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On the Turán Numbers of Linear Forests in Bipartite Graphs*

Tianying XIE[1] Longtu YUAN[2]

Abstract A linear forest is a graph consisting of paths. In this paper, the authors determine the maximum number of edges in an (m, n)-bipartite graph which does not contain a linear forest consisting of paths on at least four vertices for $n \geq m$ when m is sufficiently large.

Keywords Turán number, Linear forest, Bipartite graph
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1 Introduction

In this paper, only finite graphs without loops and multiple edges will be considered. Let K_n and P_n be the clique and path on n vertices, respectively. An even path (odd path) is a path on even (odd) number of vertices. Let $K_{m,n}$ be the complete bipartite graph with two parts of size m and n. A linear forest is a forest whose components are paths. For a given graph $G = (V(G), E(G))$, if $v \in V(G)$ is a vertex of G, let $N_G(v)$ and $d_G(v)$ be the neighborhood and degree of v in graph G, respectively. For a vertex set $U \subseteq V$, let $N_U(v) = N_G(v) \cap U$, $N_G(U) = \bigcup\limits_{u \in U} N_G(u)$, $N_G^c(U) = \bigcap\limits_{u \in U} N_G(u)$ and $d_G^c(U) = |N_G^c(U)|$. For a subset $U \subset V(G)$, if $G[U]$ is connected, denote the connected component of G containing U by $C_G(U)$.

Given two graphs G and H, we say a graph G is H-free if G does not contain a copy of H as a subgraph. The Turán number of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex H-free graph. If an n-vertex H-free graph G has $\mathrm{ex}(n, H)$ edges, then we call G an extremal graph for H. In 1959, Erdős and Gallai [1] proved the following well-known result about the Turán numbers of paths.

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[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China. E-mail: xiety@mail.ustc.edu.cn
[2]School of Mathematical Sciences, Key Laboratory of MEA (Ministry of Education) & Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China. E-mail: ltyuan@math.ecnu.edu.cn
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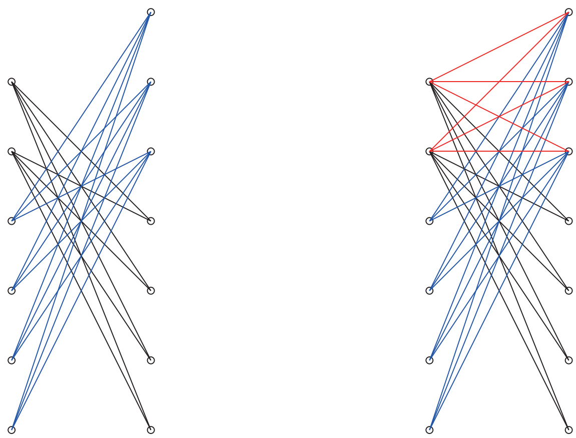
Theorem 1.1 (see Erdős and Gallai [1]) *Let $n \geq t$. Then*

$$\mathrm{ex}(n, P_t) \leq \frac{1}{2}(t-2)n.$$

For a given graph H, we use kH to denote the vertex-disjoint union of k copies of H. Recently, many researchers focus on determining the Turán numbers for linear forests. Gorgol [2] first studied the functions $\mathrm{ex}(n, 2P_3)$ and $\mathrm{ex}(n, 3P_3)$. For more results concerning the Turán numbers for linear forests, we refer the readers to [3–8, 12–17].

We use $\mathrm{ex}(m, n; H)$ to denote the maximum number of edges in an H-free (m, n)-bipartite graph, and call G an bipartite extremal graph for H if G is an H-free (m, n)-bipartite graph with $\mathrm{ex}(m, n; H)$ edges, and denote the set of all bipartite extremal graphs for H by $\mathrm{EX}(m, n; H)$. In 1984, Gyárfás, Rousseau and Schelp [9] determined $\mathrm{ex}(m, n; P_k)$ and characterized all bipartite extremal graphs for all values of m, n, k.

Denote by $K^{a,b}_{m,n}$ the (m, n)-bipartite graph consists of $K_{a,n-b}$ and $K_{m-a,b}$. Denote by $L^{a,b}_{m,n}$ the (m, n)-bipartite graph obtained from $K^{a,b}_{m,n}$ by joining each vertex of the class with size a to each vertex of the class with size b (see Figure 1). In particular, we say $L^{1,1}_{m,n}$ is a double star.



(a) (b)

Figure 1 (a)$K^{2,3}_{6,7}$, (b)$L^{2,3}_{6,7}$.

Theorem 1.2 (see Gyárfás, Rousseau and Schelp [9]) *Let $\ell = \lfloor \frac{k}{2} \rfloor - 1$ and $n \geq m$.*
(1) *If $k \geq 2$ is even, then*

$$\mathrm{ex}(m, n; P_k) = \begin{cases} mn, & \text{for } m \leq \ell; \\ \ell n, & \text{for } \ell + 1 \leq m \leq 2\ell - 1; \\ (n-\ell)\ell + (m-\ell)\ell, & \text{for } m \geq 2\ell. \end{cases}$$

Moreover,

$$\mathrm{EX}(m,n;P_k)=\begin{cases}\{K_{m,n}\}, & \text{for } m\le \ell;\\ \{K_{\ell,n}\}, & \text{for } \ell+1\le m\le 2\ell-1;\\ \bigcup_{j=0}^{\lfloor \frac{n}{2}\rfloor}\{K_{m,n}^{\ell,j}\}, & \text{for } m=2\ell;\\ \{K_{m,n}^{\ell,\ell}\}, & \text{for } m\ge 2\ell+1.\end{cases}$$

(2) *For* $k=3$,

$$\mathrm{ex}(m,n;P_3)=m.$$

Moreover, the unique extremal graph is $mP_2\cup\overline{K}_{n-m}$.

(3) *For* $k=5$,

$$\mathrm{ex}(m,n;P_5)=\begin{cases}m+n, & \text{for } m=n\ge 2 \text{ is even};\\ m+n-1, & \text{otherwise.}\end{cases}$$

Moreover, the extremal graphs consist of at most one double star and copies of $K_{2,2}$.

(4) *If* $k\ge 7$ *is odd, then*

$$\mathrm{ex}(m,n;P_k)=\begin{cases}mn, & \text{for } m\le\ell \text{ or } m=n=\ell+1;\\ \ell(n-1)+m, & \text{for } n>m=\ell+1 \text{ or } \ell+2\le m\le 2\ell+1;\\ 2\ell^2, & \text{for } m=n=2\ell+2;\\ (n-\ell)\ell+(m-\ell)\ell, & \text{for } n>m=2\ell+2 \text{ or } m\ge 2\ell+3.\end{cases}$$

Moreover,

$$\mathrm{EX}(m,n;P_k)=\begin{cases}\{K_{m,n}\}, & \text{for } m\le\ell \text{ or } m=n=\ell+1;\\ \{K_{\ell+1,\ell+1}\cup K_1, L_{m,n}^{\ell,1}\}, & \text{for } m=\ell+1 \text{ and } n=\ell+2;\\ \{K_{m,n}^{\ell+1,1}, L_{m,n}^{\ell,1}\}, & \text{for } m=n=\ell+2;\\ \{L_{m,n}^{\ell,1}\}, & \text{for } n-1>m=\ell+1 \text{ or } n\ge m=\ell+2\\ & \text{or } \ell+3\le m\le 2\ell;\\ \{K_{m,n}^{\ell,\ell+1}, L_{m,n}^{\ell,1}\}, & \text{for } m=2\ell+1;\\ \{K_{m,n}^{\ell+1,\ell+1}\}, & \text{for } m=n=2\ell+2;\\ \{K_{m,n}^{\ell,\ell}\}, & \text{for } n>m=2\ell+2 \text{ or } m\ge 2\ell+3.\end{cases}$$

In order to use it easily later, we will give a simple version of Theorem 1.2.

Corollary 1.1 *Let* $\ell=\lfloor\frac{k}{2}\rfloor-1$ *and* $n\ge m$. *If* $m\le k$, *then*

$$\mathrm{ex}(m,n;P_k)\le \ell(m+n);$$

if $m\ge k$, *then*

$$\mathrm{ex}(m,n;P_k)=\begin{cases}m, & \text{for } k=3;\\ n+m-i, & \text{for } k=5;\\ (n-\ell)\ell+(m-\ell)\ell, & \text{otherwise,}\end{cases}$$

where $i=0$ *when* $n=m$ *is even, and* $i=1$ *otherwise. In particular, for any* $n\ge m\ge 0$ *and* $k\ge 2$,

$$\mathrm{ex}(m,n;P_k)\le \ell(m+n).$$

From now on, let F_j be the linear forest consisting of $P_{k_1}, \cdots, P_{k_j}$ with $k_1 \geq k_2 \geq \cdots \geq k_j \geq 2$ and $j \geq 2$. Letting $X_j = \{k_1, \cdots, k_j\}$, we say that X_j is odd if all numbers in it are odd and X_j is not odd otherwise. Let $r_i = \lfloor \frac{k_i}{2} \rfloor$ for $i \in \{1, 2, \cdots, \ell\}$, $s_j = \sum\limits_{i=1}^{j} r_i$ and $x_j = \sum\limits_{i=1}^{j} k_i$.

Vert recently, Chen, Wang, Yuan and Zhang [10] determined $\mathrm{ex}(m, n; F_\ell)$ when n is sufficiently larger than m and s_ℓ, and characterized the extremal graphs.

Theorem 1.3 (see Chen, Wang, Yuan and Zhang [10]) *If $n \geq \sigma m$ and $m \geq s_\ell$, where $\sigma = \sigma(k_1, \cdots, k_\ell)$, then the following hold.*

(1) *If X_ℓ is not odd, then*

$$\mathrm{ex}(m, n; F_\ell) = \begin{cases} (s_\ell - 1)n, & \text{for } s_\ell \leq m \leq 2s_\ell - 2; \\ (s_\ell - 1)(n - r_\ell + 1) + (m - s_\ell + 1)(r_\ell - 1), & \text{for } m \geq 2s_\ell - 1. \end{cases}$$

Moreover, the extremal graphs are $K_{s_\ell - 1, n} \cup \overline{K}_{m - s_\ell + 1}$ for $s_\ell \leq m \leq 2s_\ell - 3$, $K_{m,n}^{s_\ell - 1, n - i}$ with $0 \leq i \leq r_\ell - 1$ for $m = 2s_\ell - 2$ and $K_{m,n}^{s_\ell - 1, r_\ell - 1}$ for $m \geq 2s_\ell - 1$.

(2) *Let $p = 2s_\ell - 2 + \frac{(s_\ell - 1)}{(r_\ell - 2)}$. If X_ℓ is odd and $k_\ell \notin \{3, 5\}$, then*

$$\mathrm{ex}(m, n; F_\ell) = \begin{cases} (s_\ell - 1)n + m - s_\ell + 1, & \text{for } s_\ell \leq m \leq p; \\ (s_\ell - 1)(n - r_\ell + 1) + (m - s_\ell + 1)(r_\ell - 1), & \text{for } m \geq p. \end{cases}$$

Moreover, for $\ell \geq 2$, the extremal graphs are $L_{m,n}^{s_\ell - 1, 1}$ for $s_\ell \leq m \leq p$ and $K_{m,n}^{s_\ell - 1, r_\ell - 1}$ for $m \geq p$.

(3) *If X_ℓ is odd and $k_\ell = 3$, then*

$$\mathrm{ex}(m, n; F_\ell) = \begin{cases} (s_\ell - 1)n + m - s_\ell + 1, & \text{for } k_1 = k_2 = \cdots = k_\ell = 3; \\ (s_\ell - 1)n + 1, & \text{otherwise}. \end{cases}$$

Moreover, the extremal graphs are the graph obtained from $K_{s_\ell - 1, n}$ by joining $m - s_\ell + 1$ independent edges connecting new $m - s_\ell + 1$ isolated vertices to $m - s_\ell + 1$ vertices with degree $s_\ell - 1$ in $K_{s_\ell - 1, n}$ respectively for $k_1 = k_2 = \cdots = k_\ell = 3$, and $L_{s_\ell, n}^{s_\ell - 1, 1} \cup \overline{K}_{m - s_\ell}$ otherwise.

(4) *If X_ℓ is odd and $k_\ell = 5$, then*

$$\mathrm{ex}(m, n; F_\ell) = (s_\ell - 1)n + m - s_\ell + 1.$$

Moreover, the unique extremal graph is $L_{m,n}^{s_\ell - 1, 1}$ for $\ell \geq 2$.

The above two theorems show that the extremal graphs for linear forests are very complicated when m is small. In this paper, by using Theorem 1.3 and a key lemma, we will determine $\mathrm{ex}(m, n; F_\ell)$ for $k_\ell \geq 4$ and $n \geq m$ when m is sufficiently large. Moreover, all the extremal graphs will be characterized.

We define

$$f(m, n; X_j) = \begin{cases} (s_j - 1)(n - 1) + m, & \text{if } X_j \text{ is odd with } k_j = 5; \\ 6m - 9, & \text{if } F_j = P_9 \cup P_4 \text{ with } m = n; \\ (s_j - 1)(n - r_j + 1) + (r_j - 1)(m - s_j + 1), & \text{otherwise}. \end{cases}$$

Denote by $\mathcal{F}(m, n; X_j)$ the F_j-free graphs with $f(m, n; X_j)$ edges:

$$\mathcal{F}(m, n; X_j) = \begin{cases} \{L_{m,n}^{s_j - 1, 1}\}, & \text{if } X_j \text{ is odd with } k_j = 5; \\ \{L_{m,m}^{3,3}\}, & \text{if } F_j = P_9 \cup P_4 \text{ with } m = n; \\ \{L_{m,m}^{4,3}, K_{m,m}^{6,1}\}, & \text{if } F_j = P_{11} \cup P_4 \text{ with } m = n; \\ \{K_{m,n}^{s_j - 1, r_j - 1}\}, & \text{otherwise}. \end{cases}$$

Furthermore, we define

$$F(m,n;X_j)=\max\{\mathrm{ex}(m,n;P_{k_1}),f(m,n;X_2),\cdots,f(m,n;X_j)\}.$$

Theorem 1.4 *Let $n\ge m$ and m be sufficiently large. If $F_\ell=P_{k_1}\cup P_{k_1}\cup\cdots\cup P_{k_\ell}$ is a linear forest with $k_1\ge k_2\ge\cdots\ge k_\ell\ge 4$, then*

$$\mathrm{ex}(m,n;F_\ell)=F(m,n;X_\ell)$$

and all extremal graphs belong to $\mathrm{EX}(m,n;P_{k_1})\cup\big(\bigcup_{i=2}^{\ell}\mathcal{F}(m,n;X_i)\big)$.

2 Several Lemmas and Proof of Main Theorem

First, we introduce a result concerning cycles in bipartite graphs. Jackson [11] proved the following result.

Theorem 2.1 (see Jackson [11]) *Let $G(A,B)$ be a bipartite graph with $|A|=m$, $|B|=n$ and $n\ge m$. If each vertex of B has degree at least k, then G contains a cycle of length at least $2k$.*

The following lemma is widely used in extremal problems.

Lemma 2.1 *Let $G=G(A,B)$ be a bipartite graph with $|A|=a$ and $|B|=n$, where a is a constant and n is sufficiently large. If $e(G)=bn-o(n)$, then there exists a vertex set $A'\subseteq A$ with size $t=\lceil b\rceil$ and a constant $\delta>0$ such that the number of common neighbors of A' is at least δn.*

Proof Let X be the set of vertices of B with degree less than t. Since $e(G)\ge bn-o(n)$, we have $(t-1)|X|+a(n-|X|)\ge bn-o(n)$. Thus, we have $|X|\le\frac{a-b}{a-t+1}n-o(n)$. Hence, there are at least $n-|X|\ge\frac{b-t+1}{a-t+1}n-o(n)$ vertices of B with degree at least t. Since there are $\binom{a}{t}$ t-sets in A, by the pigeonhole principle, there exists a vertex set $A'\subseteq A$ with size t such that the number of common neighbors of A is at least δn, where $\delta=\frac{b-t+1}{(a-t+1)\binom{a}{t}+1}>0$. The proof is complete.

Given a graph G, for any two vertices x,y of G, we use $e_G(x,y)$ to denote the number of edges incident to x or y in G. Thus, if x is adjacent to y then $e_G(x,y)=d(x)+d(y)-1$ and if x is not adjacent to y then $e_G(x,y)=d(x)+d(y)$, where $d(x)$ and $d(y)$ are the degrees of x and y in G, respectively.

Lemma 2.2 *Let $G=G(A,B)$ be an F_ℓ-free bipartite graph with $|A|=m$ and $|B|=n$. Let $\ell\ge 2$ and $k_\ell\ge 4$. Let $n\ge m$ and $m\ge m_1=m_1(k_1,\cdots,k_\ell)$. Suppose that G contains a copy of $F_{\ell-1}=P_{k_1}\cup\cdots\cup P_{k_{\ell-1}}$. Let $A_0=A-V(F_{\ell-1})$, $B_0=B-V(F_{\ell-1})$. If $e_G(x,y)\ge s_\ell+r_\ell-2$ for each $x\in A_0$ and $y\in B_0$, then*

$$e(G)\le f(m,n;X_\ell)$$

where the equality holds if and only if $G \in \mathcal{F}(m, n; X_\ell)$.

Now we will show that Lemma 2.2 and Theorem 1.3 imply Theorem 1.4.

Proof of Theorem 1.4 Let $G(A, B)$ be an F_ℓ-free (m, n)-bipartite graph. Let $n \geq m$ and $m \geq \rho m_1$, where m_1 is from Lemma 2.2, and $\rho > 1$ will be defined later. Let $k_\ell \geq 4$. Assume that

$$e(G) \geq F(m, n; X_\ell). \tag{2.1}$$

Then we will prove this theorem by induction on ℓ. It is trivial for $\ell = 1$ (by Theorem 1.1), so we may assume that $\ell \geq 2$ and the theorem holds for $\ell - 1$. Let $F_{\ell-1} = P_{k_1} \cup \cdots \cup P_{k_{\ell-1}}$. If G is $F_{\ell-1}$-free, then by the definitions of $F(m, n; X_\ell)$ and $\mathcal{F}(m, n; X_\ell)$, we have $e(G) \leq \mathrm{ex}(m, n; F_{\ell-1}) = F(m, n; X_{\ell-1}) \leq F(m, n; X_\ell)$. So $e(G) = \mathrm{ex}(m, n; F_{\ell-1}) = F(m, n; X_\ell)$ and $G \in \mathrm{EX}(m, n; P_{k_1}) \cup \big(\bigcup_{i=2}^{\ell-1} \mathcal{F}(m, n; X_i)\big)$, hence Theorem 1.4 holds. So we assume that G contains a copy of $F_{\ell-1}$, and let $A_0 = A - V(F_{\ell-1})$, $B_0 = B - V(F_{\ell-1})$.

If $e_G(x, y) \geq s_\ell + r_\ell - 2$ for each $x \in A_0$ and $y \in B_0$, then Theorem 1.4 holds by Lemma 2.2. Suppose that there exist $x_1 \in A_0$ and $y_1 \in B_0$ such that $e_G(x_1, y_1) \leq s_\ell + r_\ell - 3$. Let $G^1 = G - \{x_1, y_1\}$. Then we can construct G^{i+1} from G^i if there exist $x_i \in A_0 \cap V(G^i)$ and $y_i \in B_0 \cap V(G^i)$ such that $e_{G^i}(x, y) \leq s_\ell + r_\ell - 3$. Note that $f(m, n; X_\ell) \geq (s_\ell - 1)n + (r_\ell - 1)m - O(1)$. Thus for any t,

$$\begin{aligned} e(G^t) &\geq F(m, n; X_\ell) - (s_\ell + r_\ell - 3)t \\ &\geq f(m, n; X_\ell) - (s_\ell + r_\ell - 3)t \\ &\geq (s_\ell - 1)(n - t) + (r_\ell - 1)(m - t) + t - O(1). \end{aligned} \tag{2.2}$$

Since $e(G^t) \leq (n - t)(m - t)$, combining with (2.2), we have $(s_\ell - 1)(n - t) + (r_\ell - 1)(m - t) + t - O(1) \leq (n - t)(m - t)$ implying $t \leq m - s_\ell$.

So the process will be stopped in $1 \leq t_0 \leq m - s_\ell$ steps. Furthermore, let $r' = \lfloor \frac{x_\ell}{2} \rfloor$. Then G^{t_0} is P_{x_ℓ}-free (since P_{x_ℓ} contains a copy of F_ℓ). By Corollary 1.1,

$$e(G^{t_0}) \leq \mathrm{ex}(m - t_0, n - t_0; P_{x_\ell}) \leq (r' - 1)(n - t_0) + (r' - 1)(m - t_0).$$

By (2.2), we have

$$e(G^{t_0}) \geq (s_\ell - 1)n + (r_\ell - 1)m - (s_\ell + r_\ell - 3)t_0 - O(1).$$

Hence, combining the above two inequalities, we have

$$t_0 \leq \frac{r' - s_\ell}{2r' - s_\ell - r_\ell + 1} n + \frac{r' - r_\ell}{2r' - s_\ell - r_\ell + 1} m + O(1) \leq \frac{2r' - s_\ell - r_\ell}{2r' - s_\ell - r_\ell + 1} n + O(1).$$

Since n is sufficient large, there is a constant $0 < q_1 < 1$ such that $q_1 n - t_0 \geq m_1$.

Suppose that $m - t_0 < m_1$, i.e., $m < m_1 + t_0 \leq q_1 n$. Let $\rho = \frac{\sigma}{\frac{1}{q_1} - 1} > 1$ (since q_1 can be chosen close to one), where σ is the constant from Theorem 1.3. Now by $m \geq \rho m_1 \geq \rho(m - t_0)$, we have

$$n - t_0 \geq \Big(\frac{1}{q_1}\Big)m - t_0 \geq \Big(\frac{1}{q_1} - 1\Big)m \geq \Big(\frac{1}{q_1} - 1\Big)\rho(m - t_0) = \sigma(m - t_0).$$

Recall that $m - t_0 \geq s_\ell$. Let $p = 2s_\ell - 2 + \frac{s_\ell - 1}{r_\ell - 2}$. Theorem 1.3 implies

$$e(G^{t_0}) \leq \begin{cases} f(m, n; X_\ell) - (s_\ell + r_\ell - 2)t_0, & \text{if } m - t_0 \geq p; \\ f(m, n; X_\ell) - (s_\ell + r_\ell - 2)t_0 + O(1), & \text{if } m - t_0 \leq p. \end{cases}$$

Hence if $m - t_0 \geq p$, then $f(m, n; X_\ell) \leq e(G) \leq e(G^{t_0}) + (s_\ell + r_\ell - 3)t_0 \leq f(m, n; X_\ell) - t_0 \leq f(m, n; X_\ell) - (m - m_1) < F(m, n; X_\ell)$, a contradiction to (2.1). If $m - t_0 \leq p$, then since m is sufficient large, $f(m, n; X_\ell) \leq e(G) \leq e(G^{t_0}) + (s_\ell + r_\ell - 3)t_0 \leq f(m, n; X_\ell) - t_0 + O(1) \leq f(m, n; X_\ell) - (m - m_1) + O(1) < f(m, n; X_\ell) \leq F(m, n; X_\ell)$, contradicting (2.1).

Suppose that $m - t_0 \geq m_1$. Note that we only delete vertices in $A_0 \cup B_0$. Hence, G^{t_0} contains a copy of $F_{\ell-1}$. By the termination condition, $e_{G^{t_0}}(x, y) \geq s_\ell + r_\ell - 2$ for each $x \in V(G^{t_0}) \cap A_0$ and $y \in V(G^{t_0}) \cap B_0$. It follows from Lemma 2.2 that $e(G^{t_0}) \leq f(n - t_0, n - t_0; X_\ell) = f(m, n; X_\ell) - (s_\ell + r_\ell - 2)t_0$. Therefore, $e(G) \leq e(G^{t_0}) + (s_\ell + r_\ell - 3)t_0 \leq f(m, n; X_\ell) - t_0 < f(m, n; X_\ell) \leq F(m, n; X_\ell)$, a contradiction to (2.1). This completes the proof of Theorem 1.4.

3 Proof of Lemma 2.2

We first introduce a stability result for paths in bipartite graphs.

Lemma 3.1 *Let $G = G(A, B)$ be a P_t-free bipartite graph with $|A| = m$, $|B| = n$ and $t \geq 4$. Let Δ be a constant and $n \geq m$ be sufficiently large. If*

$$e(G) \geq \mathrm{ex}(m, n; P_t) - \Delta, \tag{3.1}$$

then there is a constant $0 < \varepsilon \leq 1$ depending on Δ and t such that

(1) *if $t = 5$, then there are two vertices $u \in A$ and $v \in B$ such that $d_G(u) \geq \varepsilon(n - 2q)$ and $d_G(v) \geq \varepsilon(m - 2q)$, where q is the number of copies of $K_{2,2}$ in G;*

(2) *if $t \neq 5$, then there are two subset $A_1 \subset A$ and $B_1 \subset B$ with $|A_1| = |B_1| = \lfloor \frac{t}{2} \rfloor - 1$ such that $d^c_G(A_1) \geq \varepsilon n$ and $d^c_G(B_1) \geq \varepsilon m$.*

Proof If $t = 4$, then G consists of stars[1]. By (3.1) and Corollary 1.1, we have $e(G) \geq n + m - \Delta - 2$. Thus G consists of at most $\Delta + 2$ stars, and hence there is a star whose center belongs to A with at least $\frac{n}{\Delta+2}$ leaves and a star whose center belongs to B with at least $\frac{m}{\Delta+2}$ leaves. The results follow from setting $\varepsilon = \frac{1}{\Delta+2}$.

If $t = 5$, then G consists of stars and copies of $K_{2,2}$. Let q be the number of copies of $K_{2,2}$ in G. The rest proof of this case is similar to that of $t = 4$ and be omitted.

Now, let $t \geq 6$, $n \geq m$ and m be sufficient large. Let $r = \lfloor \frac{t}{2} \rfloor$. Deleting vertices of degree at most $r - 2$ in G until that the resulting graph $G^* = G(A^*, B^*)$ has no such vertex. Let

[1] We view isolated vertices and edges as trivial stars.

$|A^*| = m^*$, $|B^*| = n^*$ and $\beta = m + n - m^* - n^*$. By (3.1) and Corollary 1.1,

$$e(G^*) \geq e(G) - (r-2)\beta \geq (r-1)(m^* + n^*) - 2(r-1)^2 - \Delta + \beta. \tag{3.2}$$

Without loss generality, we may assume that $m^* = \min\{m^*, n^*\}$. Since G^* is P_t-free, by Corollary 1.1,

$$e(G^*) \leq \begin{cases} (r-1)(m^* + n^*), & \text{if } m^* \leq t-1; \\ (r-1)(m^* + n^*) - 2(r-1)^2, & \text{if } m^* \geq t. \end{cases} \tag{3.3}$$

Combining (3.2) and (3.3), we have that $\beta \leq \Delta + 2(r-1)^2$ if $m^* \leq t-1$ and that $\beta \leq \Delta$ if $m^* \geq t$. However, if $m^* \leq t-1$, then $m \leq \beta + m^* \leq t - 1 + \Delta + 2(r-1)^2$, contradicting that m is sufficient large. Hence $m^* \geq t$ and $\beta \leq \Delta$, that is, $m^* + n^* \geq m + n - \Delta$. So

$$m^* \geq m - \Delta \quad \text{and} \quad n^* \geq n - \Delta. \tag{3.4}$$

Since $n \geq m$ and m is sufficient large, both m^* and n^* are sufficient large.

Since the minimum degree of G^* is at least $r - 1 \geq 2$, by Theorem 2.1, any connected component of G^* must contains a cycle of length at least $2(r-1)$. Furthermore, since G^* is P_t-free, the length of longest cycle of G^* is at most $2r$.

Claim 3.1 Any connected component of G^* must be one of the following four types:

- Type 1 A subgraph of $K_{r,r}$ containing a cycle of length $2r$;
- Type 2 A complete bipartite graph $K_{r-1,r-1}$;
- Type 3 A complete bipartite graph $K_{r-1,r-1+t}$ with $t \geq 1$ and the part of size $r-1$ belongs to A^*;
- Type 4 A complete bipartite graph $K_{r-1,r-1+t}$ with $t \geq 1$ and the part of size $r-1$ belongs to B^*.

Proof If a connected component G_1^* of G^* contains a cycle C_1 of length $2r$, since G^* is P_t-free, $N_{G_1^*}(V(C_1)) \subset V(C_1)$. So $V(G_1^*) = V(C_1)$, i.e., G_1^* is of Type 1.

If a connected component G_2^* of G^* contains no cycle of length $2r$, then G_2^* must contains a cycle $C_2 = u_1v_1u_2v_2 \cdots u_{r-1}v_{r-1}u_1$ of length $2r-2$. If $V(G_2^*) = V(C_2)$, then G_2^* is a complete graph $K_{r-1,r-1}$. So G_2^* is of Type 2.

Suppose that $V(G_2^*) \neq V(C_2)$. If there is a vertex $x_1 \notin V(C_2)$ of G_2^* such that $N_{V(C_2)\cap B^*}(x_1) \neq \emptyset$, then we claim that $N_{G_2^*}(x_1) = V(C_2) \cap B^*$. Otherwise, there is a vertex $y_1 \in N_{G_2^*}(x_1) \backslash V(C_2)$. Since G^* is P_t-free, $N_{G_2^*}(y_1) \subset V(C_2) \cup \{x_1\}$. Thus y_1 is adjacent to at least $r-2$ vertices of $V(C_2)$. Let $v_i \in N_{V(C_2)}(x_1)$. So at least one vertex of u_i and u_{i+1} is adjacent to y_1, say $u_iy_1 \in E(G_2^*)$. Thus $u_1v_1u_2v_2 \cdots u_iy_1x_1v_i \cdots u_{r-1}v_{r-1}u_1$ is a cycle of length $2r$ in G_2^*, a contradiction.

Then for any vertex $u_j \in V(C_2) \cap A$, $G[(V(C_2) \backslash \{u_j\}) \cup \{x_1\}]$ contains a cycle of length $2r-2$ in G_2^*. By the above proof, u_j has no neighbor in $B^* - V(C_2)$. Hence $V(G_2^*) \cap B^* = V(C_2) \cap B^*$, and G_2^* is a complete bipartite graph such that the class of order $r-1$ belongs to B^*. Similarly, if $N_{V(C_2)\cap A^*}(x_1) \neq \emptyset$, then G_2^* is a complete bipartite graph such that the class of order $r-1$ belongs to A^*. Therefore, G_2^* is of either Type 3 or Type 4.

Let q_i be the number of connected components in G^* with Type i for $i \in \{1,2,3,4\}$. Then we will consider two cases. Set $\varepsilon = \frac{(r-1)^2}{2(\Delta+(r-1)^2)}$.

Case 1 t is even. Clearly, we have $q_1 = 0$. Hence

$$\begin{aligned} e(G^*) &= (r-1)^2 q_2 + (r-1)(n^* - (r-1)(q_2+q_4)) + (r-1)(m^* - (r-1)(q_2+q_3)) \\ &= (r-1)(m^*+n^*) - (r-1)^2(q_2+q_3+q_4). \end{aligned}$$

Combining with (3.2), we have $q_2+q_3+q_4 \leq \frac{\Delta-\beta}{(r-1)^2} + 2$.

If $q_3 = 0$, then $e(G^*) \leq (r-1)m^*$, contradicting (3.2); if $q_4 = 0$, then $e(G^*) \leq (r-1)n^*$, contradicting (3.2).

Thus $1 \leq q_3, q_4 \leq \frac{\Delta-\beta}{(r-1)^2} + 1$. By (3.4), there are two subsets $A_0 \subset A^*$ and $B_0 \subset B^*$ with $|A_0| = |B_0| = r-1$ such that $d_G^c(A_0) \geq d_{G^*}^c(A_0) \geq \frac{n^*-(r-1)(q_2+q_4)}{q_3} \geq \varepsilon n$ and $d_G^c(B_0) \geq d_{G^*}^c(B_0) \geq \frac{m^*-(r-1)(q_2+q_3)}{q_4} \geq \varepsilon m$.

Case 2 t is odd. Note that both m^* and n^* are sufficient large. Hence

$$\begin{aligned} e(G^*) &\leq (r-1)(n^* - (r-1)(q_2+q_4) - rq_1) + (r-1)(m^* - (r-1)(q_2+q_3) - rq_1) \\ &\quad + (r-1)^2 q_2 + r^2 q_1 \\ &= (r-1)(m^*+n^*) - (r-1)^2(q_2+q_3+q_4) - (r-2)rq_1. \end{aligned}$$

Combining with (3.2), we have $(q_2+q_3+q_4) + \frac{(r-2)rq_1}{(r-1)^2} \leq 2 + \frac{\Delta-\beta}{(r-1)^2}$.

If $q_3 = 0$, then $e(G^*) \leq (r-1)m^* + rq_1$, contradicting (3.2); if $q_4 = 0$, then $e(G^*) \leq (r-1)n^* + rq_1$, contradicting (3.2).

Thus $1 \leq q_3, q_4 \leq \frac{\Delta-\beta}{(r-1)^2} + 2 - \frac{(r-2)rq_1}{(r-1)^2}$. By (3.4), there are two subsets $A_1 \subset A^*$ and $B_1 \subset B^*$ with $|A_1| = |B_1| = r-1$ such that $d_G^c(A_1) \geq d_{G^*}^c(A_1) \geq \frac{n^*-(r-1)(q_2+q_4)-rq_1}{q_3} \geq \varepsilon n$ and $d_G^c(B_1) \geq d_{G^*}^c(B_1) \geq \frac{m^*-(r-1)(q_2+q_3)-rq_1}{q_4} \geq \varepsilon m$.

This completes the proof of Lemma 3.1.

Given a multi-set of integers $D = \{q_1, \cdots, q_\ell\}$, we say that an integer a is D-sum-free if there is no subset of D whose sum is a. For any $1 \leq j \leq \ell$, denote $R_j = \{r_1, r_2, \cdots, r_j\}$. For any integer $a \geq 1$ which is R_j-sum-free, denote

$$\mathcal{R}_j(a) = \{(i,y) : 1 \leq y \leq r_i - 1, a = y \text{ or } (a-y) \text{ is not } (R_j \setminus \{r_i\})\text{-sum-free}\}.$$

Denote by $D_{m,n}^{a,b}$ the (m,n)-bipartite graph obtained from $K_{m,n}^{a,b}$ by adding an edge between the class with size a and the class with size b. If $a \geq 1$, denote by $D_{m,n}^{a,b,1}$ the $(m+1, n+1)$-bipartite graph by identifying an end vertex of P_3 and the vertex with degree $n-b+1$ in $D_{m,n}^{a,b}$. Furthermore, if $a \geq 2$, denote by $D_{m,n}^{a,b,0}$ the $(m+1, n+1)$-bipartite graph by identifying an end vertex of P_3 and one vertex with degree $n-b$ in $D_{m,n}^{a,b}$ (see Figure 2).

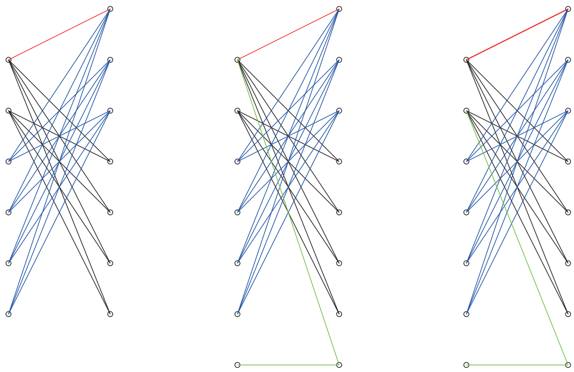


Figure 2 (a)$D_{6,7}^{2,3}$, (b)$D_{6,7}^{2,3,1}$, (c)$D_{6,7}^{2,3,0}$.

Lemma 3.2 (1) *If $a+b \geq s_\ell + 1$, then $D_{x_\ell,x_\ell}^{a,b}$ contains a copy of F_ℓ.*

(2) *Let $a+b = s_\ell$. If there is a pair $(i,1) \in \mathcal{R}_\ell(a)$, then $D_{x_\ell,x_\ell}^{a,b,1}$ contains a copy of F_ℓ. If there is a pair $(i,y) \in \mathcal{R}_\ell(a)$ with $y \geq 2$, then $D_{x_\ell,x_\ell}^{a,b,0}$ contains a copy of F_ℓ.*

(3) *If $a+b = s_\ell$, then the following three statements are equivalent.*

(i) *$L_{x_\ell,x_\ell}^{a,b}$ is F_ℓ-free;*

(ii) *$D_{x_\ell,x_\ell}^{a,b}$ is F_ℓ-free;*

(iii) *a is R_ℓ-sum-free and $k_i \in X_\ell$ is odd for any pair $(i,y) \in \mathcal{R}_\ell(a)$.*

Proof Let A_1 and B_1 be the vertex set of $D_{x_\ell,x_\ell}^{a,b}$ with degree at least $x_\ell - b$ and $x_\ell - a$, respectively. Let A_2 and B_2 be the vertex set of $L_{x_\ell,x_\ell}^{a,b}$ with degree x_ℓ and x_ℓ in two different classes, respectively. Let $F_i^* = F_\ell - P_{k_i}$.

Note that $\bigcup\limits_{j \neq i} K_{r_j,k_j}$ contains a copy of F_i^*. The definitions of $D_{2r,2r}^{a,r-a}$, $L_{2r,2r}^{a,r-a}$, $D_{2r,2r}^{a,r-a,1}$, $D_{2r,2r}^{a,r-a,0}$ and $\mathcal{R}_\ell(a)$ imply the following observations.

Observation 3.1 Let $r \geq 2$ and $1 \leq a \leq r-1$. The lengths of longest paths of both $D_{2r,2r}^{a,r-a}$ and $L_{2r,2r}^{a,r-a}$ are $2r$. The length of longest path of $D_{2r,2r}^{1,r-1,1}$ is $2r+1$ and the length of longest path of $D_{2r,2r}^{a,r-a,0}$ with $a \geq 2$ is $2r+1$.

Observation 3.2 For any pair $(i,y) \in \mathcal{R}_\ell(a)$, $D_{x_\ell,x_\ell}^{a,b}$ (or $D_{x_\ell,x_\ell}^{a,b,1}$, $D_{x_\ell,x_\ell}^{a,b,0}$) with $a+b \geq s_\ell$ contains a copy of $F_i^* \cup D_{k_i,k_i}^{y,z}$ (or $F_i^* \cup D_{k_i,k_i}^{y,z,1}$, $F_i^* \cup D_{k_i,k_i}^{y,z,0}$, respectively), where $z = (a+b-s_\ell) + (r_i - y)$.

(1) Let $a+b \geq s_\ell + 1$. If a is not R_ℓ-sum-free, then $D_{x_\ell,x_\ell}^{a,b}$ contains a copy of $\bigcup\limits_{i \in [\ell]} K_{r_i,r_i+1}$ implying that $D_{x_\ell,x_\ell}^{a,b}$ contains a copy of F_ℓ, we are done. So we may assume that a is R_ℓ-sum-free, whence $\mathcal{R}_\ell(a)$ is not empty. Let $(j_1,y_1) \in \mathcal{R}_\ell(a)$. By Observation 3.2, $D_{x_\ell,x_\ell}^{a,b}$ contains a copy of $F_{j_1}^* \cup D_{k_{j_1},k_{j_1}}^{y_1,r_{j_1}-y_1+1}$. By Observation 3.1, $D_{k_{j_1},k_{j_1}}^{y_1,r_{j_1}-y_1+1}$ contains a copy of $P_{k_{j_1}}$. Thus $D_{x_\ell,x_\ell}^{a,b}$ contains a copy of F_ℓ.

(2) Let $a+b=s_\ell$. If there is a pair $(j_2,1)\in\mathcal{R}_\ell(a)$, then by Observation 3.2, $D^{a,b,1}_{x_\ell,x_\ell}$ contains a copy of $F^*_{j_2}\cup D^{1,r_{j_2}-1,1}_{k_{j_2},k_{j_2}}$. By Observation 3.1, $D^{1,r_{j_2}-1,1}_{k_{j_2},k_{j_2}}$ contains a copy of $P_{k_{j_2}}$. Thus $D^{a,b,1}_{x_\ell,x_\ell}$ contains a copy of F_ℓ. The proof for the rest case is similar to that of the first case and be omitted.

(3) Let $a+b=s_\ell$. Assume that $L^{a,b}_{x_\ell,x_\ell}$ is F_ℓ-free. Clearly, $D^{a,b}_{x_\ell,x_\ell}$ is F_ℓ-free (by $D^{a,b}_{x_\ell,x_\ell}\subset L^{a,b}_{x_\ell,x_\ell}$). Thus (i) implies (ii).

Assume that $D^{a,b}_{x_\ell,x_\ell}$ is F_ℓ-free. Then a is R_ℓ-sum-free and $k_i\in X_\ell$ is odd for any pair $(i,y)\in\mathcal{R}_\ell(a)$. Otherwise, a is not R_ℓ-sum-free or there is a pair $(j_3,y_3)\in\mathcal{R}_\ell(a)$ with k_{j_3} is even.

If a is not R_ℓ-sum-free, then there is a partition (I_a,I_b) of $[\ell]$ such that $a=\sum\limits_{i\in I_a} r_i$ and $b=\sum\limits_{i\in I_b} r_i$. Then $D^{a,b}_{x_\ell,x_\ell}$ contains a copy of $\bigcup\limits_{\{i\in I_a\}} P_{k_i}$ containing A_1 and a copy of $\bigcup\limits_{\{i:r_i\in I_b\}} P_{k_i}$ containing B_1, and the two linear forests are disjoint. Hence $D^{a,b}_{x_\ell,x_\ell}$ contains a copy of F_ℓ.

If there is a pair $(j_3,y_3)\in\mathcal{R}_\ell(a)$ with k_{j_3} is even, then by Observation 3.2, $D^{a,b}_{x_\ell,x_\ell}$ contains a copy of $F^*_{j_3}\cup D^{y_3,r_{j_3}-y_3}_{k_{j_3},k_{j_3}}$. Then by Observation 3.1, $D^{y_3,r_{j_3}-y_3}_{k_{j_3},k_{j_3}}$ contains a copy of $P_{2r_{j_3}}=P_{k_{j_3}}$, so $D^{a,b}_{x_\ell,x_\ell}$ contains a copy of F_ℓ. Thus (ii) implies (iii).

Assume that a is R_ℓ-sum-free and $k_i\in X_\ell$ is odd for any pair $(i,y)\in\mathcal{R}_\ell(a)$. Then $L^{a,b}_{x_\ell,x_\ell}$ is F_ℓ-free. Otherwise, $L^{a,b}_{x_\ell,x_\ell}$ contains a copy of F_ℓ, then each P_i for $i\in[\ell]$ contains exact r_i vertices of $A_2\cup B_2$ (since $a+b=s_\ell$). So if k_i is odd, by Observation 3.1, P_{k_i} contains r_i vertices of A_2 or B_2. Thus either a is not R_ℓ-sum-free or $k_i\in X_\ell$ is even for some pair $(i,y)\in\mathcal{R}_\ell(a)$. Hence, (iii) implies (i).

Therefore, the above three statements are equivalent.

Lemma 3.3 *Let $n>m$ and m be sufficient large. Let $k_\ell\geq 4$. If $L^{a,b}_{m,n}$ with $a+b=s_\ell+r_\ell-2$ is F_ℓ-free, then $k_\ell\in\{4,5\}$. Moreover, $e(L^{a,b}_{m,n})\leq f(m,n;X_\ell)$ and equality holds only when $L^{a,b}_{m,n}\in\mathcal{F}(m,n,X_\ell)$.*

Proof Suppose that $L^{a,b}_{m,n}$ is F_ℓ-free with $a+b=s_\ell+r_\ell-2$. Let A_1, B_1 be the vertex sets of $L^{a,b}_{m,n}$ with degree n and m respectively.

If $k_\ell\geq 6$, then $s_\ell+r_\ell-2\geq s_\ell+1$. By Lemma 3.2(1), $D^{a,b}_{m,n}$ contains a copy of F_ℓ, so $L^{a,b}_{m,n}$ contains a copy of F_ℓ. Thus $k_\ell\in\{4,5\}$ and $a+b=s_\ell$.

Note that $L^{a,b}_{m,n}$ is F_ℓ-free if and only if $L^{b,a}_{m,n}$ is F_ℓ-free. Since $n\geq m$ and m is sufficient large, $a\geq b$ implies that $e(L^{a,b}_{m,n})\geq e(L^{b,a}_{m,n})$. Thus we may assume that $a\geq b$, so $b\leq\lfloor\frac{s_\ell}{2}\rfloor$. Since m is sufficiently large,

$$e(L^{a,b}_{m,n})=an+bm-ab \text{ is monotone decrease respect to } b.$$

Since $L^{a,b}_{m,n}$ is F_ℓ-free, by Lemma 3.2(3), b is R_ℓ-sum-free and k_i is odd for any pair $(i,y)\in\mathcal{R}_\ell(b)$. Thus $b\neq 2$ (since $r_\ell=2$). Moreover, if X_ℓ is not odd, then $b\neq 1$ (since k_i is odd for any pair $(i,y) \in \mathcal{R}_\ell(b)$). Hence,

$$b \geq \begin{cases} 3, & \text{if } k_\ell = 4; \\ 1, & \text{if } k_\ell = 5 \text{ and } X_\ell \text{ is odd;} \\ 3, & \text{if } k_\ell = 5 \text{ and } X_\ell \text{ is not odd.} \end{cases} \tag{3.5}$$

If X_ℓ is odd with $k_\ell = 5$, then $b \geq 1$ (by (3.5)), $e(L^{a,b}_{m,n}) \leq e(L^{s_\ell-1,1}_{m,n}) = f(m,n;X_\ell)$ and equality holds only when $L^{a,b}_{m,n} = L^{s_\ell-1,1}_{m,n} \in \mathcal{F}(m,n;X_\ell)$.

If $F_\ell = P_9 \cup P_4$ with $m = n$, then $b \geq 3$ (by (3.5)), $e(L^{a,b}_{m,n}) \leq e(L^{3,3}_{m,n}) = f(m,n;X_\ell)$ and equality holds only when $L^{a,b}_{m,n} = L^{3,3}_{m,n} \in \mathcal{F}(m,n;X_\ell)$.

Otherwise, by the definition of $f(m,n;X_\ell)$, we have $f(m,n;X_\ell) = e(K^{s_\ell-1,r_\ell-1}_{m,n})$. If $e(L^{a,b}_{m,n}) \geq e(K^{s_\ell-1,r_\ell-1}_{m,n})$, then since $r_\ell = 2$,

$$e(L^{a,b}_{m,n}) = s_\ell n - b(n-m) - b(s_\ell - b) \geq (s_\ell - 1)(n-1) + m - s_\ell + 1 = e(K^{s_\ell-1,1}_{m,n}) \tag{3.6}$$

that is, $(b-1)(n-m) + b(s_\ell - b) - 2(s_\ell - 1) \leq 0$. Thus $b(s_\ell - b) \leq 2(s_\ell - 1)$ since $n \geq m$. (a) If $k_\ell = 4$, then $b \geq 3$ (by (3.5)) implies $s_\ell \geq 6$ and $3(s_\ell - 3) \leq b(s_\ell - b) \leq 2(s_\ell - 1)$. Thus $6 \leq s_\ell \leq 7$. If $s_\ell = 6$, then $b = 3$ ($9 \leq b(6-b) \leq 10$). So $2(n-m) - 1 \leq 0$ implies $n = m$. Furthermore, by Lemma 3.2(3), $k_i \in X_\ell$ is odd for any pair $(i,y) \in \mathcal{R}_\ell(3)$, so $F_\ell = P_9 \cup P_4$ with $m = n$, a contradiction (in this case, we suppose that $F_\ell \neq P_9 \cup P_4$ with $m = n$). If $s_\ell = 7$, then $12 \leq b(7-b) \leq 12$ implies $b = 3$. Thus $2(n-m) \leq 0$ implies $m = n$. And we also have $k_i \in X_\ell$ is odd for any pair $(i,y) \in \mathcal{R}_\ell(3)$, so $F_\ell = P_{11} \cup P_4$. Hence, if $k_\ell = 4$, then $F_\ell = P_{11} \cup P_4$ with $m = n$. Moreover, the equality of (3.6) holds when $F_\ell = P_{11} \cup P_4$ with $m = n$. (b) If $k_\ell = 5$, then by (3.5), $b \geq 3$. By the similar proof as that of case $k_\ell = 4$, we have $6 \leq s_\ell \leq 7$ and $b = 3$. However, X_ℓ is not odd, it implies that there is an even integer $k_i \in X_\ell$ such that $(i,y) \in \mathcal{R}_\ell(3)$. By Lemma 3.2(3), $L^{a,b}_{m,n}$ contains a copy of F_ℓ, a contradiction.

Therefore, $e(L^{a,b}_{m,n}) \leq e(K^{s_\ell-1,r_\ell-1}_{m,n}) = f(m,n;X_\ell)$, and equality holds only when $F_\ell = P_{11} \cup P_4$ with $m = n$ and $L^{a,b}_{m,n} = L^{4,3}_{m,n} \in \mathcal{F}(m,n;X_\ell)$.

This completes the proof of Lemma 3.3.

Let $G = G(A,B)$ be an F_ℓ-free bipartite graph with $|A| = m$ and $|B| = n$. We call a pair of subsets (A^*, B^*) in G with $A^* \subset A$ and $B^* \subset B$ a (p,q)-core of G if $d^c_G(A^*) \geq p$, $d^c_G(B^*) \geq q$ and one of the following holds:

(a) $|A^*| + |B^*| = s_\ell$ with $e_G(A^*, B^*) \geq 1$ and $r_\ell = 2$;

(b) $|A^*| = \sum_{i \in I_a} r_i + r_\ell - 1$ and $|B^*| = \sum_{i \in I_b} r_i + r_\ell - 1$ with $e_G(A^*, B^*) = 0$, where (I_a, I_b) is a partition of $[\ell - 1]$.

We call a (p,q)-core of G satisfying (a) Type A, and satisfying (b) Type B.

Lemma 3.4 *Let $G = G(A,B)$ be an F_ℓ-free bipartite graph with $|A| = m$ and $|B| = n$. Let $\ell \geq 2$ and $k_\ell \geq 4$. Suppose that $e(G) \geq f(m,n;X_\ell) - \Delta_1$, where $\Delta_1 \geq 0$. Let $n \geq m$ and m be sufficient large compare with x_ℓ and Δ_1. If (A^*, B^*) with $A^* \subset A$ and $B^* \subset B$ is an (x_ℓ, x_ℓ)-core of G, then there is a pair of subsets (A^*_1, B^*_1) with $A^*_1 \subset A$ and $B^*_1 \subset B$*

and a constant $0 < \xi = \xi(X_\ell, \Delta_1, |A^*|, |B^*|) \le 1$ *such that* (A_1^*, B_1^*) *is an* $(\xi n, \xi m)$*-core of* G*. Moreover,* $e(G) \le f(m, n; X_\ell)$*, where equality holds only when* $G \in \mathcal{F}(m, n; X_\ell)$*.*

Proof Let $G = G(A, B)$ be an F_ℓ-free bipartite graph with $|A| = m$ and $|B| = n$. Suppose that

$$e(G) \ge f(m, n; X_\ell) - \Delta_1. \tag{3.7}$$

Let $n \ge m$ and m be sufficient large than x_ℓ and Δ_1. Let (A^*, B^*) with $A^* \subset A$ and $B^* \subset B$ be an (x_ℓ, x_ℓ)-core of G. Let $|A^*| = a^*$, $|B^*| = b^*$ and $G^* = G - A^* \cup B^*$. Let $A_0 = N_G(B^*) - A^*$ and $B_0 = N_G(A^*) - B^*$.

We first show that

(1) G^* is P_{k_ℓ}-free. Suppose that G^* contains a copy of P_{k_ℓ}. If $a^* + b^* = s_\ell$ with $e_G(A^*, B^*) \ge 1$ and $r_\ell = 2$, then $G - P_{k_\ell}$ contains a copy of $D^{a^*, b^*}_{x_{\ell-1}, x_{\ell-1}}$. By Lemma 3.2(1), $D^{a^*, b^*}_{x_{\ell-1}, x_{\ell-1}}$ contains a copy of $F_{\ell-1}$ since $a^* + b^* = s_{\ell-1} + 2$, so G contains a copy of F_ℓ, a contradiction. If $a^* = \sum\limits_{i \in I_a} r_i + r_\ell - 1$ and $b^* = \sum\limits_{i \in I_b} r_i + r_\ell - 1$ with $e_G(A^*, B^*) = 0$, where (I_a, I_b) is a partition of $[\ell - 1]$, then $G - P_{k_\ell}$ contains a copy of $\bigcup\limits_{i \in I_a} P_{k_i}$ containing $a^* - r_\ell + 1$ vertices of A^* and a copy of $\bigcup\limits_{i \in I_b} P_{k_i}$ containing $b^* - r_\ell + 1$ vertices of B^*. Thus G contains a copy of F_ℓ, a contradiction.

(2) a^* is R_ℓ-sum-free. Otherwise, there is a partition (I_{a^*}, I_{b^*}) of $[\ell]$ such that $a^* = \sum\limits_{i \in I_{a^*}} r_i$ and $b^* \ge \sum\limits_{i \in I_{b^*}} r_i$. Then G contains a copy of $\bigcup\limits_{i \in I_{a^*}} P_{k_i}$ containing A^* and a copy of $\bigcup\limits_{i \in I_{b^*}} P_{k_i}$ containing $\sum\limits_{i \in I_{b^*}} r_i$ vertices of B^*, and those two linear forests are disjoint, a contradiction.

(3) There is no path between A_0 and B_0 in G^*. Otherwise, there is a path P_{t_1} with $t_1 \ge 2$ between B_0 and A_0 in G^*. Since a^* is R_ℓ-sum-free, $\mathcal{R}_\ell(a^*)$ is not empty. For any pair $(i, y) \in \mathcal{R}_\ell(a^*)$, G contains a copy of a path with order $2y + t_1 + 2(r_i - y) \ge 2r_i + 2 > k_i$ containing y vertices of A^*, $r_i - y$ vertices of B^* and the path P_{t_1}. By the definition of $\mathcal{R}_\ell(a^*)$ and $a^* + b^* \ge s_\ell$, $G - P_{k_i}$ contains a copy of $F_\ell - P_{k_i}$. Thus, G contains a copy of F_ℓ, a contradiction.

Let $G_a^* = C_{G - B^*}(A^*)$ and $G_b^* = C_{G - A^*}(B^*)$. Let $G_r = G - (V(G_a^*) \cup V(G_b^*))$. Thus $e(G) = e(G[A^*, B^*]) + e(G_a^*) + e(G_b^*) + e(G_r)$.

Let $A_1 = N_G^c(B^*) \cap A_0$, $A_2 = A_0 - A_1$, $A_3 = V(G_a^*) \cap A$, $A_4 = V(G_b^*) \cap A - A_0$ and $A_r = V(G_r) \cap A$. Let $B_1 = N_G^c(A^*) \cap B_0$, $B_2 = B_0 - B_1$, $B_3 = V(G_b^*) \cap B$, $B_4 = V(G_a^*) \cap B - B_0$ and $B_r = V(G_r) \cap B$ (see Figure 3). Let $|A_i| = a_i$ for $i = 1, 2, 3, 4, r$ and $|B_i| = b_i$ for $i = 1, 2, 3, 4, r$. Thus we have $a^* + a_1 + a_2 + a_3 + a_4 + a_r = |A| = m$ and $b^* + b_1 + b_2 + b_3 + b_4 + b_r = |B| = n$.

Since (A^*, B^*) is an (x_ℓ, x_ℓ)-core of G, we consider two cases as following.

Case 1 (A^*, B^*) is an (x_ℓ, x_ℓ)-core of G with Type A. Thus, $a^* + b^* = s_\ell$ with $e_G(A^*, B^*) \ge 1$ and $r_\ell = 2$.

Then $k_\ell \in \{4, 5\}$, $a^*, b^* \ge 1$ and $a^* + b^* \ge 4$. Since $e_G(A^*, B^*) \ge 1$ and $d_G^c(A^*), d_G^c(B^*) \ge x_\ell$, G contains a copy of $D^{a^*, b^*}_{x_\ell, x_\ell}$. Thus $D^{a^*, b^*}_{x_\ell, x_\ell}$ is F_ℓ-free (since G is F_ℓ-free). Hence, Lemma

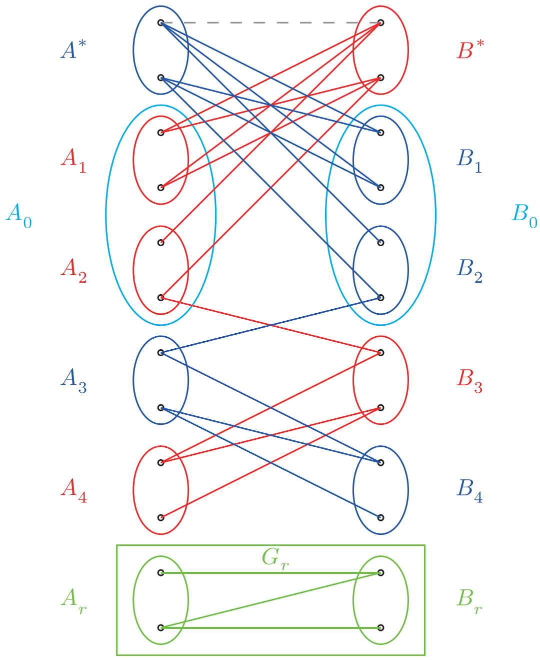


Figure 3 Graph G: G_a^* consists of blue parts and lines, G_b^* consist of red parts and lines and G_r consists of green parts and lines.

3.2(3) implies that $L_{m,n}^{a^*,b^*}$ is F_ℓ-free, a^* and b^* are R_ℓ-sum-free and k_i is odd for any pair $(i,y) \in \mathcal{R}_\ell(a^*) \cup \mathcal{R}_\ell(b^*)$. Therefore, since $r_\ell = 2$, we have

$$a^* \neq 2 \quad \text{and} \quad b^* \neq 2. \tag{3.8}$$

We claim that

$$N_{G^*}(A_1 \cup B_1) = \emptyset. \tag{3.9}$$

Otherwise, G contains both a copy of $D_{x_\ell,x_\ell}^{a^*,b^*,1}$ and a copy of $D_{x_\ell,x_\ell}^{a^*,b^*,0}$. Since a^* is R_ℓ-sum-free, $\mathcal{R}_\ell(a^*) \neq \emptyset$. Thus Lemma 3.2(2) implies that G contains a copy of F_ℓ, a contradiction.

Claim 3.2 $e(G_a^*) \leq a^* b_1 + (a^*-1)b_2 + a_3 + b_4$.

Proof Assume that there is a pair $(i_1, y_1) \in \mathcal{R}_\ell(a^*)$ with $y_1 \geq 2$. If at least two vertices of A^* are adjacent to B^*, then B_0 has no neighbor in G^*. Otherwise, G contains a copy of $D_{x_\ell,x_\ell}^{a^*,b^*,0}$, by Lemma 3.2(2), G contains a copy of F_ℓ, a contradiction. Thus $a_3 = b_4 = 0$ and $e(G_a^* - A^*) = 0$. Hence $e(G_a^*) \leq a^* b_1 + (a^*-1)b_2$, the claim holds. If only one vertex $u^* \in A^*$ is adjacent to B^*, then $N_G(A^* - \{u^*\}) \cap B_0$ has no neighbor in G^*. Otherwise, G contains a copy of $D_{x_\ell,x_\ell}^{a^*,b^*,0}$, by Lemma 3.2(2), it is a contradiction. By (3.8), we have $a^* \geq 3$ (since $y_1 \geq 2$ and $a^* \neq 2$). Let b_2' be the number of vertices of B_0 with degree at least one in G^*. Since G^* is P_{k_ℓ}-free and $r_\ell = 2$, by Corollary 1.1, we have $e(G_a^* - A^*) \leq b_2' + a_3 + b_4$. Thus by $a^* \geq 3$ and $b_2' \geq b_2$, $e(G_a^*) \leq a^* b_1 + (a^*-1)(b_2 - b_2') + b_2' + (a_3 + b_2' + b_4) \leq a^* b_1 + (a^*-1)b_2 + a_3 + b_4$, the claim holds.

Assume that $y = 1$ for any pair $(i, y) \in \mathcal{R}_\ell(a^*)$. If $a^* = 1$, then $b_2 = a_3 = b_4 = 0$ (by (3.9)). Thus $e(G_a^*) \le b_1$, the claim holds. If $a^* \ge 3$, then any neighbor of B_0 in G^* has degree exact one in G^*. Otherwise, suppose that $u_1 \in N_{G^*}(B_0)$ has two neighbors $v_1, v_2 \in V(G^*)$ with $v_1 \in B_0$. Since $a^* \ge 3$ and $(i, 1) \in \mathcal{R}_\ell(a^*)$, there is a subset $\emptyset \ne I_1 \subset [\ell] \setminus \{i\}$ such that $a^* = 1 + \sum_{i \in I_1} r_i$. For any integer $j \in I_1$, G contains a path P^1 of order $2(r_j - 1) + 1$ whose an end vertex is v_1 such that $|V(P^1) \cap A^*| = r_j - 1$. So G contains a path $P^2 = P^1 v_1 u_1 v_2$ of order $2r_j + 1 \ge k_j$ with $|V(P^2) \cap A^*| = r_j - 1$. It is clearly that $G - P^2$ contains a copy of $D_{x_\ell - k_j, x_\ell - k_j}^{a^* - r_j + 1, b^*}$, so $G - P^2$ contains a copy of $F_\ell - P_{k_j}$ (since $D_{x_\ell - k_j, x_\ell - k_j}^{a^* - r_j + 1, b^*}$ contains a copy of $F_\ell - P_{k_j}$ by Lemma 3.2(1)). Hence, G contains a copy of F_ℓ, a contradiction. Thus, $b_4 = 0$ and $e(G_a^* - A^*) \le a_3$. Hence $e(G_a^*) \le a^* b_1 + (a^* - 1) b_2 + a_3$, the claim holds.

Similarly, we have $e(G_b^*) \le b^* a_1 + (b^* - 1) a_2 + b_3 + a_4$. Then since G_r is P_{k_ℓ}-free and $r_\ell = 2$, $e(G_r) \le a_r + b_r$ by Corollary 1.1. Thus since $L_{m,n}^{a^*, b^*}$ is F_ℓ-free, by Lemma 3.3, we have

$$\begin{aligned} e(G) &= e(G[A^*, B^*]) + e(G_a^*) + e(G_b^*) + e(G_r) \\ &\le a^* b^* + a^* b_1 + (a^* - 1) b_2 + a_3 + b_4 + b^* a_1 + (b^* - 1) a_2 + b_3 + a_4 + e(G_r) \\ &= a^* n + b^* m - a^* b^* - (a_2 + b_2) - (a^* - 1)(b_3 + b_4) - (b^* - 1)(a_3 + a_4) \\ &\quad - a^* b_r - b^* a_r + e(G_r) \qquad (3.10a) \\ &\le a^* n + b^* m - a^* b^* - (a_2 + b_2) - (a^* - 1)(b_3 + b_4 + b_r) - (b^* - 1)(a_3 + a_4 + a_r) \\ &\le e(L_{m,n}^{a^*, b^*}) \le f(m, n; X_\ell). \qquad (3.10) \end{aligned}$$

Moreover, the equality holds only when $a_2 = b_2 = a_3 = b_3 = a_4 = b_4 = a_r = b_r = 0$, and $G = L_{m,n}^{a^*, b^*} \in \mathcal{F}(m, n; X_\ell)$. This completes the moreover part of the lemma in Case 1.

Since $L_{m,n}^{a^*, b^*}$ is F_ℓ-free, Lemma 3.3 implies that

$$e(G) = f(m, n; X_\ell) - \Delta_1 \ge a^* n + b^* m - a^* b^* - \Delta_1. \qquad (3.11)$$

Thus by (3.10a) and (3.10)–(3.11),

$$\begin{cases} a_2 + b_2 + (a^* - 1)(b_3 + b_4) + (b^* - 1)(a_3 + a_4) + a^* b_r + b^* a_r \le \Delta_1, & \text{if } e(G_r) = 0; \\ a_2 + b_2 + (a^* - 1)(b_3 + b_4 + b_r) + (b^* - 1)(a_3 + a_4 + a_r) \le \Delta_1, & \text{if } e(G_r) \ge 0. \end{cases} \qquad (3.12)$$

Hence, if $a^*, b^* \ge 2$, then by (3.12), $b_2 + b_3 + b_4 + b_r + a_2 + a_3 + a_4 + a_r \le \Delta_1$. Thus since $n \ge m$ and m is sufficient large, $a_1 \ge m - a^* - \Delta_1 \ge \frac{m}{2}$ and $b_1 \ge n - b^* - \Delta_1 \ge \frac{n}{2}$, this is, (A^*, B^*) is an $(\frac{n}{2}, \frac{m}{2})$-core of G with Type A. The lemma holds by setting $A_2^* = A^*$, $B_2^* = B^*$ and $\xi = \frac{1}{2}$.

If $a^* = 1$ or $b^* = 1$, without loss of generality, we may assume that $a^* = 1$ and $b^* = s_\ell - 1$. Then $b_2 = a_3 = b_4 = 0$. Since k_i is odd for any pair $(i, y) \in \mathcal{R}_\ell(1)$ (by Lemma 3.2(3)), $k_\ell = 5$ and X_ℓ is odd. By (3.12) and $b^* = s_\ell - 1 \ge 3$, we have $a_2 + b_2 + a_3 + a_4 + a_r \le \Delta_1$. Thus $a_1 \ge m - a^* - \Delta_1 \ge \frac{m}{2}$.

We will show that any neighbor of A_0 in G^* has degree exact one in G^*. Otherwise, G^* contains a copy of P_3 with an end vertex in A_0. Hence G contains a copy of P_5 containing one vertex of B^* and a copy of $F_\ell - P_5$ containing $s_\ell - 2 = s_{\ell-1}$ vertices of B^*, a contradiction. Thus $a_4 = 0$.

Furthermore, for any vertex $v^* \in B^*$, there is at most one vertex in $N_G(v^*) \cap A_0$ which has neighbors in G^*. Otherwise, G contains a copy of P_5 containing only one vertex v^* in B^* and four vertices of G^*, then G contains a copy of F_ℓ, a contradiction. Thus there are at most $b^* = s_\ell - 1$ vertices of A_0 have neighbors in G^*. Hence, there is one vertex of A_0 has degree at least $\frac{b_3}{(s_\ell-1)}$.

(i) If $b_r = 0$ or $a_r = 0$, then $e(G_r) = 0$. Since $a^* = 1$ and $b_2 = a_3 = b_4 = a_4 = 0$, by (3.12), $a_2 + a^* b_r + b^* a_r \le \Delta_1$. Then there is a vertex $u_1^* \in A$ has degree at least $\max\{b_1, \frac{b_3}{s_\ell - 1}\} \ge \frac{n-a^*-\Delta_1}{s_\ell} \ge \frac{n}{s_\ell+1}$ (since n is sufficient large). Thus $(\{u_1^*\}, B^*)$ is an $(\frac{n}{s_\ell+1}, \frac{m}{2})$-core of G with Type A. The lemma holds by setting $A_2^* = \{u_1^*\}$, $B_2^* = B^*$ and $\xi = \frac{1}{s_\ell+1}$.

(ii) If $b_r \ge 1$ and $a_r \ge 1$, then $e(G_r) \ge 0$. Since $a^* = 1$ and $b_2 = a_3 = b_4 = a_4 = 0$, by (3.12), $a_2 + a_r \le \Delta_1$.

By (3.10)–(3.12), we have

$$\begin{aligned} e(G_r) &= e(G) - (e(G[A^*, B^*]) + e(G_a^*) + e(G_b^*)) \\ &\ge a^* n + b^* m - a^* b^* - \Delta_1 - (a^* n + b^* m - a^* b^*) + a_2 + b_2 + (a^* - 1)(b_3 + b_4) \\ &\quad + (b^* - 1)(a_3 + a_4) + a^* b_r + b^* a_r \\ &\ge a_r + b_r - (\Delta_1 - (a_2 + b_2 + (a^* - 1)(b_3 + b_4 + b_r) + (b^* - 1)(a_3 + a_4 + a_r))) \\ &\ge \mathrm{ex}(a_r, b_r; P_{k_\ell}) - \Delta_1. \end{aligned}$$

Since the number of $K_{2,2}$ in G_r is at most $\frac{a_r}{2} \le \frac{\Delta_1}{2}$, Lemma 3.1 implies that there is a constant ε such that there is a vertex of $V(G_r) \cap A$ with degree at least $\varepsilon(b_r - \Delta_1)$. Thus there is a vertex $u_2^* \in A$ with $d_G(u_2^*) \ge \max\left\{b_1, \frac{b_3}{s_\ell-1}, \varepsilon(b_r - \Delta_1)\right\} \ge (n - a^* - \Delta_1)\frac{\varepsilon}{\varepsilon(s_\ell-1)+1} \ge \frac{\varepsilon n}{\varepsilon s_\ell + 2}$ (since n is sufficient large). Hence $(\{u_2^*\}, B^*)$ is an $\left(\frac{\varepsilon n}{\varepsilon s_\ell+2}, \frac{m}{2}\right)$-core with Type A or Type B. The lemma holds by setting $A_2^* = \{u_2^*\}$, $B_2^* = B^*$ and $\xi = \frac{\varepsilon}{\varepsilon s_\ell + 2}$.

Case 2 (A^*, B^*) is an (x_ℓ, x_ℓ)-core of G with Type B. That is, $a^* = \sum_{i \in I_a} r_i + r_\ell - 1$ and $b^* = \sum_{i \in I_b} r_i + r_\ell - 1$ with $e_G(A^*, B^*) = 0$, where (I_a, I_b) is a partition of $[\ell - 1]$.

Then any neighbor of B_0 in G^* has degree exact one in G^*. Otherwise, G^* contains a copy of P_3 with one end vertex in B_0, then one can find a copy of P_{k_ℓ} containing $r_\ell - 1$ vertices of A^*. By the condition of Case 2, G contains a copy of F_ℓ, a contradiction. Thus $b_4 = 0$, and $e(G_a^*) \le a^* b_1 + (a^* - 1) b_2 + a_3$. Similarly, we have that any neighbor of A_0 in G^* has degree exact one in G^*, $a_4 = 0$ and $e(G_b^*) \le b^* a_1 + (b^* - 1) a_2 + b_3$.

By Corollary 1.1, we have $e(G_r) \le \mathrm{ex}(a_r, b_r; P_{k_\ell}) \le (r_\ell - 1)(a_r + b_r)$ for any $a_r \ge 0$, $b_r \ge 0$ and $k_\ell \ge 4$. Thus

$$\begin{aligned} e(G) &= e(G[A^*, B^*]) + e(G_a^*) + e(G_b^*) + e(G_r) \\ &\le a^* b_1 + (a^* - 1) b_2 + a_3 + b^* a_1 + (b^* - 1) a_2 + b_3 + e(G_r) \end{aligned}$$

$$= a^*n + b^*m - 2a^*b^* - (a_2 + b_2) - (a^* - 1)b_3 - (b^* - 1)a_3$$
$$- a^*b_r - b^*a_r + e(G_r) \tag{3.13a}$$
$$\leq a^*n + b^*m - 2a^*b^* - (a_2 + b_2) - (a^* - 1)b_3 - (a^* - r_\ell + 1)b_r - (b^* - 1)a_3$$
$$- (b^* - r_\ell + 1)a_r$$
$$\leq e(K_{m,n}^{a^*,b^*}) \leq f(m, n; X_\ell). \tag{3.13}$$

Moreover, the equality holds only when $a_2 = b_2 = a_3 = b_3 = a_r = b_r = 0$ and $G = K_{m,n}^{s_\ell-1,r_\ell-1} \in \mathcal{F}(m, n; X_\ell)$. This completes the moreover part of the lemma in Case 2.

Since

$$e(G) = f(m, n; X_\ell) - \Delta_1 \geq a^*n + b^*m - 2a^*b^* - \Delta_1, \tag{3.14}$$

by (3.13a) and (3.13),

$$\begin{cases} a_2 + b_2 + (a^* - 1)b_3 + (b^* - 1)a_3 + a^*b_r + b^*a_r \leq \Delta_1, & \text{if } e(G_r) = 0; \\ a_2 + b_2 + (a^* - 1)b_3 + (a^* - r_\ell + 1)b_r + (b^* - 1)a_3 & \\ +(b^* - r_\ell + 1)a_r \leq \Delta_1, & \text{if } e(G_r) \geq 0. \end{cases} \tag{3.15}$$

Hence, if $a^*, b^* \geq r_\ell \geq 2$, then by (3.15), $b_2 + b_3 + b_r + a_2 + a_3 + +a_r \leq \Delta_1$. Thus since $n \geq m$ and m is sufficient large, $a_1 \geq m - a^* - \Delta_1 \geq \frac{m}{2}$ and $b_1 \geq n - b^* - \Delta_1 \geq \frac{n}{2}$. Hence (A^*, B^*) is an $\left(\frac{n}{2}, \frac{m}{2}\right)$-core of G with Type B. The lemma holds by setting $A_2^* = A^*$, $B_2^* = B^*$ and $\xi = \frac{1}{2}$.

If $a^* = r_\ell - 1$ or $b^* = r_\ell - 1$, without loss of generality, we may assume that $b^* = r_\ell - 1$ and $a^* = s_\ell - 1$. By (3.15), $a_2 + b_2 + b_3 + b_r \leq \Delta_1$, so $b_1 \geq n - a^* - \Delta_1 \geq \frac{n}{2}$ (since n is sufficient large). Then we consider the following two subcases.

Subcase 2.1 If $k_\ell \in \{4, 5\}$, then $b^* = 1$ and $a_2 = 0$. (a) If X_ℓ is not odd, then B_0 has no neighbor in G^*. Otherwise, G contains a copy of an even path P_{k_i} containing exact $r_i - 1$ vertices of A^*, then $G - P_{k_i}$ contains a copy of $F_\ell - P_{k_i}$ containing $s_\ell - 1 - r_i + 1 = s_\ell - r_i$ vertices of A^*. Thus G contains a copy of F_ℓ, a contradiction. Hence $a_3 = 0$. (b) If X_ℓ is odd, then $k_\ell = 5$. We claim that for any vertex $u^* \in A^*$, there is at most one vertex in $N_G(u^*) \cap B_0$ with neighbors in G^*. Otherwise, G contains a copy of P_5 containing only one vertex u^* in A^* and four vertices of G^*, then G contains a copy of F_ℓ, a contradiction. Thus there are at most $a^* = s_\ell - 1$ vertices of B_0 with neighbors in G^*. Hence, there is one vertex of B_0 has degree at least $\frac{a_3}{s_\ell - 1}$.

(i) If $b_r = 0$ or $a_r = 0$, then $e(G_r) = 0$. Since $b^* = r_\ell - 1 = 1$ and $a_2 = a_4 = b_4 = 0$, by (3.15), $b_2 + b_3 + b_r + a_r \leq \Delta_1$. If X_ℓ is not odd, then $a_3 = 0$ implies $a_1 \geq m - a^* - \Delta_1 \geq \frac{m}{2}$ (since m is sufficient large). So (A^*, B^*) is an $\left(\frac{n}{2}, \frac{m}{2}\right)$-core of G of Type B. If X_ℓ is odd with $k_\ell = 5$, then there is a vertex v_1^* of B with degree at least $\max\left\{a_1, \frac{a_3}{s_\ell - 1}\right\} \geq \frac{m}{s_\ell + 1}$ since m is sufficient large. Thus $(A^*, \{v_1^*\})$ is an $\left(\frac{n}{2}, \frac{m}{s_\ell + 1}\right)$-core of G with Type A or Type B. The lemma holds by setting $A_2^* = A^*$, $B_2^* = \{v_1^*\}$ and $\xi = \frac{1}{s_\ell + 1}$.

(ii) If $b_r \geq 1$ and $a_r \geq 1$, then $e(G_r) \geq 0$. By (3.15), $a_2+b_2+b_3+b_r \leq \Delta_1$. By (3.13)–(3.15),

$$\begin{aligned} e(G_r) &= e(G) - (e(G[A^*, B^*]) + e(G_a^*) + e(G_b^*)) \\ &\geq a^*n + b^*m - 2a^*b^* - \Delta_1 - (a^*n + b^*m - 2a^*b^*) + a_2 + b_2 + (a^* - 1)b_3 \\ &\quad + (b^* - 1)a_3 + a^*b_r + b^*a_r \\ &\geq a_r + b_r - (\Delta_1 - (a_2 + b_2 + (a^* - 1)(b_3 + b_r) + (b^* - 1)(a_3 + a_r))) \\ &\geq \mathrm{ex}(a_r, b_r; P_{k_\ell}) - \Delta_1. \end{aligned}$$

When $k_\ell = 5$, the number of $K_{2,2}$ in G_r is at most $\frac{b_r}{2} \leq \frac{\Delta_1}{2}$. So, by Lemma 3.1, there is a constant ε and a vertex of $V(G_r) \cap B$ with degree at least $\varepsilon(a_r - \Delta_1)$. Thus if X_ℓ is not odd, then there is a vertex $v_2^* \in B$ with $d_G(v_2^*) \geq \max\{a_1, \varepsilon(a_r - \Delta_1)\} \geq \frac{(m-a^*-\Delta_1)\varepsilon}{\varepsilon+1} \geq \frac{\varepsilon m}{\varepsilon+2}$ since m is sufficient large. Thus $(A^*, \{v_2^*\})$ is an $(\frac{n}{2}, \frac{\varepsilon m}{\varepsilon+2})$-core of G of Type B, the lemma holds by setting $A_2^* = A^*$, $B_2^* = \{v_2^*\}$ and $\xi = \frac{\varepsilon}{\varepsilon+2}$. If X_ℓ is odd with $k_\ell = 5$, then there is a vertex $v_3^* \in B$ with $d_G(v_3^*) \geq \max\left\{a_1, \frac{a_3}{s_\ell - 1}, \varepsilon(a_r - \Delta_1)\right\} \geq \frac{(m-a^*-\Delta_1)\varepsilon}{\varepsilon(s_\ell-1)+1} \geq \frac{\varepsilon m}{\varepsilon s_\ell+2}$ (since m is sufficient large). Thus $(A^*, \{v_3^*\})$ is an $(\frac{n}{2}, \frac{\varepsilon m}{\varepsilon s_\ell+2})$-core of G of Type A or Type B, the lemma holds by setting $A_2^* = A^*$, $B_2^* = \{v_3^*\}$ and $\xi = \frac{\varepsilon}{\varepsilon s_\ell+2}$.

Subcase 2.2 If $k_\ell \geq 6$, then $b^* = r_\ell - 1 \geq 2$.

(i) If $b_r = 0$ or $a_r = 0$, then $e(G_r) = 0$. By (3.15), $a_2 + b_2 + b_3 + b_r + a_3 + a_r \leq \Delta_1$. Since $n \geq m$ and m is sufficient large, $a_1 \geq m - a^* - \Delta_1 \geq \frac{m}{2}$ and $b_1 \geq n - b^* - c \geq \frac{n}{2}$. This is, (A^*, B^*) is an $(\frac{n}{2}, \frac{m}{2})$-core of G of Type B. The lemma holds by setting $A_2^* = A^*$, $B_2^* = B^*$ and $\xi = \frac{1}{2}$.

(ii) If $b_r \geq 1$ and $a_r \geq 1$, then by (3.15), $a_2 + b_2 + b_3 + b_r + a_3 \leq \Delta_1$. By (3.13)–(3.15),

$$\begin{aligned} e(G_r) &= e(G) - (e(G[A^*, B^*]) + e(G_a^*) + e(G_b^*)) \\ &\geq a^*n + b^*m - 2a^*b^* - \Delta_1 - (a^*n + b^*m - 2a^*b^*) + a_2 + b_2 + (a^* - 1)b_3 \\ &\quad + (b^* - 1)a_3 + a^*b_r + b^*a_r \\ &\geq (r_\ell - 1)(a_r + b_r) - (\Delta_1 - (a_2 + b_2 + (a^* - 1)b_3 + (a^* - r_\ell + 1)b_r \\ &\quad + (b^* - 1)a_3 + (b^* - r_\ell + 1)a_r)) \\ &\geq \mathrm{ex}(a_r, b_r; P_{k_\ell}) - \Delta_1. \end{aligned}$$

Then by Lemma 3.1, there is a subset $B_1^* \subset B$ with $|B_1^*| = r_\ell - 1$ and a constant ε such that $d_{G^*}(B_1^*) \geq \max\{a_1, \varepsilon a_r\} \geq \frac{m-a^*-\Delta_1}{\varepsilon+1} \geq \frac{m}{\varepsilon+2}$ (since m is sufficient large). Hence (A^*, B_1^*) is an $(\frac{n}{2}, \frac{m}{\varepsilon+2})$-core of G of Type B, the lemma holds by setting $A_2^* = A^*$, $B_2^* = B_1^*$ and $\xi = \frac{1}{\varepsilon+2}$.

This completes the proof of Lemma 3.4.

Lemma 3.5 *Let $G = G(A, B)$ be an F_ℓ-free bipartite graph with $|A| = m$ and $|B| = n$. Let $\ell \geq 2$ and $k_\ell \geq 4$. Let $\Delta_2 \geq 0$ and $\delta_2 \geq 0$. Let $n \geq m$ and m be sufficiently large than x_ℓ, Δ_2 and δ_2. Suppose that G contains a copy of $F_{\ell-1} = P_{k_1} \cup \cdots \cup P_{k_{\ell-1}}$. Let $A_0 = A - V(F_{\ell-1})$,*

$B_0 = B - V(F_{\ell-1})$. If $e_G(x,y) \geq s_\ell + r_\ell - 2$ for each $x \in A_0 - A_0'$ and $y \in B_0 - B_0'$ with $|A_0'|, |B_0'| \leq \delta_2$ and

$$e(G) \geq f(m,n;X_\ell) - \Delta_2,$$

then there is a pair of subsets (A_2^*, B_2^*) with $A_2^* \subset A$ and $B_2^* \subset B$ which is an (x_ℓ, x_ℓ)-core of G.

Proof Let $G = G(A,B)$ be an F_ℓ-free bipartite graph with $|A| = m$ and $|B| = n$. Let $\ell \geq 2$ and $k_\ell \geq 4$. Let $\Delta_2 \geq 0$ and $\delta_2 \geq 0$. Let

$$n \geq m \text{ and } m \text{ be larger enough than } x_\ell, \Delta_2 \text{ and } \delta_2. \tag{3.16}$$

Let

$$e(G) \geq f(m,n;X_\ell) - \Delta_2. \tag{3.17}$$

Suppose that G contains a copy of $F_{\ell-1}$. Let $A_0 = A - V(F_{\ell-1})$, $B_0 = B - V(F_{\ell-1})$, $A_1 = A - A_0$, $B_1 = B - B_0$ and $G^0 = G(A_0, B_0)$. Let $|A_0| = m_0$ and $|B_0| = n_0$. Let $A_0' \subset A_0$ and $B_0' \subset B_0$ with $|A_0'|, |B_0'| \leq \delta_2$. Assume that

$$e_G(x,y) \geq s_\ell + r_\ell - 2 \text{ for each } x \in A_0 - A_0' \text{ and } y \in B_0 - B_0'. \tag{3.18}$$

Since G is F_ℓ-free, G^0 is P_{k_ℓ}-free. Noting that $m_0 \geq m - x_{\ell-1}$ and $n_0 \geq n - x_{\ell-1}$, by (3.16), m_0 and n_0 are sufficient large. It follows from Corollary 1.1 that

$$e(G^0) \leq \mathrm{ex}(m_0, n_0; P_{k_\ell}) \leq (r_\ell - 1)(m_0 + n_0). \tag{3.19}$$

Note that

$$f(m,n;X_\ell) = \begin{cases} (s_\ell - 1)(n-1) + m, & \text{if } X_\ell \text{ is odd with } k_\ell = 5; \\ 6m - 9, & \text{if } F_\ell = P_9 \cup P_4 \text{ with } m = n; \\ (s_\ell - 1)(n - r_\ell + 1) + (r_\ell - 1)(m - s_\ell + 1), & \text{otherwise.} \end{cases} \tag{3.20}$$

Since $e(G(A_1,B_0)) + e(G(A_0,B_1)) = e(G) - e(G^0) - e(G(A_1,B_1)) \geq f(m,n;X_\ell) - \Delta_2 - e(G^0) - e(G(A_1,B_1))$, combining (3.19)–(3.20) and $e(G(A_1,B_1)) \leq \left(\frac{x_{\ell-1}}{2}\right)^2$, we have

$$e(G(A_1,B_0)) + e(G(A_0,B_1)) \geq s_{\ell-1} n - \eta_1,$$

where $\eta_1 = 2(r_\ell - 1)(s_\ell - 1) + \left(\frac{x_{\ell-1}}{2}\right)^2 + \Delta_2$. By (3.16), we may assume that $e(G(A_1,B_0)) = c_1 n - \eta_1$ and $e(G(B_1,A_0)) = c_2 n - \eta_1$ with $c_1 + c_2 \geq s_{\ell-1}$. By Lemma 2.1, there exist $A_2 \subseteq A_1$ with $|A_2| = \lceil c_1 \rceil$ and $B_2 \subseteq B_1$ with $|B_2| = s_{\ell-1} - \lceil c_1 \rceil$ and a constant $0 < \delta \leq 1$ from Lemma 2.1 such that

$$d_G^c(A_2) \geq \delta n \quad \text{and} \quad d_G^c(B_2) \geq \delta n. \tag{3.21}$$

Let $|A_2| = a_2$, $|B_2| = b_2$ and $G' = G(A - A_2, B - B_2)$. So $a_2 + b_2 = s_{\ell-1}$, and $|V(G') \cap A| = m - a_2$ and $|V(G') \cap B| = n - b_2$ are sufficient large by (3.16).

Let $X_1 = N_G^c(A_2) \cap (B_0 - B_0')$ and $Y_1 = N_G^c(B_2) \cap (A_0 - A_0')$. Thus

$$|X_1| \geq \delta n - x_{\ell-1} - \delta_2 \quad \text{and} \quad |Y_1| \geq \delta n - x_{\ell-1} - \delta_2. \tag{3.22}$$

Claim 3.3 If G' is P_{k_ℓ}-free, then there is a constant η_2 such that $|X_1|, |Y_1| \geq n - \eta_2$ and $m \leq n \leq m + \eta_2$. Moreover, there are two subsets of $A_2' \subset A - A_2$ and $B_2' \subset B - B_2$ with $|A_2'| = |B_2'| = r_\ell - 1$ such that $d_G^c(A_2 \cup A_2') \geq x_\ell$ and $d_G^c(B_2 \cup B_2') \geq x_\ell$.

Proof Suppose that G' is P_{k_ℓ}-free, by Corollary 1.1, we have

$$e(G') \leq \mathrm{ex}(m - a_2, n - b_2; P_{k_\ell}) \leq (r_\ell - 1)(m + n - s_{\ell-1}). \tag{3.23}$$

Then

$$e(G(A_2, B - B_2)) + e(G(A - A_2, B_2)) \geq e(G) - e(G') - e(G(A_2, B_2)) \geq s_{\ell-1}n - \eta_2',$$

where $\eta_2' \leq 2(r_\ell - 1)(s_\ell - 1) + \left(\frac{s_{\ell-1}}{2}\right)^2 + \Delta_2$. By $|A - A_2| \leq m$, $|B - B_2| \leq n$ and $a_2 + b_2 = s_{\ell-1}$, each vertex in A_2 and B_2 has degree at least $n - \eta_2'$. Let $\eta_2 = \eta_2' + \delta_2$. Thus $|X_1| \geq n - \eta_2$ and $|Y_1| \geq n - \eta_2$, and $m \leq n \leq m + \eta_2$.

Since $e(G') = e(G) - e(G(A_2, B)) - e(G(A - A_2, B_2)) \geq e(G) - s_{\ell-1}n + b_2(n - m + a_2)$, by (3.20) and Corollary 1.1,

$$\begin{aligned} e(G') &\geq (r_\ell - 1)(m + n - s_{\ell-1}) - (2(r_\ell - 1)(s_\ell - 1) + \Delta_2) \\ &\geq \mathrm{ex}(m - a_2, n - b_2; P_{k_\ell}) - (2(r_\ell - 1)(s_\ell - 1) + \Delta_2). \end{aligned} \tag{3.24}$$

Note that for $k_\ell = 5$, there are at most one copies of $K_{2,2}$ in $G(X_1, Y_1)$ (otherwise one can find a copy fo F_ℓ). Thus there are at most $\frac{\eta_2}{2}$ copies of $K_{2,2}$ in G'. Thus by Lemma 3.1 and (3.24), we have that for $k_\ell \geq 4$, there are two subsets $A_2' \subset A - A_2$ and $B_2' \subset B - B_2$ with $|A_2'| = |B_2'| = r_\ell - 1$ and a constant γ such that $d_{G'}^c(A_2') \geq \gamma n$ and $d_{G'}^c(B_2') \geq \gamma n$.

Moreover, noting that $d_G^c(A_2) \geq |X_1| \geq n - \eta_2$ and $d_G^c(A_2') \geq d_{G'}^c(A_2') \geq \gamma n$, by (3.16), $d_G^c(A_2 \cup A_2') \geq \gamma n - \eta_2 \geq x_\ell$. Similarly, we have $d_G^c(B_2 \cup B_2') \geq \gamma n - \eta_2 \geq x_\ell$.

Claim 3.4 a_2 is $R_{\ell-1}$-sum-free.

Proof Suppose that a_2 is not $R_{\ell-1}$-sum-free. Then there is a partition (I_a^2, I_b^2) of $[\ell - 1]$ such that $a = \sum_{i \in I_a^2} r_i$ and $b_2 = \sum_{i \in I_b^2} r_i$. And we claim that G' is P_{k_ℓ}-free. Otherwise, G' contains a copy of P_{k_ℓ}, then $G - P_{k_\ell}$ contains a copy of $D_{\delta n, \delta n}^{a_2, b_2}$ (when $e(G[A_2, B_2]) \geq 1$) or a copy of $K_{\delta n, \delta n}^{a_2, b_2}$ (when $e(G[A_2, B_2]) = 0$). Since a_2 is not $R_{\ell-1}$-sum-free, by Lemma 3.2(3) and the definition of $K_{\delta n, \delta n}^{a_2, b_2}$, we have that $G - P_{k_\ell}$ contains a copy of $F_{\ell-1}$. That is, G contains a copy of F_ℓ, a contradiction.

By Claim 3.1, we have that there are two subsets of $U_1 \subset A - A_2$ and $U_1 \subset B - B_2$ with $|U_1| = |V_1| = r_\ell - 1$ such that $d_G^c(A_2 \cup U_1) \geq x_\ell$ and $d_G^c(B_2 \cup V_1) \geq x_\ell$.

Note that $|A_2 \cup U_1| = \sum_{i\in I_a^2} r_i + r_\ell - 1$ and $|B_2 \cup V_1| = \sum_{i\in I_b^2} r_i + r_\ell - 1$. If $e(G[A_2, B_2]) \geq 1$, then G contains a copy of $D_{x_\ell,x_\ell}^{a_2+r_\ell-1,b_2+r_\ell-1}$. If $r_\ell \geq 3$, then $a_2+r_\ell-1+b_2+r_\ell-1 = s_\ell+r_\ell-2 \geq s_\ell+1$. By Lemma 3.2(1), $D_{x_\ell,x_\ell}^{a_2+r_\ell-1,b_2+r_\ell-1}$ contains a copy of F_ℓ. Thus $r_\ell = 2$. Hence $(A_2\cup U_1, B_2\cup V_1)$ is a (x_ℓ, x_ℓ)-core of G with Type A. The lemma holds by setting $A_2^* = A_2\cup U_1$ and $B_2^* = B_2\cup V_1$. If $e(G[A_2, B_2]) = 0$, then $(A_2 \cup U_1, B_2 \cup V_1)$ is a (x_ℓ, x_ℓ)-core of G with Type B. The lemma holds by setting $A_2^* = A_2 \cup U_1$ and $B_2^* = B_2 \cup V_1$.

Claim 3.5 $e(G[A_2, B_2]) = 0$.

Proof Let $\theta = \binom{\lceil k_\ell \rceil}{r_\ell - 1}$ and $\mu = x_\ell\theta$. Suppose that $e(G[A_2, B_2]) \geq 1$. By (3.22) and (3.16), we have $|X_1| \geq \mu$ and $|Y_1| \geq \mu$. We consider two cases.

(a) G' is P_{k_ℓ}-free. By Claim 3.3, there is a constant η_2 such that $|X_1|, |Y_1| \geq n - \eta_2$, $n = m + \eta_2$ and there are two subsets $U_2 \subset A - A_2$ and $V_2 \subset B - B_2$ with $|U_2| = |V_2| = r_\ell - 1$ such that $d_G^c(A_2 \cup U_2) \geq x_\ell$ and $d_G^c(B_2 \cup V_2) \geq x_\ell$.

If $k_\ell \in \{4, 5\}$, then since $e(G[A_2, B_2]) \geq 1$, $(A_2 \cup U_2, B_2 \cup V_2)$ is an (x_ℓ, x_ℓ)-core of G with Type A. Thus the lemma holds by setting $A_2^* = A_2 \cup U_2$ and $B_2^* = B_2 \cup V_2$. If $k_\ell \geq 6$, then $a_2^* + b_2^* = s_\ell + r_\ell - 2 \geq s_\ell + 1$. Since G contains a copy of $D_{x_\ell,x_\ell}^{a_2^*,b_2^*}$, Lemma 3.2(1) implies that G contains a copy of F_ℓ, a contradiction.

(b) G' contains a copy of P_{k_ℓ}. Then $G - V(P_{k_\ell})$ is $F_{\ell-1}$-free. Since $e(G[A_2, B_2]) \geq 1$, each vertex of X_1 can only be adjacent to vertices of $V(P_{k_\ell}) \cup A_2$ and each vertex of Y_1 can only be adjacent to vertices of $V(P_{k_\ell}) \cup B_2$. Otherwise, $G - V(P_{k_\ell})$ contains both a copy of $D_{x_\ell,x_\ell}^{a_2,b_2,1}$ and a copy of $D_{x_\ell,x_\ell}^{a_2,b_2,0}$. By Claim 3.4, a_2 is $R_{\ell-1}$-sum-free, so $\mathcal{R}_{\ell-1}(a) \neq \emptyset$. Lemma 3.2(2) implies that $G - V(P_{k_\ell})$ contains a copy of $F_{\ell-1}$. Thus G contains a copy of F_ℓ, it is a contradiction. Hence, by (3.18), each pair (x, y) with $x \in X_1$ and $y \in Y_1$ is adjacent to at least $2r_\ell - 2$ vertices of $V(P_{k_\ell})$.

If $k_\ell \in \{4, 5\}$, then $2r_\ell - 2 = 2$. Thus there are two subsets $U_3 \subset V(P_{k_\ell}) \cap A$ and $V_3 \subset V(P_{k_\ell}) \cap B$ with $|U_3| + |V_3| = 2$ such that $|N_G^c(U_3) \cap X_1| \geq \frac{|X_1|}{\theta} \geq x_\ell$ and $|N_G^c(V_3) \cap Y_1| \geq \frac{|Y_1|}{\theta} \geq x_\ell$. Thus $d_G^c(A_2 \cup U_3) \geq |N_G^c(U_3) \cap X_1| \geq x_\ell$ and $d_G^c(B_2 \cup V_3) \geq |N_G^c(V_3) \cap Y_1| \geq x_\ell$. Hence $(A_2 \cup U_3, B_2 \cup V_3)$ is an (x_ℓ, x_ℓ)-core of G with Type A. The lemma holds by setting $A_2^* = A_2 \cup U_3$ and $B_2^* = B_2 \cup V_3$.

We may assume that $k_\ell \geq 6$. If there are more than r_ℓ vertices of X_1 have degree at least r_ℓ vertices in $V(P_{k_\ell}) \cap A$, then G contains a path of order k_ℓ (say P'_{k_ℓ}) containing r_ℓ vertices in $V(P_{k_\ell}) \cap A$ and $k_\ell - r_\ell$ vertices in X_1. Since $2r_\ell - 2 \geq r_\ell + 1$, $X_1 \cup Y_1$ has at least one neighbor in $V(G^*) \setminus V(P'_{k_\ell})$. Then $G - P'_{k_\ell}$ contains a copy of $F_{\ell-1}$ and G contains a copy of F_ℓ, a contradiction. Similarly, the number of vertices in Y_1 which are adjacent to at least r_ℓ vertices of $V(P_{k_\ell}) \cap B$ is at most r_ℓ.

However, if there is a vertex $x \in X_1$ such that $d_{V(P_{k_\ell})}(x) \leq r_\ell - 2$, then each vertex of Y_1 is adjacent to at least r_ℓ vertices in $V(P_{k_\ell})$. Thus any vertex of $X_1 \cup Y_1$ is adjacent to at least $r_\ell - 1$ vertices of $V(P_{k_\ell})$. So there exist subsets $U_4 \subset V(P_{k_\ell}) \cap A$ and $V_4 \subset V(P_{k_\ell}) \cap B$ with

$|U_4| = |V_4| = r_\ell - 1$ such that $|N_G^c(U_4) \cap X_1| \geq \frac{|X_1|}{\theta} \geq x_\ell$ and $|N_G^c(V_4) \cap Y_1| \geq \frac{|Y_1|}{\theta} \geq x_\ell$. Then G contains a copy of $D_{x_\ell, x_\ell}^{a_2+r_\ell-1, b_2+r_\ell-1}$. Since $a_2 + b_2 + 2r_\ell - 2 \geq s_\ell + 1$, by Lemma 3.2(1), G contains a copy of F_ℓ, a contradiction. The proof of Claim 3.5 is complete.

Claim 3.6 There is no edge between $N_{V(G')}(A_2)$ and $N_{V(G')}(B_2)$.

Proof Suppose that there are two vertices $v_0 \in N_{V(G')}(A_2)$ and $u_0 \in N_{V(G')}(B_2)$ such that u_0v_0 is an edge of G'. Let $H = G - (A_2 \cup B_2 \cup \{u_0, v_0\})$. We claim that H is P_{k_ℓ}-free. Otherwise, H contains a copy of P_{k_ℓ}. By Claim 3.4, a_2 is $R_{\ell-1}$-sum-free, so $\mathcal{R}_{\ell-1}(a_2) \neq \emptyset$. Thus there is a pair $(i, y) \in \mathcal{R}_{\ell-1}(a_2)$. Then G contains a path of order $2r_i + 2 > k_i$ containing the edge u_0v_0 and y vertices of A_2 and $r_i - y$ vertices of B_2. Hence by (3.16) and (3.21), $G - P_{k_\ell}$ contains a copy of $F_{\ell-1}$. Therefore, G contains a copy of F_ℓ, a contradiction.

If there is a subset $A_3 \subset A_2$ with $|A_3| = a_2 - 1$ such that $|N_G^c(A_3 \cup \{u_0\}) \cap (A_0 - A_0')| \geq \mu$ (or there is a subset $B_3 \subset B_2$ with $|B_3| = b_2 - 1$ such that $|N_G^c(B_3 \cup \{v_0\}) \cap (B_0 - B_0')| \geq \mu$), then by Claim 3.5 with $A_2 = A_3 \cup \{u_0\}$ and $B_2 = B_2$ (or $A_2 = A_2$ and $B_2 = B_3 \cup \{v_0\}$), a contradiction.

Then we may assume that for any subset $A_3 \subset A_2$ with $|A_3| = a_2 - 1$, $|N_G^c(A_3 \cup \{u_0\}) \cap (A_0 - A_0')| \leq \mu$ and for any subset $B_3 \subset B_2$ with $|B_3| = b_2 - 1$, $|N_G^c(B_3 \cup \{v_0\}) \cap (B_0 - B_0')| \leq \mu$. Since $|V(H) \cap A| - |A_0 - A_0'| \leq x_\ell + \delta_2$ and $|V(H) \cap B| - |B_0 - B_0'| \leq x_\ell + \delta_2$, $|N_G^c(A_2 \cup \{u_0\}) \cap V(H)| \leq \mu + x_\ell + \delta_2$ and $|N_G^c(B_2 \cup \{v_0\}) \cap V(H)| \leq \mu + x_\ell + \delta_2$. Thus

$$e(G[A_2 \cup \{u_0\}, B]) + e(G[A - (A_2 \cup u_0), B_2 \cup \{v_0\}]) \leq a_2 n + b_2 m + 2(\mu + x_\ell + \delta_2). \quad (3.25)$$

Furthermore, since H is P_{k_ℓ}-free, by Corollary 1.1, $e(H) \leq (r_\ell - 1)(m + n - s_{\ell-1} - 2)$. Thus by (3.17) and (3.20),

$$\begin{aligned}
&e(G[A_2 \cup \{u_0\}, B]) + e(G[A - (A_2 \cup u_0), B_2 \cup \{v_0\}]) = e(G) - e(H) \\
&\geq (s_\ell - 1)n + (r_\ell - 1)m - 2(r_\ell - 1)(s_\ell - 1) - \Delta_2 - (r_\ell - 1)(m + n - s_{\ell-1} - 2) \\
&\geq s_{\ell-1}n - (\Delta_2 + 2(r_\ell - 1)(s_\ell - 1)).
\end{aligned}$$

Hence

$$|N_G^c(A_2) \cap V(H)| \geq n - \eta_3 \quad \text{and} \quad |N_G^c(B_2) \cap V(H)| \geq n - \eta_3, \quad (3.26)$$

where $\eta_3 = \Delta_2 + 2(r_\ell - 1)(s_\ell - 1) + \mu + x_\ell + \delta_2 + a_2\mu + b_2\mu$.

Moreover, by (3.17), (3.20), (3.25) and Corollary 1.1,

$$\begin{aligned}
e(H) &= e(G) - e(G[A_2 \cup \{u_0\}, B]) - e(G[A - (A_2 \cup u_0), B_2 \cup \{v_0\}]) \\
&\geq (s_\ell - 1)n + (r_\ell - 1)m - 2(r_\ell - 1)(s_\ell - 1) - \Delta_2 - a_2 n - b_2 m - 2(\mu + x_\ell + \delta_2) \\
&\geq (r_\ell - 1)(m + n - s_\ell - 2) - (\Delta_2 + 2(r_\ell - 1)(s_\ell - 1) + 2(\mu + x_\ell + \delta_2)) \\
&\geq \mathrm{ex}(m - a_2 - 1, n - b_2 - 1; P_{k_\ell}) - \eta_4, \quad (3.27)
\end{aligned}$$

where $\eta_4 = \Delta_2 + 2(r_\ell - 1)(s_\ell - 1) + 2(\mu + x_\ell + \delta_2)$.

Note that for $k_\ell = 5$, there are at most one copy of $K_{2,2}$ in $G[N^c_G(A_2)\cap V(H), N^c_G(B_2)\cap V(H)]$ (otherwise, one can find a copy of F_ℓ in G). Thus by (3.26), there are at most $\frac{\eta_3}{2}$ copies of $K_{2,2}$ in H. By (3.27) and Lemma 3.1, there are two subsets $U_5 \subset A - (A_2 \cup \{u_0\})$ and $V_5 \subset B - (B_2 \cup \{v_0\})$ with $|U_5| = |V_5| = r_\ell - 1$ such that $d^c_H(U_5), d^c_H(V_5) \geq \varepsilon n$.

Thus by (3.26) and (3.16), $d^c_G(A_2 \cup U_5) \geq \varepsilon n - \eta_3$ and $d^c_G(B_2 \cup V_5) \geq \varepsilon n - \eta_3 \geq x_\ell$. Since $|A_2 \cup U_5| + |B_2 \cup V_5| = s_\ell + r_\ell - 2 \geq s_\ell$ and there is the edge u_0v_0 between $N_{V(G')}(A_2)$ and $N_{V(G')}(B_2)$, G contains a copy of F_ℓ, it is a contradiction.

Combining Claims 3.4–3.6, we conclude that a_2 is $R_{\ell-1}$-sum-free, $e(G[A_2, B_2]) = 0$ and there is no edge between $N_{V(G')}(A_2)$ and $N_{V(G')}(B_2)$.

Now we will give the proof of Lemma 3.5 by induction. When $\ell = 2$, if a_2 is $R_{\ell-1}$-sum-free, then either $e(G[A_2, B_2]) \geq 1$ or there is an edge between $N_{V(G')}(A_2)$ and $N_{V(G')}(B_2)$ with $e(G[A_2, B_2]) = 0$ (since $a_2 + b_2 = r_1$, $A_2 \subset A_1$ and $B_2 \subset B_1$). Thus the lemma holds.

Let $\ell \geq 3$. We claim that there exists an $i^* \in [\ell - 1]$ such that $|V(P_{i^*}) \cap A_2| \geq r_{i^*} + 1$ or $|V(P_{i^*}) \cap B_2| \geq r_{i^*} + 1$. Otherwise, $|V(P_{K_i}) \cap (A_2 \cup B_2)| = r_i$ for any $i \in [\ell - 1]$ (since $a_2 + b_2 = s_{\ell-1}$, $A_2 \subset A_1$ and $B_2 \subset B_1$). Then since a_2 is $R_{\ell-1}$-sum-free, there is $i_1 \in [\ell - 1]$ such that $V(P_{k_{i_1}}) \cap A_2 \neq \emptyset$ and $V(P_{k_{i_1}}) \cap B_2 \neq \emptyset$. Thus either $e(G[A_2, B_2]) \geq 1$ or there is an edge between $N_{V(G')}(A_2)$ and $N_{V(G')}(B_2)$ with $e(G[A_2, B_2]) = 0$, a contradiction.

We may assume that $|V(P_{i^*}) \cap A_2| \geq r_{i^*} + 1$. Let $C_1 \subset V(P_{i^*}) \cap A_2$ with $|C_1| = r_{i^*}$ and $G^{i^*} = G - C_1$. Let $F'_{\ell-1} = \bigcup_{i\in[\ell]\setminus i^*} P_{k_i}$ and $F'_{\ell-2} = \bigcup_{i\in[\ell-1]\setminus i^*} P_{k_i}$.

Since $d^c_G(C_1) \geq \delta n$, G^{i^*} is $F'_{\ell-1}$-free. Since $C_1 \subset V(P_{i^*})$ and G contains a copy of $F_{\ell-1}$, G^{i^*} contains a copy of $F'_{\ell-2}$. Thus by (3.17) and (3.20),

$$\begin{aligned} e(G^{i^*}) &\geq e(G) - r_{i^*}n \\ &\geq f(m, n; X_\ell) - \Delta_2 - r_{i^*}n \\ &\geq f(m - r_{i^*}, n; X_\ell \setminus \{i^*\}) - \eta_5, \end{aligned} \tag{3.28}$$

where $\eta_5 = \Delta_2 + r_{i^*}(r_\ell - 1)$, and for any $x \in A_0 - A'_0$ and $y \in B_0 - V(P_{k_{i_0}}) - B'_0$, $e_G(x, y) \geq s_\ell - r_{i_0} + r_\ell - 2$.

By (3.16), $m - r_{i^*}$ is sufficient large than $|V(P_{k_{i_0}}) - B'_0| \leq \delta_2 + r_{i_0}$ and η_5. By the induction hypothesis, Lemma 3.5 holds for G^{i^*}. Thus there is a pair of subsets (A^{i^*}, B^{i^*}) with $A^{i^*} \subset V(G^{i^*}) \cap A$ and $B^{i^*} \subset V(G^{i^*}) \cap B$ being an $(x_\ell - r_{i^*}, x_\ell - r_{i^*})$-core of G^{i^*}. From (3.28), Lemma 3.4 implies that $e(G^{i^*}) \leq f(m - r_{i^*}, n; X_\ell \setminus \{k_{i^*}\})$ and there is a pair of subsets (C^*, D^*) with $C^* \subset A$ and $D^* \subset B$ and a constants ξ such that (C^*, D^*) is an $(\xi n, \xi(m - r_{i^*}))$-core of G^{i^*}.

Since $e_G(C_1, B) \geq e(G) - e(G^{i^*}) \geq f(m, n; X_\ell) - \Delta_2 - f(m - r_{i^*}, n; X_\ell \setminus \{k_{i^*}\}) \geq r_{i^*}n - \eta_5$, $d^c_G(C_1) \geq n - \eta_5$. Hence $d^c_G(C_1 \cup C^*) \geq \xi n - \eta_5 \geq x_\ell$ and $d^c_G(D^*) \geq \xi(m - r_{i^*}) \geq x_\ell$. Therefore, $(C_1 \cup C^*, D^*)$ is an (x_ℓ, x_ℓ)-core of G. Lemma 3.5 holds by setting $A^*_2 = C_1 \cup C^*$ and $B^*_2 = D^*$. This completes the proof of Lemma 3.5.

Proof of Lemma 2.2 Let $G = G(A, B)$ be a bipartite extremal graph for F_ℓ with classes

A, B. Let $|A| = m$ and $|B| = n$ with $n \geq m \geq m_1 = m_1(k_1, \cdots, k_\ell)$. Let $k_\ell \geq 4$. From the definitions of $f(m, n; X_\ell)$ and $\mathcal{F}(m, n; X_\ell)$, we have

$$e(G) \geq f(m, n; X_\ell).$$

Suppose that G contains a copy of $F_{\ell-1}$. Let $A_0 = A - V(F_{\ell-1})$, $B_0 = B - V(F_{\ell-1})$, $A_1 = A - A_0$, $B_1 = B - B_0$ and $G^0 = G(A_0, B_0)$. Let $|A_0| = m_0$ and $|B_0| = n_0$. Assume that

$$e_G(x, y) \geq s_\ell + r_\ell - 2 \text{ for each } x \in A_0 \text{ and } y \in B_0.$$

Then by Lemma 3.5 with setting $\Delta_2 = \delta_2 = 0$, we have that there is a pair of subsets (A^*, B^*) with $A^* \subset A$ and $B^* \subset B$ being an (x_ℓ, x_ℓ)-core of G. Then by Lemma 3.4, we have $e(G) \leq f(m, n; X_\ell)$, where equality holds only when $G \in \mathcal{F}(m, n; X_\ell)$. This completes the proof of Lemma 2.2.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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