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Holomorphic Connections and Problems of Lifts

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Abstract Considering the bundle of 2-jets as a realization of the holomorphic manifold over 3-dimensional nilpotent algebra, the authors introduce a new class of lifts of connections in the bundle of 2-jets which is a generalization of the complete lifts.

Keywords Holomorphic functions, Bundle of 2-jets, Deformed lift, Pure connection
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1 Introduction

Theory of lifts in the tangent bundles of different order has experienced an extraordinary development in the last decades. The fundamental references in the area are represented by the treatises of Yano and Ishihara [7]. In recent years, there has been a growing interest in the study of lifts on tangent bundle of higher order. Among the interesting works of this type of liftings, we can cite for example [1–5]. The theory of lifts is closely related to the theory of holomorphic connections on tangent bundles. In the present paper a new connection in the tangent bundle of order 2 is introduced and the relation between holomorphic connections and lifts of connections is studied in detail.

Let $\Pi = \{ \underset{\alpha}{J}{}^{i}_{j} \}$, $\alpha = 1, \cdots, m;\ i, j = 1, \cdots, n$ be a Π-structure (see [1]) on a smooth manifold M_n. If there exists a frame $\{\partial_i = \frac{\partial}{\partial x^i}\}$, $i = 1, \cdots, n$, $x = (x^i) \in M_n$ such that $\partial_i \underset{\alpha}{J}{}^{k}_{j} = 0$, then the Π-structure is said to be integrable. Let $\mathcal{A}_m$ be an associative and commutative algebra with the unit element $e_1 = 1$. An algebraic structure on M_n is an integrable Π-structure such that $\underset{\alpha}{J}{}^{m}_{j} \underset{\beta}{J}{}^{i}_{m} = C^{\gamma}_{\alpha\beta} \underset{\gamma}{J}{}^{i}_{j}$, i.e., if there exists an isomorphism $\mathcal{A}_m \leftrightarrow \Pi$, where $C^{\gamma}_{\alpha\beta}$ are structure constants of $\mathcal{A}_m$. An algebraic structure is said to be an r-regular Π-structure if the matrices $(\underset{\alpha}{J}{}^{i}_{j})$ of order $n \times n$, $\alpha = 1, \cdots, m$, simultaneously reduced to the form

$$\left(\underset{\alpha}{J}{}^{i}_{j}\right) = \begin{pmatrix} C_\alpha & 0 & \cdots & 0 \\ 0 & C_\alpha & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & C_\alpha \end{pmatrix}, \quad \alpha = 1, \cdots, m,\ i, j = 1, \cdots, n \tag{1.1}$$

with respect to the adapted frame $\{\partial_i\}$, where $C_\alpha = (C^{\gamma}_{\alpha\beta})$ is the regular representation of $\mathcal{A}_m$ and r is a number of C_α-blocks. We note that the r-regular Π-structure is integrable if a structure-preserving connection with free-torsion exists on M_n.

From (1.1) it follows that $n = rm$ and therefore the structure tensors $\underset{\sigma}{J}$ have the components $\underset{\sigma}{J}{}^{i}_{j} = \underset{\sigma}{J}{}^{u\alpha}_{v\beta} = \delta^u_v C^{\alpha}_{\sigma\beta}$, $u, v = 1, \cdots, r$, where δ^u_v is the Kronecker delta and $u\alpha = (u-1)m + \alpha$, $v\beta = (v-1)m + \beta$.

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An $\mathcal{A}$-holomorphic manifold (see [6]) $X_r(\mathcal{A})$ over algebra $\mathcal{A}_m$ of dimension r is a Hausdorff space with a fixed complete atlas compatible with a group of $\mathcal{A}$-holomorphic transformations of space $\mathcal{A}_m^r$, where $\mathcal{A}_m^r = \mathcal{A}_m \times \cdots \times \mathcal{A}_m$ is the space of r-tuples of algebraic numbers $(z^1, z^2, \cdots, z^r)$ with $z^u = x^{u\alpha} e_\alpha \in \mathcal{A}_m, x^{u\alpha} = x^i \in R, i = 1, \cdots, n; u = 1, \cdots, r; \alpha = 1, \cdots, m$. Let now $\Pi = \{ \underset{\sigma}{J} \}$ be an integrable r-regular structure on M_{rm}. The transformation $z^{u'} = z^{u'}(z^u)$ of local coordinates on $X_r(\mathcal{A})$ is $\mathcal{A}$-holomorphic if and only if the transformation $x^{i'} = x^{i'}(x^i)$ of local coordinates on M_{rm} is a structure-preserving transformation (an admissible transformation), i.e.,

$$\underset{\alpha}{J} A = A \underset{\alpha}{J}, \quad A = \left(\frac{\partial x^j}{\partial x^{j}}\right), \quad \underset{\alpha}{J} = \left(\underset{\alpha}{J}{}_j^i\right).$$

Thus the real smooth manifold M_{rm} with an integrable r-regular Π-structure and with a structure-preserving transformations of local coordinates is a real modeling of an $\mathcal{A}$-holomorphic manifold $X_r(\mathcal{A})$ over algebra $\mathcal{A}_m$.

Let $R(\varepsilon^2)$ be an algebra of order 3 with a canonical basis $\{e_1, e_2, e_3\} = \{1, \varepsilon, \varepsilon^2\}$, $\varepsilon^3 = 0$. From $e_\alpha e_\beta = C_{\alpha\beta}^\gamma e_\gamma$, the (3×3)-matrices $C_\sigma = (C_{\sigma\beta}^\gamma)$, $\sigma = 1, 2, 3$ of regular representation of $R(\varepsilon^2)$ have respectively the following forms

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $z = x^1 + \varepsilon x^2 + \varepsilon^2 x^3$. Then the generalized Cauchy-Riemann conditions (i.e., the Scheffers conditions) (see [6])

$$C_{\sigma\beta}^\alpha \frac{\partial y^\beta}{\partial x^\gamma} = \frac{\partial y^\alpha}{\partial x^\beta} C_{\sigma\gamma}^\beta$$

for $\mathcal{A}$-holomorphic function

$$w = w(z) = y^1(x^1, x^2, x^3) + \varepsilon y^2(x^1, x^2, x^3) + \varepsilon^2 y^3(x^1, x^2, x^3),$$

reduces to the following equations:

$$\begin{aligned} &\text{(i)} \ \frac{\partial y^1}{\partial x^2} = \frac{\partial y^1}{\partial x^3} = \frac{\partial y^2}{\partial x^3} = 0, \\ &\text{(ii)} \ \frac{\partial y^2}{\partial x^2} = \frac{\partial y^1}{\partial x^1} = \frac{\partial y^3}{\partial x^3}, \\ &\text{(iii)} \ \frac{\partial y^3}{\partial x^2} = \frac{\partial y^2}{\partial x^1}. \end{aligned}$$

From (i)–(iii), we have

$$\begin{aligned} &y^1 = y^1(x^1), \\ &y^2 = y^2(x^1, x^2), \\ &y^2(x^1, x^2) = x^2 \frac{\mathrm{d}y^1}{\mathrm{d}x^1} + G(x^1), \\ &y^3(x^1, x^2, x^3) = x^3 \frac{\mathrm{d}y^1}{\mathrm{d}x^1} + \frac{1}{2}(x^2)^2 \frac{\mathrm{d}^2 y^1}{(\mathrm{d}x^1)^2} + x^2 \frac{\mathrm{d}G}{\mathrm{d}x^1} + H(x^1), \end{aligned}$$

where $G = G(x^1)$ and $H = H(x^1)$ are arbitrary functions. Thus the $R(\varepsilon^2)$-holomorphic function $w = w(z)$ has the following expression:

$$w(z) = y^1(x^1) + \varepsilon\Big(x^2 \frac{\mathrm{d}y^1}{\mathrm{d}x^1} + G(x^1)\Big) + \varepsilon^2\Big(x^3 \frac{\mathrm{d}y^1}{\mathrm{d}x^1} + \frac{1}{2}(x^2)^2 \frac{\mathrm{d}^2 y^1}{(\mathrm{d}x^1)^2} + x^2 \frac{\mathrm{d}G}{\mathrm{d}x^1} + H(x^1)\Big).$$

Similarly, if

$$w(z^1,\cdots,z^n)=y^1(x^1,\cdots,x^n)+\varepsilon y^2(x^1,\cdots,x^n)+\varepsilon^2 y^3(x^1,\cdots,x^n),$$

where $z^i=x^i+\varepsilon x^{n+i}+\varepsilon^2 x^{2n+i}$, $i=1,\cdots,n$, is a multi-variable $R(\varepsilon^2)$-holomorphic function, then the function $w=w(z^1,\cdots,z^n)$ has the following specific form:

$$\begin{aligned} w(z^1,\cdots,z^n)&=y^1(x^1,\cdots,x^n)+\varepsilon(x^{n+i}\partial_i y^1+G(x^1,\cdots,x^n))\\ &\quad+\varepsilon^2\Big(x^{2n+i}\frac{\partial y^1}{\partial x^i}+\frac{1}{2}x^{n+i}x^{n+j}\frac{\partial^2 y^1}{\partial x^i\partial x^j}+x^{n+i}\frac{\partial G}{\partial x^i}+H(x^1,\cdots,x^n)\Big). \end{aligned}\tag{1.2}$$

From (1.2) we see that if

$$G(x^1,\cdots,x^n)=H(x^1,\cdots,x^n)=0$$

and

$$y^1(x^1,\cdots,x^n)=f(x^1,\cdots,x^n),$$

then the function

$$w(z^1,\cdots,z^n)=f(x^1,\cdots,x^n)+\varepsilon x^{n+i}\partial_i f+\varepsilon^2\Big(x^{2n+i}\frac{\partial f}{\partial x^i}+\frac{1}{2}x^{n+i}x^{n+j}\frac{\partial^2 f}{\partial x^i\partial x^j}\Big)\tag{1.3}$$

is said to be natural extension of the real C^∞-functions $f=f(x^1,\cdots,x^n)$ to $\mathbb{R}(\varepsilon^2)$.

2 Holomorphic Connections

In this section, we always assume that the regular Π-structure is integrable and we consider only local adapted coordinates with respect to the structure.

Let ∇ be a Π-connection on M_{mr}, i.e., $\nabla \underset{\sigma}{\varphi}=0$ for any $\underset{\sigma}{\varphi}\in\Pi$. Since the components of $\underset{\sigma}{\varphi}$ with respect to the local adapted coordinates $x^1,\cdots,x^{mr}$ are constant, we have

$$\nabla\underset{\sigma}{\varphi}=0\Leftrightarrow \Gamma^i_{km}\underset{\sigma}{\varphi}{}^m_j=\Gamma^m_{kj}\underset{\sigma}{\varphi}{}^i_m\,.\tag{2.1}$$

We see that the Π-connection has components of the form (see [3])

$$\Gamma^i_{kj}=\Gamma^{u\alpha}_{w\gamma v\beta}=\overset{\sigma}{\tau}{}^{u}_{w\gamma v}C^\alpha_{\sigma\beta},\quad i=u\alpha,\ j=v\beta,\ k=w\gamma,\tag{2.2}$$

where $\overset{\sigma}{\tau}{}^{u}_{w\gamma v}=\Gamma^{u\sigma}_{w\gamma v\beta}\varepsilon^\beta$ are any functions in the adapted chart $U\subset M_{mr}$.

With Π-connection of type (2.2) we can associate a hypercomplex objects from $\mathcal{A}_m$:

$$\overset{*}{\Gamma}{}^{u}_{wv}=\Gamma^{u\alpha}_{w\gamma v\beta}\varepsilon^\gamma\varepsilon^\beta e_\alpha=\overset{\sigma}{\tau}{}^{u}_{w\gamma v}\varepsilon^\gamma e_\sigma.\tag{2.3}$$

If the hypercomplex objects $\overset{*}{\Gamma}{}^{u}_{wv}$ satisfies the following condition:

$$\overset{*}{\Gamma}{}^{u'}_{w'v'}=\frac{\partial z^{u'}}{\partial z^u}\frac{\partial z^w}{\partial z^{w'}}\frac{\partial z^v}{\partial z^{v'}}\overset{*}{\Gamma}{}^{u}_{wv}+\frac{\partial^2 z^u}{\partial z^{v'}\partial z^{w'}}\frac{\partial z^{u'}}{\partial z^u},$$

i.e., if $\overset{*}{\Gamma}{}^{u}_{wv}$ is a component of the hypercomplex connection $\overset{*}{\nabla}$ in $X_r(\mathcal{A}_m)$, then we say that the Π-connection ∇ is a pure connection.

Let Π be a regular integrable Π-structure on M_{mr}. The Π-connection ∇ is pure if and only if $\overset{\sigma}{\tau}{}^{u}_{w\gamma\nu}$ satisfies the condition

$$\overset{\alpha}{\tau}{}^{u}_{w\gamma v}=\overset{\sigma}{\tau}{}^{u}_{wv}C^\alpha_{\sigma\gamma},\tag{2.4}$$

where $\overset{\sigma}{\tau}{}^{u}_{wv} = \overset{\sigma}{\tau}{}^{u}_{w\eta v}\varepsilon^{\eta}$.

Using (2.4), we get from (2.2)–(2.3), respectively,

$$\Gamma^{i}_{kj} = \Gamma^{u\alpha}_{w\gamma v\beta} = \overset{\sigma}{\tau}{}^{u}_{wv} C^{\mu}_{\sigma\gamma} C^{\alpha}_{\mu\beta} = \overset{\sigma}{\tau}{}^{u}_{wv} B^{\alpha}_{\sigma\gamma\beta} \tag{2.5}$$

and

$$\overset{*}{\Gamma}{}^{u}_{wv} = \overset{\sigma}{\tau}{}^{u}_{wv} e_{\sigma}, \tag{2.6}$$

where $B^{\alpha}_{\sigma\gamma\beta} = C^{\mu}_{\sigma\gamma} C^{\alpha}_{\mu\beta}$.

Thus, a pure Π-connection ∇ has the components (2.5) with respect to the adapted coordinates and it is a realization of the hypercomplex connection $\overset{*}{\nabla}$ with components (2.6).

From here and (2.1) it follows that the pure Π-connection is defined by

$$\Gamma^{i}_{km} \underset{\alpha}{\varphi}{}^{m}_{j} = \Gamma^{m}_{kj} \underset{\alpha}{\varphi}{}^{i}_{m} = \Gamma^{i}_{mj} \underset{\alpha}{\varphi}{}^{m}_{k}, \qquad \alpha = 1, \cdots, m \tag{2.7}$$

with respect to the adapted charts.

3 Bundle of 2-Jets as a Realization of $X_r(R(\varepsilon^2))$

Let now $T^2(M_r)$ be the bundle of 2-jets, i.e., the tangent bundle of order 2 over C^{∞}-manifold M_r, $\dim T^2(M_r) = 3r$, and let

$$(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}}) = (x^i, x^{r+i}, x^{2r+i}), \quad x^i = x^i(t), \quad x^{\bar{i}} = \frac{\mathrm{d}x^i}{\mathrm{d}t}, \quad x^{\bar{\bar{i}}} = \frac{1}{2}\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}, \quad t \in \mathbb{R},\ i = 1, \cdots, r$$

be an induced local coordinates in $T^2(M_r)$. It is clear that there exists an affinor field (a tensor field of type $(1,1)$) φ on $T^2(M_r)$ which has components of the form

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \tag{3.1}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}, \frac{\partial}{\partial x^{\bar{\bar{i}}}}\right\}$, $i = 1, \cdots, r$, where I denotes the $r \times r$ identity matrix. From (3.1), we have

$$\varphi^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}, \quad \varphi^3 = 0, \tag{3.2}$$

i.e., $T^2(V_r)$ has a natural integrable structure $\Pi = \{I, \varphi, \varphi^2\}, I = id_{T^2(M_r)}$, which is an isomorphic representation of the algebra $R(\varepsilon^2), \varepsilon^3 = 0$. Using

$$\varphi\partial_i = \partial_{\bar{i}}, \quad \varphi^2\partial_i = \varphi\partial_{\bar{i}} = \partial_{\bar{\bar{i}}},$$

we have $\{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\} = \{\partial_i, \varphi\partial_i, \varphi^2\partial_i\}$. Also, using a frame

$$\{\partial_1, \varphi\partial_1, \varphi^2\partial_1, \partial_2, \varphi\partial_2, \varphi^2\partial_2, \cdots, \partial_r, \varphi\partial_r, \varphi^2\partial_r\} = \{\partial_1, \partial_{\bar{1}}, \partial_{\bar{\bar{1}}}, \partial_2, \partial_{\bar{2}}, \partial_{\bar{\bar{2}}}, \cdots, \partial_r, \partial_{\bar{r}}, \partial_{\bar{\bar{r}}}\},$$

which is obtained from $\{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\} = \{\partial_i, \varphi\partial_i, \varphi^2\partial_i\}$ by changing of numbers of frame elements, we see that structure affinors I, φ and φ^2 have the following components, respectively,

$$I = \begin{pmatrix} C_1\, 0 \, \cdots 0 \\ 0\;\, C_1 \cdots 0 \\ \vdots \\ 0\; 0\; \cdots C_1 \end{pmatrix}, \quad \varphi = \begin{pmatrix} C_2\, 0 \, \cdots \, 0 \\ 0\;\, C_2 \cdots 0 \\ \vdots \\ 0\; 0\; \cdots C_2 \end{pmatrix}, \quad \varphi^2 = \begin{pmatrix} C_3\, 0 \, \cdots \, 0 \\ 0\;\, C_3 \cdots 0 \\ \vdots \\ 0\; 0\; \cdots C_3 \end{pmatrix}$$

with respect to the frame $\{\partial_1, \partial_{\bar{1}}, \partial_{\bar{\bar{1}}}, \partial_2, \partial_{\bar{2}}, \partial_{\bar{\bar{2}}}, \cdots, \partial_r, \partial_{\bar{r}}, \partial_{\bar{\bar{r}}}\}$, where the block matrices C_σ of order 3, $\sigma = 1, 2, 3$, are the regular representation of algebra $R(\varepsilon^2)$. Thus the bundle $T^2(M_r)$ has a natural integrable structure $\Pi = \{I, \varphi, \varphi^2\}$, which is an r-regular representation of $R(\varepsilon^2)$. On the other hand, the transformation of induced coordinates $(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}})$ in $T^2(M_r)$ is given

$$
\begin{aligned}
x^{i'} &= x^{i'}(x^i),\\
x^{\bar{i}'} &= \frac{\mathrm{d}x^{i'}}{\mathrm{d}t} = \frac{\partial x^{i'}}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial x^{i'}}{\partial x^i} x^{\bar{i}},\\
x^{\bar{\bar{i}}'} &= \frac{1}{2}\frac{d^2 x^{i'}}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial x^{i'}}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big) = \frac{1}{2}\frac{\partial x^{i'}}{\partial x^i}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial x^{i'}}{\partial x^i}x^{\bar{\bar{i}}} + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}x^{\bar{i}}x^{\bar{j}},
\end{aligned}
$$

and its Jacobian matrix is

$$
A = (A_\alpha^{\alpha'}) = \begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & \frac{\partial x^{i'}}{\partial x^{\bar{i}}} & \frac{\partial x^{i'}}{\partial x^{\bar{\bar{i}}}} \\ \frac{\partial x^{\bar{i}'}}{\partial x^i} & \frac{\partial x^{\bar{i}'}}{\partial x^{\bar{i}}} & \frac{\partial x^{\bar{i}'}}{\partial x^{\bar{\bar{i}}}} \\ \frac{\partial x^{\bar{\bar{i}}'}}{\partial x^i} & \frac{\partial x^{\bar{\bar{i}}'}}{\partial x^{\bar{i}}} & \frac{\partial x^{\bar{\bar{i}}'}}{\partial x^{\bar{\bar{i}}}} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & 0 & 0 \\ \frac{\partial^2 x^{i'}}{\partial x^i \partial x^s}x^{\bar{s}} & \frac{\partial x^{i'}}{\partial x^i} & 0 \\ \frac{\partial^2 x^{i'}}{\partial x^i \partial x^s}x^{\bar{\bar{s}}} + \frac{\partial^3 x^{i'}}{\partial x^i \partial x^s \partial x^t}x^{\bar{s}}x^{\bar{t}} & \frac{\partial^2 x^{i'}}{\partial x^i \partial x^s}x^{\bar{s}} & \frac{\partial x^{i'}}{\partial x^i} \end{pmatrix}.
$$

From here it follows that $A^{-1}\varphi A = \varphi$, $A^{-1}\varphi^2 A = \varphi^2$, i.e., the transformation of local coordinates $(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}})$ in $T^2(M_r)$ is a structure-preserving transformation. Then the transition functions

$$
z^{i'}(z^i) = x^{i'} + \varepsilon x^{\bar{i}'} + \varepsilon^2 x^{\bar{\bar{i}}'} = x^{i'}(x^i) + \varepsilon \frac{\partial x^{i'}}{\partial x^i}x^{\bar{i}} + \varepsilon^2\Big(\frac{\partial x^{i'}}{\partial x^i}x^{\bar{\bar{i}}} + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}x^{\bar{i}}x^{\bar{j}}\Big)
$$

of charts on $X_r(R(\varepsilon^2))$ are $R(\varepsilon^2)$-holomorphic functions by virtue of (1.3), i.e., the bundle $T^2(M_r)$ is a real modeling of $R(\varepsilon^2)$-holomorphic manifold $X_r(R(\varepsilon^2))$.

4 Deformed Complete Lifts of Connections

Let $\widetilde{\nabla}$ be a projectable connection with projection $\Gamma_{ij}^k(x^1, \cdots, x^r)$ and with the components $\widetilde{\Gamma}_{\alpha\beta}^\gamma$, $\alpha = (i, \bar{i}, \bar{\bar{i}})$, $\beta = (j, \bar{j}, \bar{\bar{j}})$, $\gamma = (k, \bar{k}, \bar{\bar{k}})$ in the tangent bundle $T^2(M_r)$ preserving the structure $\Pi = \{I, \varphi, \varphi^2\}$. That connection is called a pure connection by definition if (see (2.7))

$$
\begin{aligned}
\widetilde{\Gamma}_{\alpha\beta}^\sigma \varphi_\sigma^\gamma &= \widetilde{\Gamma}_{\sigma\beta}^\gamma \varphi_\alpha^\sigma = \widetilde{\Gamma}_{\alpha\sigma}^\gamma \varphi_\beta^\sigma,\\
\widetilde{\Gamma}_{\alpha\beta}^\sigma (\varphi^2)_\sigma^\gamma &= \widetilde{\Gamma}_{\sigma\beta}^\gamma (\varphi^2)_\alpha^\sigma = \widetilde{\Gamma}_{\alpha\sigma}^\gamma (\varphi^2)_\beta^\sigma.
\end{aligned}
$$

Using (3.1)–(3.2), from here it follows that

$$
(\widetilde{\Gamma}_{\alpha\beta}^k) = \begin{pmatrix} \widetilde{\Gamma}_{ij}^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\widetilde{\Gamma}_{\alpha\beta}^{\bar{k}}) = \begin{pmatrix} \widetilde{\Gamma}_{ij}^{\bar{k}} & \widetilde{\Gamma}_{ij}^k & 0 \\ \widetilde{\Gamma}_{ij}^k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\widetilde{\Gamma}_{\alpha\beta}^{\bar{\bar{k}}}) = \begin{pmatrix} \widetilde{\Gamma}_{ij}^{\bar{\bar{k}}} & \widetilde{\Gamma}_{ij}^{\bar{k}} & \widetilde{\Gamma}_{ij}^k \\ \widetilde{\Gamma}_{ij}^{\bar{k}} & \widetilde{\Gamma}_{ij}^k & 0 \\ \widetilde{\Gamma}_{ij}^k & 0 & 0 \end{pmatrix}. \tag{4.1}
$$

The pure connection $\widetilde{\nabla}$ with components $\widetilde{\Gamma}_{\alpha\beta}^\gamma$ is called a holomorphic connection, if (see [3])

$$
\begin{aligned}
(\Phi_\varphi \Gamma)_{\tau\alpha\beta}^\gamma &= \varphi_\tau^\sigma \partial_\sigma \widetilde{\Gamma}_{\alpha\beta}^\gamma - \varphi_\alpha^\sigma \partial_\tau \widetilde{\Gamma}_{\sigma\beta}^\gamma = 0,\\
(\Phi_{\varphi^2} \Gamma)_{\tau\alpha\beta}^\gamma &= (\varphi^2)_\tau^\sigma \partial_\sigma \widetilde{\Gamma}_{\alpha\beta}^\gamma - (\varphi^2)_\alpha^\sigma \partial_\tau \widetilde{\Gamma}_{\sigma\beta}^\gamma = 0.
\end{aligned} \tag{4.2}
$$

It is well known that, such a connection is a real image of corresponding holomorphic connection from $X_n(R(\varepsilon^2))$.

(1) Let $\gamma = k$. Since $\sigma = (m, \bar{m}, \bar{\bar{m}})$, from (4.2) we have

$$\begin{aligned}(\Phi_\varphi \Gamma)^k_{\tau\alpha\beta} &= \varphi^m_\tau \partial_m \widetilde{\Gamma}^k_{\alpha\beta} + \varphi^{\bar{m}}_\tau \partial_{\bar{m}} \widetilde{\Gamma}^k_{\alpha\beta} + \varphi^{\bar{\bar{m}}}_\tau \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^k_{\alpha\beta} \\ &\quad - \varphi^m_\alpha \partial_\tau \widetilde{\Gamma}^k_{m\beta} - \varphi^{\bar{m}}_\alpha \partial_\tau \widetilde{\Gamma}^k_{\bar{m}\beta} - \varphi^{\bar{\bar{m}}}_\alpha \partial_\tau \widetilde{\Gamma}^k_{\bar{\bar{m}}\beta} = 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}(\Phi_{\varphi^2} \Gamma)^k_{\tau\alpha\beta} &= (\varphi^2)^m_\tau \partial_m \widetilde{\Gamma}^k_{\alpha\beta} + (\varphi^2)^{\bar{m}}_\tau \partial_{\bar{m}} \widetilde{\Gamma}^k_{\alpha\beta} + (\varphi^2)^{\bar{\bar{m}}}_\tau \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^k_{\alpha\beta} \\ &\quad - (\varphi^2)^m_\alpha \partial_\tau \widetilde{\Gamma}^k_{m\beta} - (\varphi^2)^{\bar{m}}_\alpha \partial_\tau \widetilde{\Gamma}^k_{\bar{m}\beta} - (\varphi^2)^{\bar{\bar{m}}}_\alpha \partial_\tau \widetilde{\Gamma}^k_{\bar{\bar{m}}\beta} = 0. \end{aligned} \tag{4.4}$$

For $\tau = (t, \bar{t}, \bar{\bar{t}})$, $\alpha = (i, \bar{i}, \bar{\bar{i}})$, $\beta = (j, \bar{j}, \bar{\bar{j}})$, from (4.3) by virtue of (3.1)–(3.2) and (4.1) we have

$$\begin{aligned}(\Phi_\varphi \Gamma)^k_{tij} &= \varphi^m_t \partial_m \widetilde{\Gamma}^k_{ij} + \varphi^{\bar{m}}_t \partial_{\bar{m}} \widetilde{\Gamma}^k_{ij} + \varphi^{\bar{\bar{m}}}_t \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^k_{ij} - \varphi^m_i \partial_t \widetilde{\Gamma}^k_{mj} \\ &\quad - \varphi^{\bar{m}}_i \partial_t \widetilde{\Gamma}^k_{\bar{m}j} - \varphi^{\bar{\bar{m}}}_i \partial_t \widetilde{\Gamma}^k_{\bar{\bar{m}}j} = 0 \Leftrightarrow \varphi^{\bar{m}}_t \partial_{\bar{m}} \widetilde{\Gamma}^k_{ij} = 0 \Rightarrow \widetilde{\Gamma}^k_{ij} = \widetilde{\Gamma}^k_{ij}(x^m, x^{\bar{\bar{m}}}). \end{aligned}$$

Similarly, from (4.4) we obtain

$$\begin{aligned}(\Phi_{\varphi^2} \Gamma)^k_{tij} &= (\varphi^2)^m_t \partial_m \widetilde{\Gamma}^k_{ij} + (\varphi^2)^{\bar{m}}_t \partial_{\bar{m}} \widetilde{\Gamma}^k_{ij} + (\varphi^2)^{\bar{\bar{m}}}_t \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^k_{ij} - (\varphi^2)^m_i \partial_t \widetilde{\Gamma}^k_{mj} \\ &\quad - (\varphi^2)^{\bar{m}}_i \partial_t \widetilde{\Gamma}^k_{\bar{m}j} - (\varphi^2)^{\bar{\bar{m}}}_i \partial_t \widetilde{\Gamma}^k_{\bar{\bar{m}}j} = 0 \Leftrightarrow (\varphi^2)^{\bar{\bar{m}}}_t \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^k_{ij} = 0 \Rightarrow \widetilde{\Gamma}^k_{ij} \\ &= \widetilde{\Gamma}^k_{ij}(x^m) = \Gamma^k_{ij}(x^m). \end{aligned}$$

Thus

$$(\widetilde{\Gamma}^k_{\alpha\beta}) = \begin{pmatrix} \Gamma^k_{ij}(x^m) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = ({}^C\Gamma^k_{\alpha\beta}), \quad x^m = (x^1, \cdots, x^r), \tag{4.5}$$

where ${}^C\Gamma^k_{\alpha\beta}$ denote the first components of $\widetilde{\Gamma}^\gamma_{\alpha\beta}$, which coincides with first components of complete lift of connection Γ^k_{ij}.

(2) Let $\gamma = \bar{k}$. In this case, we have

$$\begin{aligned}(\Phi_\varphi \Gamma)^{\bar{k}}_{\tau\alpha\beta} &= \varphi^m_\tau \partial_m \widetilde{\Gamma}^{\bar{k}}_{\alpha\beta} + \varphi^{\bar{m}}_\tau \partial_{\bar{m}} \widetilde{\Gamma}^{\bar{k}}_{\alpha\beta} + \varphi^{\bar{\bar{m}}}_\tau \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^{\bar{k}}_{\alpha\beta} - \varphi^m_\alpha \partial_\tau \widetilde{\Gamma}^{\bar{k}}_{m\beta} \\ &\quad - \varphi^{\bar{m}}_\alpha \partial_\tau \widetilde{\Gamma}^{\bar{k}}_{\bar{m}\beta} - \varphi^{\bar{\bar{m}}}_\alpha \partial_t \widetilde{\Gamma}^{\bar{k}}_{\bar{\bar{m}}\beta} = 0 \Rightarrow \varphi^{\bar{m}}_\tau \partial_{\bar{m}} \widetilde{\Gamma}^{\bar{k}}_{\alpha\beta} + \varphi^{\bar{\bar{m}}}_\tau \partial_{\bar{\bar{m}}} \widetilde{\Gamma}^{\bar{k}}_{\alpha\beta} - \varphi^{\bar{m}}_\alpha \partial_\tau \widetilde{\Gamma}^{\bar{k}}_{\bar{m}\beta} \\ &= 0. \end{aligned}$$

From here it follows that

$$
\begin{aligned}
(\Phi_\varphi\Gamma)^{\bar k}_{t\bar i j} &= \varphi_t^m\partial_m\widetilde\Gamma^{\bar k}_{\bar i j} + \varphi_t^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\bar i j} + \varphi_t^{\bar{\bar m}}\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{\bar i j} - \varphi_{\bar i}^m\partial_t\widetilde\Gamma^{\bar k}_{mj}\\
&\quad - \varphi_{\bar i}^{\bar m}\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} - \varphi_{\bar i}^{\bar{\bar m}}\partial_t\widetilde\Gamma^{\bar k}_{\bar{\bar m} j} = 0 \Rightarrow \varphi_t^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\bar i j} = 0 \Rightarrow \widetilde\Gamma^{\bar k}_{\bar i j} = \widetilde\Gamma^{\bar k}_{\bar i j}(x^m, x^{\bar{\bar m}}),\\
(\Phi_\varphi\Gamma)^{\bar k}_{ti\bar j} &= \varphi_t^m\partial_m\widetilde\Gamma^{\bar k}_{i\bar j} + \varphi_t^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{i\bar j} + \varphi_t^{\bar{\bar m}}\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{i\bar j} - \varphi_i^m\partial_t\widetilde\Gamma^{\bar k}_{m\bar j}\\
&\quad - \varphi_i^{\bar m}\partial_t\widetilde\Gamma^{\bar k}_{\bar m\bar j} - \varphi_i^{\bar{\bar m}}\partial_t\widetilde\Gamma^{\bar k}_{\bar{\bar m}\,\bar j}\\
&= 0 \Rightarrow \varphi_t^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{i\,\bar j} = 0 \Rightarrow \widetilde\Gamma^{\bar k}_{i\,\bar j} = \widetilde\Gamma^{\bar k}_{i\,\bar j}(x^m, x^{\bar{\bar m}})\\
&= \Gamma^k_{mj}\ ((\Phi_\varphi\Gamma)^{\bar k}_{\bar t\, i\,\bar j} = 0 \Rightarrow \widetilde\Gamma^{\bar k}_{i\,\bar j} = \widetilde\Gamma^{\bar k}_{i\,\bar j}(x^m)),\\
(\Phi_\varphi\Gamma)^{\bar k}_{\bar t ij} &= \varphi_{\bar t}^m\partial_m\widetilde\Gamma^{\bar k}_{ij} + \varphi_{\bar t}^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} + \varphi_{\bar t}^{\bar{\bar m}}\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{ij} - \varphi_i^m\partial_{\bar t}\widetilde\Gamma^{\bar k}_{mj}\\
&\quad - \varphi_i^{\bar m}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m j} - \varphi_i^{\bar{\bar m}}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar{\bar m} j} = 0 \Rightarrow \varphi_{\bar t}^{\bar{\bar m}}\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{ij} - \varphi_i^{\bar m}\partial_{\bar t}\widetilde\Gamma^{\bar k}_{\bar m j} = 0,\\
(\Phi_\varphi\Gamma)^{\bar k}_{\bar{\bar t} ij} &= \varphi_{\bar{\bar t}}^m\partial_m\widetilde\Gamma^{\bar k}_{ij} + \varphi_{\bar{\bar t}}^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} + \varphi_{\bar{\bar t}}^{\bar{\bar m}}\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{ij} - \varphi_i^m\partial_{\bar{\bar t}}\widetilde\Gamma^{\bar k}_{mj}\\
&\quad - \varphi_i^{\bar m}\partial_{\bar{\bar t}}\widetilde\Gamma^{\bar k}_{\bar m j} - \varphi_i^{\bar{\bar m}}\partial_{\bar{\bar t}}\widetilde\Gamma^{\bar k}_{\bar{\bar m} j} = 0 \Rightarrow -\varphi_i^{\bar m}\partial_{\bar{\bar t}}\widetilde\Gamma^{\bar k}_{\bar m j}\\
&= 0 \Rightarrow \widetilde\Gamma^{\bar k}_{\bar m j} = \widetilde\Gamma^{\bar k}_{\bar m j}(x^m, x^{\bar m}) \Rightarrow \widetilde\Gamma^{\bar k}_{\bar m j} = \widetilde\Gamma^{\bar k}_{\bar m j}(x^m) = \Gamma^k_{mj},\\
(\Phi_\varphi\Gamma)^{\bar k}_{tij} &= \varphi_t^m\partial_m\widetilde\Gamma^{\bar k}_{ij} + \varphi_t^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} + \varphi_t^{\bar{\bar m}}\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{ij} - \varphi_i^m\partial_t\widetilde\Gamma^{\bar k}_{mj}\\
&\quad - \varphi_i^{\bar m}\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} - \varphi_i^{\bar{\bar m}}\partial_t\widetilde\Gamma^{\bar k}_{\bar{\bar m} j} = 0 \Rightarrow \varphi_t^{\bar m}\partial_{\bar m}\widetilde\Gamma^{\bar k}_{ij} - \varphi_i^{\bar m}\partial_t\widetilde\Gamma^{\bar k}_{\bar m j} = 0\\
&\Rightarrow \widetilde\Gamma^{\bar k}_{ij} = x^t\partial_t\widetilde\Gamma^{\bar k}_{\bar i j} + G^k_{ij}(x^m)\\
&= x^t\partial_t\Gamma^k_{ij} + G^k_{ij}(x^m)
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi_{\varphi^2}\Gamma)^{\bar k}_{\tau\alpha\beta} &= (\varphi^2)^m_\tau\partial_m\widetilde\Gamma^{\bar k}_{\alpha\beta} + (\varphi^2)^{\bar m}_\tau\partial_{\bar m}\widetilde\Gamma^{\bar k}_{\alpha\beta} + (\varphi^2)^{\bar{\bar m}}_\tau\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{\alpha\beta} - (\varphi^2)^m_\alpha\partial_\tau\widetilde\Gamma^{\bar k}_{m\beta}\\
&\qquad - (\varphi^2)^{\bar m}_\alpha\partial_\tau\widetilde\Gamma^{\bar k}_{\bar m\beta} - (\varphi^2)^{\bar{\bar m}}_\alpha\partial_\tau\widetilde\Gamma^{\bar k}_{\bar{\bar m}\beta} = 0,\\
&(\varphi^2)^{\bar{\bar m}}_\tau\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{\alpha\beta} - (\varphi^2)^{\bar{\bar m}}_\alpha\partial_\tau\widetilde\Gamma^{\bar k}_{\bar{\bar m}\beta} = 0,\\
&(\varphi^2)^{\bar{\bar m}}_t\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{ij} = 0 \Rightarrow \widetilde\Gamma^{\bar k}_{ij} = \widetilde\Gamma^{\bar k}_{ij}(x^t, x^{\bar t}),\\
&(\varphi^2)^{\bar{\bar m}}_t\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{\bar i j} = 0 \Rightarrow \widetilde\Gamma^{\bar k}_{\bar i j} = \widetilde\Gamma^{\bar k}_{\bar i j}(x^t, x^{\bar t}),\\
&(\varphi^2)^{\bar{\bar m}}_t\partial_{\bar{\bar m}}\widetilde\Gamma^{\bar k}_{i\,\bar j} = 0 \Rightarrow \widetilde\Gamma^{\bar k}_{i\,\bar j} = \widetilde\Gamma^{\bar k}_{i\,\bar j}(x^t, x^{\bar t}).
\end{aligned}
$$

Thus

$$
(\widetilde\Gamma^{\bar k}_{\alpha\beta}) = \begin{pmatrix} x^t\partial_t\Gamma^k_{ij} + G^k_{ij}(x^m) & \Gamma^k_{ij}(x^m) & 0\\ \Gamma^k_{ij}(x^m) & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} = ({}^C\Gamma^{\bar k}_{\alpha\beta}) + ({}^C G^k_{\alpha\beta}),\quad x^m = (x^1, \cdots, x^r)\,, \tag{4.6}
$$

where $G = (G^k_{ij})$ is an arbitrary tensor field of type (1, 2) on M_r, ${}^C G^k_{\alpha\beta}$ is first component of complete lift of G and ${}^C\Gamma^{\bar k}_{\alpha\beta}$ denotes the second component of complete lift of connection Γ^k_{ij} (see [7]).

Let $\gamma = \bar{\bar k}$. Similarly, in this case we have

$$
(\widetilde\Gamma^{\bar{\bar k}}_{\alpha\beta}) = \begin{pmatrix} x^{\bar{\bar s}}\partial_s\Gamma^k_{ij} + \frac{1}{2}x^{\bar t}x^{\bar s}\partial_t\partial_s\Gamma^k_{ij} + x^{\bar s}\partial_s G^k_{ij} + H^k_{ij}(x^m) & x^{\bar t}\partial_t\Gamma^k_{ij} + G^k_{ij}(x^m) & \Gamma^k_{ij}\\ x^{\bar t}\partial_t\Gamma^k_{ij} + G^k_{ij}(x^m) & \Gamma^k_{ij} & 0\\ \Gamma^k_{ij} & 0 & 0 \end{pmatrix},
$$

$$
x^m = (x^1, \cdots, x^r),
$$

where $H=(H^k_{ij})$ is also an arbitrary tensor field of type (1, 2) on M_r. From here, it follows that

$$(\widetilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}) = \begin{pmatrix} x^{\bar{\bar{s}}}\partial_s\Gamma^k_{ij} + \frac{1}{2}x^{\bar{t}}x^{\bar{s}}\partial_t\partial_s\Gamma^k_{ij} + x^{\bar{s}}\partial_s G^k_{ij} + H^k_{ij} & x^{\bar{t}}\partial_t\Gamma^k_{ij} + G^k_{ij} & \Gamma^k_{ij} \\ x^{\bar{t}}\partial_t\Gamma^k_{ij} + G^k_{ij} & \Gamma^k_{ij} & 0 \\ \Gamma^k_{ij} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x^{\bar{\bar{s}}}\partial_s\Gamma^k_{ij} + \frac{1}{2}x^{\bar{t}}x^{\bar{s}}\partial_t\partial_s\Gamma^k_{ij} & x^{\bar{t}}\partial_t\Gamma^k_{ij} & \Gamma^k_{ij} \\ x^{\bar{t}}\partial_t\Gamma^k_{ij} & \Gamma^k_{ij} & 0 \\ \Gamma^k_{ij} & 0 & 0 \end{pmatrix} + \begin{pmatrix} x^{\bar{s}}\partial_s G^k_{ij} & G^k_{ij} & 0 \\ G^k_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} H^k_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= ({}^C\Gamma^{\bar{\bar{k}}}_{\alpha\beta}) + ({}^C G^{\bar{k}}_{\alpha\beta}) + ({}^c H^k_{\alpha\beta}), \tag{4.7}$$

where ${}^C H^k_{\alpha\beta}$ is first component of complete lift of H, ${}^C G^{\bar{k}}_{\alpha\beta}$ is second component of complete lift of G and ${}^C\Gamma^{\bar{\bar{k}}}_{\alpha\beta}$ denotes the third component of complete lift of connection Γ^k_{ij} (see [7]). If $H=G=0$, then we have $\widetilde{\nabla} = \nabla^C$, where ∇^C is the complete lift of ∇ to $T^2(M_r)$. Therefore the connection $\widetilde{\nabla}$ on $T^2(M_r)$ is called the deformed complete lift of ∇ to $T^2(M_r)$. Thus, from (4.5)–(4.7), we have the following theorem.

Theorem 4.1 *Let ∇ be a connection on M_r. Then the deformed complete lift $\widetilde{\nabla}$ of ∇ to tangent bundle of 2-jets $T^2(M_r)$ has components*

$$\widetilde{\nabla} = (\widetilde{\Gamma}^k_{\alpha\beta}, \widetilde{\Gamma}^{\bar{k}}_{\alpha\beta}, \widetilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}) = ({}^C\Gamma^k_{\alpha\beta}, {}^C\Gamma^{\bar{k}}_{\alpha\beta} + {}^C G^k_{\alpha\beta}, {}^C\Gamma^{\bar{\bar{k}}}_{\alpha\beta} + {}^C G^{\bar{k}}_{\alpha\beta} + {}^C H^k_{\alpha\beta}),$$

where ${}^C G^k_{\alpha\beta}$ and ${}^C H^k_{\alpha\beta}$ are first components of complete lift of $(1,2)$-tensor fields G and H respectively, ${}^C G^{\bar{k}}_{\alpha\beta}$ is second component of complete lift of G and ${}^C\Gamma^k_{\alpha\beta}$, ${}^C\Gamma^{\bar{k}}_{\alpha\beta}$, ${}^C\Gamma^{\bar{\bar{k}}}_{\alpha\beta}$ denote the all components of complete lift ${}^C\nabla$ of connection $\nabla = (\Gamma^k_{ij})$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Aslanci, S., Deformed intermadiate and complete lifts of 1-forms to the bundle of 2-jets, *Filomat*, **37**(4), 2023, 1123–1131.

[2] Aslanci, S. and Sultanova, T., Deformed lifts in the bundle of 2-jets, *Tbilisi Math. J.*, **14**(3), 2021, 155–162.

[3] Salimov, A., Applications of Holomorphic Functions in Geometry, Springer-Verlag, Birkh*ä*user, Singapore, 2023.

[4] Salimov, A., Iscan, M. and Akbulut, K., Some remarks concerning hyperholomorphic B-manifolds, *Chin. Ann. Math. Ser. B.,* **29**(6), 2008, 631–640.

[5] Sultanova, T., Salimov, A., On holomorphic metrics of 2-jet bundles, *Mediterr. J. Math.*, **19**(29), 2021, 12 pp.

[6] Vishnevskii, V.V., Shirokov, A.P. and Shurygin V.V., Spaces over Algebras, Kazanskii Gosudarstvennyi Universitet, Kazan, 1985.

[7] Yano, K. and Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973.