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Bresse Beam with Damping and Logarithmic Source*

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Abstract This paper investigates the stabilization of a Bresse system with internal damping and logarithmic source. The authors use the potential well theory. For initial data in the stability set created by the Nehari surface, the existence of a global solution is proved by using Faedo-Galerkin's approximation. The Nakao theorem gives the exponential decay. A numerical approach is presented to illustrate the results obtained.

Keywords Bresse beam, Logarithmic source, Global solution, Exponential decay, Numerical approach
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1 Introduction

The Bresse system is known as the circular arch problem. Its mathematical formulation is given by a system of three partial differential equations representing vertical displacement, shear, and longitudinal motion, given by,

$$\rho A\varphi_{tt} - Q_x - \ell N = \mathcal{F}_1,$$
$$\rho I\psi_{tt} - M_x + Q = \mathcal{F}_2,$$
$$\rho A\omega_{tt} - N_x + \ell Q = \mathcal{F}_3,$$

where M is the bending moment, N is the axial force and Q is the shear force. $\mathcal{F}_i,\ i = 1, 2, 3,$ are external sources. The coefficient ρ is the density of the beam, and $\ell = \frac{1}{R}$, where R is the radius of the arch. The functions φ, ψ and ω depending on $(x, t) \in (0, L) \times (0, T)$ and describes, respectively, the vertical displacement, shear angle, and longitudinal displacements.

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We take

$$
\begin{aligned}
M(x,\,t) &= EI\psi_x,\\
N(x,\,t) &= EA(\omega_x - \ell\varphi),\\
Q(x,\,t) &= kAG(\varphi_x + \psi + \ell\omega),
\end{aligned}
$$

where E, G, k, A, and I are the modulus of elasticity, the shear modulus, the shear factor, the cross-sectional area, and the moment of inertia of the cross-section.

We consider logarithmic source

$$
\begin{aligned}
\mathcal{F}_1(x,t) &= \mu_1\,\varphi \ln|\varphi|^2_{\mathbb{R}},\\
\mathcal{F}_2(x,t) &= \mu_2\,\psi \ln|\psi|^2_{\mathbb{R}},\\
\mathcal{F}_3(x,t) &= \mu_3\,\omega \ln|\omega|^2_{\mathbb{R}},
\end{aligned}
$$

where $\mu_j > 0$, $j = 1, 2, 3$ and $|\cdot|_{\mathbb{R}}$ denotes the absolute value of a real number. The physical setting is represented in Figure 1.

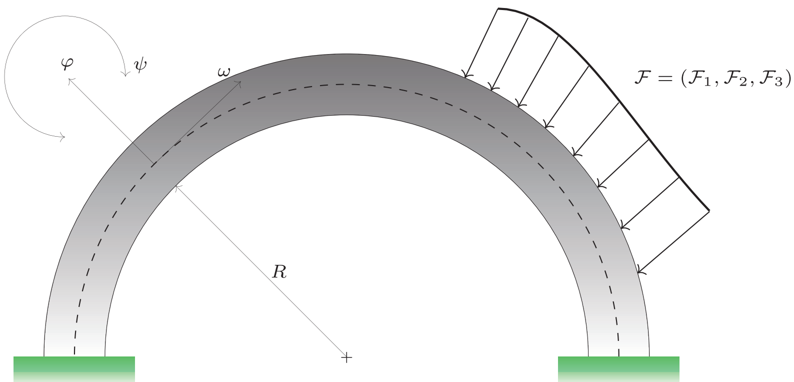

Figure 1 The circular arch beam.

We are interested in studying the competition between internal damping and the logarithmic source. To simplify the notation let us denote by $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = kAG$, $b = EI$, and $\kappa_0 = EA$. Under these conditions, we get the following initial-boundary problem

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + \ell\omega)_x - \ell\kappa_0(\omega_x - \ell\varphi) + \gamma_1\varphi_t = \mu_1\,\varphi \ln|\varphi|^2_{\mathbb{R}} \quad \text{in } (0,\,L)\times(0,\,\infty), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + \ell\omega) + \gamma_2\psi_t = \mu_2\,\psi \ln|\psi|^2_{\mathbb{R}} \quad \text{in } (0,\,L)\times(0,\,\infty), \tag{1.2}$$

$$\rho_1\omega_{tt} - \kappa_0(\omega_x - \ell\varphi)_x + \ell\kappa(\varphi_x + \psi + \ell\omega) + \gamma_3\omega_t = \mu_3\,\omega \ln|\omega|^2_{\mathbb{R}} \quad \text{in } (0,\,L)\times(0,\,\infty), \tag{1.3}$$

$$\varphi(x,\,0) = \varphi_0(x), \quad \varphi_t(x,\,0) = \varphi_1(x), \quad x \in (0,\,L), \tag{1.4}$$

$$\psi(x,\,0) = \psi_0(x), \quad \psi_t(x,\,0) = \psi_1(x), \quad x \in (0,\,L), \tag{1.5}$$

$$\omega(x,\,0) = \omega_0(x), \quad \omega_t(x,\,0) = \omega_1(x), \quad x \in (0,\,L), \tag{1.6}$$

$$\varphi(0,\,t)=\varphi(L,\,t)=\psi(0,\,t)=\psi(L,\,t)=\omega(0,\,t)=\omega(L,\,t)=0,\quad t\geq 0. \tag{1.7}$$

Shear deformation effects were first introduced by Rankine [28] in 1858. Rotary inertia effects were discovered independently by Bresse [9] in 1859, and Rayleigh [29] in 1945. One contributor to developing the theory that considers both effects was Paul Ehrenfest, cited by Timoshenko [31] in the footnote of his book in 1916. Nowadays, this celebrated Timoshenko's theory is due to pioneer work [32] of 1921. For more detailed historical context, see [13–15] with references therein.

The internal damping is associated with an oscillating system and produces a loss of energy to overcome external sources that act in the mechanical resistance of the material. Logarithmic nonlinearity is a class of nonlinearities distinguished by several interesting physical properties. It appears, for instance, in dynamics of Q-ball in theoretical physics (see [18]), theories of quantum gravity (see [35]), inflationary models (see [6]), quantum mechanics (see [8]).

There are several studies on this competition, that is, stability analysis of the global solution taking into account the effect provoked by the presence of both stabilizing mechanism and source term. Below we cite a few. [12] studied the existence and exponential stability of the global solution to a Klein-Gordon equation of Kirchhoff-Carrier type with a strong damping and logarithmic source term. An extensible beam equation of Kirchhoff type with internal damping and source term was investigated in [26]. Kirchhoff plate equations with internal damping and logarithmic nonlinearity were considered in [25]. The general decay result for a plate equation with nonlinear damping and a logarithmic source term was established in [3]. For global solution and blow-up of logarithmic Klein-Gordon equation, see [34]. The global existence and asymptotic behavior of a Timoshenko system with internal damping and logarithmic source terms were considered in [11].

About Bresse beams, below we gather some results in the literature intending to awaken the reader to the importance of the subject. [16] showed the exponential stability of the Bresse system with temperature taking into account that speeds of the wave propagation in the three equations of the system are equal. [23] studied the energy decay rate of the Bresse system with one locally internal distributed dissipation law acting on the equation about the shear angle displacement. Under the equal speed wave propagation condition, it was shown that the system is exponentially stable. For non-equal speed waves, it was established a polynomial energy decay rate. [2] proved the stability of the Bresse system with one discontinuous local internal KelvinVoigt damping on the first equation of the system. [4] considered a one-dimensional linear Bresse system with only one infinite memory acting in the second equation of the system and proved the asymptotic stability. [7] studied uniform and weak stability of the Bresse system with one infinite memory in the shear angle displacements.

Motivated by the above research, in this paper, we prove the global existence and obtain the exponential decay of solution. Furthermore, we develope a numerical algorithm to obtain the numerical solution to the system.

This paper is organized as follows: In the next section, we will give some preliminaries. Section 3 deals with the potential well theory introduced by Payne and Sattinger [24] and Sattinger [30], and we introduce the stability set. In Section 4, we prove the existence of a global solution. In Section 5, we study exponential decay. Finally, Section 6 is devoted to the numerical approach.

2 Preliminaries

We denote $L^2(0, L)$ the Hilbert's space of square-integrable function on the interval $(0, L)$, with the inner product

$$(u, v) = \int_0^L uv\mathrm{d}x, \quad \forall u, v \in L^2(0, L)$$

and norm

$$|u|^2 = (u, u), \quad \forall u \in L^2(0, L).$$

We use the notation and properties of Sobolev space as in [1]. We denote

$$H^1(0, L) = \{u : \; u \in L^2(0, L), \; u_x \in L^2(0, L)\}$$

and

$$H_0^1(0, L) = \{u \in H^1(0, L) : \; u(0) = u(L) = 0\}.$$

In this section, we present some results needed for to obtain our results. We start defining the energy functional associated with the problem (1.1)–(1.7),

$$\begin{aligned} E(t) =& \frac{1}{2}\Big[\rho_1|\varphi_t(t)|^2 + \rho_2|\psi_t(t)|^2 + \rho_3|\omega_t(t)|^2 + \kappa|\varphi_x(t) + \psi(t) + \ell\omega(t)|^2 \\ &+ \; b|\psi_x(t)|^2 + \kappa_0|\omega_x - \ell\varphi|^2 + \mu_1|\varphi(t)|^2 + \mu_2|\psi(t)|^2 + \mu_3|\omega(t)|^2 \\ &- \mu_1\int_0^L \varphi^2(t)\, \ln|\varphi(t)|_{\mathbb{R}}^2\mathrm{d}x - \mu_2\int_0^L \psi^2(t)\, \ln|\psi(t)|_{\mathbb{R}}^2\mathrm{d}x \\ &- \mu_3\int_0^L \omega^2(t)\, \ln|\omega(t)|_{\mathbb{R}}^2\mathrm{d}x\Big]. \end{aligned} \tag{2.1}$$

Direct differentiation of (2.1) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\gamma_1|\varphi_t(t)|^2 - \gamma_2|\psi_t(t)|^2 - \gamma_3|\omega_t(t)|^2. \tag{2.2}$$

Now, consider the following lemmas.

Lemma 2.1 (Sobolev-Poincaré inequality) *Let p be a number with in $2 < p < \infty$ if $n = 1, 2$ or $2 \le p \le \frac{2n}{n-2}$ if $n \ge 3$, then there exist a constant $C > 0$ such that*

$$\|u\|_p \le C|u_x|, \quad \forall u \in H_0^1(0, L). \tag{2.3}$$

Lemma 2.2 (Aubin-Lions compactness Theorem) (see [20, Theorem 5.1]) *Let $T > 0$, $1 < p_0, p_1 < \infty$. Consider $B_0 \subset B \subset B_1$ Banach spaces, where B_0, B_1 reflexive, B_0 with compact embedding in B. Define*

$$W = \{u : \ u \in L^{p_0}(0, T; B_0), u_t \in L^{p_1}(0, T; B_1)\}$$

equipped with the norm $\|u\|_W = \|u\|_{L^{p_0}(0, T; B_0)} + \|u_t\|_{L^{p_1}(0, T; B_1)}$. Then, W has compact embedding in $L^{p_0}(0, T; B)$.

Lemma 2.3 (see [20, lemma 1.3]) *Let $Q = \Omega \times (0, T)$, $T > 0$ be a bounded open set of $\mathbb{R}^n \times \mathbb{R}$ and $g_m, g : Q \to \mathbb{R}$ be functions of $L^p(0, T; L^p(\Omega)) = L^p(Q)$, $1 < p < \infty$ such that $\|g_m\|_{L^p(Q)} \leq C$, $g_m \to g$ a.e. in Q. Then $g_m \rightharpoonup g$ in $L^p(Q)$ as $m \to \infty$.*

Lemma 2.4 (Nakao's lemma) (see [21]) *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\sup_{t \leq s \leq t+1} \text{ess}\ \phi(s) \leq C_0[\phi(t) - \phi(t+1)]$$

for any $t \geq 0$, where C_0 is a positive constant. Then,

$$\phi(t) \leq C\mathrm{e}^{-\alpha t}, \quad \forall\ t \geq 0,$$

where C and α are positive constants.

3 The Potential Well

In this section, we present the potential well corresponding to the equations (1.1)–(1.2). We define the operator $J : (H_0^1(0, L))^3 \to \mathbb{R}$ by

$$\begin{aligned} J(\varphi, \psi, \omega) \overset{\text{def}}{=} \frac{1}{2}\Big[&\kappa|\varphi_x + \psi + \ell\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - \ell\varphi|^2 + \mu_1|\varphi|^2 + \mu_2|\psi|^2 + \mu_3|\omega|^2 \\ &- \mu_1 \int_0^L \varphi^2 \ln|\varphi|_{\mathbb{R}}^2 \mathrm{d}x - \mu_2 \int_0^L \psi^2 \ln|\psi|_{\mathbb{R}}^2 \mathrm{d}x - \mu_3 \int_0^L \omega^2 \ln|\omega|_{\mathbb{R}}^2 \mathrm{d}x\Big]. \end{aligned}$$

For $(\varphi, \psi, \omega) \in (H_0^1(0, L))^3$ and $\lambda > 0$ we have

$$\begin{aligned} &J(\lambda\varphi, \lambda\psi, \lambda\omega) \\ \overset{\text{def}}{=}& \frac{\lambda^2}{2}\Big[\kappa|\varphi_x+\psi+\ell\omega|^2+b|\psi_x|^2 + \kappa_0|\omega_x-\ell\varphi| + \mu_1|\varphi|^2 + \mu_2|\psi|^2 + \mu_3|\omega|^2 - 2\mu_1 \ln\lambda\int_0^L \varphi^2 \mathrm{d}x \\ &- \mu_1 \int_0^L \varphi^2 \ln|\varphi|_{\mathbb{R}}^2 \mathrm{d}x - 2\mu_2 \ln\lambda \int_0^L \psi^2 \mathrm{d}x - \mu_2 \int_0^L \psi^2 \ln|\psi|_{\mathbb{R}}^2 \mathrm{d}x - 2\mu_3 \ln\lambda \int_0^L \omega^2 \mathrm{d}x \\ &- \mu_3 \int_0^L \omega^2 \ln|\omega|_{\mathbb{R}}^2 \mathrm{d}x\Big]. \end{aligned}$$

Associated with J, we have the well-known Nehari manifold

$$\mathcal{N} \overset{\text{def}}{=} \Big\{(\varphi, \psi, \omega) \in (H_0^1(0, L))^3/\{0\};\ \Big[\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda\varphi, \lambda\psi, \lambda\omega)\Big]_{\lambda=1} = 0\Big\}.$$

Equivalently,

$$\mathcal{N} = \Big\{(\varphi,\, \psi,\, \omega) \in (H_0^1(0,\, L))^3; \kappa|\varphi_x + \psi + \ell\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - \ell\varphi|^2 = \mu_1 \int_0^L \varphi^2 \ln|\varphi|_{\mathbb{R}}^2 \mathrm{d}x + \mu_2 \int_0^L \psi^2 \ln|\psi|_{\mathbb{R}}^2 \mathrm{d}x + \mu_3 \int_0^L \omega^2 \ln|\omega|_{\mathbb{R}}^2 \mathrm{d}x \Big\}.$$

We define, as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [5],

$$d \stackrel{\text{def}}{=} \inf_{(\varphi,\, \psi,\, \omega)\in(H_0^1(0,\, L))^3/\{0\}} \sup_{\lambda>0} J(\lambda u).$$

According to Willem [33] and Theorem 4.2, the depth of the well d is a strictly positive constant given by

$$0 < d = \inf_{\varphi,\, \psi,\, \omega\in\mathcal{N}} J(\lambda u).$$

Now, we introduce

$$W = \{(\varphi,\, \psi,\, \omega) \in H_0^1(0,\, L)^3;\; J(\varphi,\, \psi,\, \omega) < d\} \cup \{0\}$$

and a partition of this into two sets as follows

$$W_1 = \Big\{(\varphi,\, \psi,\, \omega) \in W;\; \kappa|\varphi_x + \psi + \ell\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - \ell\varphi| > \mu_1 \int_0^l \varphi^2 \ln|\varphi|_{\mathbb{R}}^2 \mathrm{d}x + \mu_2 \int_0^l \psi^2 \ln|\psi|_{\mathbb{R}}^2 \mathrm{d}x + \mu_3 \int_0^l \omega^2 \ln|\omega|_{\mathbb{R}}^2 \mathrm{d}x \Big\} \cup \{0\}$$

and

$$W_2 = \Big\{(\varphi,\, \psi,\, \omega) \in W; \kappa|\varphi_x + \psi + \ell\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - \ell\varphi| < \mu_1 \int_0^l \varphi^2 \ln|\varphi|_{\mathbb{R}}^2 \mathrm{d}x + \mu_2 \int_0^l \psi^2 \ln|\psi|_{\mathbb{R}}^2 \mathrm{d}x + \mu_3 \int_0^l \omega^2 \ln|\omega|_{\mathbb{R}}^2 \mathrm{d}x \Big\}.$$

Then, we define by W_1 the set of stability for the problem (1.1)–(1.7).

4 Existence of Global Weak Solutions

In this section we prove the existence of global weak solutions.

Theorem 4.1 *Let* $(\varphi_0, \psi_0, \omega_0) \in W_1, J(\varphi_0, \psi_0, \omega_0) < d$ *and* $(\varphi_1, \psi_1, \omega_1) \in (L^2(0, L))^3$. *Then the problem* (1.1)–(1.7) *admits a weak solution* (φ, ψ, ω) *in the class*

$$(\varphi, \psi, \omega) \in (L^\infty_{\text{loc}}(0, \infty; H_0^1(0, L)))^3, \tag{4.1}$$

$$(\varphi_t, \psi_t, \omega_t) \in (L^\infty_{\text{loc}}(0, \infty; L^2(0, L)))^3 \tag{4.2}$$

satisfying $u, y, z \in H_0^1(0, L)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_1\varphi_t(t),u)+(\kappa(\varphi_x+\psi+\ell\omega)(t),u_x)$$
$$-(\kappa_0\ell(\omega_x-\ell\varphi)(t),u)+(\gamma_1\varphi_t(t),u)-(\mu_1\varphi(t)\ln|\varphi(t)|^2_{\mathbb{R}},u)=0, \tag{4.3}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_2\psi_t(t),y)+(b\psi_x(t),y_x)+(\kappa(\varphi_x+\psi+\ell\omega)(t),y)$$
$$+(\gamma_2\psi_t(t),y)-(\mu_2\psi(t)\ln|\psi(t)|^2_{\mathbb{R}},y)=0, \tag{4.4}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_1\omega_t(t),z)+(\kappa_0(\omega_x-\ell\varphi)(t),z_x)+(\kappa\ell(\varphi_x+\psi+\ell\omega)(t),z)$$
$$+(\gamma_3\omega_t(t),z)-(\mu_3\omega(t)\ln|\omega(t)|^2_{\mathbb{R}},z)=0, \tag{4.5}$$
$$(\varphi,\psi,\omega)(x,0)=(\varphi_0,\psi_0,\omega_0), \tag{4.6}$$
$$(\varphi_t,\psi_t,\omega_t)(x,0)=(\varphi_1,\psi_1,\omega_1) \tag{4.7}$$

in $\mathcal{D}'(0,T)$.

We use the Faedo-Galerkin's method. The proof of the global existence of solutions will be made in three steps: Approximated problem, a priori estimates, and passage to the limit.

4.1 Approximated problem

Let $(u_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H^1_0(0,L)$ from the eigenvectors of the operator $-\Delta$, and

$$V_m=\mathrm{span}\{u_1,u_2,\cdots,u_m\}.$$

Consider

$$\varphi^m(t)=\sum_{j=1}^m g_{jm}(t)u_j,\quad \psi^m(t)=\sum_{j=1}^m h_{jm}(t)u_j,\quad \omega^m(t)=\sum_{j=1}^m l_{jm}(t)u_j$$

a solution of the approximated problem

$$(\rho_1\varphi^m_{tt}(t),u)+(\kappa(\varphi^m_x(t)+\psi^m(t)+\ell\omega^m(t)),u_x)$$
$$-(\kappa_0\ell(\omega^m_x-\ell\varphi^m)(t),u)+(\gamma_1\varphi^m_t(t),u)-(\mu_1\varphi^m(t)\ln|\varphi^m(t)|^2,u)=0, \tag{4.8}$$
$$(\rho_2\psi^m_{tt}(t),y)+(b\psi^m_x(t),y_x)+(\kappa(\varphi^m_x(t)+\psi^m(t)+\ell\omega^m(t)),y)$$
$$+(\gamma_2\psi^m_t(t),y)-(\mu_2\psi^m(t)\ln|\psi^m(t)|^2,y)=0, \tag{4.9}$$
$$(\rho_1\omega^m_{tt}(t),z)+(\kappa_0(\omega^m_x-\ell\varphi^m)(t),z_x)+(\kappa\ell(\varphi^m_x(t)+\psi^m(t)+\ell\omega^m(t)),z)$$
$$+(\gamma_3\omega^m_t(t),z)-(\mu_3\omega^m(t)\ln|\omega^m(t)|^2,z)=0, \tag{4.10}$$
$$(\varphi^m(0),\psi^m(0),\omega^m(0))=(\varphi_{0m},\psi_{0m},\omega_{0m})\to(\varphi_0,\psi_0,\omega_0)\text{ strongly in }(H^1_0(0,l))^3, \tag{4.11}$$
$$(\varphi^m_t(0),\psi^m_t(0),\omega^m_t(0))=(\varphi_{1m},\psi_{1m},\omega_{1m})\to(\varphi_1,\psi_1,\omega_1)\text{ strongly in }(L^2(0,l))^3, \tag{4.12}$$

$\forall\ u,y,z\in V_m$. By virtue of Carathéodory's theorem (see [10]), the system (4.8) has a local solution in $[0,t_m)$, $0<t_m\le T$. The extension of the solution to the whole interval $[0,T]$ is a consequence of the following a priori estimates.

4.2 A priori estimates

Let $u=\varphi_t^m(t)$, $y=\psi_t^m(t)$ and $z=\omega_t^m(t)$ in (4.8), (4.9) and (4.10), respectively. Then we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho_1|\varphi_t^m(t)|^2+\rho_2|\psi_t^m(t)|^2+\rho_1|\omega_t^m(t)|^2+\kappa|\varphi_x^m(t)+\psi^m(t)+\ell\omega^m(t)|^2\\&+b|\psi_x^m(t)|^2+\kappa_0|\omega_x^m(t)-\ell\varphi^m(t)|^2+\mu_1|\varphi^m(t)|^2+\mu_2|\psi^m(t)|^2\\&+\mu_3|\omega^m(t)|^2-\mu_1\int_0^L\varphi^m(t)^2\ln|\varphi^m(t)|_{\mathbb{R}}^2\mathrm{d}x\\&-\mu_2\int_0^L\psi^m(t)^2\ln|\psi^m(t)|_{\mathbb{R}}^2\mathrm{d}x-\mu_3\int_0^L\omega^m(t)^2\ln|\omega^m(t)|_{\mathbb{R}}^2\mathrm{d}x\Big]\\&+\gamma_1|\varphi_t^m(t)|^2+\gamma_2|\psi_t^m(t)|^2+\gamma_3|\omega_t^m(t)|^2=0.\end{aligned}$$

From (2.1) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t)+\gamma_1|\varphi_t^m(t)|^2+\gamma_2|\psi_t^m(t)|^2+\gamma_3|\omega_t^m(t)|^2=0, \tag{4.13}$$

where $E_m(t)$ is the approximated energy of the problem (4.8). Now, integrating (4.13) from 0 to t, $0\le t\le t_m$, we obtain

$$E_m(t)+\gamma_1\int_0^t|\varphi_t^m(t)|^2\mathrm{d}s+\gamma_2\int_0^t|\psi_t^m(s)|^2\mathrm{d}s+\gamma_3\int_0^t|\omega_t^m(s)|^2\mathrm{d}s=E_m(0). \tag{4.14}$$

Thus

$$\begin{aligned}&E_m(t)+\gamma_1\int_0^t|\varphi_t^m(s)|^2\mathrm{d}s+\gamma_2\int_0^t|\psi_t^m(s)|^2\mathrm{d}s+\gamma_3\int_0^t|\omega_t^m(s)|^2\mathrm{d}s\\&=\rho_1|\varphi_{1m}|^2+\rho_2|\psi_{1m}|^2+\rho_1|\omega_{1m}|^2\\&\quad+\kappa|\varphi_{0mx}+\psi_{0m}+\ell\omega_{0m}|^2+\kappa_0|\omega_{0mx}-\ell\varphi_{0m}|^2\\&\quad+b|\psi_{0mx}|^2+\mu_1|\varphi_{0m}|^2+\mu_2|\psi_{0m}|^2+\mu_3|\omega_{0m}|^2\\&\quad-\mu_1\int_0^L\varphi_{0m}^2\ln|\varphi_{0m}|_{\mathbb{R}}^2\mathrm{d}x-\mu_2\int_0^L\psi_{0m}^2\ln|\psi_{0m}|_{\mathbb{R}}^2\mathrm{d}x-\mu_3\int_0^L\omega_{0m}^2\ln|\omega_{0m}|_{\mathbb{R}}^2\mathrm{d}x,\end{aligned}$$

which gives us the following estimate

$$\begin{aligned}&E_m(t)+\gamma_1\int_0^t|\varphi_t^m(s)|^2\mathrm{d}s+\gamma_2\int_0^t|\psi_t^m(s)|^2\mathrm{d}s+\gamma_3\int_0^t|\omega_t^m(s)|^2\mathrm{d}s\\&\le\rho_1|\varphi_{1m}|^2+\rho_2|\psi_{1m}|^2+\rho_1|\omega_{1m}|^2+J(\varphi_{0m},\psi_{0m},\omega_{0m}).\end{aligned}$$

We have that $J(\varphi_{0m},\psi_{0m},\omega_{0m})<d$, then by (4.8) we get

$$E_m(t)+\mu_1\int_0^t|\varphi_t^m(s)|^2\mathrm{d}s+\mu_2\int_0^t|\psi_t^m(s)|^2\mathrm{d}s+\gamma_3\int_0^t|\omega_t^m(s)|^2\mathrm{d}s\le C_1, \tag{4.15}$$

where C_1 is a positive constant independent of m and t.

These estimates imply that the approximated solution $(\varphi^m,\psi^m,\omega^m)$ exists globally in $[0,\infty)$ (see [17]). Then by estimate (4.15) we have

$$(\varphi^m),(\psi^m),(\omega^m)\ \text{ are bounded in }\ L^\infty_{\mathrm{loc}}(0,T;H_0^1(0,L)), \tag{4.16}$$

$$(\varphi_t^m), (\psi_t^m), (\omega_t^m) \text{ are bounded in } L^\infty_{\rm loc}(0,T;L^2(0,L)). \tag{4.17}$$

Now by the logarithmic inequality

$$|t^2 \ln t| \le C(1+|t|^3),$$

we get

$$\begin{aligned}
&\mu_1 \int_0^L |\varphi^m(t)\ln|\varphi^m(t)|_{\mathbb{R}}^2|^2 \mathrm{d}x\\
&= 4\mu_1 \int_0^L |\varphi^m(t)|_{\mathbb{R}}^2 \ln|\varphi^m(t)|_{\mathbb{R}}^2 \mathrm{d}x\\
&= 4\mu_1 \int_{x\in(0,L);|\varphi^m|<1} |\varphi^m(t)|_{\mathbb{R}}^2 \ln|\varphi^m(t)|^2 \mathrm{d}x + 4\mu_1 \int_{x\in(0,L);|\varphi^m|\ge 1} |\varphi^m(t)|_{\mathbb{R}}^2 \ln|\varphi^m(t)|^2 \mathrm{d}x\\
&\le 4\mu_1 \int_0^L |\varphi^m(t)|_{\mathbb{R}}^2 \mathrm{d}x + 4\mu_1 \int_0^L |\varphi^m(t)|_{\mathbb{R}}^4 \ln|\varphi^m(t)|_{\mathbb{R}}^2 \mathrm{d}x\\
&\le 4\mu_1 |\varphi^m(t)|^2 + 4\mu_1 C \int_0^L (1+|\varphi^m(t)|_{\mathbb{R}}^6) \mathrm{d}x\\
&= 4\mu_1 |\varphi^m(t)|^2 + 4\mu_1 CL + C|\varphi^m(t)|_2^6\\
&\le \mu_1 |\varphi^m(t)|^2 + CL + C|\varphi^m(t)|^6 \le \widetilde{C}_1.
\end{aligned} \tag{4.18}$$

Analogously we have

$$\mu_2 \int_0^L |\psi^m(t)\ln|\psi^m(t)|_{\mathbb{R}}^2|^2 \mathrm{d}x \le \widetilde{C}_2, \tag{4.19}$$

$$\mu_3 \int_0^L |\omega^m(t)\ln|\omega^m(t)|_{\mathbb{R}}^2|^2 \mathrm{d}x \le \widetilde{C}_3, \tag{4.20}$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$ are constant independent of m and t. From (4.18)–(4.20), we get

$$\varphi^m \ln|\varphi|_{\mathbb{R}}^2 \text{ are bounded in } L^2_{\rm loc}(0,\infty;L^2(0,L)), \tag{4.21}$$

$$\psi^m \ln|\psi|_{\mathbb{R}}^2 \text{ are bounded in } L^2_{\rm loc}(0,\infty;L^2(0,L)), \tag{4.22}$$

$$\omega^m \ln|\omega|_{\mathbb{R}}^2 \text{ are bounded in } L^2_{\rm loc}(0,\infty;L^2(0,L)). \tag{4.23}$$

4.3 Passage to the limit

From estimates (4.16)–(4.17), there exists a subsequence of (φ^m), (ψ^m) and (ω^m) also denoted by (φ^m), (ψ^m) and (ω^m), such that

$$(\varphi^m), (\psi^m), (\omega^m) \overset{*}{\rightharpoonup} \varphi, \psi, \omega \quad \text{weakly star in } L^\infty_{\rm loc}(0,\infty;H_0^1(0,L)), \tag{4.24}$$

$$(\varphi_t^m), (\psi_t^m), (\omega_t^m) \overset{*}{\rightharpoonup} \varphi_t, \psi_t, \omega_t \text{ weakly in } L^\infty_{\rm loc}(0,\infty;L^2(0,L)). \tag{4.25}$$

Applying the Aubin-Lions compactness theorem (see Lemma 2.2), we get from (4.24)–(4.25),

$$(\varphi^m), (\psi^m), (\omega^m) \to \varphi, \psi, \omega \text{ strongly in } L^2_{\rm loc}(0,\infty;L^2(0,L)), \tag{4.26}$$

and, for all $T > 0$,

$$(\varphi^m) \to \varphi \quad \text{a.e in } (0,L)\times(0,T), \tag{4.27}$$

$$(\psi^m) \to \psi \quad \text{a.e in } (0,L)\times(0,T), \tag{4.28}$$

$$(\omega^m) \to \omega \quad \text{a.e in } (0,L)\times(0,T). \tag{4.29}$$

Now, since that $f(s) = s\ln|s|^2$ is continuous, we have the convergence

$$\mu_1\varphi^m \ln|\varphi^m|^2_{\mathbb{R}} \to \mu_1\varphi\ln|\varphi|^2_{\mathbb{R}} \quad \text{a.e in } (0,L)\times(0,T), \tag{4.30}$$

$$\mu_2\psi^m \ln|\psi^m|^2_{\mathbb{R}} \to \mu_2\psi\ln|\psi|^2_{\mathbb{R}} \quad \text{a.e in } (0,L)\times(0,T), \tag{4.31}$$

$$\mu_3\omega^m \ln|\omega^m|^2_{\mathbb{R}} \to \mu_3\omega\ln|\omega|^2_{\mathbb{R}} \quad \text{a.e in } (0,L)\times(0,T). \tag{4.32}$$

From (4.21)–(4.23), (4.30)–(4.32) using the Lions's lemma (Lemma 2.3), we obtain

$$\mu_1\varphi^m \ln|\varphi^m|^2_{\mathbb{R}} \rightharpoonup \mu_1\varphi\ln|\varphi|^2_{\mathbb{R}} \quad \text{weakly in } L^2_{\text{loc}}(0,\infty;L^2(0,L)), \tag{4.33}$$

$$\mu_2\psi^m \ln|\psi^m|^2_{\mathbb{R}} \rightharpoonup \mu_2\psi\ln|\psi|^2_{\mathbb{R}} \quad \text{weakly in } L^2_{\text{loc}}(0,\infty;L^2(0,L)), \tag{4.34}$$

$$\mu_3\omega^m \ln|\omega^m|^2_{\mathbb{R}} \rightharpoonup \mu_3\omega\ln|\omega|^2_{\mathbb{R}} \quad \text{weakly in } L^2_{\text{loc}}(0,\infty;L^2(0,L)). \tag{4.35}$$

By the convergences (4.16)–(4.17) and (4.30)–(4.32), we can pass to the limit in the approximate system (4.8)–(4.10) and obtain that for all $u, y, z \in H^1_0(0,L)$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\rho_1\varphi_t(t),u) + (\kappa(\varphi_x+\psi+\ell\omega)(t),u_x)\\ &-(\kappa_0\ell(\omega_x-\ell\varphi)(t),u) + (\gamma_1\varphi_t(t),u) - (\mu_1\varphi(t)\ln|\varphi(t)|^2_{\mathbb{R}},u) = 0,\end{aligned} \tag{4.36}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\rho_2\psi_t(t),y) + (b\psi_x(t),y_x) + (\kappa(\varphi_x+\psi+\ell\omega)(t),y)\\ &+(\gamma_2\psi_t(t),y) - (\mu_2\psi(t)\ln|\psi(t)|^2_{\mathbb{R}},y) = 0,\end{aligned} \tag{4.37}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\rho_1\omega_t(t),z) + (\kappa_0(\omega_x-\ell\varphi)(t),z_x) + (\kappa\ell(\varphi_x+\psi+\ell\omega)(t),z) + (\gamma_3\omega_t(t),z)\\ &-(\mu_3\omega(t)\ln|\omega(t)|^2_{\mathbb{R}},z) = 0\end{aligned} \tag{4.38}$$

in $\mathcal{D}'(0,T)$.

The verification of the initial data is obtained in a standard way.

5 Exponential Decay

In this section, we provide the exponential decay of the energy associated with the system solution (1.1)–(1.7).

Theorem 5.1 *Under the hypothesis of Theorem* 4.1. *The energy associated to problem* (1.1)–(1.7) *satisfies*

$$E(t) \le C_0\mathrm{e}^{-\alpha t}, \quad \forall t \ge 0,$$

where C_0 and α are positive constants.

Proof Let $u = \varphi_t(t)$, $y = \psi_t(t)$ and $z = \omega_t(t)$ in (4.36) and (4.38), respectively, and summing up the result, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho_1|\varphi_t(t)|^2 + \rho_2|\psi_t(t)|^2 + \rho_1|\omega_t(t)|^2 + \kappa|\varphi_x(t) + \psi(t) + \ell\omega(t)|^2 \\ &+ b|\psi_x(t)|^2 + \kappa_0|\omega_x(t) - \ell\varphi(t)|^2 + \mu_1|\varphi(t)|^2 + \mu_2|\psi(t)|^2 + \mu_3|\omega(t)|^2 \\ &- \mu_1\int_0^L |\varphi(t)|_{\mathbb{R}}^2 \ln|\varphi(t)|_{\mathbb{R}}^2 \mathrm{d}x - \mu_2\int_0^L |\psi(t)|_{\mathbb{R}}^2 \ln|\psi(t)|_{\mathbb{R}}^2 \mathrm{d}x \\ &- \mu_3\int_0^L |\omega(t)|_{\mathbb{R}}^2 \ln|\omega(t)|_{\mathbb{R}}^2 \mathrm{d}x\Big] + \gamma_1|\varphi_t(t)|^2 + \gamma_2|\psi_t(t)|^2 + \gamma_3|\omega_t(t)|^2 = 0,\end{aligned} \tag{5.1}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \gamma_1|\varphi_t(t)|^2 + \gamma_2|\psi_t(t)|^2 + \gamma_3|\omega_t(t)|^2 \le 0, \tag{5.2}$$

where $E(t)$ is defined in (2.1). Integrating (5.1) from t to $t+1$, we obtain

$$\int_t^{t+1} [\gamma_1|\varphi_t(s)|^2 + \gamma_2|\psi_t(s)|^2 + \gamma_3|\omega_t(t)|^2]\mathrm{d}s \le E(t) - E(t+1) \overset{\text{def}}{:=} F^2(t), \tag{5.3}$$

therefore, there exist $t_1 \in \big[t, t+\frac{1}{4}\big]$ and $t_2 \in \big[t+\frac{3}{4}, t+1\big]$ such that

$$\gamma_1|\varphi_t(t_i)|^2 + \gamma_2|\psi_t(t_i)|^2 + \gamma_3|\omega_t(t_i)|^2 \le 4F(t_i), \quad i = 1, 2. \tag{5.4}$$

Let $u = \varphi(t)$, $y = \psi(t)$ and $z = \omega(t)$ in (4.36)–(4.38), respectively. Summing the result, we get

$$\begin{aligned}&b|\psi_x(t)|^2 + \kappa|\varphi_x(t) + \psi(t) + \ell\omega t|^2 + \kappa_0\ell|\omega_x(t) - \ell\varphi(t)|^2 - \mu_1\int_0^L (\varphi(t))^2 \ln|\varphi(t)|_{\mathbb{R}}^2 \mathrm{d}x \\ &- \mu_2\int_0^L (\psi(t))^2 \ln|\psi(t)|_{\mathbb{R}}^2 \mathrm{d}x - \mu_3\int_0^L (\omega(t))^2 \ln|\omega(t)|_{\mathbb{R}}^2 \mathrm{d}x \\ =& -\frac{\mathrm{d}}{\mathrm{d}t}\rho_1(\varphi_t(t), \varphi(t)) + \rho_1|\varphi_t(t)|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\rho_2(\psi_t(t), \psi(t)) + \rho_2|\psi_t(t)|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\rho_1(\omega_t(t), \omega(t)) \\ &+ \rho_1|\omega_t(t)|^2 - \gamma_1(\varphi_t(t), \varphi(t)) - \gamma_2(\psi_t(t), \psi(t)) - \gamma_3(\omega_t(t), \omega(t)).\end{aligned} \tag{5.5}$$

Integration (5.5) from t_1 to t_2, and using (5.4), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2} \Big[b|\psi_x(t)|^2 + \kappa|\varphi_x(t) + \psi(t) + \ell\omega(t)|^2 + \kappa_0\ell|\omega_x(t) - \ell\varphi(t)|^2 \\ &- \mu_1\int_0^L (\varphi(t))^2 \ln|\varphi(t)|_{\mathbb{R}}^2 \mathrm{d}x - \mu_2\int_0^L (\psi(t))^2 \ln|\psi(t)|_{\mathbb{R}}^2 \mathrm{d}x - \mu_3\int_0^L (\omega(t))^2 \ln|\omega(t)|_{\mathbb{R}}^2 \mathrm{d}x\Big]\mathrm{d}s \\ \le& \rho_1|\varphi_t(t_1)||\varphi(t_1)| + \rho_1|\varphi_t(t_2)||\varphi(t_2)| + \rho_2|\psi_t(t_1)||\psi(t_1)| \\ &+ \rho_2|\psi_t(t_2)||\psi(t_2)| + \rho_1|\omega_t(t_1)||\omega(t_1)| + \rho_1|\omega_t(t_2)||\omega(t_2)| \\ &+ \rho_1\int_{t_1}^{t_2} |\varphi_t(s)|^2\mathrm{d}s + \rho_2\int_{t_1}^{t_2} |\psi_t(s)|^2\mathrm{d}s + \rho_1\int_{t_1}^{t_2} |\omega_t(s)|^2\mathrm{d}s + \gamma_1\int_{t_1}^{t_2} |\varphi_t(s)||\varphi(s)|\mathrm{d}s \\ &+ \gamma_2\int_{t_1}^{t_2} |\psi_t(s)||\psi(s)|\mathrm{d}s + \gamma_3\int_{t_1}^{t_2} |\omega_t(s)||\omega(s)|\mathrm{d}s,\end{aligned}$$

therefore,

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big[b|\psi_x(t)|^2 + \kappa|\varphi_x(t) + \psi(t) + \ell\omega(t)|^2 + \kappa_0\ell|\omega_x(t) - \ell\varphi(t)|^2 \\
&- \mu_1 \int_0^L (\varphi(t))^2 \ln|\varphi(t)|^2_{\mathbb{R}} \mathrm{d}x - \mu_2 \int_0^L (\psi(t))^2 \ln|\psi(t)|^2_{\mathbb{R}} \mathrm{d}x \\
&- \mu_3 \int_0^L (\omega(t))^2 \ln|\omega(t)|^2_{\mathbb{R}} \mathrm{d}x\Big] \mathrm{d}s \\
\leq\ & C_1\Big[F(t) \operatorname*{sup\,ess}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s) + \frac{1}{4} \operatorname*{sup\,ess}_{t\leq s\leq t+1} E(s) + F^2(t)\Big] \stackrel{\text{def}}{:=} G^2(t),
\end{aligned} \tag{5.6}$$

where $C_1 = C_1(\rho_1, \rho_2, \gamma_1, \gamma_2, \gamma_3) > 0$ is a constant. Now, from (5.3) and (5.6), we get

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big[b|\psi_x(t)|^2 + \kappa|\varphi_x(t) + \psi(t) + \ell\omega(t)|^2 + \kappa_0\ell|\omega_x(t) - \ell\varphi(t)|^2 \\
&- \mu_1 \int_0^L (\varphi(t))^2 \ln|\varphi(t)|^2_{\mathbb{R}} \mathrm{d}x - \mu_2 \int_0^L (\psi(t))^2 \ln|\psi(t)|^2_{\mathbb{R}} \mathrm{d}x \\
&- \mu_3 \int_0^L (\omega(t))^2 \ln|\omega(t)|^2_{\mathbb{R}} \mathrm{d}x\Big] \mathrm{d}s \\
\leq\ & 2[F^2(t) + G^2(t)],
\end{aligned} \tag{5.7}$$

thus, there exists $t^* \in [t_1, t_2]$ such that

$$\begin{aligned}
&\rho_1|\varphi_t(t^*)|^2 + \rho_2|\psi_t(t^*)|^2 + \rho_1|\omega_t(t^*)|^2 + b|\psi_x(t^*)|^2 \\
&+ \kappa|\varphi_x(t^*) + \psi(t^*) + \ell\omega(t^*)|^2 + \kappa_0\ell|\omega_x(t^*) - \ell\varphi(t^*)|^2 \\
&- \mu_1 \int_\Omega (\varphi(t^*))^2 \ln|\varphi(t^*)|^2_{\mathbb{R}} \mathrm{d}x - \mu_2 \int_0^L (\psi(t^*))^2 \ln|\psi(t^*)|^2_{\mathbb{R}} \mathrm{d}x \\
&- \mu_3 \int_0^L (\omega(t^*))^2 \ln|\omega(t^*)|^2_{\mathbb{R}} \mathrm{d}x \\
\leq\ & C_2[F^2(t) + G^2(t)].
\end{aligned} \tag{5.8}$$

We deduce

$$\begin{aligned}
&|\varphi(t^*)|^2 + |\psi(t^*)|^2 + |\omega(t^*)|^2 \\
\leq\ & C_3[|\varphi_x(t^*) + \psi(t^*) + \ell|\omega(t^*)|^2|^2 \\
&+ |\psi_x(t^*)|^2 + |\omega_x(t^*) - \ell\varphi(t^*)|^2].
\end{aligned} \tag{5.9}$$

By (5.8)–(5.9), we have

$$E(t^*) \leq C_4[F^2(t) + G^2(t)]. \tag{5.10}$$

Since $E(t)$ is increasing, by (5.3) and (5.9)–(5.10) we obtain

$$\operatorname*{sup\,ess}_{t\leq s\leq t+1} E(s) \leq E(t^*) + \int_t^{t+1} [\gamma_1|\varphi_t(s)|^2 + \gamma_2|\psi_t(s)|^2 + \gamma_3|\omega_t(s)|^2] \mathrm{d}s$$

$$\begin{aligned}&\le C_5[F^2(t)+G^2(t)]\\&\le C_6\Big[F(t)\sup_{t\le s\le t+1}\text{ess}\, E^{\frac{1}{2}}(s)+F^2(t)+\frac{1}{4}\sup_{t\le s\le t+1}\text{ess}\, E(s)\Big]\\&\le C_7F^2(t)+\frac{1}{2}\sup_{t\le s\le t+1}\text{ess}\, E(s).\end{aligned}$$

Hence, by Nakao's lemma (see Lemma 2.4),

$$\sup_{t\le s\le t+1}\text{ess}\, E(s)\le C_8F^2(t)=C_9[E(t)-E(t+1)],$$

where $C_i = 1, 2, \cdots, 9$ are positive constants. By Lemma (2.4), we conclude

$$E(t)\le C_0\mathrm{e}^{-\alpha t},\quad \forall t\ge 0,$$

where C_0 and α are positive constants.

6 Numerical Approach

In this section, we develope an algorithms numerical to obtain the numerical solution to system (1.1)–(1.7). Here, we use the Newmark's methods (see [22]).

6.1 Variational formulation

Here, we use a representation to the functions φ, ψ and ω in the form by component vectorial $\mathbf{u} = [\varphi, \psi, \omega]^{\mathrm{T}}$. Thus, from (1.1)–(1.3) we get the following variational problem

$$(\mathbf{u}_{tt}(t),\widetilde{\mathbf{u}})+a(\mathbf{u}^{\epsilon}(t),\widetilde{\mathbf{u}})+(\mathbf{u}_t(t),\widetilde{\mathbf{u}})=(\mathcal{F}(\mathbf{u}),\widetilde{\mathbf{u}}),\quad \forall\ \widetilde{\mathbf{u}}\in[H_0^1(0,L)]^3, \tag{6.1}$$

with $\mathbf{u}$ satisfying the initial conditions

$$(\mathbf{u}(0),\widetilde{\mathbf{u}})=(\mathbf{u}_0,\widetilde{\mathbf{u}}),\quad (\mathbf{u}_t(0),\widetilde{\mathbf{u}})=(\mathbf{u}_1,\widetilde{\mathbf{u}}),$$

where

$$\begin{aligned}a(\mathbf{u}(t),\widetilde{\mathbf{u}})&=\kappa(\varphi_x+\psi+\ell\omega,u_{1,x}+u_2+\ell u_3)+b(\psi_x,u_{2,x})+\kappa_0(\omega_x-\ell\varphi,u_{3,x}-\ell u_1),\\(\mathbf{u}_t(t),\widetilde{\mathbf{u}})&=\gamma_1(\varphi_t(t),u_1)+\gamma_2(\psi_t(t),u_2)+\gamma_3(\omega_t(t),u_3),\\(\mathcal{F}(\mathbf{u}(t)),\widetilde{\mathbf{u}})&=(\varphi(t)\ln|\varphi(t)|_{\mathbb{R}}^2,u_1)+(\psi(t)\ln|\psi(t)|_{\mathbb{R}}^2,u_2)+(\omega(t)\ln|\omega(t)|_{\mathbb{R}}^2,u_3)\end{aligned}$$

and

$$(\mathbf{u}_{tt}(t),\widetilde{\mathbf{u}})=\rho_1(\varphi_{tt},u_1)+\rho_2(\psi_{tt},u_2)+\rho_1(\omega_{tt},u_3).$$

6.2 Algorithms and numerical approximation

To obtain the full discretization to problem (6.1), firstly, we consider a partition X_h over the interval $\Omega = (0, L)$, that is,

$$X_h = \{0 = x_0 < x_1 < \cdots < x_N = L\}, \quad \Omega_{j+1} = (x_j, x_{j+1}),$$

being,

$$\Omega_i \cap \Omega_j = \emptyset, \quad i \neq j, \quad \Omega = \bigcup_{e=1}^{Ne} \overline{\Omega}_e,$$

where N_e is the number of the elements obtained of partition.

Let $S^h(0, L)$ be a finite-dimensional subspaces of $C(0, L)$ piecewise polinomial finite element interpolation of degree 1. Also, we consider the following finite-dimensional subspaces

$$\mathcal{H}_0^1(0, L) = S^h(0, L) \cap H_0^1(0, L).$$

Analogously to continuous case, in the finite dimensional problem we consider the functions φ^h, ψ^h and ω^h in the form by component vectorial $[\varphi^h, \psi^h, \omega^h]^{\mathrm{T}}$. Then

$$\mathbf{u}^h(t, x) = \sum_{i=1}^{3N} d_i(t)\phi_i(x),$$

where $3N$ is the numbers total of degrees of freedom of the finite element approximation to displacement and $\phi_i(x),\ i = 1, \cdots, 3N$ are the global vector interpolation functions.

Thus the semi-discrete finite approximation of the variational problem (6.1) is characterized as the following finite-dimensional problem

$$(\mathbf{u}_{tt}^h(t), \widetilde{\mathbf{u}}^h) + a(\mathbf{u}^h(t), \widetilde{\mathbf{u}}^h) + (\mathbf{u}_t^h(t), \widetilde{\mathbf{u}}^h) = (\mathcal{F}(\mathbf{u}^h), \widetilde{\mathbf{u}}^h), \quad \forall\, \widetilde{\mathbf{u}} \in [\mathcal{H}_0^1(0, L)]^3, \tag{6.2}$$

where $\mathbf{u}^h(t)$ satisfies the initial conditions

$$(\mathbf{u}^h(0), \widetilde{\mathbf{u}}^h) = (\mathbf{u}_0^h, \widetilde{\mathbf{u}}^h), \quad (\mathbf{u}_t^h(0), \widetilde{\mathbf{u}}^h) = (\mathbf{u}_1^h, \widetilde{\mathbf{u}}^h).$$

Therefore, from the finite dimensional problem (6.2) we obtain the following dynamical problem in $\mathbb{R}^{3N}$

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{K}\mathbf{d}(t) &= \mathbf{F}(\mathbf{d}(t)), \\ \mathbf{d}(0) \ = \ \mathbf{d}_0, \quad \dot{\mathbf{d}}(0) &= \dot{\mathbf{d}}_1, \end{aligned}$$

where $\mathbf{M}$ is a consistent mass matrix, $\mathbf{C}$ is a damping matrix, $\mathbf{K}$ is a vector of consistent nodal elastic stiffness at time t, and $\mathbf{F}(\mathbf{d}(t)$ is a vector of consistent nodal applied forces generalized at time t.

To solve this system above we introduce a partition P of the time domain $[0,T]$ into M intervals of lenght Δt such that $0 = t_0 < t_1 < \cdots < t_M = T$, with $t_{n+1} - t_n = \Delta t$. Considering the non-linearity in our work, follows that our numerical scheme becomes

$$\begin{aligned}
&\mathbf{M}\ddot{\mathbf{d}}_{n+1} + \mathbf{C}\dot{\mathbf{d}}_{n+1} + \mathbf{K}\mathbf{d}_{n+1} = \mathbf{F}(\mathbf{d}_{n+1}),\\
&\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t\dot{\mathbf{d}}_n + \frac{\Delta t^2}{2}[(1-2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1}],\\
&\dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_n + \Delta t[(1-\gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1}],
\end{aligned}$$

where, β and γ are parameters that govern the stability and accuracy of the method.

The matrices, from the above system, are obtained from the standard finite element method assembly (see [19]), that is,

$$\mathbf{K} = \bigcup_{e=1}^{Ne}(\mathbf{k}_Q^e + \mathbf{k}_M^e + \mathbf{k}_N^e),$$

where $\mathbf{k}_Q^e, \mathbf{k}_M^e, \mathbf{k}_N^e$ are the elementar matrices obtained by element.

For instance, considering linear functions, we have

$$\mathbf{k}_M^e = \frac{b^h}{h}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{k}_Q^e = \frac{k}{6h}\begin{bmatrix} 6 & -3h & -3h\ell & -6 & -3h & -3h\ell\\ -3h & 2h^2 & 2\ell h^2 & 3k & h^2 & \ell h^2\\ -3h\ell & 2\ell h^2 & 2\ell^2 h^2 & 3\ell h & \ell h^2 & \ell^2 h^2\\ -6 & 3h & 3\ell h & 6 & 3h & 3\ell h\\ -3h & h^2 & \ell h^2 & 3h & 2h^2 & 2\ell h^2\\ -3lh & \ell h^2 & \ell^2 h^2 & 3\ell h & 2\ell h^2 & 3\ell^2 h^2 \end{bmatrix}.$$

Due to its non-linearity we have a vector $\mathbf{F}(\mathbf{d}(t))$ with entries for each element of

$$\mathbf{F^e} = \Big[\ \mu_1\int_{\Omega_e}(\mathbf{u}^h(t))\ln|\mathbf{u}^h(t)|^2\phi_i^e\ \mathrm{d}x, \mu_2\int_{\Omega_e}(\mathbf{u}^h(t))\ln|\mathbf{u}^h(t)|^2\phi_i^e\ \mathrm{d}x, \mu_3\int_{\Omega_e}(\mathbf{u}^h(t))\ln|\mathbf{u}^h(t)|^2\phi_i^e\ \mathrm{d}x\Big]^{\mathrm{T}}.$$

These vectorial components are obtained by Gaussian-Quadrature using two points.

Remark 6.1 We point out the numerical pathology which occurs in penalized systems the locking problem, in particular to Bresse system it's the shear locking. Numerical alternatives to this problem was performed in the literature and to more details we indicate the classical reference by Hughes et al [19], Prathap and Bhashyam [27].

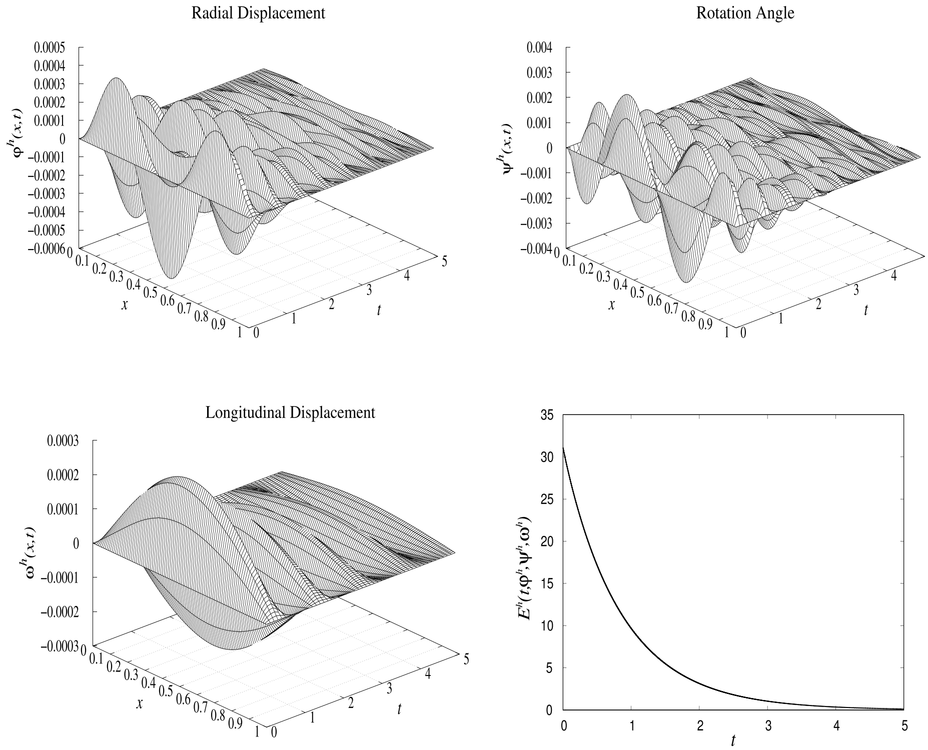

Figure 2 Evolution of the beam's numerical solution: Transversal displacement $\varphi^h(x,t)$, evolution of rotation $\psi^h(x,t)$ and evolution of the longitudinal displacement $\omega^h(x,t)$. The asymptotic behaviour of the numerical energy $E^h(t,\varphi,\psi,\omega)$ at 5 s.

Remark 6.2 To get computational results, we use the implemented code in Language C. The graphics were developed using GNUplot.

In the sequel we realize some numerical experiments to highlight our theorical results.

6.3 Numerical experiments

In our performed numerical experiments to view the asymptotic properties we consider an uniform mesh $h = 0.01$ m, $\Delta t = 10^{-5}$ s. The parameters Newmark's rules algorithms are $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$.

Experiment We consider a rectangular arch beam with $L = 1.0$ m, thickness 0.09 m, width 0.09 m, $E = 69 \cdot 10^8 \text{N/m}^2$ $\rho = 7680$ Kg/m^3, $\kappa = \frac{5}{6}, r = 0.30$ (Poisson ratio). Futhermore we have $\mu_1 = \mu_2 = \mu_3 = 1.0$, $\gamma_1 = 62.2$, $\gamma_2 = 0.42$, $\gamma_3 = 62.2$ and the following initial conditions: $\varphi(x,0) = 0$, $\varphi_t(x,0) = \sin 3\pi x, \psi(x,0) = 0$, $\psi_t(x,0) = \sin 5\pi x, \omega(x,0) = 0$, $\omega_t(x,0) = \sin \pi x$.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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