Chin. Ann. Math. Ser. B
45(5), 2024, 703–708
DOI: 10.1007/s11401-024-0035-3

Chinese Annals of Mathematics, Series B

Fixed Subgroups are not Compressed in Direct Products of Surface Groups*

Qiang ZHANG[1] Jianchun WU[2] Fanling GU[3]

Abstract By constructing counterexamples, the authors show that the fixed subgroups are not compressed in direct products of free and surface groups, and hence negate a conjecture in [Zhang, Q., Ventura, E. and J. Wu, Fixed subgroups are compressed in surface groups, *Internat. J. Algebra Comput.*, **25**, 2015, 865–887].

Keywords Fixed subgroup, Free group, Surface group, Direct product, Inertia, Compression
2000 MR Subject Classification 20F65, 20F34, 57M07

1 Introduction

For a finitely generated group G, let $\mathrm{rk}(G)$ denote the rank (i.e., the minimal number of the generators) of G, and let $\mathrm{End}(G)$ (resp. $\mathrm{Aut}(G)$) denote the set of endomorphisms (resp. automorphisms) of G. For an arbitrary family $\mathcal{B} \subseteq \mathrm{End}(G)$, the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\,\mathcal{B} := \{g \in G \mid \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi\in\mathcal{B}} \mathrm{Fix}\,\phi \leqslant G.$$

In the past fifty years, the studies on fixed subgroups of various groups were very active (see [10] for a survey and [5] for some new progress). For free groups, a celebrated result was due to Bestvina and Handel [2]. They introduced train track maps for automorphisms of free groups and then proved the famous Scott's conjecture: For any automorphism ϕ of a free group F_n of rank n, $\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant \mathrm{rk}(F_n)$. For a surface group G, i.e., the fundamental group of a connected closed surface, the earliest study of fixed subgroups may belong to Nielsen [7–8], and an analogue of Scott's conjecture was proved in [4].

Manuscript received March 11, 2023. Revised November 16, 2023.
[1]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China.
E-mail: zhangq.math@mail.xjtu.edu.cn
[2]Corresponding author. Department of Mathematics, Soochow University, Suzhou 215006, Jiangsu, China. E-mail: wujianchun@suda.edu.cn
[3]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China; School of General Education, Guangzhou College of Technology and Business, Guangzhou 510850, China.
E-mail: 321702015@qq.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12271385, 12471066) and the Shaanxi Fundamental Science Research Project for Mathematics and Physics (No. 23JSY027).

In [3], Dicks and Ventura introduced the notions of inertia and compression. A subgroup $H \leqslant G$ is said to be inert in G if $\mathrm{rk}(K \cap H) \leqslant \mathrm{rk}(K)$ for every (finitely generated) subgroup $K \leqslant G$. A subgroup $H \leqslant G$ is said to be compressed in G if $\mathrm{rk}(H) \leqslant \mathrm{rk}(K)$ for every (finitely generated) subgroup K with $H \leqslant K \leqslant G$. It is obvious that inertia means compression. Researches for these two properties can be set for any finitely generated group G and for some special subgroups, especially for fixed subgroups. In [6], Martino and Ventura showed that the fixed subgroups of any family of endomorphisms are compressed in free groups, and later, their result was extended to surface groups by Zhang, Ventura and Wu [12].

For inertia, Dicks and Ventura [3] proved the following theorem which extended Bestvina-Handel's result: The fixed subgroup $\mathrm{Fix}\,\mathcal{B}$ is inert in F_n for any family $\mathcal{B}$ of injective endomorphisms of F_n. Moreover, they conjectured (the so-called inertia conjecture) that $\mathrm{Fix}\,\mathcal{B}$ is inert in either a free or a surface group for any family of general endomorphisms. After some partial proofs, the inertia conjecture has finally been fully proved by Antolín and Jaikin-Zapirain [1] recently.

A surface group is the fundamental group of a connected closed surface. To fix the notation, we shall denote Σ_g the closed orientable surface of genus $g \geqslant 1$, and

$$S_g = \pi_1(\Sigma_g) = \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

its fundamental group. Here, we use the notation $[x, y] = xyx^{-1}y^{-1}$. And for the non-orientable case, we shall denote $N\Sigma_k$ the connected sum of $k \geqslant 1$ projective planes, and

$$NS_k = \pi_1(N\Sigma_k) = \langle a_1, a_2, \cdots, a_k \mid a_1^2 \cdots a_k^2 \rangle$$

its fundamental group.

In [12], the authors proved the following theorem.

Theorem 1.1 (see [12, Theorem 4.9]) *Let $G = G_1 \times \cdots \times G_n$ be a direct product, where each G_i is either a free group F_r, $r \geqslant 1$, or an orientable surface group S_g, $g \geqslant 1$, or a non-orientable surface group NS_k, $k \geqslant 1$. If $\mathrm{Fix}\,\phi$ is compressed in G for every $\phi \in \mathrm{Aut}(G)$, then G must be of one of the following forms:*

(euc1) $G = \mathbb{Z}^p \times \mathbb{Z}_2^q$ *for some* $p, q \geqslant 0$; *or*

(euc2) $G = NS_2 \times \mathbb{Z}_2^q$ *for some* $q \geqslant 0$; *or*

(euc3) $G = NS_2 \times \mathbb{Z}^p \times \mathbb{Z}_2$ *for some* $p \geqslant 1$; *or*

(euc4) $G = NS_2^p \times \mathbb{Z}^q$ *for some* $p \geqslant 1$, $q \geqslant 0$; *or*

(hyp1) $G = F_r \times NS_3^\ell$ *for some* $r \geqslant 2$, $\ell \geqslant 0$; *or*

(hyp2) $G = S_g \times NS_3^\ell$ *for some* $g \geqslant 2$, $\ell \geqslant 0$; *or*

(hyp3) $G = NS_k \times NS_3^\ell$ *for some* $k \geqslant 3$, $\ell \geqslant 0$.

Moreover, they have the following conjecture.

Conjecture 1.1 (see [12, Conjecture 4.11]) If G is one of the above seven types, then for any $\phi \in \mathrm{Aut}(G)$, $\mathrm{Fix}\,\phi$ is compressed in G.

In [11], Wu, Ventura and Zhang constructed a counterexample for the case (euc3), and proved that $\mathrm{Fix}\,\phi$ is compressed if G is in the cases (euc1), (euc2), (euc4). In this note, we consider the cases (hyp1), (hyp2), (hyp3), and construct counterexamples for all G having at least 5 factors, and for some G having at least 4 factors. See the next section for more details.

2 Main Result

At first, we have a lemma as follows.

Lemma 2.1 *Suppose that G is $F_r(r \geqslant 2)$, $S_g(g \geqslant 2)$ or $NS_k(k \geqslant 3)$. Then there exists $\phi \in \mathrm{Aut}(G)$ such that $\mathrm{Fix}\,\phi = \langle h \rangle$ where h is a commutator.*

Proof Suppose that the commutator subgroup G' of G is a free group of infinite rank (see [9]). Take a nontrivial element $h = aba^{-1}b^{-1} \in G'$ where a, b are distinct elements in a basis of G. Let ϕ be the inner automorphism of G defined by $\phi(g) = hgh^{-1}$. Then $\mathrm{Fix}\,\phi = \langle h \rangle$. The proof is completed.

Now, in the cases (hyp1)–(hyp3), we construct a counterexample for any G having at least 5 factors.

Proposition 2.1 *Suppose that $G = G_1 \times \cdots \times G_\ell (\ell \geqslant 5)$, each G_i is $F_r(r \geqslant 2)$, $S_g(g \geqslant 2)$ or $NS_k(k \geqslant 3)$. Then there exists $\phi \in \mathrm{Aut}(G)$ such that $\mathrm{Fix}\,\phi$ is not compressed in G.*

Proof By Lemma 2.1, for each $G_i (i = 1, 2, \cdots, 5)$, there exists $\phi_i \in \mathrm{Aut}(G_i)$ such that $\mathrm{Fix}\,\phi_i = \langle h_i \rangle$ where $h_i = [s_i, t_i] = s_i t_i s_i^{-1} t_i^{-1}$ is a commutator in G_i. Let

$$\phi = \phi_1 \times \cdots \times \phi_5 \times Id_6 \times \cdots \times Id_\ell \in \mathrm{Aut}(G),$$

where Id_j is the identity of $G_j (j = 6, \cdots, \ell)$. Then we have

$$\mathrm{Fix}\,\phi = \langle s_1 t_1 s_1^{-1} t_1^{-1} \rangle \times \cdots \times \langle s_5 t_5 s_5^{-1} t_5^{-1} \rangle \times G_6 \times \cdots \times G_\ell \cong \mathbb{Z}^5 \times G_6 \times \cdots \times G_\ell,$$

and $\mathrm{Fix}\,\phi \leqslant H = \langle s_1 s_2 s_4, t_1 t_3 t_5, t_2 s_3, s_5 t_4 \rangle \times G_6 \times \cdots \times G_\ell$. Because

$$\begin{aligned} [s_1 s_2 s_4, t_1 t_3 t_5] &= [s_1, t_1], \\ [s_1 s_2 s_4, t_2 s_3] &= [s_2, t_2], \\ [t_2 s_3, t_1 t_3 t_5] &= [s_3, t_3], \\ [s_1 s_2 s_4, s_5 t_4] &= [s_4, t_4], \\ [s_5 t_4, t_1 t_3 t_5] &= [s_5, t_5], \end{aligned}$$

we have $\mathrm{rk}(\mathrm{Fix}\,\phi) > \mathrm{rk}(H)$, and hence $\mathrm{Fix}\,\phi$ is not compressed in G. The proof is completed.

For some G having at least 4 factors, we have the follwing theorem.

Theorem 2.1 *Suppose that* $G = G_0 \times NS_3^{\ell}(\ell \geqslant 3)$, G_0 *is* $F_r(r \geqslant 2)$, $S_g(g \geqslant 2)$ *or* $NS_k(k \geqslant 4)$. *Then there exists* $\phi \in \mathrm{Aut}(G)$ *such that* $\mathrm{Fix}\,\phi$ *is not compressed in* G.

Proof Let $G = G_0 \times G_1 \times G_2 \times G_3 \times \cdots \times G_\ell(\ell \geqslant 3)$ where $G_1 \cong \cdots \cong G_\ell \cong NS_3$. By Lemma 2.1, for $i = 1, 2, 3$, there exists $\phi_i \in \mathrm{Aut}(G_i)$ such that $\mathrm{Fix}\,\phi_i = \langle h_i\rangle$ where $h_i = [s_i, t_i] = s_i t_i s_i^{-1} t_i^{-1}$ is a commutator in G_i.

Case 1 If $G_0 = F_r = \langle a_1, \cdots, a_r\rangle(r \geqslant 2)$, let

$$\phi_0 \colon G_0 \to G_0, \quad a_2 \mapsto a_2, \quad a_1 \mapsto a_1 a_2, \quad a_i \mapsto a_i, \quad i = 3, \cdots, r.$$

Then $\phi_0 \in \mathrm{Aut}(G_0)$ and $\mathrm{Fix}\,\phi_0 = \langle a_2, a_1 a_2 a_1^{-1}, a_3, \cdots, a_r\rangle$ with $\mathrm{rk}(\mathrm{Fix}\,\phi_0) = r$. Now we have

$$\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3 \leqslant H = \langle a_2 a_1 s_1 s_2, s_3 t_3 t_1 t_2, s_3 s_1 t_2, a_1 t_3 t_1 s_2, a_3, \cdots, a_r\rangle,$$

which is because

$$\begin{aligned}
&[a_2 a_1 s_1 s_2, s_3 s_1 t_2] = [s_2, t_2],\\
&[a_2 a_1 s_1 s_2, s_3 t_3 t_1 t_2] = [s_1, t_1][s_2, t_2],\\
&[a_2 a_1 s_1 s_2, a_1 t_3 t_1 s_2] = [a_2, a_1][s_1, t_1],\\
&[s_3 s_1 t_2, a_1 t_3 t_1 s_2] = [s_3, t_3][s_1, t_1][s_2, t_2]^{-1},\\
&a_2 a_1 s_1 s_2 \cdot s_3 t_3 t_1 t_2 \cdot (s_3 s_1 t_2)^{-1} \cdot (a_1 t_3 t_1 s_2)^{-1} = a_2 [s_1, t_1][s_3, t_3].
\end{aligned}$$

Note that $\mathrm{rk}(\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3) = r + 3$ while $\mathrm{rk}(H) \leqslant r + 2$.

Case 2 If $G_0 = S_g = \langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g]\rangle$, with $g \geqslant 2$, let $\phi_0 \colon G_0 \to G_0$ as follows:

$$a_1 \mapsto a_1 b_1,\ b_1 \mapsto b_1,\ a_2 \mapsto a_2 b_2,\ b_2 \mapsto b_2,\ a_i \mapsto a_i,\ b_i \mapsto b_i,\ i = 3, \cdots, g.$$

Then $\phi_0 \in \mathrm{Aut}(G_0)$ and

$$\begin{aligned}
\mathrm{Fix}\,\phi_0 &= \langle b_1, a_1 b_1 a_1^{-1}, b_2, a_2 b_2 a_2^{-1}, a_3, b_3, \cdots, a_g, b_g\rangle\\
&= \langle b_1, a_1 b_1 a_1^{-1}, b_2, a_3, b_3, \cdots, a_g, b_g\rangle
\end{aligned}$$

is a free subgroup of rank $2g - 1$ because of the defining relation for S_g. Let

$$H = \langle b_1 a_1 s_1 s_2, s_3 t_3 t_1 t_2, s_3 s_1 t_2, a_1 t_3 t_1 s_2, b_2, a_3, b_3, \cdots, a_g, b_g\rangle.$$

Then $\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3 \leqslant H$ for similar reason in Case 1. Note that

$$\mathrm{rk}(\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3) = 2g + 2,$$

while $\mathrm{rk}(H) \leqslant 2g + 1$.

Case 3 If $G_0 = NS_k = \langle a_1, a_2, \cdots, a_k \mid a_1^2 a_2^2 \cdots a_k^2 \rangle$ with $k \geqslant 4$, then G can also be presented as $\langle a, b, c, d, a_5, \cdots, a_k \mid aba^{-1}bcdc^{-1}da_5^2 \cdots a_k^2 \rangle$ (the isomorphism being $a_1 \mapsto a$, $a_2 \mapsto a^{-1}b$, $a_3 \mapsto c$, $a_4 \mapsto c^{-1}d$, $a_i \mapsto a_i$ for $i = 5, \cdots, k$). With this new presentation, let

$$\phi_0 \colon G_0 \to G_0, \quad a \mapsto ab, \quad b \mapsto b, \quad c \mapsto cd, \quad d \mapsto d, \quad a_i \mapsto a_i, \quad i = 5, \cdots, k.$$

Then $\phi_0 \in \mathrm{Aut}(G_0)$, and

$$\mathrm{Fix}\,\phi_0 = \langle b, aba^{-1}, d, cdc^{-1}, a_5, \cdots, a_k \rangle = \langle b, aba^{-1}, d, a_5, \cdots, a_k \rangle \cong F_{k-1},$$

because of the defining relation for NS_k. Let

$$H = \langle bas_1s_2, s_3t_3t_1t_2, s_3s_1t_2, at_3t_1s_2, d, a_5, \cdots, a_k \rangle.$$

Then $\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3 \leqslant H$ for similar reason in Case 1. Note that

$$\mathrm{rk}(\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3) = k + 2,$$

while $\mathrm{rk}(H) \leqslant k + 1$.

So, in any of the above three cases, we have $\mathrm{rk}(\mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3) > \mathrm{rk}(H)$. Now, let us consider the automorphism

$$\phi = \phi_0 \times \phi_1 \times \phi_2 \times \phi_3 \times Id \times \cdots \times Id \in \mathrm{Aut}(G).$$

It happens that

$$\mathrm{Fix}\,\phi = \mathrm{Fix}\,\phi_0 \times \cdots \times \mathrm{Fix}\,\phi_3 \times G_4 \times \cdots \times G_\ell \leqslant H \times G_4 \times \cdots \times G_\ell,$$

while $\mathrm{rk}(\mathrm{Fix}\,\phi) > \mathrm{rk}(H \times G_4 \times \cdots \times G_\ell)$. So $\mathrm{Fix}\,\phi$ is not compressed in G. The proof is completed.

Note that $\mathrm{Fix}\,\phi$ is inert (and hence compressed) in G for any endomorphism ϕ of a single compact surface group G (see [1]). Now for a direct product group G, by Proposition 2.1 and Theorem 2.1, solutions of the same problem unsolved are as follows.

Question 2.1 Suppose that $G = NS_3^4$, or $G = G_0 \times NS_3^\ell$ ($\ell = 1, 2$; $G_0 = F_r (r \geqslant 2)$, $S_g (g \geqslant 2)$ or $NS_k (k \geqslant 3)$). Does there exist $\phi \in \mathrm{Aut}(G)$ such that $\mathrm{Fix}\,\phi$ is not compressed in G?

Acknowledgement The authors thank the referee for valuable and detailed comments.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Antolín, Y. and Jaikin-Zapirain, A., The Hanna Neumann conjecture for surface groups, *Compos. Math.,* **158**(9), 2022, 1850–1877.

[2] Bestvina, M. and Handel, M., Train tracks and automorphisms of free groups, *Ann. Math.,* **135**, 1992, 1–51.

[3] Dicks, W. and Ventura, E., The group fixed by a family of injective endomorphisms of a free group, Contemporary Mathematics, 195, AMS, Providence , 1996.

[4] Jiang, B., Wang, S. and Zhang, Q., Bounds for fixed points and fixed subgroups on surfaces and graphs, *Alg. Geom. Topology,* **11**, 2011, 2297–2318.

[5] Lei, J. and Zhang, Q., Explicit bounds for fixed subgroups of endomorphisms of free products, *J. Algebra,* **633**, 2023, 773–787.

[6] Martino, A. and Ventura, E., Fixed subgroups are compressed in free groups, *Commun. Algebra,* **32**(10), 2004, 3921–3935.

[7] Nielsen, J., Untersuchungen zur Topologie der geschlossenen zweiseitigen Flachen, *Acta Math.,* **50**, 1927, 189–358.

[8] Nielsen, J., Untersuchungen zur Topologie der geschlossenen zweiseitigen Flachen. II., *Acta Math.,* **53**, 1929, 1–76.

[9] Putman, A., The commutator subgroups of free groups and surface groups, *Enseign. Math,* **68**(3–4), 2022, 389–408.

[10] Ventura, E., Fixed subgroups in free groups: A survey, *Contemp. Math.,* **296**, 2002, 231–255.

[11] Wu, J., Ventura, E. and Zhang, Q., Fixed subgroups in direct products of surface groups of Euclidean type, *Commun. Algebra,* **48**(7), 2020, 3003–3010.

[12] Zhang, Q., Ventura, E. and Wu, J., Fixed subgroups are compressed in surface groups, *Internat. J. Algebra Comput.,* **25**, 2015, 865–887.