Big[\int_0^T \delta H(t;u_t^k)\,\mathrm{d}t\Big] = \mathbb{E}\Big[\int_0^T \delta H(t;u_t)\,\mathrm{d}t\Big]. \tag{3.15}
$$

Proof Since the right-hand side of (3.15) is integrable, we only need to prove

$$\lim_{k\to\infty}\mathbb{E}\Big[\int_0^T \delta H(t;u_t^k)-\delta H(t;u_t)\,\mathrm{d}t\Big]=0. \tag{3.16}$$

From the definition of Hamiltonian in (2.1), we have

$$\delta H(t;u_t^k)=p_t^*B_t(u_t^k-\overline{u}_t)+\sum_{j=1}^d q_t^{j,*}D_t^j(u_t^k-\overline{u}_t)+l(t,\overline{X}_t,u_t^k)-l(t,\overline{X}_t,\overline{u}_t)$$

and

$$\delta H(t;u_t)=p_t^*B_t(u_t-\overline{u}_t)+\sum_{j=1}^d q_t^{j,*}D_t^j(u_t-\overline{u}_t)+l(t,\overline{X}_t,u_t)-l(t,\overline{X}_t,\overline{u}_t).$$

Thus we only need to prove the corresponding term converges to 0. From the definition of u^k in (3.14), we have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T p_t^*B_t((u_t^k-\overline{u}_t)-(u_t-\overline{u}_t))\,\mathrm{d}t\Big]=\mathbb{E}\Big[\int_0^T p_t^*B_t((u-\overline{u})_t^k-(u_t-\overline{u}_t))\,\mathrm{d}t\Big]\\&\le\Big\{\mathbb{E}\Big[\int_0^T |p_t^*B_t|^2\,\mathrm{d}t\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\int_0^T |(u-\overline{u})_t^k-(u_t-\overline{u}_t)|^2\,\mathrm{d}t\Big]\Big\}^{\frac12}\to 0,\quad\text{as } k\to\infty.\end{aligned}$$

The last limit follows from the strong convergence of $(u-\overline{u})^k$ to $u-\overline{u}$ in $\mathcal{L}^2_{\mathscr{F}}(0,T)$. Similarly, for some $j=1,\cdots,d$, we have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T q_t^{j,*}D_t^j((u_t^k-\overline{u}_t)-(u_t-\overline{u}_t))\,\mathrm{d}t\Big]=\mathbb{E}\Big[\int_0^T q_t^{j,*}D_t^j((u-\overline{u})_t^k-(u_t-\overline{u}_t))\,\mathrm{d}t\Big]\\&\le\Big\{\mathbb{E}\Big[\int_0^T |q_t^{j,*}D_t^j|^2\,\mathrm{d}t\Big]\Big\}^{\frac12}\Big\{\mathbb{E}\Big[\int_0^T |(u-\overline{u})_t^k-(u_t-\overline{u}_t)|^2\,\mathrm{d}t\Big]\Big\}^{\frac12}\to 0,\quad\text{as } k\to\infty.\end{aligned}$$

Since $|l(t,x,u)|\le C(1+|x|^2+|u|^2)$, we have

$$\begin{aligned}|l(t,\overline{X}_t,u_t^k)-l(t,\overline{X}_t,u_t)|&\le 2C(1+|\overline{X}_t|^2+|u_t^k|^2+|u_t|^2)\\&\le 4C(1+|\overline{X}_t|^2+|u_t^k-\overline{u}_t|^2+|\overline{u}_t|^2+|u_t|^2).\end{aligned}$$

Then according to the definition of u^k in (3.14) and the sequence $\{(u-\overline{u})^k\}_{k=1}^\infty$ satisfies the condition (ii), we have

$$|u_t^k-\overline{u}_t|=|(u-\overline{u})_t^k|\le|u_t-\overline{u}_t|.$$

So

$$|l(t,\overline{X}_t,u_t^k)-l(t,\overline{X}_t,u_t)|\le 4C(1+|\overline{X}_t|^2+|u_t-\overline{u}_t|^2+|\overline{u}_t|^2+|u_t|^2).$$

Then using dominated convergence theorem, we have

$$\mathbb{E}\Big[\int_0^T |l(t,\overline{X}_t,u_t^k)-l(t,\overline{X}_t,u_t)|\,\mathrm{d}t\Big]=0.$$

Above all, we end the proof.

Since P satisfies linear backward stochastic differential equation (2.3), and A, B, C^j, D^j, $M_{xx}(\cdot)$, $l_{xx}(\cdot)$ are bounded, we have that $|P_t|$ is bounded for almost everywhere $t \in [0,T]$ and almost surely $\omega \in \Omega$. Thus the following integral exists, i.e.,

$$\mathbb{E}\Big[\int_0^T (u_t - \overline{u}_t)^* D_t^* P_t D_t^* (u_t - \overline{u}_t)\,\mathrm{d}t\Big] < \infty, \quad \forall u, \overline{u} \in \mathcal{L}^2_{\mathscr{F}}(0,T).$$

Lemma 3.5 *For u^k defined in* (3.14), *we have*

$$\lim_{k\to\infty} \mathbb{E}\Big[\int_0^T (u_t^k - \overline{u}_t)^* D_t^* P_t D_t (u_t^k - \overline{u}_t)\,\mathrm{d}t\Big] = \mathbb{E}\Big[\int_0^T (u_t - \overline{u}_t)^* D_t^* P_t D_t (u_t - \overline{u}_t)\,\mathrm{d}t\Big].$$

Proof According to the definition of u^k in (3.14) and the condition (ii) which the sequence $\{(u-\overline{u})^k\}_{k=1}^\infty$ satisfied, we have

$$\begin{aligned} & |(u_t^k - \overline{u}_t)^* D_t^* P_t D_t (u_t^k - \overline{u}_t)| \\ \le\ & |D_t^* P_t D_t| |u_t^k - \overline{u}_t|^2 \\ \le\ & |D_t^* P_t D_t| |(u - \overline{u})_t^k|^2 \\ \le\ & |D_t^* P_t D_t| |u_t - \overline{u}_t|^2. \end{aligned}$$

Since P_t and D_t are bounded, u_t and $\overline{u}_t$ are L^2-integrable, the right-hand side of the last inequality is integrable. We use the dominated convergence theorem to get the lemma.

According to the definition of u^k in (3.14), we have that u^k satisfies (3.12). Using Lemma 3.3, we have

$$\mathbb{E}\int_0^T \Big[\delta H(t; u_t^k) + \frac{1}{2}(u_t^k - \overline{u}_t)^* D_t^* P_t D_t (u_t^k - \overline{u}_t)\Big]\,\mathrm{d}t \ge 0. \tag{3.17}$$

According to Lemmas 3.4 and 3.5, we have the following result.

Lemma 3.6 $\forall u \in \mathcal{U}_{ad}$, *we have*

$$\mathbb{E}\int_0^T \Big[\delta H(t; u_t) + \frac{1}{2}(u_t - \overline{u}_t)^* D_t^* P_t D_t (u_t - \overline{u}_t)\Big]\,\mathrm{d}t \ge 0. \tag{3.18}$$

Since the last lemma holds for any u in $\mathcal{U}_{ad}$, we can deduce that the maximum condition (2.4) in Theorem 2.1.

4 The Case of Quadratic Cost Functional

We give the maximum condition for linear quadratic optimal control problem with square-integrable optimal control. The system is also (1.1) and the cost functional is

$$J(u) = \frac{1}{2}\mathbb{E}[X_T^* M X_T] + \mathbb{E}\Big[\int_0^T \Big(\frac{1}{2}X_s^* G_s X_s + \frac{1}{2}u_s^* N_s u_s\Big)\,\mathrm{d}s\Big], \tag{4.1}$$

where $G : [0, T] \to \mathbb{R}^{n\times n}$, $N : [0, T] \to \mathbb{R}^{m\times m}$, $M \in \mathbb{R}^{n\times n}$; and $G \geq 0$, $M \geq 0$, $N \geq \tilde{\delta}$ for $\tilde{\delta} > 0$. The admissible control set is

$$\mathcal{U}_{ad} = \{u \in \mathcal{L}^2_{\mathscr{F}}(0, T) : u_t \in U \text{ a.e.a.s.}\}$$

and control domain U is not necessarily convex. The optimal control problem is to find an optimal control $\overline{u}$ to minimize (4.1) over $\mathcal{U}_{ad}$. In the following two examples, we point out that the optimal control $\overline{u}$ may not satisfy (1.3).

Example 4.1 Let $n = 1$ and set

$$Y_t = \Big(\frac{W_t^2}{\sqrt{t}}\Big)^2 + \Big(\frac{W_t^3}{\sqrt{t}}\Big)^2 + \Big(\frac{W_t^4}{\sqrt{t}}\Big)^2.$$

Since W_t^j, $j = 1, \cdots, d$, are independent and have the normal distribution $N(0, t)$, we have that $\frac{W_t^1}{\sqrt{t}}, \frac{W_t^2}{\sqrt{t}}, \cdots, \frac{W_t^d}{\sqrt{t}}$ obey the standard normal law $N(0, 1)$. Then we verify that Y_t has the law χ_3^2, where 3 represents the freedom degree of a chi-square distribution (this can also be explained that Y_t is the sum of 3 independent variables (see [6, p. 31])). We set

$$\alpha_t = \beta_t^1 = \cdots = \beta_t^d := \frac{W_t^1/\sqrt{t}}{\sqrt{Y_t/3}},$$

then the law of $\alpha_t = \beta_t^1 = \cdots = \beta_t^d$ is t_3 (student distribution [6, p. 34, 12, p. 390, 18, p. 38]), we have

$$\mathbb{E}\Big[\int_0^T \alpha_t^2 \,\mathrm{d}t\Big] = 3T,$$

but

$$\mathbb{E}\Big[\int_0^T \alpha_t^4 \,\mathrm{d}t\Big] \quad \text{does not exists.}$$

Example 4.2 Let $n = 1$. We set

$$\alpha_t = \beta_t^j = t^{-\frac{1}{4}}, \quad j = 1, \cdots, d.$$

Then

$$\int_0^T |\alpha_t|^2 \,\mathrm{d}t = \int_0^T t^{-\frac{1}{2}} \,\mathrm{d}t = 2T^{\frac{1}{2}} < \infty,$$

but

$$\int_0^T |\alpha_t|^4 \,\mathrm{d}t = \int_0^T t^{-1} \,\mathrm{d}t = \infty.$$

When $U = \mathbb{R}^m$, we have

$$\overline{u}_t = -N_t^{-1}\Big(B_t^* p_t + \sum_{j=1}^d D_t^{j,*} q_t^j\Big),$$

where (p, q) is the first order adjoint process. In general, we assume

$$p_t = K_t \overline{X}_t + \varphi_t,$$

where K_t satisfies a Riccati equation, and φ_t satisfies an ordinary differential equation which can be found in [19, Chapter 6, (6.6)–(6.7), p. 314]. When coefficients A, B, C^j, D^j are deterministic (see [19]), we have

$$\overline{u}_t = -\Big(N_t + \sum_{j=1}^{d} D_t^{j,*} K_t D_t^j\Big)^{-1} \Big[\Big(B_t^* K_t + \sum_{j=1}^{d} D_t^{j,*} K_t C_t^j\Big)\overline{X}_t + B_t^* \varphi_t + \sum_{j=1}^{d} D_t^{j,*} K_t \beta_t^j\Big].$$

When coefficients A, B, C^j, D^j are stochastic, we refer to [17]. When control domain is the whole space, the optimal control $\overline{u}_t$ is a feedback of the optimal state $\overline{X}_t$.

In Examples 4.1–4.2, the optimal trajectory of $\overline{X}$ is only square-integrable. Since the optimal control $\overline{u}$ is always a feedback of optimal trajectory of $\overline{X}$, $\overline{u}$ does not satisfy (1.3). Therefore, Peng's stochastic maximum principle dose not apply both examples.

According to the linear stochastic system (1.1) and the quadratic cost functional (4.1), the Hamiltonian is

$$H(t, x, u, p, q) = \langle p, A_t x + B_t u + \alpha_t\rangle + \sum_{j=1}^{d} \langle q^j, C_t^j x + D_t^j u + \beta_t^j\rangle + \frac{1}{2} x^* G_t x + \frac{1}{2} u^* N_t u.$$

Note that the admissible control range U is not necessarily convex. Applying Theorem 2.1, we have the following theorem.

Theorem 4.1 *Let* $(\overline{X}, \overline{u})$ *be an optimal pair for the linear stochastic system* (1.1) *and the quadratic cost functional* (4.1). *Let the adapted stochastic processes* (p, q) *and* (P, Q) *solve the adjoint equations*:

$$\begin{cases} p_t = -\Big(A_t^* p_t + \sum_{j=1}^{d} C_t^{j,*} q_t^j + G_t \overline{X}_t\Big)\,\mathrm{d}t + \sum_{j=1}^{d} q_t^j \,\mathrm{d}W_t^j, \quad t \in [0, T), \\ p_T = M \overline{X}_T \end{cases}$$

and

$$\begin{cases} P_t = -\Big\{A_t^* P_t + P_t A_t + \sum_{j=1}^{d} (C_t^{j,*} P_t C_t^j + C_t^{j,*} Q_t^j + Q_t^j C_t^j) + G_t\Big\}\,\mathrm{d}t \\ \qquad + \sum_{j=1}^{d} Q_t^j \,\mathrm{d}W_t^j, \quad t \in [0, T), \\ P_T = M. \end{cases}$$

Then, we have the maximum condition: Almost surely $\omega \in \Omega$ *and almost every* $t \in [0, T]$,

$$\begin{aligned} &\min_{u \in U} \Big(\langle p_t, B_t(u - u_t)\rangle + \sum_{j=1}^{d} \langle q_t^j, D_t^j (u - u_t)\rangle + \frac{1}{2} u^* N_t u - \frac{1}{2} \overline{u}_t^* N_t \overline{u}_t \\ &+ \frac{1}{2} \sum_{j=1}^{d} [D_t^j (u - \overline{u}_t)]^* P_t [D_t^j (u - \overline{u}_t)]\Big) = 0. \end{aligned}$$

The above maximum condition also reads that $\forall u \in U$,

$$\Big(p_t^* B_t + \sum_{j=1}^d q_t^{j,*} D_t^j + \overline{u}_t^* N_t\Big)(u - \overline{u}_t) + \frac{1}{2}(u - \overline{u}_t)^* \Big(N_t + \sum_{j=1}^d D_t^{j,*} P_t D_t^j\Big)(u - \overline{u}_t) \geq 0.$$

The stochastic maximum principle of Cadenillas and Karatzas (see [5]) for linear stochastic system assumes that the cost functional is convex. On one hand, taking advantage of the convexity of the cost functional to avoid estimating the variation of state processes, they (see [5, Theorem 1.4, p. 608]) do not require the (t, ω)-joint L^2-integrability of admissible controls; in fact, their admissible control processes are only almost surely square time-integrable, that is,

$$\mathbb{P}\Big\{ \int_0^T |u_t|^2 \, \mathrm{d}t < \infty \Big\} = 1.$$

On the other hand, since their admissible control range U is required to be convex, their stochastic maximum principle does not apply to the typical situation that we consider in this section. However, when the control range U is convex, their stochastic maximum condition (see [5, Theorem 3.2, p. 608]):

$$\max_{u \in U} H(t, p_t, q_t, \widehat{X}_t, u) = H(t, p_t, q_t, \widehat{X}_t, \widehat{u}_t), \quad \text{Leb} \otimes P\text{−a.e. on } [0, T] \times \Omega$$

coincides with ours.

Acknowledgement The authors thank referees for their careful reading and helpful comments.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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