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Tilings of the Sphere by Congruent Quadrilaterals I: Edge Combination a^2bc*

Yixi LIAO[1] Pinren QIAN[1] Erxiao WANG[2] Yingyun XU[1]

Abstract Edge-to-edge tilings of the sphere by congruent a^2bc-quadrilaterals are classified as 3 classes: (1) A 1-parameter family of quadrilateral subdivisions of the octahedron with 24 tiles, and a flip modification for one special parameter; (2) a 2-parameter family of 2-layer earth map tilings with $2n$ tiles for each $n \geq 3$; (3) a 3-layer earth map tiling with $8n$ tiles for each $n \geq 2$, and two flip modifications for each odd n. The authors also describe the moduli of parameterized tilings and provide the full geometric data for all tilings.

Keywords Spherical tiling, Quadrilateral, Classification, Earth map tiling, Subdivision
2000 MR Subject Classification 52C20, 05B45

1 Introduction

Tiling has been part of human civilization for thousands of years. The mathematical study of tiling can be traced back to Platonic solids. However, a full classification of monohedral convex tilings of the plane has been completed only recently. See [7] for the hardest pentagon case and see [15] for a recent survey. There are not as many studies on spherical tilings as the planar ones. In this paper, we study edge-to-edge tilings of the sphere by congruent simple polygons, such that all vertices have degree ≥ 3. In such a tiling, the tile must be triangle, quadrilateral, or pentagon (see [10], for example). The study of triangular case was started by Sommerville [9] in 1924, initially classified by Davies [4] in 1967, and completed with full details by Ueno and Agaoka [11] in 2002. Recent works of Wang, Yan and Akama [2, 12–14] studied pentagonal case.

However, earlier explorations (see [1, 8, 10]) suggested that the quadrilateral case might be the most difficult. We will give the full classification of quadrilateral tilings in a series of three papers, of which this paper is the first one (see [5–6] for the later two). We notice the independent complete classification work by Cheung, Luk and Yan [3] using quite different strategies.

In this paper, we classify edge-to-edge tilings of the sphere by congruent simple quadrilaterals (see Figure 1) with edge lengths a, a, b, c, where a, b, c have distinct length values, and all vertices

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[1]School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.
E-mail: yixiliao@zjnu.edu.cn 201939900222@zjnu.edu.cn 201939900111@zjnu.edu.cn
[2]Corresponding author. School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, Zhejiang, China. E-mail: wang.eric@zjnu.edu.cn
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have degree ≥ 3. We will simply call such tilings a^2bc-tilings. We also denote the a^2-angle, ab-angle, ac-angle and bc-angle by $\alpha, \beta, \gamma, \delta$.

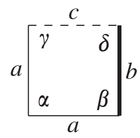


Figure 1 Quadrilaterals with the edge combination a^2bc.

Besides a^2bc-tilings, the other possible edge length combinations suitable for tilings are a^2b^2, a^3b, a^4 (see [10] or our Lemma 2.3). Sakano and Akama [8] classified a^2b^2-tilings and a^4-tilings, which can be reduced to triangular tilings in [11]. Akama and Cleemput [1] had some partial study for convex a^3b-tilings. We will classify a^3b-tilings, including non-convex ones, in the subsequent papers [5–6] of this series.

The following summarizes all a^2bc-tilings. We denote the total number of tiles by f.

Main Theorem There are exactly three classes of a^2bc-tilings:

(1) *A 1-parameter family of quadrilateral subdivisions of the octahedron* $T(8\alpha^3, 12\beta^2\gamma^2, 6\delta^4)$ *with* $f = 24$. *Moreover, for the case* $\beta = \frac{\pi}{3}$, *the tiling has a flip modification* $T(2\alpha^3, 6\alpha\gamma^2, 6\alpha^2\beta^2, 6\beta^2\gamma^2, 6\delta^4)$;

(2) *a 2-parameter family of 2-layer earth map tilings* $T(f\beta\gamma\delta, 2\alpha^{\frac{f}{2}})$, *for each even* $f \geq 6$;

(3) *a 3-layer earth map tiling* $T(\frac{f}{2}\alpha\gamma^2, \frac{f}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 2\beta^{\frac{f}{4}})$ *by a unique quadrilateral, for each* $f \geq 16$ *satisfying* $f \equiv 0 \pmod 8$. *Moreover, if* $f \equiv 8 \pmod{16}$, *the tiling has two flip modifications* $T(\frac{f}{2}\alpha\gamma^2, \frac{f-8}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 4\alpha\beta^{\frac{f+8}{8}})$, $T(\frac{f-4}{2}\alpha\gamma^2, \frac{f}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 2\alpha\beta^{\frac{f+8}{8}}, 2\beta^{\frac{f-8}{8}}\gamma^2)$.

The notation $T(f\beta\gamma\delta, 2\alpha^{\frac{f}{2}})$ means that the tiling has exactly f vertices $\beta\gamma\delta$ and 2 vertices $\alpha^{\frac{f}{2}}$, and is uniquely determined by them.

The first and third classes are related to pentagonal tilings in an interesting way. Since all vertices $\delta\cdots$ are δ^4, we may remove all b-edges and get a tiling by symmetric almost equilateral pentagons with edge lengths $a, a, a, a, 2c$. In fact, all such pentagonal tilings are obtained in this way. For example, 3-layer quadrilateral earth map tilings induce pentagonal earth map tilings in the right.

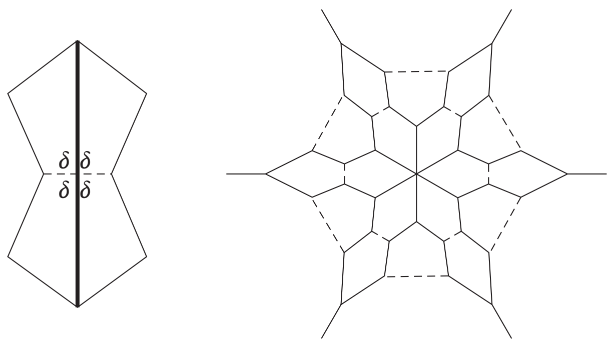


Figure 2 The induced tilings by congruent symmetric pentagons.

The a^2bc-quadrilateral with $\alpha = \frac{2\pi}{3}, \beta = \frac{\pi}{3}, \gamma = \frac{2\pi}{3}, \delta = \frac{\pi}{2}$ appears in both the first and third classes. This gives five different tilings, all inducing the regular dodecahedron tiling. In particular, the quadrilateral is half of the pentagonal face of the regular dodecahedron, which determines all edge lengths a, b, c. The second to sixth pictures of Figure 3 show these five different tilings.

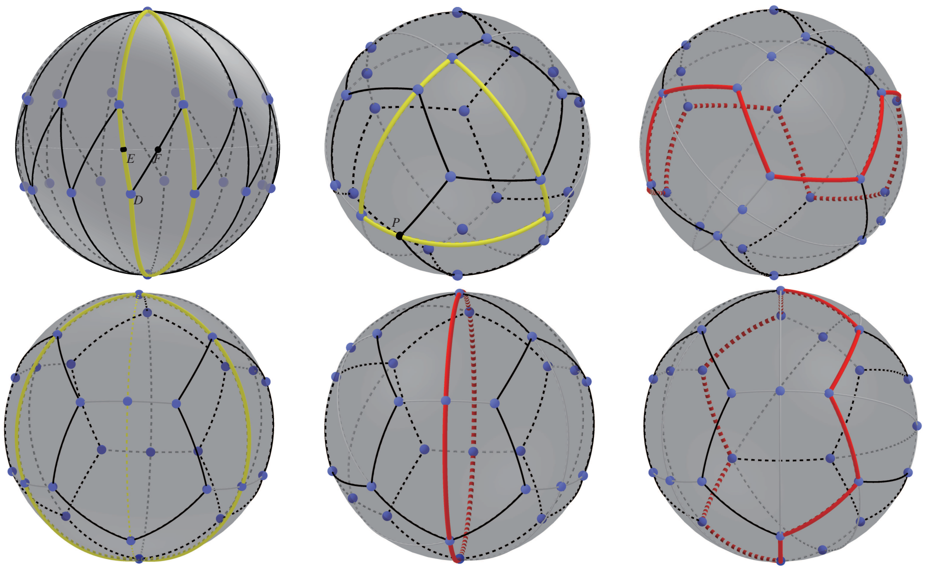


Figure 3 Six types of a^2bc-tilings with $f = 24$.

• The first picture is a 2-layer earth map tiling: One timezone is outlined by the yellow line. The picture shows 12 timezones, and in general the number of timezones can be any $n = \frac{f}{2} \geq 3$. All a^2-angles appear at the north/south poles. The $2n$ middle points of all b-edges and c-edges distribute evenly on the equator with spacing $EF = \frac{\pi}{n}$. The tiling is determined by D, E, F, and is parameterized by the location of D.

• The second is a quadrilateral subdivision of the octahedron: The yellow triangle is one face of the regular octahedron. The face is divided into 3 identical quadrilaterals, and the operation is applied to all 8 faces in a compatible way. The tiling is parameterized by the location of P on the yellow edge.

• The third is the flip modification of the second: When P in the second picture is at certain location such that the dodecahedron underlying the quadrilateral tiling is regular, the red line in the third picture divides the tiling into two identical halves. Then we may flip one half to get a new tiling.

• The fourth is a 3-layer earth map tiling: One timezone is symmetric and outlined by the yellow line. The picture shows 3 timezones, and in general the number of timezones can be any integer ≥ 2.

• The fifth and sixth are two flip modifications of the fourth: When the number of timezones is odd, two red lines divide the tiling into two identical halves in different ways. In each case, we may flip one half to get new tilings.

The key technique used in this paper is the analysis of the neighborhood of a special tile (see Lemma 2.1) with four vertices of degree $333d$, $334d$, $335d$ or $344d$. On the other hand, our subsequent papers [5–6] on the classification of a^3b-tilings rely on allowable combinations of angles at degree 3 and 4 vertices, and will apply interesting new techniques of cyclotomic field and trigonometric Diophantine equation.

This paper is organized as follows. Section 2 develops basic techniques needed for the classification work. This includes general results for all quadrilateral tilings of the sphere and some technical results specific to a^2bc. All other sections analyze the neighborhood of a special tile and complete the classification. Along the way we describe the moduli of 2-layer earth map tilings and the quadrilateral subdivisions, and also provide exact calculations for the unique quadrilaterals in the 3-layer earth map tilings.

2 Basic Facts

2.1 Vertex

Let v, e, f be the numbers of vertices, edges, and tiles, respectively. Let v_d be the number of vertices of degree d. We have Euler's formula and basic counting equalities:

$$\begin{aligned} 2 &= v - e + f, \\ 2e = 4f &= \sum_{d=3}^{\infty} d v_d = 3v_3 + 4v_4 + 5v_5 + \cdots, \\ v &= \sum_{d=3}^{\infty} v_d = v_3 + v_4 + v_5 + \cdots. \end{aligned}$$

Then it is easy to derive $v = f + 2$ and

$$f = 6 + \sum_{d=4}^{\infty}(d-3)v_d = 6 + v_4 + 2v_5 + 3v_6 + \cdots, \tag{2.1}$$

$$v_3 = 8 + \sum_{d=5}^{\infty}(d-4)v_d = 8 + v_5 + 2v_6 + 3v_7 + \cdots. \tag{2.2}$$

These equalities imply $f \geq 6$, $v_3 \geq 8$, and there are many more degree 3 vertices than vertices of degree ≥ 5.

For a^2bc-tilings, each b-edge is shared by exactly two tiles. Then f is twice of the number of b-edges, and is therefore even.

Lemma 2.1 *In an edge-to-edge quadrilateral tiling of the sphere with all vertices having degree ≥ 3, there is a tile, such that the four vertices have degree* $333d$ $(d \geq 3)$, $334d$ $(4 \leq d \leq 11)$, $335d$ $(d = 5, 6, 7)$ *or* $344d$ $(d = 4, 5)$.

Proof Denote the degrees of four vertices of any tile T by d_1, d_2, d_3, d_4. Counting the total number of vertices via each tile's contribution, we get

$$\sum_{\text{all } f \text{ tiles}} \left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4}\right) = v = f + 2 > f.$$

This implies the existence of a special tile T such that $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4} > 1$. The integer solutions $d_1, d_2, d_3, d_4 \geq 3$ of the inequality are exactly $333d$ $(d \geq 3)$, $334d$ $(4 \leq d \leq 11)$, $335d$ $(d = 5, 6, 7)$ or $344d$ $(d = 4, 5)$.

2.2 Angle

The sum of all angles (angle sum) at a vertex is 2π. The following is the angle sum for quadrilateral.

Lemma 2.2 *If all tiles in a tiling of the sphere by f quadrilaterals have the same four angles $\alpha, \beta, \gamma, \delta$, then*

$$\alpha + \beta + \gamma + \delta = \left(2 + \frac{4}{f}\right)\pi$$

ranging in $\left(2\pi, \frac{8}{3}\pi\right]$. In particular no vertex contains all four angles.

Proof The sum of all angles at a vertex is 2π, and the total sum of all angles in the tiling is $2\pi v$. The sum of all four angles in a tile is $\Sigma = \alpha + \beta + \gamma + \delta$, and the total sum of all angles in the tiling is Σf. Therefore $2\pi v = \Sigma f$. By $v = f + 2$, we get the equality in the lemma. Moreover, by $f \geq 6$, we get $2\pi < \Sigma \leq \frac{8}{3}\pi$.

Henceforth we often use this angle sum lemma without mentioning it.

2.3 Edge

The following describes all the possible edge length combinations of the quadrilateral in a tiling and their arrangements. The result appeared in Ueno and Agaoka [10].

Lemma 2.3 *In a tiling of the sphere by congruent quadrilaterals, the edge lengths of any tile are arranged in one of the four ways in Figure 4, with distinct edge lengths a, b, c.*

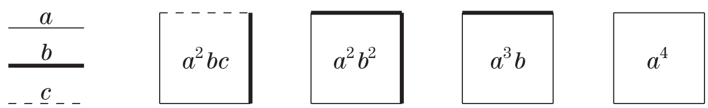


Figure 4 Edge arrangements suitable for tiling.

Proof There are five possible edge combinations (a, b, c, d are distinct)

$$abcd,\ a^2bc,\ a^2b^2,\ a^3b,\ a^4.$$

For $abcd$, without loss of generality, we may assume that the edges are arranged as in the first of Figure 5. Moreover, by $v_3 \geq 8$, we may assume that the vertex shared by b, c has degree 3. Let x be the third edge at the vertex. Then x, b are adjacent in a tile, and x, c are adjacent in another tile. Since there is no edge in the quadrilateral that is adjacent to both b and c, we get a contradiction.

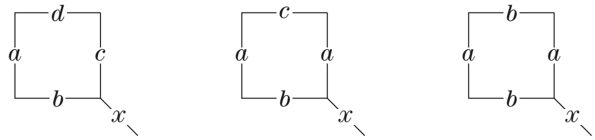


Figure 5 Not suitable for tiling.

For a^2bc, we need to consider two possible arrangements: The case that two a are adjacent is the first of Figure 4; the case that two a are separated is the second of Figure 5. In the second of Figure 5, we get a similar contradiction that there is no edge adjacent to both a and b.

For a^2b^2, the edges are arranged either as the second of Figure 4, or as the third of Figure 5. The third of Figure 5 leads to a similar contradiction.

For a^3b and a^4, the edges can only be arranged as the third and fourth of Figure 4.

2.4 Basic techniques

We use the notations and techniques in [12, Section 2], and add some discussion specific to a^2bc.

Lemma 2.4 *For an a^2bc-quadrilateral in Figure* 1, *$\beta = \gamma$ if and only if $\delta = \pi$.*

Proof If $\delta = \pi$, then the quadrilateral becomes the isosceles triangle in the first picture of Figure 6. This implies $\beta = \gamma$.

Conversely, suppose $\beta = \gamma$. By $AB = AC$, we get $\angle ABC = \angle ACB$. Then $\beta = \gamma$ implies $\angle DBC = \angle DCB$. If $\delta \neq \pi$, then this implies $b = BD = CD = c$, a contradiction.

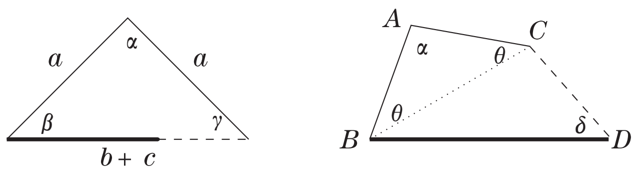


Figure 6 For the proof of Lemmas 2.4–2.5.

Lemma 2.5 *If the a^2bc-quadrilateral in Figure* 1 *is convex, then $\alpha + 2\beta > \pi$, $\alpha + 2\gamma > \pi$.*

Proof By the convexity assumption, the line BC is inside the quadrilateral in the second picture of Figure 6. Thus $\theta < \beta, \gamma$. Since the sum of three angles in a spherical triangle is $> \pi$, we get $\alpha + 2\beta > \alpha + 2\theta > \pi$ and $\alpha + 2\gamma > \alpha + 2\theta > \pi$.

Lemma 2.6 (Parity Lemma) *In an a^2bc-tiling, the respective numbers of β, γ, δ at any vertex have the same parity.*

We call a vertex even or odd whenever the numbers of β, γ, δ are even or odd. Then Lemma 2.2 implies that any vertex $\alpha \cdots$ is always even. In other words, $\alpha \cdots = \alpha^k \beta^l \gamma^m \delta^n$, where l, m, n are even.

Proof The total number of β, δ together at a vertex is twice the number of b-edges at the vertex. Then the respective numbers of β and δ must have the same parity. Similar argument applies to γ, δ.

Lemma 2.7 *In an a^2bc-tiling, a vertex without β, γ must be α^k or δ^n.*

Proof If a vertex has only a-edge, then it has only a^2-angles α. Therefore the vertex is α^k. If a vertex has no a-edge, then it has only bc-angle δ. Therefore the vertex is δ^n. In all other cases, it has ab-angle β or ac-angle γ.

The proof above uses the characterization of $\alpha, \beta, \gamma, \delta$ as a^2-angle, ab-angle, ac-angle, bc-angle. The characterization can be used to distinguish the four angles. Then each of $\alpha, \beta, \gamma, \delta$ appears f times in the tile. Therefore, if one vertex has more α than β, there must exist another vertex with more β than α. Such global counting induces many interesting and useful results.

Lemma 2.8 (Balance Lemma) *In an a^2bc-tiling, one of $\beta^2\cdots$, $\gamma^2\cdots$, $\delta^2\cdots$ is a vertex if and only if all three are vertices. Moreover, if all three are not vertices, then $\alpha^{\frac{f}{2}}$ and $\beta\gamma\delta$ are the only vertices.*

Proof If $\beta^2\cdots$ is not a vertex, then any vertex $\alpha^k\beta^l\gamma^m\delta^n$ has $l = 0, 1$. Then by Parity Lemma, we have $m \geq l$ at every vertex. Since the sum of m at all vertices is f, and the sum of l at all vertices is also f, this implies $m = l \leq 1$ at every vertex. This means that $\gamma^2\cdots$ is not a vertex. Similar argument works for any two angles from β, γ, δ, and this proves the first part of the lemma.

If $\beta^2\cdots, \gamma^2\cdots, \delta^2\cdots$ are not vertices, then $l, m, n \leq 1$. By Parity Lemma, we get $l = m = n = 0$ or $l = m = n = 1$. In the first case, the vertex is α^k. In the second case, by Lemma 2.2, we get that $k = 0$, and the vertex is $\beta\gamma\delta$. By substituting $k\alpha = \beta + \gamma + \delta = 2\pi$ into Lemma 2.2, we get $k = \frac{f}{2}$.

Lemma 2.9 *In an a^2bc-tiling, there are only four possible degree 3 vertices α^3, $\alpha\beta^2$, $\alpha\gamma^2$ and $\beta\gamma\delta$ shown in Figure 7.*

Figure 7 Four possible degree 3 vertices.

Proof Since there is neither b^2-angle nor c^2-angle, the 3 edges at any degree 3 vertex must be aaa, aab, aac or abc in Figure 7, which determine four degree 3 vertices uniquely.

Lemma 2.10 *In an a^2bc-tiling, besides α^4, β^4, γ^4, δ^4, there are only five possible degree 4 vertices $\alpha^2\beta^2$, $\alpha^2\gamma^2$, $\beta^2\gamma^2$, $\beta^2\delta^2$, $\gamma^2\delta^2$ shown in Figure 8. Each of them is uniquely determined by two different angles in it.*

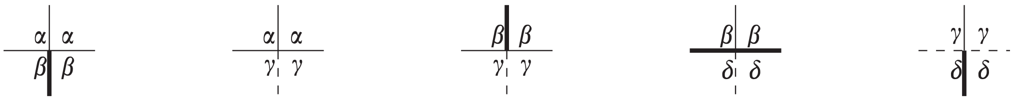


Figure 8 Five possible degree 4 vertices with two different angles.

Proof By Parity Lemma, a degree 4 odd vertex is $\alpha\beta\gamma\delta$, contradicting Lemma 2.2. Therefore a degree 4 vertex is even. This means that the vertex is θ^4 or $\theta^2\phi^2$. Moreover, by Lemma 2.7, we know that $\alpha^2\delta^2$ is not a vertex. Then we get all the degree 4 vertices as claimed in the lemma.

Proposition 2.1 *There is no a^2bc-tiling such that both $\alpha\beta^2$ and $\alpha\gamma^2$ are vertices.*

Proof If both $\alpha\beta^2$ and $\alpha\gamma^2$ are vertices, then $\beta=\gamma$. By Lemma 2.4, we get $\delta=\pi$. Therefore $\delta^2\cdots$ is not a vertex. By Balance Lemma, we know that $\alpha^{\frac{f}{2}}$ and $\beta\gamma\delta$ are the only vertices, a contradiction.

The very useful tool adjacent angle deduction (abbreviated as AAD) was introduced in [12, Section 2.5]. The following is the same as in [12, Lemma 10].

Lemma 2.11 *The AAD of α^k has the following properties*:

- *If $\Vert\beta|\beta\Vert\cdots$ or $\text{¦}\gamma|\gamma\text{¦}\cdots$ is not a vertex, then α^k has the unique AAD $|^{\beta}\alpha^{\gamma}|^{\beta}\alpha^{\gamma}|^{\beta}\alpha^{\gamma}|\cdots$.*
- *If k is odd, then we have the AAD $|^{\beta}\alpha^{\gamma}|^{\beta}\alpha^{\gamma}|$ at α^k.*

We remark that, for $\theta=|^{\alpha}\beta^{\delta}\Vert$, $|^{\alpha}\gamma^{\delta}\text{¦}$ or $\Vert^{\beta}\delta^{\gamma}\text{¦}$, the vertex θ^n has a unique AAD.

Lemma 2.12 *In an a^2bc-tiling, if $\beta>\frac{\pi}{2},\beta+\delta>\pi$, then $\Vert\beta|\beta\Vert\cdots$ is not a vertex. If $\delta>\frac{\pi}{2},\beta+\delta>\pi$, then $\Vert\delta\text{¦}\delta\Vert\cdots$ is not a vertex.*

Proof We have $\Vert\beta|\beta\Vert\cdots=\theta\Vert\beta|\beta\Vert\rho\cdots$ with $\theta,\rho=\beta$ or δ, where θ,ρ are not the same angle (i.e., the vertex is not degree 3). Then by $\beta>\frac{\pi}{2}$ and $\beta+\delta>\pi$, the angle sum is $>2\pi$, a contradiction. The case $\Vert\delta\text{¦}\delta\Vert\cdots$ is similar.

Lemma 2.13 *In an a^2bc-tiling, if $\alpha\gamma^2$ is a vertex, then $\alpha\cdots=\alpha\gamma^2$ or $\alpha^k\beta^{2t}$. Furthermore, $\alpha^k\beta^{2t}$ for some $k\ge 1,t\ge 0$ must appear.*

Proof Recall that a vertex $\alpha\cdots=\alpha^k\beta^l\gamma^m\delta^n$, where l,m,n are even. Since $\alpha\gamma^2$ is a vertex, we get $\alpha\cdots=\alpha\gamma^2$ or $\alpha^k\beta^l\delta^n$. If $n\ge 2$, then by Lemma 2.7, we get $l\ge 2$ in $\alpha^k\beta^l\delta^n$. This implies $\alpha+2\beta+2\delta\le 2\pi$. Combining $\alpha+2\gamma=2\pi$, we get $\alpha+\beta+\gamma+\delta\le 2\pi$, contradicting Lemma 2.2. Therefore $n=0$ in $\alpha^k\beta^l\delta^n$, and $\alpha\cdots=\alpha\gamma^2$ or $\alpha^k\beta^l$.

Since the total numbers of α and γ in the tiling are the same, the vertex $\alpha\gamma^2$ implies that there is a vertex with strictly more α than γ. By $\alpha\cdots=\alpha\gamma^2$ or $\alpha^k\beta^l$, this means that $\alpha^k\beta^l$ is a vertex for some $k\ge 1$ and some even $l\ge 0$.

We will use Lemma/Proposition n' to denote Lemma/Proposition n after exchanging $\beta\leftrightarrow\gamma$.

3 333*d*-Tile

This section classifies all tilings with a special 333d-tile. To facilitate discussion, we denote by T_i the tile labeled i, by E_{ij} the edge shared by T_i, T_j. We denote by θ_i the angle θ in T_i. We say a tile being determined when we know all the edges and angles of the tile.

Proposition 3.1 *For an a^2bc-tiling, the following statements are equivalent*:

(1) *Every tile is a* 333d*-tile.*

(2) *There exists a* 333d*-tile.*

(3) *The bc-angle δ appears at some degree* 3 *vertex* (*i.e., $\beta\gamma\delta$ is a vertex*).

(4) *It is the* 2*-layer earth map tiling* $T(2d\beta\gamma\delta, 2\alpha^d)$ $(d \geq 3)$ *in* Figure 9.

We remark that $d = \frac{f}{2}$.

Proof (1) ⇒ (2) is trivial.

(2) ⇒ (3) If δ does not appear at degree 3 vertices, then in a special 333d-tile, the δ-vertex has degree d. This implies that both β-vertex and γ-vertex have degree 3. Since there is no δ at degree 3 vertices and Lemma 2.9, the β-vertex is $\alpha\beta^2$, and the γ-vertex is $\alpha\gamma^2$. This contradicts Proposition 2.1.

(3) ⇒ (4) By Lemma 2.9, a degree 3 vertex $\delta\cdots = \beta\gamma\delta$.

Next we show $\beta\delta\cdots = \beta\gamma\delta$. Let $\beta\delta\cdots = \alpha^k\beta^l\gamma^m\delta^n$. If $m \geq 1$, we have $\beta\delta\cdots = \beta\gamma\delta$. If $m = 0$, by Parity Lemma, we get $l, n \geq 2$. Then we have $\beta + \delta \leq \pi$. By $\beta\gamma\delta$, we get $\gamma \geq \pi$. However, the unique AAD $\|^{\beta}\delta^{\gamma}|^{\gamma}\delta^{\beta}\|\cdots$ of $\alpha^k\beta^l\delta^n$ gives $\gamma^2\cdots$, a contradiction. Therefore, $\beta\delta\cdots = \beta\gamma\delta$.

Similarly, $\gamma\delta\cdots = \beta\gamma\delta$. In Figure 9, $\beta_1\gamma_3\delta_2$ determines T_1, T_2, T_3. Then $\gamma_2\delta_3\cdots = \beta_4\gamma_2\delta_3$ determines T_4; $\beta_3\delta_4\cdots = \beta_3\gamma_5\delta_4$ determines T_5. The argument started at $\beta_1\gamma_3\delta_2$ can be repeated at $\beta_3\gamma_5\delta_4$. More repetitions give the unique tiling of $f = 2d$ tiles with $2\alpha^d$ $(d \geq 3)$ and $2d\beta\gamma\delta$.

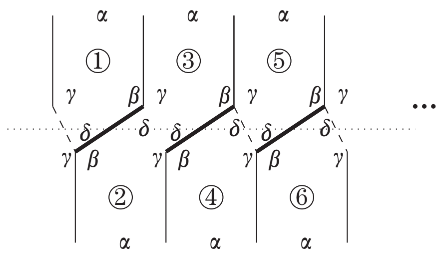


Figure 9 The 2-layer earth map tiling $T(f\beta\gamma\delta, 2\alpha^{\frac{f}{2}})$.

(4) ⇒ (1) Any tile in the 2-layer earth map tiling is a 333d-tile.

Proposition 3.2 *For an a^2bc-tiling, if $\delta = \pi$ (or equivalently $\beta = \gamma$), then it is a* 2*-layer earth map tiling.*

Proof By Lemma 2.4, $\beta = \gamma$ if and only if $\delta = \pi$. This implies that $\delta^2\cdots$ is not a vertex.

Then by Balance Lemma, $\delta\cdots=\beta\gamma\delta$. By Proposition 3.1, this determines a 2-layer earth map tiling.

After Proposition 3.1, we may assume that δ never appears in any degree 3 vertex. In this case, we have the following result.

Lemma 3.1 *In an a^2bc-tiling, if $\beta\gamma\delta$ is not a vertex (i.e., δ never appears in degree 3 vertices), then $\alpha^2\cdots$, $\beta^2\cdots$, $\gamma^2\cdots$, $\delta^2\cdots$ appear as vertices. In particular, the quadrilateral is convex and $\beta\neq\gamma$.*

Proof If $\beta\gamma\delta$ is not a vertex, then by Balance Lemma, $\beta^2\cdots$, $\gamma^2\cdots$, $\delta^2\cdots$ are all vertices. If $\alpha^2\cdots$ is not a vertex, then by Lemma 2.9 and Proposition 2.1, either $\alpha\beta^2$ is the only degree 3 vertex, or $\alpha\gamma^2$ is the only degree 3 vertex. Assume that $\alpha\gamma^2$ is a vertex. Then by Lemma 2.13 and no $\alpha^2\cdots$, we know that $\alpha\beta^{2t}$ $(t\geq 2)$ must appear. However $\alpha\beta^{2t}=\beta^{\alpha}|^{\alpha}\beta\cdots$ implies a vertex $\alpha^2\cdots$, a contradiction. The vertex $\alpha\beta^2$ leads to a similar contradiction.

The vertices $\alpha^2\cdots$, $\beta^2\cdots$, $\gamma^2\cdots$, $\delta^2\cdots$ imply that all angles are $<\pi$. Then we have $\beta\neq\gamma$ by Lemma 2.4.

3.1 Geometric realization and the moduli of $T(f\beta\gamma\delta,2\alpha^{\frac{f}{2}})$

Two poles of the 2-layer earth map tiling in Figure 9 are α^d. This implies that the $2d$ middle points of all b-edges and c-edges distribute evenly on the equator with spacing $\frac{\pi}{d}$. It suggests the following geometric construction in Figure 10. Fix a point A on the sphere as the north pole, and take two points E,F on the equator $\left(\text{i.e., } AE=AF=\frac{\pi}{2}\right)$ with $EF=\frac{\pi}{d}$. The quadrilateral is then determined by the location of D: Extend DE to B, such that E is the middle point of DB (b-edge); extend DF to C, such that F is the middle point of DC (c-edge); connect A to B, C to form the quadrilateral $\square ABDC$. The moduli of 2-layer earth map tilings is the possible locations of D such that the boundary of $\square ABDC$ has no self intersection, i.e., $\square ABDC$ is simple.

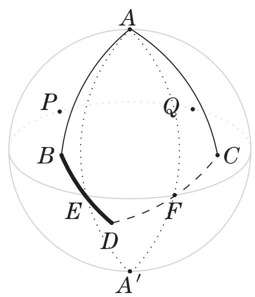


Figure 10 Quadrilateral $\square ABDC$.

In Figure 10, we denote the south pole by A'. Extend FE to P, such that $PE=\frac{\pi}{2}$. Extend EF to Q, such that $FQ=\frac{\pi}{2}$. Then we get the triangle $\triangle A'PQ$ with $PEFQ$ as one edge. We will show that $\square ABDC$ is simple if and only if D lies in the interior of $\triangle AEF\cup\triangle A'PQ$.

Figures 11–12 describe $\square ABDC$ for various locations of D. Figure 11 is the stereographic projection from the antipode of the middle point of EF. Figure 12 shows the cases that $\square ABDC$ is simple and also gives the 3D pictures of the tilings. We study all possibilities as follows.

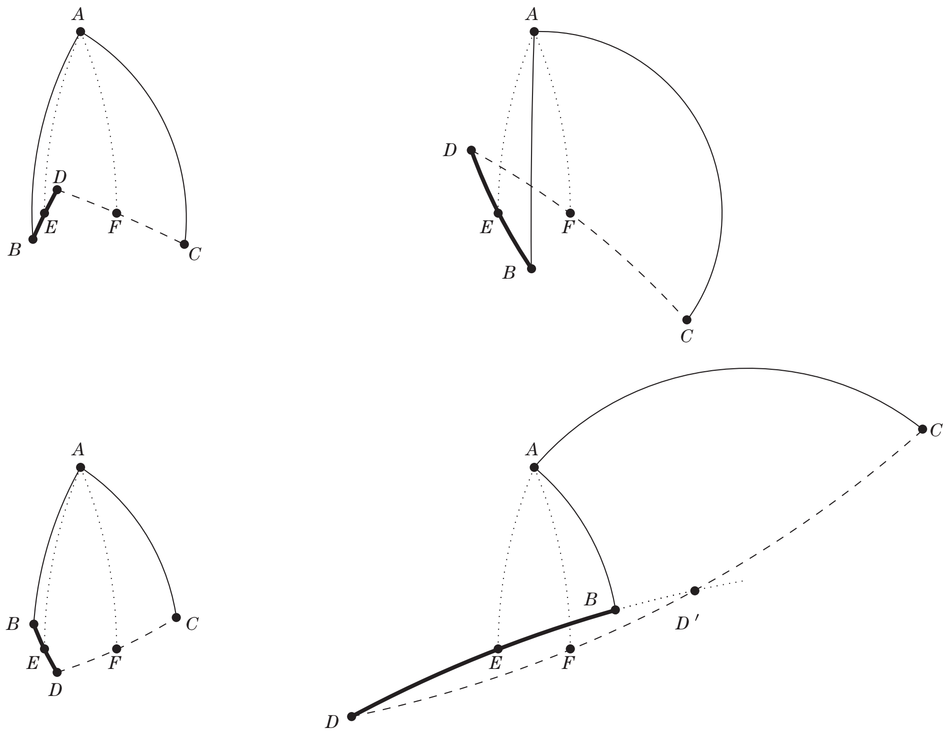


Figure 11 Quadrilaterals corresponding to 4 positions of D.

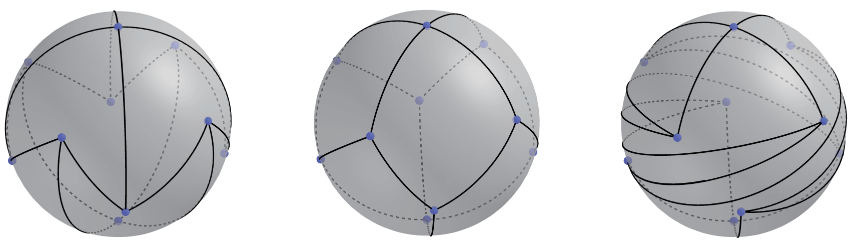


Figure 12 2-layer earth map tilings.

(1) If D lies in the interior of $\triangle AEF$, then $\square ABDC$ is simple and concave with $\delta > \pi$. See the first picture of Figures 11–12.

(2) If D is in the northern hemisphere and outside $\triangle AEF$, then either AB intersects DC, or AC intersects DB. See the second picture of Figure 11.

(3) If D lies in the interior of $\triangle A'EF$, then $\square ABDC$ is simple and convex. See the third picture of Figure 11 and the second picture of Figure 12.

(4) If D is in the southern hemisphere and outside $\triangle A'EF$, then DB and DC are the only pair of edges in $\square ABDC$ which can possibly intersect. The key fact is that any two great arcs ($< 2\pi$) starting from D either intersect at its antipode D' or never intersect. In the fourth picture of Figure 11, D is on the left of the longitude AEA', and we have $DF > DE$, $\beta > \pi$. If $DE < \frac{\pi}{2}$, then $DB < \pi$ and it is too short to reach D'. Then DB does not intersect DC. If $DE \geq \frac{\pi}{2}$, then $DB \geq \pi$ and $DC = 2DF > 2DE \geq \pi$. Then DB meets DC at D'. All such D's satisfying $DE = \frac{\pi}{2}$ form the great arc $A'P$. Then $\square ABDC$ is simple and concave with $\beta > \pi$ if and only if D lies in the interior of $\triangle A'EP$. Symmetrically, $\square ABDC$ is simple and concave with $\gamma > \pi$ if and only if D lies in the interior of $\triangle A'FQ$. The quadrilateral may have one edge $> \pi$, as shown in the third picture of Figure 12.

(5) If D lies in the interior of EF, $A'E$ or $A'F$, then $\square ABDC$ degenerates to a simple triangle with $\delta = \pi$, $\beta = \pi$ or $\gamma = \pi$, respectively.

(6) If D lies on AE, AF or $EPQF$, then $\angle DBA = 0$, $\angle DCA = 0$ or $\angle BDC = 0$, respectively. The quadrilateral is not simple.

In summary, $\square ABDC$ is simple in the cases (1) and (3)–(5) above. The cases combine to form the region in the theorem below.

Theorem 3.1 *The quadrilateral* $\square ABDC$ *in Figure* 10 *is simple if and only if* D *lies in the interior of* $\triangle AEF \cup \triangle A'PQ$ *in* Figure 13. *Furthermore,* $\square ABDC$ *degenerates to a triangle if and only if* D *lies in the interior of* EF ($\delta = \pi$), *or* $A'E$ ($\beta = \pi$), *or* $A'F$ ($\gamma = \pi$).

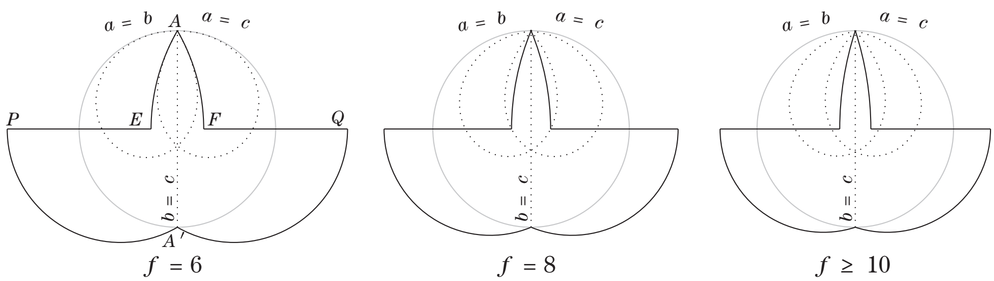


Figure 13 The moduli $(\triangle AEF \cup \triangle A'PQ)^\circ$.

Figure 13 is the stereographic projection from the antipode of the middle point of EF. The first picture shows the moduli of $T(2\alpha^3, 6\beta\gamma\delta)$, and the dotted curves inside the moduli represent reductions of the quadrilateral from type a^2bc to type a^2b^2 ($b = c$), type a^3b ($a = b$ or $a = c$) and type a^4 ($a = b = c$). The second and third pictures of Figure 13 are for $f = 8$ and $f \geq 10$, respectively, where the reduction curves have different positions inside the moduli. In the next two papers [5–6] of this series, it turns out that most a^3b-tilings of the sphere come from these 2-layer earth map tilings on the reduction curves, together with their modifications under extra conditions. Thus the detailed study of the reduction curves will be shown in [6].

We remark that three tilings in Figure 12 all have the same vertices distributed on the sphere. These three different quadrilaterals are closely related to each other, generalizing the notion "companion" in [13–14].

4 334d-Tile and 335d-Tile

We classify a^2bc-tilings under the assumption that there is a special 334d-tile or 335d-tile. Since the tiling in Proposition 3.1 has no such special tile, we know that δ does not appear in degree 3 vertices. By Lemma 2.9 and Proposition 2.1, either $\alpha^3, \alpha\beta^2$ are all degree 3 vertices, or $\alpha^3, \alpha\gamma^2$ are all degree 3 vertices.

Let us look at the neighborhood of a special 334d-tile or 335d-tile. Up to the symmetry of exchanging $\beta \leftrightarrow \gamma$, we may assume that $\alpha^3, \alpha\gamma^2$ are all degree 3 vertices. This means that β and δ do not appear in degree 3 vertices. Then we get 4 possibilities for the special tile in Figure 14. We denote the degree d vertex by H and indicate it by $\bullet$.

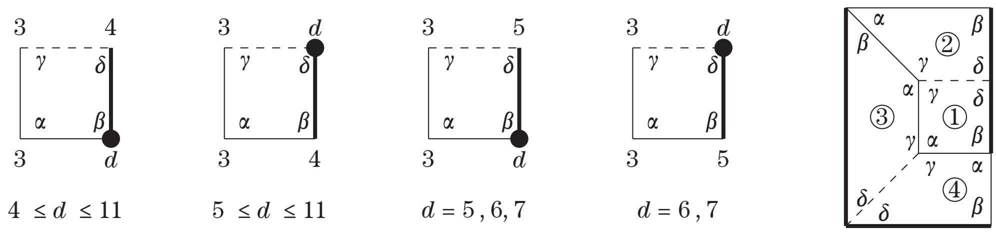


Figure 14 Special 334d, 335d-tiles and their common partial neighborhood.

The fifth picture of Figure 14 shows the common partial neighborhood of these 334d-tiles and 335d-tiles: The degree 3 vertex $\gamma_1 \cdots = \alpha\gamma^2 = \alpha_3\gamma_1\gamma_2$ determines T_2 and α_3. By α_3, the degree 3 vertex $\alpha_1 \cdots \neq \alpha^3$. Then $\alpha_1 \cdots = \alpha\gamma^2$ determines T_3, T_4.

Proposition 4.1 *Tilings with the* 1*st special tile in Figure* 14 *are the following*:

(1) *The flip modification of a unique quadrilateral subdivision of the octahedron* $T(2\alpha^3, 6\alpha\gamma^2, 6\alpha^2\beta^2, 6\beta^2\gamma^2, 6\delta^4)$ *with* 24 *tiles*;

(2) *a* 3*-layer earth map tiling* $T(\frac{f}{2}\alpha\gamma^2, \frac{f}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 2\beta^{\frac{f}{4}})$ *by a unique quadrilateral, for each* $f \geq 16$ *satisfying* $f \equiv 0 \pmod 8$. *Moreover, if* $f \equiv 8 \pmod{16}$, *the tiling has two flip modifications* $T\big(\frac{f}{2}\alpha\gamma^2, \frac{f-8}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 4\alpha\beta^{\frac{f+8}{8}}\big)$ *and* $T\big(\frac{f-4}{2}\alpha\gamma^2, \frac{f}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 2\alpha\beta^{\frac{f+8}{8}}, 2\beta^{\frac{f-8}{8}}\gamma^2\big)$.

Proof In the partial neighborhood given by the fifth picture of Figure 14, by Lemma 2.10, the degree 4 vertex $\delta_1\delta_2 \cdots = \beta^2\delta^2$ or δ^4. This determines T_5, T_6 in the two pictures in Figure 15.

By Lemma 2.13, $H = \alpha_4\beta_1\delta_6 \cdots$ in the first of Figure 15 is a contradiction. Moreover, we have $H = \alpha_4\beta_1\beta_6 \cdots = \alpha^k\beta^{2t}$ $(k, t \geq 1)$ in the second of Figure 15. If $k \geq 2$ and $t \geq 2$, we have $2\alpha + 4\beta \leq 2\pi$. By $\alpha\gamma^2$ and δ^4, we get $4(\alpha+\beta+\gamma+\delta) = 2(\alpha+2\gamma) + 4\delta + (2\alpha+4\beta) \leq 8\pi$, contradicting Lemma 2.2. Therefore $t = 1$ and $H = \alpha^{d-2}\beta^2 (4 \leq d \leq 11)$, or $k = 1$ and $H = \alpha\beta^{d-1} (d = 5, 7, 9, 11)$.

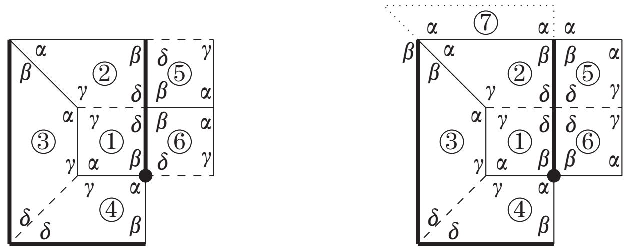


Figure 15 Partial neighborhoods of the 1st special tile.

4.1 $\mathbf{H = \alpha^{d-2}\beta^2\ (4 \le d \le 11)}$

By $\alpha\gamma^2$, δ^4 and $\alpha^{d-2}\beta^2$, we get

$$\alpha = \frac{\pi}{d-3} - \frac{8\pi}{(d-3)f}, \quad \beta = \frac{\pi}{2} - \frac{\pi}{2(d-3)} + \frac{4(d-2)\pi}{(d-3)f},$$
$$\gamma = \pi - \frac{\pi}{2(d-3)} + \frac{4\pi}{(d-3)f}, \quad \delta = \frac{\pi}{2}.$$

If $d \ge 5$, then $\alpha < \frac{\pi}{2}$, $\beta > \frac{\pi}{4}, \gamma > \frac{3\pi}{4}$. Moreover, the AAD of $H = |\beta\|\beta|\alpha|\alpha|\alpha|\cdots$ implies that $\beta|\gamma\cdots$ or $\gamma|\gamma\cdots$ is a vertex. By $\gamma > \frac{\pi}{2}$, $\gamma + \delta > \pi$ and Lemma 2.12′, we know that $\gamma|\gamma\cdots$ is not a vertex. By $\beta + \gamma > \pi$, we know that $\beta|\gamma\cdots$ is an odd vertex. Then by $R(\beta\gamma\delta\cdots) < \frac{\pi}{4} < \beta, \gamma, \delta$, and Lemma 2.2, we get $\beta|\gamma\cdots = \beta\gamma\delta$. This contradicts Proposition 3.1.

We conclude $d = 4$, and $\alpha = \big(1 - \frac{8}{f}\big)\pi$, $\beta = \frac{8\pi}{f}$, $\gamma = \big(\frac{1}{2} + \frac{4}{f}\big)\pi$, $\delta = \frac{\pi}{2}$. Now we show $f \ge 16$. If $f < 16$, we get $\alpha < \frac{\pi}{2}$, $\beta > \frac{\pi}{2}$, $\gamma > \frac{3\pi}{4}$, $\delta = \frac{\pi}{2}$. This implies $\beta^2\cdots = \alpha^2\beta^2$. Then by Lemma 2.13, we further get $\alpha\beta\cdots = \alpha\beta^2\cdots = \alpha^2\beta^2$. Then $\alpha_2\beta_3\cdots = \alpha_2\alpha_7\beta_3\beta$, and $\beta_2\beta_5\cdots = \alpha_7\alpha\beta_2\beta_5$ in the second picture of Figure 15. We get two α in T_7, a contradiction.

By $f \ge 16$, we have $\alpha \ge \frac{\pi}{2}, \beta \le \frac{\pi}{2}, \frac{\pi}{2} < \gamma \le \frac{3\pi}{4}, \delta = \frac{\pi}{2}$. If $\alpha^k\beta^l\gamma^m\delta^n$ is a vertex, then we have $k \le 4$, $m \le 3$, $n \le 4$, and

$$\Big(1 - \frac{8}{f}\Big)k + \frac{8}{f}l + \Big(\frac{1}{2} + \frac{4}{f}\Big)m + \frac{1}{2}n = 2.$$

We substitute the finitely many combinations of exponents satisfying the bounds into the equation above and solve for even $f \ge 16$. By the angle values and the edge length consideration, we get all possible vertices in Table 1. The first row "f = all" means that the vertices may appear for all f.

Claim For any a^2bc-tiling with the AVC (for "anglewise vertex combination") in Table 1, if $\beta^{\frac{f}{4}}$ is a vertex, and $\alpha^3, \alpha^4, \beta^{\frac{f}{8}}\delta^2$ are not vertices, then it is the 3-layer earth map tiling in Figure 16.

By no $\alpha^3, \alpha^4, \beta^{\frac{f}{8}}\delta^2$ and the AVC in Table 1, we get $\alpha^2\cdots = \alpha^2\beta^2$, $\alpha\gamma\cdots = \alpha\gamma^2$ and $\delta^2\cdots = \delta^4$. In Figure 16, $\beta^{\frac{f}{4}} = \|\beta_1|\beta_2\|\cdots$ determines T_1, T_2. Then $\alpha_1\alpha_2\cdots = \alpha_1\alpha_2\beta_3\beta_4$ determines T_3, T_4; $\alpha_3\gamma_1\cdots = \alpha_3\gamma_1\gamma_5$ determines T_5; $\alpha_4\gamma_2\cdots = \alpha_4\gamma_2\gamma_6$ determines T_6; $\delta_3\delta_4\cdots = \delta^4$

Table 1 The AVC for $H = \alpha^2\beta^2$ and $f \geq 16$.

f	vertex
all	$\alpha\gamma^2, \alpha^2\beta^2, \delta^4$
$16s-4, s=2,3,\cdots$	$\beta^{2s-1}\gamma\delta$
16	$\alpha^4, \beta^4, \beta^2\delta^2$
$16s, s=2,3,\cdots$	$\beta^{4s}, \beta^{2s}\delta^2$
24	$\alpha^3, \beta^2\gamma^2, \alpha\beta^4, \beta^6$
$16s+8, s=2,3,\cdots$	$\alpha\beta^{2s+2}, \beta^{4s+2}, \beta^{2s}\gamma^2$

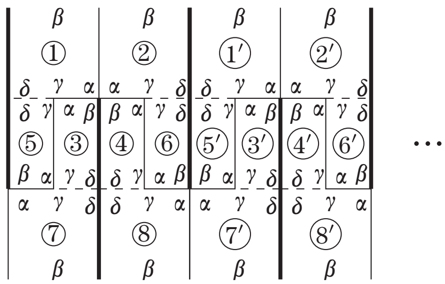


Figure 16 The 3-layer earth map tiling $T(\frac{f}{2}\alpha\gamma^2, \frac{f}{4}\alpha^2\beta^2, \frac{f}{4}\delta^4, 2\beta^{\frac{f}{4}})$.

determines T_7, T_8. The tiles $T_1, \cdots, T_8$ together form a time zone. Similarly, we can determine $T_{1'}, \cdots, T_{8'}$. By repeating the process, we get the 3-layer earth map tiling.

Remark 4.1 The proof above actually shows that $\|\beta_1|\beta_2\|$ determines $T_1, \cdots, T_8$. This fact will be very useful to deduce other possible tilings.

4.2 Calculate the quadrilaterals in 3-layer earth map tilings

By Figure 16 and $\alpha + \beta = 2\delta = \pi$, the two poles are connected by a great arc consisting of one a-edge and two b-edges. Therefore, $a + 2b = \pi$.

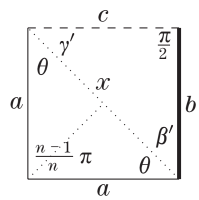


Figure 17 The quadrilateral in the 3-layer earth map tiling.

In Figure 17, we divide the quadrilateral into two triangles, where $n = \frac{f}{8} \geq 2$ is the number of timezones in Figure 16. By $\alpha\gamma^2$, $\alpha^2\beta^2$, β^{2n}, δ^4, we get

$$\alpha = \frac{(n-1)\pi}{n}, \quad \beta = \frac{\pi}{n} = \theta + \beta', \quad \gamma = \frac{(n+1)\pi}{2n} = \theta + \gamma', \quad \delta = \frac{\pi}{2}.$$

Then we have

$$\cos b \cos c = \cos x = \cos^2 a + \sin^2 a \cos \frac{(n-1)\pi}{n} = \cos^2 a - \sin^2 a \cos \frac{\pi}{n},$$

$$\begin{aligned}
&\cos a = \cot\theta\cot\frac{(n-1)\pi}{2n} = \cot\theta\tan\frac{\pi}{2n},\\
&\frac{\sin\gamma'}{\sin b} = \frac{1}{\sin x},\\
&\frac{\sin\theta}{\sin a} = \frac{\sin\frac{(n-1)\pi}{n}}{\sin x} = \frac{\sin\frac{\pi}{n}\sin\gamma'}{\sin b}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sin\theta &= \frac{\sin a}{\sin b}\sin\frac{\pi}{n}\sin\gamma' = 2\sin\frac{\pi}{n}\cos b\sin\Big(\frac{(n+1)\pi}{2n}-\theta\Big)\\
&= 2\sin\frac{\pi}{n}\cos b\Big(\cos\theta\cos\frac{\pi}{2n}+\sin\theta\sin\frac{\pi}{2n}\Big)\\
&= 2\sin\frac{\pi}{n}\cos b\Big(\sin\theta\cos a\cot\frac{\pi}{2n}\cos\frac{\pi}{2n}+\sin\theta\sin\frac{\pi}{2n}\Big)\\
&= 4\sin\theta\cos\frac{\pi}{2n}\cos b\Big(\cos a\cos^2\frac{\pi}{2n}+\sin^2\frac{\pi}{2n}\Big)\\
&= 4\sin\theta\cos\frac{\pi}{2n}\cos b\Big((1-2\cos^2 b)\cos^2\frac{\pi}{2n}+\sin^2\frac{\pi}{2n}\Big).
\end{aligned}$$

Dividing by $\sin\theta$, we get $8t^3-4t+1=0$ for $t=\cos\frac{\pi}{2n}\cos b$. Thus $t=\frac{1}{2}$, $\frac{\pm\sqrt{5}-1}{4}$. Note that $2b<a+2b=\pi$ implies $\cos b>0$. Then $t>0$. If $t=\frac{1}{2}$, then by $-\cos 2b=\cos a=\cot\theta\tan\frac{\pi}{2n}$, we get $\theta=\frac{\pi}{n}$. However $\beta'=\frac{\pi}{n}-\theta=0$, a contradiction. Therefore, we get a unique solution $t=\frac{\sqrt{5}-1}{4}$, and

$$b=\arccos\frac{\sqrt{5}-1}{4\cos\frac{\pi}{2n}},\quad a=\pi-2b,\quad c=\arccos\frac{(3-\sqrt{5})\cos^2\frac{\pi}{2n}+\sqrt{5}-2}{\cos\frac{\pi}{2n}}.$$

For $f=16$, we get $\alpha=\beta=\delta=\frac{\pi}{2}$, $\gamma=\frac{3\pi}{4}$, $a\approx 0.2879\pi$, $b\approx 0.3560\pi$, $c\approx 0.1615\pi$.

For $f=24$, we get $\alpha=\gamma=\frac{2\pi}{3}$, $\beta=\frac{\pi}{3}$, $\delta=\frac{\pi}{2}$, $a\approx 0.2323\pi$, $b\approx 0.3838\pi$, $c\approx 0.1161\pi$. This quadrilateral also gives the first tiling in Proposition 4.1.

As $f=8n\to\infty$, we get $\alpha\nearrow\pi$, $\beta\searrow 0$, $\gamma\searrow\frac{\pi}{2}$, $a\searrow\frac{\pi}{5}$, $b\nearrow\frac{2\pi}{5}$, $c\searrow 0$. In summary a,b,c are distinct for all $n\ge 2$ and the quadrilateral is indeed of type a^2bc.

Let us deduce all possible tilings based on the AVC of Table 1. If AVC $=\{\alpha\gamma^2,\alpha^2\beta^2,\delta^4\}$, then we get the following contradiction:

$$f=\#\alpha=\#\alpha\gamma^2+2\#\alpha^2\beta^2=\frac{1}{2}\#\gamma+\#\beta=\frac{3}{2}f.$$

Therefore, we just need to consider the following three cases.

Case $f=16s-4$ We know AVC $=\{\alpha\gamma^2,\alpha^2\beta^2,\delta^4,\beta^{2s-1}\gamma\delta\}$. Since there is no tiling for AVC $=\{\alpha\gamma^2,\alpha^2\beta^2,\delta^4\}$, $\beta^{2s-1}\gamma\delta$ must appear. In the first of Figure 18, $\beta^{2s-1}\gamma\delta=\Vert\beta_1\vert\gamma_2\vdots\delta_3\Vert\cdots$ determines T_1,T_2,T_3. Then $\alpha_1\alpha_2\cdots=\alpha_1\alpha_2\beta_4\beta$ determines T_4; $\alpha_4\beta_2\cdots=\alpha_4\alpha\beta_2\beta_5$ determines T_5. We get $\gamma_3\delta_2\delta_5\cdots$, contradicting the AVC.

Case $f=16s$, including $f=16$ We know AVC $=\{\alpha\gamma^2,\alpha^4,\alpha^2\beta^2,\delta^4,\beta^{4s},\beta^{2s}\delta^2\}$. If α^4 appears, then its AAD gives a vertex $\beta\vert\gamma\cdots$ or $\gamma\vert\gamma\cdots$, contradicting the AVC.

If $\beta^{2s}\delta^2$ appears, then $\beta^{2s}\delta^2=\vert\beta_1\Vert\delta_2\vdots\delta_3\Vert\beta_4\vert\cdots$ in the second of Figure 18, which determines T_1,T_2,T_3,T_4. Then $\vdots\delta_1\Vert\beta_2\vert\cdots=\vdots\delta_1\Vert\beta_2\vert\beta_5\Vert\cdots$ determines T_5; $\vdots\delta_4\Vert\beta_3\vert\cdots=\vdots\delta_4\Vert\beta_3\vert\beta_6\Vert\cdots$

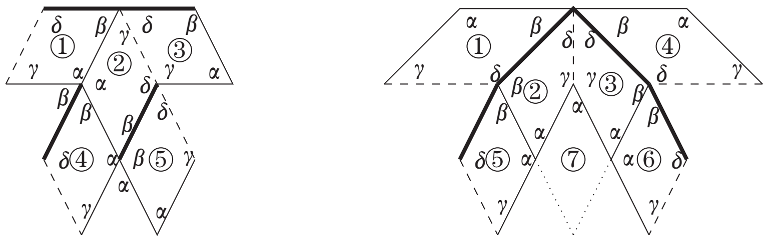


Figure 18 $\beta^{2s-1}\gamma\delta$ or $\beta^{2s}\delta^2$ appears.

determines T_6. By the AVC, $\gamma_2\gamma_3\cdots = \alpha_7\gamma_2\gamma_3$. By α_7, either $\alpha_2\alpha_5\cdots$ or $\alpha_3\alpha_6\cdots$ is $\alpha^2\gamma\cdots$, contradicting the AVC.

Therefore, AVC $= \{\alpha\gamma^2, \alpha^2\beta^2, \delta^4, \beta^{4s}\}$. Since there is no tiling for AVC $= \{\alpha\gamma^2, \alpha^2\beta^2, \delta^4\}$, β^{4s} must appear. By the Claim after Table 1, we get the 3-layer earth map tiling in Figure 16.

Case $f = 16s+8$, including $f = 24$ We know AVC $= \{\alpha^3, \alpha\gamma^2, \alpha^2\beta^2, \delta^4, \alpha\beta^{2s+2}, \beta^{4s+2}, \beta^{2s}\gamma^2\}$. We divide our discussions into two subcases.

Subcase α^3 appears This subcase means $f = 24$. By no $¦\gamma|\gamma¦\cdots$ in the AVC, we get the unique AAD $|^\gamma\alpha^\beta|^\gamma\alpha^\beta|^\gamma\alpha^\beta|$ of α^3. This determines T_1, T_2, T_3 in Figure 19. Then $\delta_1\cdots = \delta_2\cdots = \delta_3\cdots = \delta^4$ determines $T_4, T_5, \cdots, T_{12}$.

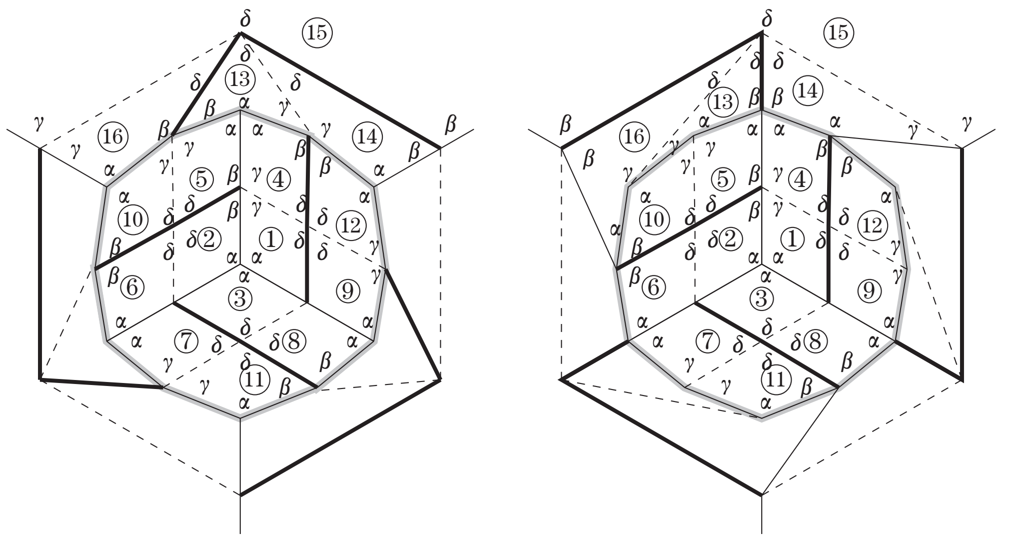


Figure 19 Tilings when α^3 appears.

We have $\alpha_4\alpha_5\cdots = \alpha^3$ or $\alpha^2\beta^2$. In the first of Figure 19, $\alpha_4\alpha_5\cdots = \alpha^3$. Then the unique AAD of α^3 determines T_{13}. Then $\delta_{13}\cdots = \delta^4$ determines T_{14}, T_{15}, T_{16}. In the second of Figure 19, $\alpha_4\alpha_5\cdots = \alpha_4\alpha_5\beta_{13}\beta_{14}$ determines T_{13}, T_{14}. Then $\delta_{13}\delta_{14}\cdots = \delta^4$ determines T_{15}, T_{16}.

Similarly, we have $\alpha_6\alpha_7\cdots = \alpha^3$ or $\alpha^2\beta^2$, and $\alpha_8\alpha_9\cdots = \alpha^3$ or $\alpha^2\beta^2$. We also get the induced four tiles similar to $T_{13}, T_{14}, T_{15}, T_{16}$. For all the induced tiles to be compatible and produce a tiling, we must have $\alpha_4\alpha_5\cdots = \alpha_6\alpha_7\cdots = \alpha_8\alpha_9\cdots = \alpha^3$, or $\alpha_4\alpha_5\cdots = \alpha_6\alpha_7\cdots = \alpha_8\alpha_9\cdots = \alpha^2\beta^2$. Then we get two tilings in Figure 19. The first tiling is the quadrilateral subdivision of the octahedron. Each tile is a 3444-tile, and the tiling actually belongs to later Proposition 5.3. The tiling is divided into two identical halves along the shaded edges. The left of Figure 20 gives the angles along the shaded edges.

The second tiling is also divided into two identical halves along the shaded edges. The right of Figure 20 gives the angles along the shaded edges. The two tilings have the same inside halves, and two outside halves are related by the flip with respect to the line L. In fact, the second tiling can also be obtained from the first by the flip of the inside half with respect to the line L.

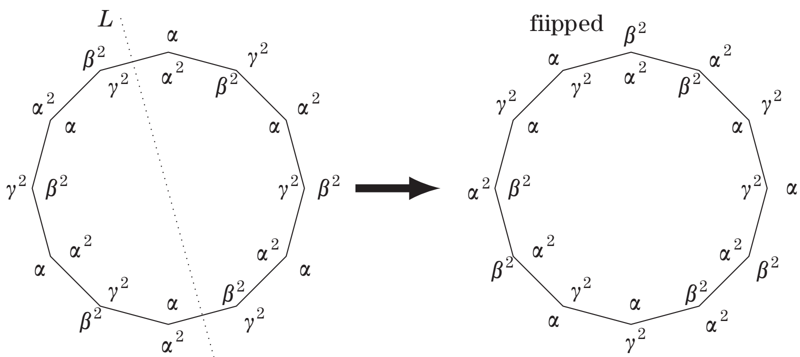


Figure 20 Flip modification of a quadrilateral subdivision.

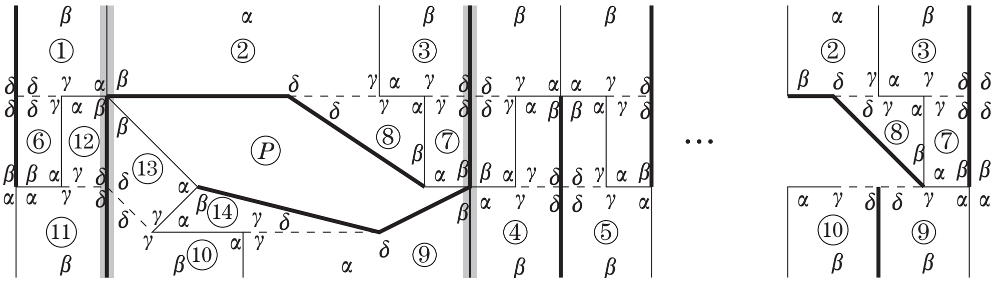
We will calculate the geometric data for the quadrilateral in the proof of Proposition 5.3.

Subcase α^3 is not vertex We know AVC $= \{\alpha\gamma^2, \alpha^2\beta^2, \delta^4, \alpha\beta^{2s+2}, \beta^{4s+2}, \beta^{2s}\gamma^2\}$.

If β^{4s+2} appears, then by the Claim after Table 1, the tiling is the 3-layer earth map tiling in Figure 16.

If β^{4s+2} is not a vertex, then AVC $= \{\alpha\gamma^2, \alpha^2\beta^2, \delta^4, \alpha\beta^{2s+2}, \beta^{2s}\gamma^2\}$. Assume that $\alpha\beta^{2s+2}$ appears. Its AAD is $\Vert\beta\vert\alpha\vert\beta\Vert\beta\vert\beta\Vert\cdots\Vert\beta\vert\beta\Vert$. The $\Vert\beta\vert\alpha\vert\beta\Vert$ part of the vertex determines T_1, T_2, T_3 in the left of Figure 21. By the remark after the Claim, the remaining part $\Vert\beta\vert\beta\Vert\cdots\Vert\beta\vert\beta\Vert$ of the vertex determines s timezones consisting of $8s$ tiles, including T_4, T_5. Moreover, $\delta_1\cdots = \delta_3\cdots = \delta^4$ determines T_6, T_7, and $\alpha_3\gamma_2\cdots = \alpha\gamma^2$ determines T_8. Then $\beta_4\beta_5\cdots = \beta^{2s}\cdots = \alpha\beta^{2s+2}$ or $\beta^{2s}\gamma^2$, shown in Figure 21 and 23, respectively.

In Figure 21, $\beta_4\beta_5\cdots = \alpha\beta^{2s+2}$. Then we get $E_{9,10} = a$ or b. If $E_{9,10} = b$, then we determine T_9, T_{10} in the right of Figure 21. Then we have $\alpha_7\beta_8\gamma_9\cdots$, contradicting the AVC. Therefore, $E_{9,10} = a$ as in the left picture. Then by $\beta_4\beta_5\cdots = \alpha\beta^{2s+2}$ and no $\alpha_7\beta_8\gamma_9\cdots$, we can determine T_9, T_{10}, T_{11}. Then $\delta_{10}\delta_{11}\cdots = \delta^4$ determines T_{12}, T_{13}, and $\alpha_{10}\gamma_9\cdots = \alpha\gamma^2$ determines T_{14}. Moreover $\Vert\beta_9\vert\alpha_4\vert\beta\Vert\beta_7\vert\cdots = \alpha\beta^{2s+2}$. The complement $\beta_7\cdots$ of $\Vert\beta_9\vert\alpha_4\vert\beta\Vert$ determines $8s$ tiles consisting of T_2, T_3, T_7, T_8 and P. We obtain the first flip modification of the 3-layer earth map tiling, as explained in Figure 22.



$T((8s+4)\ \alpha^2\gamma, 4s\,\alpha^2\beta^2, (4s+2)\ \delta^4, 4\,\alpha\,\beta^{2s+2})$ $E_{9,10} = b$

Figure 21 $\alpha\beta^{2s+2}$ appears, $\beta_4\beta_5\cdots = \alpha\beta^{2s+2}$.

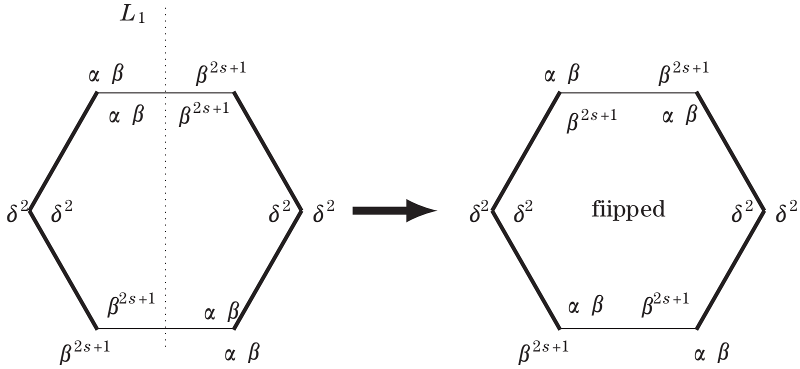
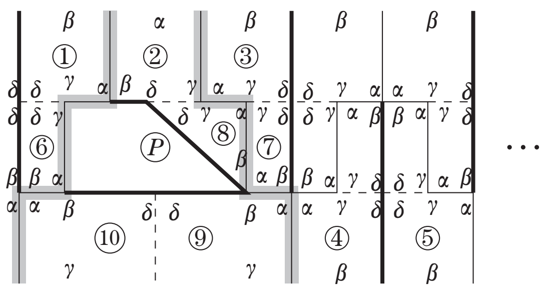


Figure 22 First flip of 3-layer earth map tiling with $f = 16s + 8$.

The shaded edges in Figure 21 form a full great circle, and the angles along it are indicated in the right of Figure 22. The left of Figure 22 gives the angles along the circle for the 3-layer earth map tiling in Figure 16. The two tilings have the same outside hemispheres, and the two inside hemispheres are related by the flip with respect to the line L_1.



$T((8s+2)\ \alpha^2\gamma, (4s+2)\ \alpha^2\beta^2, (4s+2)\ \delta^4, 2\,\alpha\,\beta^{2s+2}, 2\,\beta^{2s}\gamma^2)$

Figure 23 $\alpha\beta^{2s+2}$ appears, and $\beta_4\beta_5\cdots = \beta^{2s}\gamma^2$.

In Figure 23, $\beta_4\beta_5\cdots = \beta^{2s}\gamma^2$. This determines T_9, T_{10}. Then $\| \beta_8 | \alpha_7 | \beta_9 \| \cdots = \alpha\beta^{2s+2}$. This determines the part of $8s$ tiles labeled by P. We obtain the second flip modification of the 3-layer earth map tiling, as explained in Figure 24.

The tiling in Figure 23 is divided into two identical halves along the shaded edges. The right of Figure 24 gives the angles along the shaded edges. The left of Figure 24 gives the angles along the shaded edges for the 3-layer earth map tiling in Figure 16. The two tilings have the same outside halves, and the two inside halves are related by the flip with respect to the line L_2.

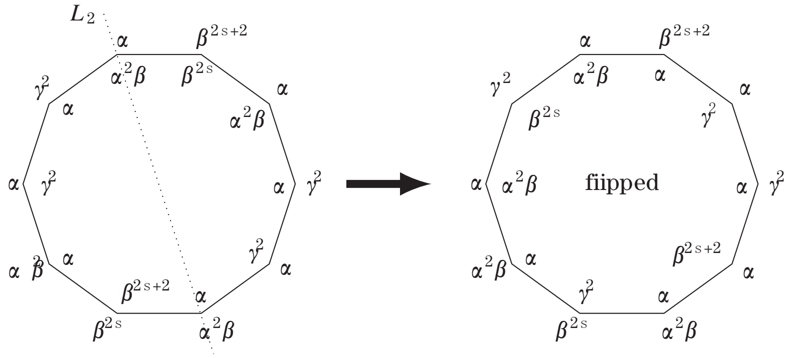


Figure 24 Second flip of 3-layer earth map tiling with $f = 16s + 8$.

If β^{4s+2} and $\alpha\beta^{2s+2}$ are not vertices, then AVC $= \{\alpha\gamma^2, \alpha^2\beta^2, \delta^4, \beta^{2s}\gamma^2\}$. Since there is no tiling for AVC $= \{\alpha\gamma^2, \alpha^2\beta^2, \delta^4\}$, $\beta^{2s}\gamma^2$ must appear. In Figure 25, $\beta^{2s}\gamma^2 = \Vert\beta_1|\gamma_3\vdots\gamma_4|\beta_2\Vert\cdots$ determines T_1, T_2, T_3, T_4. Then $\delta_3\delta_4\cdots = \delta^4$ determines T_5, T_6; $\alpha_1\alpha_3\cdots = \alpha_1\alpha_3\beta_7\beta$ determines T_7, and $\alpha_7\beta_3\beta_5\cdots = \alpha_7\alpha_8\beta_3\beta_5$. By α_8 and no $\gamma|\gamma\cdots$, we get $\vdots\gamma_7\Vert\cdots = \vdots\gamma_7|\beta_8\Vert\cdots$, which determines T_8. Then $\alpha_5\gamma_8\cdots = \alpha_5\gamma_8\gamma_9$ determines T_9; $\alpha_2\alpha_4\cdots = \alpha_2\alpha_4\beta_{10}\beta$ determines T_{10}, and $\alpha_{10}\beta_4\beta_6\cdots = \alpha_{10}\alpha_{11}\beta_4\beta_6$. By α_{11}, we have $\alpha_6\beta_9\cdots \neq \alpha^2\beta^2$, contradicting the AVC.

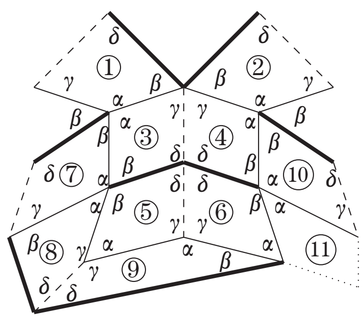


Figure 25 $\beta^{2s}\gamma^2$ appears.

4.3 $\mathbf{H = \alpha\beta^{d-1}}$ (d = 5, 7, 9, 11)

By $\alpha\gamma^2$, δ^4, $\alpha\beta^{d-1}$, we get

$$\alpha = \pi - \frac{2\pi}{d-3} + \frac{8(d-1)\pi}{(d-3)f}, \quad \beta = \frac{\pi}{d-3} - \frac{8\pi}{(d-3)f}, \quad \gamma = \frac{\pi}{2} + \frac{\pi}{d-3} - \frac{4(d-1)\pi}{(d-3)f}, \quad \delta = \frac{\pi}{2}.$$

By Lemma 3.1, we have $\alpha < \pi$, which implies $f > 4(d-1)$. Then $\alpha+2\beta > \pi$, $\frac{\pi}{d-1} < \beta < \frac{\pi}{d-3}$, $\frac{\pi}{2} < \gamma < \frac{\pi}{2} + \frac{\pi}{d-3}$.

Suppose $\gamma^2\cdots = \alpha^k\beta^l\gamma^m\delta^n$. By $R(\gamma^2\cdots) < \gamma+\delta$ and Parity Lemma, we know that $\gamma^2\cdots$ is even. By $\gamma > \frac{\pi}{2}$ and $2\gamma+2\delta > 2\pi$, we get $m = 2$ and $n = 0$. By $\alpha+2\gamma = 2\pi$, we get $k = 0$ or 1. If $k = 1$, then $\gamma^2\cdots = \alpha\gamma^2$. If $k = 0$, by $\frac{\pi}{d-1} < \beta < \frac{\pi}{d-3}$, $\frac{\pi}{2} < \gamma < \frac{\pi}{2} + \frac{\pi}{d-3}$, we deduce that $d-5 < l < d-1$, which forces $l = d-3$. Therefore, $\gamma^2\cdots = \alpha\gamma^2$ or $\beta^{d-3}\gamma^2$. If $\beta^{d-3}\gamma^2$ is a vertex, we get $\alpha = \frac{(d-3)\pi}{d-2}$, $\beta = \frac{\pi}{d-2}$, $\gamma = \frac{(d-1)\pi}{2(d-2)}$ for $d = 5,7,9,11$. Then $\frac{(\alpha,\beta,\gamma)}{\pi} = \left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$, $\left(\frac{4}{5},\frac{1}{5},\frac{3}{5}\right)$, $\left(\frac{6}{7},\frac{1}{7},\frac{4}{7}\right)$ or $\left(\frac{8}{9},\frac{1}{9},\frac{5}{9}\right)$. These cases have exactly the same AVC as the Case $f = 16s+8$ in page 749 for $s = 1,2,3,4$, which has been classified. Therefore, we may assume that $\beta^{d-3}\gamma^2$ is not a vertex. Then $\gamma^2\cdots = \alpha\gamma^2$. By Lemma 2.13, $\alpha\cdots = \alpha\gamma^2$ or $\alpha^k\beta^{2t}$. Therefore, $\alpha\gamma\cdots = \alpha\gamma^2$. If $k \ge 2$, then by $\alpha+2\beta > \pi$, we get $t = 0$ or 1. Therefore, $\alpha^2\cdots = \alpha^k$ or $\alpha^k\beta^2$.

By Parity Lemma and $\gamma > \delta = \frac{\pi}{2}$, we get $\vdots\delta\|\delta\vdots\cdots = \delta^4$.

We extend the 2nd picture of Figure 15 to Figure 26. We have $\gamma_5\gamma_6\cdots = \alpha_7\gamma_5\gamma_6$. By $H = \alpha\beta^{d-1}$, we determine T_8, T_9. By α_7, $\alpha_6\alpha_8\cdots$ is not α^k and must be $\alpha^k\beta^2 (k \ge 2)$. We discuss two cases $k = 2$ and $k \ge 3$.

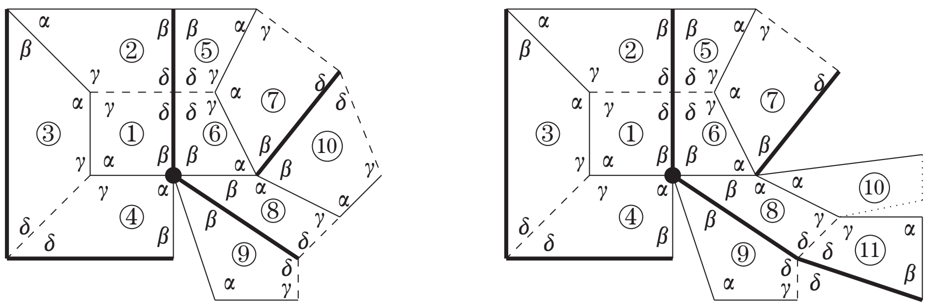


Figure 26 $H = \alpha\beta^{d-1}$.

In the first of Figure 26, $\alpha_6\alpha_8\cdots = \alpha^2\beta^2$. This determines T_7, T_{10}. Then $\alpha_5\gamma_7\cdots = \alpha\gamma^2$, and $\vdots\delta_7\|\delta_{10}\vdots\cdots = \delta^4$. Therefore, T_7 is a special 3344-tile, which has been discussed in page 746 for $H = \alpha^{d-2}\beta^2$ with $d = 4$.

In the second of Figure 26, $\alpha_6\alpha_8\cdots = \alpha^k\beta^2$ $(k \ge 3)$. By α_7, we determine T_7 and get α_{10}. Then $\vdots\delta_8\|\delta_9\vdots\cdots = \delta^4$ determines T_{11}. By α_{10}, we have $\gamma_8\gamma_{11}\cdots \ne \alpha\gamma^2$, a contradiction.

Proposition 4.2 *There is no tiling with the* 2*nd special tile in Figure* 14.

Proof Let the second of Figure 14 be the center tile T_1 in the partial neighborhood in Figure 27. By Lemma 2.10, the degree 4 vertex $\alpha_4\beta_1\cdots = \alpha^2\beta^2 = \alpha_4\alpha_5\beta_1\beta_6$. This determines T_6 and α_5.

Suppose $H = \delta_1\delta_2\delta_6\cdots = \alpha^k\beta^l\gamma^m\delta^n$. Since $\alpha^2\beta^2$ implies $\alpha+\beta = \pi$ and $\gamma+\delta = \left(1+\frac{4}{f}\right)\pi$, we get $m \le 1$. If $m = 1$, H is odd and $l \ge 1$. Then we have $\beta+\gamma+3\delta \le 2\pi$. By $\alpha^2\beta^2$ and $\alpha\gamma^2$,

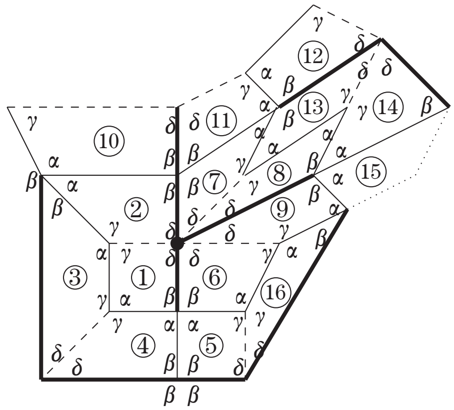


Figure 27 Partial neighborhood of the 2nd special tile.

we get $3(\alpha+\beta+\gamma+\delta)=(2\alpha+2\beta)+(\alpha+2\gamma)+(\beta+\gamma+3\delta)\le 6\pi$, contradicting Lemma 2.2. Therefore, we have $m=0$, H is even and $n\ge 4$. If $l\ge 2$, we have $2\beta+4\delta\le 2\pi$. By $\alpha^2\beta^2$ and $\alpha\gamma^2$, we get $4(\alpha+\beta+\gamma+\delta)=(2\alpha+2\beta)+2(\alpha+2\gamma)+(2\beta+4\delta)\le 8\pi$, contradicting Lemma 2.2. Therefore, we have $l=0$. By Lemma 2.7, we get $H=\delta^d$, $d=6,8,10$. This determines T_7,T_9.

The angle sum at $H=\delta^d$ further implies

$$\alpha=\frac{4\pi}{d}-\frac{8\pi}{f},\quad \beta=\pi-\frac{4\pi}{d}+\frac{8\pi}{f},\quad \gamma=\pi-\frac{2\pi}{d}+\frac{4\pi}{f},\quad \delta=\frac{2\pi}{d}.$$

Then we have $\beta>\frac{\pi}{3},\gamma>\frac{2\pi}{3},\frac{\pi}{5}\le\delta\le\frac{\pi}{3}$. By the angle values, the edge length consideration and Lemma 2.13, we get

$$\text{AVC}=\{\alpha\gamma^2,\alpha^2\beta^2,\beta^4,\beta^2\delta^2,\alpha^k,\delta^d\}.$$

Then $\alpha_2\beta_3\cdots=\alpha_2\alpha_{10}\beta_3\beta$. By α_{10}, $\beta_2\beta_7\cdots=\beta_2\beta_7\beta_{10}\beta_{11}$ determines T_{10},T_{11}. By $\alpha\gamma^2$, $\alpha^2\beta^2$, β^4, we get $\alpha=\beta=\frac{\pi}{2}$, $\gamma=\frac{3\pi}{4}$. This implies $\delta=\frac{2\pi}{d}=\left(\frac{1}{4}+\frac{4}{f}\right)\pi>\frac{\pi}{4}$. Then we get $d=6$, $\delta=\frac{\pi}{3}$ and

$$\text{AVC}=\{\alpha\gamma^2,\alpha^4,\alpha^2\beta^2,\beta^4,\delta^6\}.$$

Then $H=\delta^6$ determines T_8, and $\gamma_7\gamma_8\cdots=\alpha_{13}\gamma_7\gamma_8$. By α_{13}, $\alpha_7\alpha_{11}\cdots=\alpha_7\alpha_{11}\beta_{12}\beta_{13}$ determines T_{12},T_{13}; $\alpha_8\gamma_{13}\cdots=\alpha_8\gamma_{13}\gamma_{14}$ determines T_{14}. Then $\alpha_{14}\beta_8\beta_9\cdots=\alpha_{14}\alpha_{15}\beta_8\beta_9$. By α_5, we get $\rvert\beta_4\rvert\cdots=\beta^4$, which determines T_5. Then $\alpha_6\gamma_5\cdots=\alpha_6\gamma_5\gamma_{16}$ determines T_{16}. By α_{15}, we have $\alpha_9\beta_{16}\cdots\ne\alpha^2\beta^2$, contradicting the AVC.

Proposition 4.3 *There is no tiling with the* 3*rd special tile in Figure* 14.

Proof Let the third of Figure 14 be the center tile T_1 in the partial neighborhood in Figure 28.

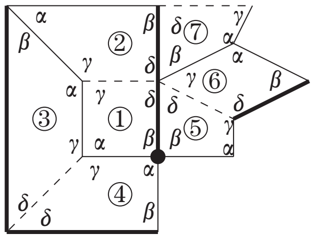


Figure 28 Partial neighborhood of the 3rd special tile.

By the edge length consideration, the degree 5 vertex $\delta_1\delta_2\cdots = \alpha\beta^2\delta^2$ or $\beta\gamma\delta^3$. If $\alpha\beta^2\delta^2$ is a vertex, by $\alpha\gamma^2$, then $2(\alpha+\beta+\gamma+\delta) = 4\pi$, contradicting Lemma 2.2. Therefore, $\delta_1\delta_2\cdots = \beta\gamma\delta^3$. By Lemma 2.13 and $\alpha\gamma^2$, we get $\alpha_4\beta_1\cdots = \alpha^k\beta^{2t}$. This determines T_5, T_6, T_7. If $k \ge 2$, then $2\alpha+2\beta \le 2\pi$. By $\alpha\gamma^2$ and $\beta\gamma\delta^3$, we get $3(\alpha+\beta+\gamma+\delta) = (\alpha+2\gamma)+(\beta+\gamma+3\delta)+(2\alpha+2\beta) \le 6\pi$, contradicting Lemma 2.2. Therefore, $H = \alpha\beta^{d-1}$, $d = 5, 7$.

By Lemma 3.1, $\alpha^2\cdots$ is a vertex and must be even. By $\alpha\gamma^2, \beta\gamma\delta^3$ and Lemma 2.2, we get $\alpha+\beta = \big(1+\frac{6}{f}\big)\pi > \pi$ and $\alpha+\gamma > \pi$. This implies $R(\alpha^2\cdots) < 2\beta, 2\gamma$. Then by Lemma 2.7, we get $\alpha^2\cdots = \alpha^k$. By $\alpha^k, \alpha\beta^{d-1}$, we get $3\alpha \le 2\pi$ and $\alpha+4\beta \le 2\pi$. This implies $\alpha+\beta \le \pi$, contradicting $\alpha+\beta = \big(1+\frac{6}{f}\big)\pi$.

Proposition 4.4 *There is no tiling with the* 4*th special tile in Figure* 14.

Proof Let the fourth of Figure 14 be the center tile T_1 in the partial neighborhoods in Figure 29. By the edge length consideration and Lemma 2.13, the degree 5 vertex $\alpha_4\beta_1\cdots = \alpha^3\beta^2$ or $\alpha\beta^4$, shown in the first and second pictures of Figure 29.

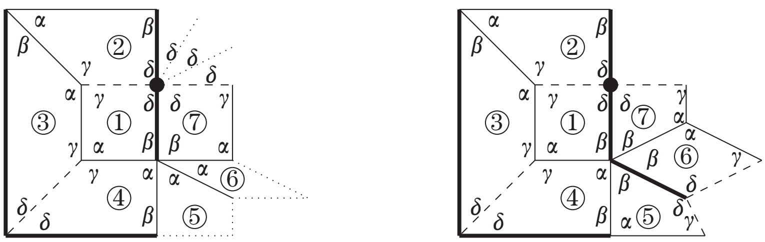


Figure 29 Partial neighborhoods of the 4th special tile.

Case $\boldsymbol{\alpha_4\beta_1\cdots = \alpha^3\beta^2}$ The vertex $|\alpha_4|\beta_1\|\cdots = |\beta_1\|\beta_7|\alpha_6|\alpha_5|\alpha_4|$ determines T_7. Suppose $H = \delta_1\delta_2\delta_7\cdots = \alpha^k\beta^l\gamma^m\delta^n$. If $m \ge 1$, by Parity Lemma, we get $H = \beta\gamma\delta^3\cdots$ or $\gamma^2\delta^4\cdots$. Then by $\alpha\gamma^2, \alpha^3\beta^2$, we have

$$3(\alpha+\beta+\gamma+\delta) < (\alpha+2\gamma)+(3\alpha+2\beta)+(\beta+\gamma+3\delta) \le 6\pi,$$

$$2(\alpha+\beta+\gamma+\delta) < (3\alpha+2\beta)+(2\gamma+4\delta) \le 4\pi.$$

Both contradict Lemma 2.2. Therefore, $m = 0$ and H must be even. If $l > 0$, we get $H = \beta^2\delta^4\cdots$. Then we get

$$4(\alpha+\beta+\gamma+\delta) < 2(\alpha+2\gamma)+(3\alpha+2\beta)+(2\beta+4\delta) \le 8\pi,$$

contradicting Lemma 2.2. Therefore, $l = 0$. By Lemma 2.7, we have $H = \delta^6$. By $\alpha\gamma^2, \alpha^3\beta^2, \delta^6$, we get $\alpha = (\frac{1}{3} - \frac{4}{f})\pi, \beta = (\frac{1}{2} + \frac{6}{f})\pi, \gamma = (\frac{5}{6} + \frac{2}{f})\pi, \delta = \frac{\pi}{3}$.

The AAD of $|\alpha_4|\alpha_5|\alpha_6|\cdots$ implies $\Vert\beta|\gamma\vdots\cdots$ or $\vdots\gamma|\gamma\vdots\cdots$. By $R(\Vert\beta|\gamma\vdots\cdots) < \frac{2\pi}{3} < \gamma$, we get $\Vert\beta|\gamma\vdots\cdots = \Vert\beta|\gamma\vdots\delta\Vert\cdots$. Then by $R(\beta\gamma\delta) = (\frac{1}{3} - \frac{8}{f})\pi <$ all angles, we know that $\beta\gamma\delta$ is a vertex, a contradiction. Moreover, by $R(\vdots\gamma|\gamma\vdots\cdots) < \frac{\pi}{3} \le \gamma,\ \delta$, we also get a contradiction.

Case $\boldsymbol{\alpha_4\beta_1\cdots = \alpha\beta^4}$ The vertex $|\alpha_4|\beta_1\Vert\cdots = \alpha\beta^4$ determines T_5, T_6, T_7. By Lemma 3.1, we have $\alpha < \pi$. Then by $\alpha\beta^4, \alpha\gamma^2$, we get $\beta > \frac{\pi}{4}$, $\gamma > \frac{\pi}{2}$. Moreover by $\alpha\gamma^2$, $\alpha\beta^4$ and Lemma 2.2, we get $\delta = \frac{\pi}{2} - \frac{\alpha}{4} + \frac{4\pi}{f} > \frac{\pi}{4}$.

By Lemma 2.13, $H = \delta^3\cdots$ has no α. If $\deg H = 7$, then by Parity Lemma, $H = \delta_1\delta_2\delta_7\cdots = \beta^3\gamma\delta^3, \beta\gamma^3\delta^3$ or $\beta\gamma\delta^5$. By $\beta, \delta > \frac{\pi}{4}$ and $\gamma > \frac{\pi}{2}$, all have angle sums $> 2\pi$, a contradiction.

Therefore, $\deg H = 6$. By Parity Lemma and $2\gamma + 4\delta > 2\pi$, we get $H = \beta^2\delta^4$ or δ^6. By $\alpha\gamma^2, \alpha\beta^4$ and H, we get

$$H = \beta^2\delta^4 : \alpha = \Big(\frac{2}{3} + \frac{32}{3f}\Big)\pi, \quad \beta = \Big(\frac{1}{3} - \frac{8}{3f}\Big)\pi, \quad \gamma = \Big(\frac{2}{3} - \frac{16}{3f}\Big)\pi, \quad \delta = \Big(\frac{1}{3} + \frac{4}{3f}\Big)\pi;$$
$$H = \delta^6 : \alpha\Big(\frac{2}{3} + \frac{16}{f}\Big)\pi, \quad \beta = \Big(\frac{1}{3} - \frac{4}{f}\Big)\pi, \quad \gamma = \Big(\frac{2}{3} - \frac{8}{f}\Big)\pi, \quad \delta = \frac{\pi}{3}.$$

Both imply $R(\alpha_6\alpha_7\cdots) < \alpha, 2\beta$, contradicting Lemma 2.13.

5 344d-Tile

We classify a^2bc-tilings under the assumption that there is a special 344d-tile, $d = 4, 5$. By Proposition 3.1, δ does not appear in degree 3 vertices. Then up to the symmetry of exchanging $\beta \leftrightarrow \gamma$, there are 7 different configurations in Figure 30. The first two cases have $d = 4$, and all other five cases have $d = 5$. We first prove two useful propositions before studying each special tile.

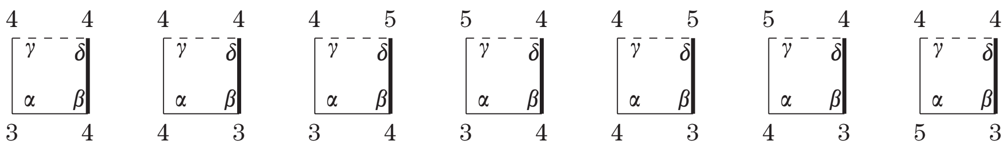


Figure 30 Special tiles with vertex degrees 344d, $d = 4, 5$.

Proposition 5.1 *There is no a^2bc-tiling with vertices $\alpha^3, \beta^4, \gamma^2\delta^2$.*

Proof If $\alpha^3, \beta^4, \gamma^2\delta^2$ are vertices, we have $\alpha = \frac{2\pi}{3}, \beta = \frac{\pi}{2}, \gamma + \delta = \pi$. By Lemma 2.5, we have $\gamma > \frac{\pi}{6}$. By Lemma 2.11, the AAD of α^3 gives a vertex $\beta\gamma\cdots = \alpha^k\beta^l\gamma^m\delta^n$. If $k \ge 1$, then $\beta\gamma\cdots = \alpha\beta\gamma\cdots$ must be even. Therefore, $l, m \ge 2$. By $\alpha + 2\beta + 2\gamma > 2\pi$, we get a contradiction.

Therefore, $k = 0$. If $n = 0$, the vertex $\beta\gamma\cdots$ is even. Then by $\beta = \frac{\pi}{2}, \gamma > \frac{\pi}{6}$, we get $\beta\gamma\cdots = \beta^2\gamma^2$ or $\beta^2\gamma^4$. However $\beta^2\gamma^2$ and β^4 imply $\beta = \gamma$, contradicting Lemma 3.1. Therefore, $\beta\gamma\cdots = \beta^2\gamma^4$. The angle sum of $\beta^2\gamma^4$ further implies $\delta = \frac{3\pi}{4} > \frac{\pi}{2}, \beta + \delta = \frac{5\pi}{4} > \pi$. By Lemma 2.12, $\Vert\delta\vdots\delta\Vert\cdots$ is not a vertex. However $\beta^2\gamma^4 = \vert^{\alpha}\gamma^{\delta}\vdots^{\delta}\gamma^{\alpha}\vert\cdots$ gives a vertex $\Vert\delta\vdots\delta\Vert\cdots$, a contradiction.

Therefore, $n > 0$. This means $\beta\gamma\cdots = \beta\gamma\delta\cdots$. We know that the vertex has no α. Then by $R(\beta\gamma\delta\cdots) = \frac{\pi}{2} = \beta < \gamma + \delta$ and Parity Lemma, we get $\beta\gamma\cdots = \beta\gamma^m\delta$ or $\beta\gamma\delta^n$. Since δ does not appear in degree 3 vertices, we have $m, n \ge 3$.

If $\beta\gamma^m\delta$ is a vertex, by $\beta = \frac{\pi}{2}, \gamma + \delta = \pi$, we get $\gamma = \frac{\pi}{2(m-1)} \le \frac{\pi}{4}$ and $\delta \ge \frac{3\pi}{4}$. By Lemma 2.12, $\Vert\delta\vdots\delta\Vert\cdots$ is not a vertex. However $\beta\gamma^m\delta = \gamma^{\delta}\vdots^{\delta}\gamma\cdots$ gives a vertex $\Vert\delta\vdots\delta\Vert\cdots$, a contradiction.

If $\beta\gamma\delta^n$ is a vertex, then similarly we get $\gamma \ge \frac{3\pi}{4}, \delta \le \frac{\pi}{4}$. Then $\beta\gamma\delta^n = \Vert^{\beta}\delta^{\gamma}\vdots^{\gamma}\delta^{\beta}\Vert\cdots$ gives a vertex $\vert\gamma\vdots\gamma\vert\cdots = \theta\vert\gamma\vdots\gamma\vert\rho\cdots$, where $\theta, \rho = \alpha, \beta$ or γ. By $\alpha = \frac{2\pi}{3}, \beta = \frac{\pi}{2}, \gamma \ge \frac{3\pi}{4}$, we get a contradiction.

Proposition 5.2 *There is no a^2bc-tiling with vertices $\alpha^3, \beta^2\gamma^2, \gamma^2\delta^2, \Vert\beta\vert\beta\Vert\cdots$.*

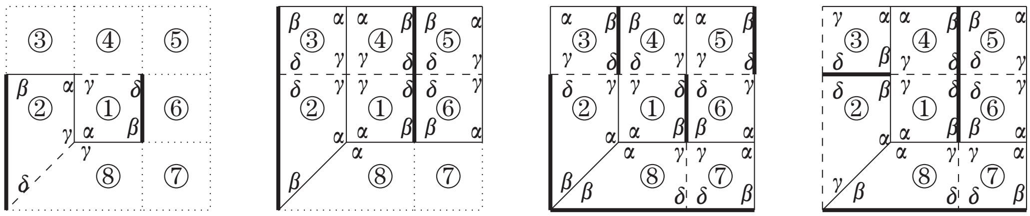
Proof If $\alpha^3, \beta^2\gamma^2$ and $\gamma^2\delta^2$ are vertices, we have $\alpha = \frac{2\pi}{3}, \gamma = \left(\frac{2}{3} - \frac{4}{f}\right)\pi, \beta = \delta = \left(\frac{1}{3} + \frac{4}{f}\right)\pi$. Let $\Vert\beta\vert\beta\Vert\cdots = \alpha^k\beta^l\gamma^m\delta^n$. By $\beta^2\gamma^2 = \vert\beta\Vert\beta\vert\gamma\vdots\gamma\vert$, we get $m \le 1$. If $m = 1$, then it is odd and $l \ge 3, n \ge 1$. Then we get $3\beta + \gamma + \delta = \left(2 + \frac{12}{f}\right)\pi > 2\pi$, a contradiction. Therefore, $m = 0$ and the vertex is even. By edge length consideration, we have $l + n \ge 4$. By $\alpha + 4\beta = \alpha + 2\beta + 2\delta = \left(2 + \frac{16}{f}\right)\pi > 2\pi$, we get $k = 0$. Then we have $\Vert\beta\vert\beta\Vert\cdots = \beta^l\delta^n$ for some even l, n. If $l + n \ge 6$, we get $l\beta + n\delta = (l + n)\left(\frac{1}{3} + \frac{4}{f}\right)\pi > 2\pi$, a contradiction. Therefore $l + n = 4$ and $\beta = \delta = \frac{\pi}{2}$. By $\beta^2\gamma^2$, we get $\gamma = \frac{\pi}{2} = \beta$, contradicting Lemma 3.1.

Proposition 5.3 *Tilings with the* 1*st special tile in Figure* 30*, and without* 334*d-tile, is a* 1*-parameter family of quadrilateral subdivisions of the octahedron $T(8\alpha^3, 12\beta^2\gamma^2, 6\delta^4)$ with* 24 *tiles.*

Proof Let the first of Figure 30 be the center tile T_1 in the partial neighborhoods in Figure 31. In the first picture, we assume $E_{28} \ne a$. If $E_{28} = c$, then T_2 is determined. By Lemma 2.10, the degree 4 vertex $\alpha_2\gamma_1\cdots = \alpha^2\gamma^2$, contradicting $\alpha_1\gamma_2\gamma_8$. If $E_{28} = b$, we get a similar contradiction at $\beta_1\cdots$. Therefore, $E_{28} = a$, and $\alpha_1\cdots = \alpha^3$. Then $E_{23} = b$ or c.

Case $E_{23} = c$ This edge determines T_2. By Lemma 2.10, the degree 4 vertex $\gamma_1\gamma_2\cdots = \gamma^4$ or $\gamma^2\delta^2$. This determines T_3, T_4 in the second and third pictures.

In the second picture, by Lemma 2.10, the degree 4 vertex $\Vert\delta_1\vdots\delta_4\Vert\cdots = \beta^2\delta^2$ or δ^4. By Proposition 5.1′, we know $\beta^2\delta^2$ is not a vertex. Then $\delta_1\delta_4\cdots = \delta^4$, which determines T_5, T_6. By α_8, the degree 4 vertex $\beta_1\beta_6\cdots = \beta^4$ or $\beta^2\gamma^2$. By γ^4, we get $\beta = \gamma$, contradicting Lemma



$E_{28} = c$ $E_{23} = c$ $E_{23} = c$ $E_{23} = b$

Figure 31 Partial neighborhoods of the 1st special tile.

3.1.

In the third picture, by Lemma 2.10, the degree 4 vertex $\gamma_4\delta_1\cdots = \gamma^2\delta^2$ determines T_5, T_6. By α_8, the degree 4 vertex $\beta_1\beta_6\cdots = \beta^4$ or $\beta^2\gamma^2$. By Proposition 5.1, we know β^4 is not a vertex. Then $\beta_1\beta_6\cdots = \beta^2\gamma^2$, which determine T_7, T_8. Therefore, $\alpha^3, \beta^2\gamma^2, \gamma^2\delta^2, \Vert\beta_2|\beta_8\Vert\cdots$ are vertices, contradicting Proposition 5.2.

Case $\boldsymbol{E_{23} = b}$ This edge determines T_2 in the fourth picture. By Lemma 2.10, the degree 4 vertex $\beta_2\gamma_1\cdots = \beta^2\gamma^2$ determines T_3, T_4. By the symmetry of exchanging $\beta \leftrightarrow \gamma$, the case $E_{78} = b$ is also discussed as the previous case $E_{23} = c$. Therefore it remains to consider $E_{78} = c$. This determines T_8, which further determines T_7, T_6, T_5.

By $\alpha^3, \beta^2\gamma^2, \delta^4$, we get $\alpha = \frac{2\pi}{3}, \beta+\gamma = \pi, \delta = \frac{\pi}{2}, f = 24$. By $\alpha = \frac{2\pi}{3}, \beta, \gamma, \delta < \pi$ and Lemma 2.5, we have $\beta, \gamma > \frac{\pi}{6}$. By the symmetry of exchanging $\beta \leftrightarrow \gamma$ and Lemma 3.1, we may assume $\beta < \gamma$. Then the AVC is derived as shown in Table 2.

Table 2 AVC for $\alpha = \frac{2\pi}{3}, \beta + \gamma = \pi$ with $\frac{\pi}{6} < \beta < \gamma, \delta = \frac{\pi}{2}$.

β	vertex
all	$\alpha^3, \beta^2\gamma^2, \delta^4$
$\frac{\pi}{5}$	β^{10}
$\frac{2\pi}{9}$	$\alpha\beta^6$
$\frac{\pi}{4}$	$\beta^3\gamma\delta, \beta^4\delta^2, \beta^8$
$\frac{\pi}{3}$	$\alpha\gamma^2, \alpha^2\beta^2, \alpha\beta^4, \beta^6$

For $\beta = \frac{\pi}{4}$, we have

$$\#\beta = 2\,\#\beta^2\gamma^2 + 3\,\#\beta^3\gamma\delta + 4\,\#\beta^4\delta^2 + 8\,\#\beta^8,$$
$$\#\gamma = 2\,\#\beta^2\gamma^2 + \#\beta^3\gamma\delta.$$

By $\#\beta = f = \#\gamma$, the equalities above imply $\#\beta^3\gamma\delta = \#\beta^4\delta^2 = \#\beta^8 = 0$. This means that $\beta^3\gamma\delta, \beta^4\delta^2, \beta^8$ are not vertices. By a similar argument, we know that β^{10} is not a vertex for

$\beta = \frac{\pi}{5}$, and $\alpha\beta^6$ is not a vertex for $\beta = \frac{2\pi}{9}$. Moreover, for $\beta = \frac{\pi}{3}$, we know that $\alpha\gamma^2$ is a vertex if and only if one of $\alpha^2\beta^2, \alpha\beta^4, \beta^6$ is a vertex.

In the first of Figure 32, $\alpha_1\gamma_2\gamma_3 = |^{\gamma}\alpha_1^{\beta}|^{\alpha}\gamma_2^{\delta}\|^{\delta}\gamma_3^{\alpha}|$ determines T_1, T_2, T_3. Then by $\alpha_3\gamma_1\cdots = \alpha\gamma^2$, $\delta_2\delta_3\cdots = \delta^4$, $\beta_3\cdots = \alpha^2\beta^2, \beta^2\gamma^2$, $\alpha\beta^4$ or β^6, we know that T_3 is a $334d$-tile, which has been discussed in Section 4.

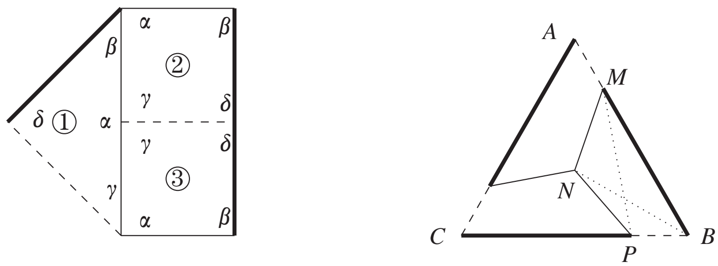


Figure 32 $\alpha\gamma^2$ and quadrilateral subdivision of a triangle.

In conclusion, we only need to consider AVC $= \{\alpha^3, \beta^2\gamma^2, \delta^4\}$. By the argument in "Subcase α^3 appears" on page 1.7, we get the quadrilateral subdivision of the octahedron in the first picture of Figure 19.

5.1 Moduli of the quadrilateral subdivision of the octahedron

The second picture of Figure 32 shows a quadrilateral subdivision of one triangular face of the regular octahedron. We have $MN = NP = a, BM = b, BP = c$, and

$$\alpha = \frac{2\pi}{3}, \quad \beta + \gamma = \pi, \quad \delta = \frac{\pi}{2}, \quad b + c = \frac{\pi}{2}.$$

By $\beta < \gamma$ and $\beta+\gamma = \pi$, we get $\beta < \frac{\pi}{2} < \gamma$. By $\angle NPM = \angle NMP$, we get $\angle BPM > \angle BMP$. Therefore, $b > c$. By $b+c = \frac{\pi}{2}$, we have $c \in (0, \frac{\pi}{4})$. Then by applying the cosine law to $\triangle MNP$ and $\triangle BMP$, we get

$$\cos^2 a + \sin^2 a \cos\alpha = \cos MP = \cos b \cos c = \sin c \cos c.$$

Solving the above equation, we get

$$\sqrt{3}\cos a = \sin c + \cos c. \tag{5.1}$$

Similarly, the cosine laws for $\triangle BMN$ and $\triangle BNP$ give

$$\cos a \cos b + \sin a \sin b \cos\beta = \cos BN = \cos a \cos c + \sin a \sin c \cos\gamma.$$

Then by $\beta + \gamma = \pi, b + c = \frac{\pi}{2}$, this implies

$$\cos\beta = \frac{\cos a(\cos c - \sin c)}{\sin a(\cos c + \sin c)} = \frac{\cos c - \sin c}{\sqrt{2 - \sin 2c}}. \tag{5.2}$$

The flip modification in Proposition 4.1 requires $\beta = \frac{\pi}{3}$. Then by (5.2), we get $\sin 2c = \frac{2}{3}$. Then

$$c = \frac{1}{2}\arcsin\frac{2}{3} \approx 0.116\pi, \quad b = \frac{\pi}{2} - c \approx 0.383\pi, \quad a = \arccos\frac{\sqrt{5}}{3} \approx 0.232\pi.$$

The quadrilateral reduces to type a^3b if $a = c$. By (5.1), we get

$$a = c = \arctan(\sqrt{3} - 1) \approx 0.201\pi, \quad b = \frac{\pi}{2} - c \approx 0.298\pi.$$

By (5.2), we get $\beta = \arccos \frac{3-\sqrt{3}}{6} \approx 0.432\pi$, $\gamma = \pi - \beta \approx 0.567\pi$. The reduced case $a = b$ is obtained from the case $a = c$ by exchanging $b \leftrightarrow c$.

We remark that the quadrilateral reduces to type a^2b^2 for $b = c = \frac{\pi}{4}$. In this case we have $\beta = \gamma = \frac{\pi}{2}$ and $a = \arccos \frac{\sqrt{6}}{3} \approx 0.195\pi$.

We conclude that a^2bc-tilings with the first special tile in Figure 30 are parameterized by

$$c \in \Big(0, \frac{\pi}{4}\Big) \backslash \{\arctan(\sqrt{3} - 1)\}.$$

Proposition 5.4 *There is no tiling with the* 2nd *special tile in Figure* 30*, and without* 3344*-tile.*

Proof Let the second of Figure 30 be the center tile T_1 in the partial neighborhoods in Figure 33. By Lemma 2.9, the degree 3 vertex $\beta_1 \cdots = \alpha_3\beta_1\beta_2$ determines T_2. This implies that $\alpha^2\beta^2$ is not a vertex. Then by α_3 and Lemma 2.10, the degree 4 vertex $\alpha_1 \cdots = \alpha^2\gamma^2 = \alpha_1\alpha_5\gamma_3\gamma_4$, which determines T_3, T_4. Then $E_{56} = b$ or c.

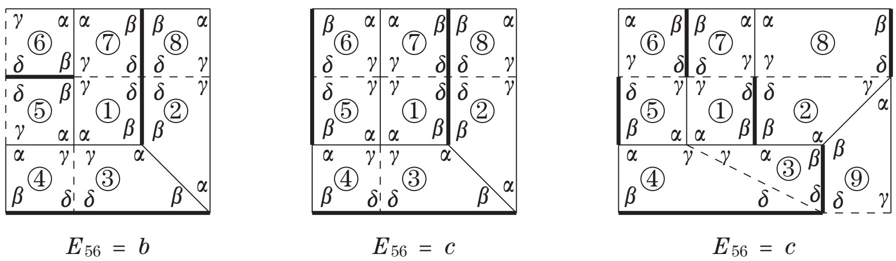


Figure 33 Partial neighborhoods of the 2nd special tile.

Case $E_{56} = b$ This edge determines T_5 in the first picture. By Lemma 2.10, the degree 4 vertex $\beta_5\gamma_1 \cdots = \beta^2\gamma^2$ determines T_6, T_7. Then the degree 4 vertex $\delta_1\delta_2\delta_7 \cdots = \delta^4$ determines T_8. By $\alpha\beta^2$, $\alpha^2\gamma^2$, $\beta^2\gamma^2$, δ^4, we get $\alpha = \beta = \frac{2\pi}{3}, \gamma = \frac{\pi}{3}, \delta = \frac{\pi}{2}$. Then by the edge length consideration, we get $\alpha_2\beta_3 \cdots = \alpha\beta^2$, $\delta_3\delta_4 \cdots = \delta^4$. Then T_3 is a special 3344-tile, which has been discussed in Section 4.

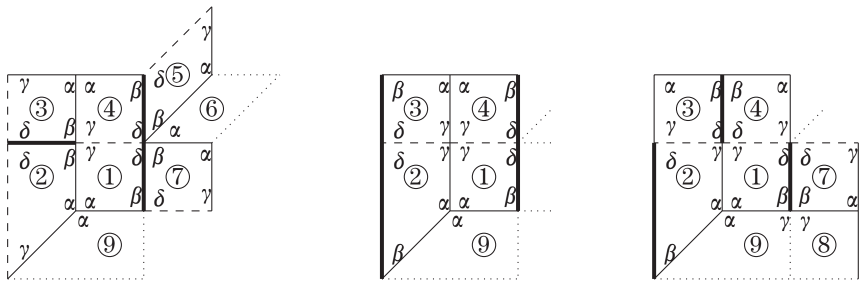
Case $E_{56} = c$ This edge determines T_5. By Lemma 2.10, the degree 4 vertex $\gamma_1\gamma_5 \cdots = \gamma^4$ or $\gamma^2\delta^2$. This determines T_6, T_7 in the second and third pictures.

In the second picture, the degree 4 vertex $\delta_1\delta_2\delta_7 \cdots = \delta^4$ determines T_8. By $\alpha\beta^2, \alpha^2\gamma^2, \gamma^4, \delta^4$, we get $\alpha = \gamma = \delta = \frac{\pi}{2}, \beta = \frac{3\pi}{4}$. Then by the edge length consideration, we get $\alpha_2\beta_3 \cdots = \alpha\beta^2$, $\|\delta_5 \vdots \delta_6\| \cdots = \delta^4$. Then T_3 is a special 3344-tile, which has been discussed in Section 4.

In the third picture, the degree 4 vertex $\gamma_7\delta_1\delta_2 \cdots = \gamma^2\delta^2$ determines T_8. By $\alpha\beta^2$ and Parity Lemma, we get $\alpha_2\beta_3 \cdots = \alpha_2\beta_3\beta_9$, which determines T_9. Then $\alpha_9\gamma_2\delta_8 \cdots$ contradicts Lemma 2.13′.

Proposition 5.5 *There is no tiling with the* 3rd *special tile in* Figure 30.

Proof Let the third of Figure 30 be the center tile T_1 in the partial neighborhoods in Figure 34. For the same reason as Proposition 5.3, we have $E_{29} = a, \alpha_1 \cdots = \alpha^3$. Then $E_{23} = b$ or c.



$E_{23} = b$ $E_{23} = c$ $E_{23} = c$

Figure 34 Partial neighborhoods of the 3rd special tile.

Case $E_{23} = b$ This edge determines T_2 in the first picture. By Lemma 2.10, the degree 4 vertex $\beta_2\gamma_1 \cdots = \beta^2\gamma^2$ determines T_3, T_4. By the edge length consideration, the degree 5 vertex $\Vert\delta_1\vdots\delta_4\Vert \cdots = \alpha\beta^2\delta^2$ or $\beta\gamma\delta^3$. However $\alpha^3, \beta^2\gamma^2$ and $\beta\gamma\delta^3$ imply $3(\alpha+\beta+\gamma+\delta) = 6\pi$, contradicting Lemma 2.2. Then $\delta_1\delta_4 \cdots = \alpha\beta^2\delta^2$, which determines T_5, T_7. Then by Lemma 2.10, the degree 4 vertex $\beta_1\delta_7 \cdots = \beta^2\delta^2$, which contradicts $\alpha\beta^2\delta^2$.

Case $E_{23} = c$ This edge determines T_2. By Lemma 2.10, the degree 4 vertex $\gamma_1\gamma_2 \cdots = \gamma^4$ or $\gamma^2\delta^2$. This determines T_3, T_4 in the second and third pictures.

In the second picture, by Lemma 2.10, the degree 4 vertex $\beta_1 \cdots = \beta^4, \beta^2\gamma^2$ or $\beta^2\delta^2$. If β^4 or $\beta^2\gamma^2$ is a vertex, by γ^4, we get $\beta = \gamma$, contradicting Lemma 3.1. Then $\beta_1 \cdots = \beta^2\delta^2$. Therefore, $\alpha^3, \beta^2\delta^2, \gamma^4$ are vertices, contradicting Proposition 5.1′.

In the third picture, by Lemma 2.10, the degree 4 vertex $\beta_1 \cdots = \beta^4, \beta^2\gamma^2$ or $\beta^2\delta^2$. If β^4 is a vertex, then $\alpha^3, \beta^4, \gamma^2\delta^2$ contradicts Proposition 5.1. If $\beta^2\delta^2$ is a vertex, by $\gamma^2\delta^2$, we get $\beta = \gamma$, contradicting Lemma 3.1. Then $\beta_1 \cdots = \beta^2\gamma^2$, which determines T_7. Then the degree 5 vertex $\vert\gamma_4\vdots\delta_1\Vert\delta_7\vdots \cdots = \alpha\gamma^2\delta^2$ or $\beta\gamma\delta^3$. By $\gamma^2\delta^2$, we get $\gamma_4\delta_1\delta_7 \cdots = \beta\gamma\delta^3$. By $\alpha^3, \beta^2\gamma^2, \gamma^2\delta^2$ and $\beta\gamma\delta^3$, we get $\alpha = \gamma = \frac{2\pi}{3}$, $\beta = \delta = \frac{\pi}{3}$, contradicting Lemma 2.2.

Proposition 5.6 *There is no tiling with the* 4*th special tile in* Figure 30.

Proof Let the fourth of Figure 30 be the center tile T_1 in the partial neighborhoods in Figure 35. If $E_{27} = c$ in the first picture, then the degree 3 vertex $\alpha_1 \cdots = \alpha_1\gamma_2\gamma_7$. This determines T_7. By Parity Lemma, the degree 5 vertex $\alpha_7\gamma_1 \cdots$ must be even. This implies $\alpha + 2\gamma < 2\pi$, contradicting $\alpha_1\gamma_2\gamma_7$. If $E_{27} = b$, we get a similar contradiction at the degree 4 vertex $\alpha_2\beta_1 \cdots$. Therefore, $E_{27} = a, \alpha_1 \cdots = \alpha^3$. Then $E_{23} = b$ or c.

Case $E_{23} = b$ This edge determines T_2. By Lemma 2.10, the degree 4 vertex $\Vert\beta_1\vert\beta_2\Vert \cdots = \beta^4$ or $\beta^2\delta^2$. This determines T_3, T_4 in the second and third pictures.

In the second picture, by Lemma 2.10, the degree 4 vertex $\vdots\delta_1\Vert\delta_4\vdots \cdots = \gamma^2\delta^2$ or δ^4. However $\alpha^3, \beta^4, \gamma^2\delta^2$ contradicts Proposition 5.1. Then $\delta_1\delta_4 \cdots = \delta^4$, which determines T_5, T_6. By $\alpha^3, \beta^4, \delta^4$, we get $\alpha = \frac{2\pi}{3}$, $\beta = \delta = \frac{\pi}{2}$, $\gamma = \left(\frac{1}{3} + \frac{4}{f}\right)\pi$. By α_7 and Parity Lemma, the degree 5 vertex $\vert\gamma_1\vdots\gamma_6\vert \cdots = \alpha\beta^2\gamma^2, \alpha\gamma^4$ or $\beta\gamma^3\delta$. All have angle sums $> 2\pi$, a contradiction.

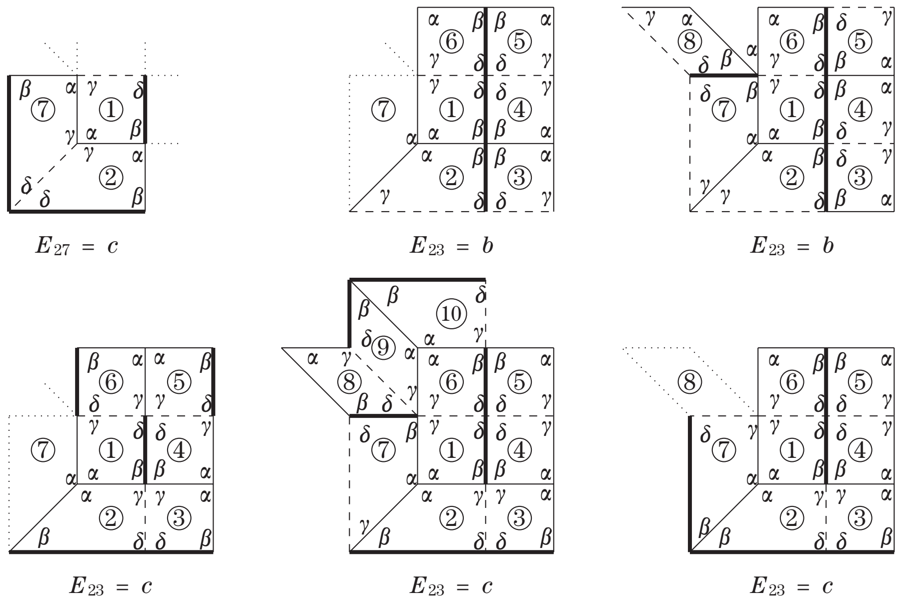


Figure 35 Partial neighborhoods of the 4th special tile.

In the third picture, by Lemma 2.10, the degree 4 vertex $\beta_4\delta_1\cdots = \beta^2\delta^2$. This determines T_5, T_6. By α_7 and Parity Lemma, the degree 5 vertex $\Vert\gamma_1\vdots\gamma_6\Vert\cdots = \alpha\beta^2\gamma^2, \alpha\gamma^4$ or $\beta\gamma^3\delta$. By $\alpha^3, \beta^2\delta^2$, we get $\alpha = \frac{2\pi}{3}$, $\beta+\delta=\pi$, $\gamma = \left(\frac{1}{3}+\frac{4}{f}\right)\pi$. Then $\alpha+4\gamma > 2\pi$ and $\beta+3\gamma+\delta > 2\pi$. Therefore, $\gamma_1\gamma_6\cdots = \alpha\beta^2\gamma^2 = \Vert\gamma_6\vdots\gamma_1\vert\beta_7\Vert\beta_8\vert\alpha\Vert$. This determines T_7, T_8. By $\alpha\beta^2\gamma^2$, we get $\beta = \left(\frac{1}{3}-\frac{4}{f}\right)\pi$, $\delta = \left(\frac{2}{3}+\frac{4}{f}\right)\pi$. Then by $\gamma+\delta > \pi, \delta > \frac{\pi}{2}$ and Lemma 2.12$'$, $\vdots\delta\Vert\delta\vdots\cdots$ is not a vertex, contradicting $\vdots\delta_7\Vert\delta_8\vdots\cdots$.

Case $E_{23} = c$ This edge determines T_2. By Lemma 2.10, the degree 4 vertex $\beta_1\gamma_2\cdots = \beta^2\gamma^2$ determines T_3, T_4. Then the degree 4 vertex $\vdots\delta_1\Vert\delta_4\vdots\cdots = \gamma^2\delta^2$ in the fourth picture, or $\vdots\delta_1\Vert\delta_4\vdots\cdots = \delta^4$ in the fifth and sixth pictures. This determines T_5, T_6.

In the fourth picture, by $\alpha^3, \beta^2\gamma^2, \gamma^2\delta^2$, we get $\alpha = \frac{2\pi}{3}$, $\beta=\delta=\left(\frac{1}{3}+\frac{4}{f}\right)\pi$, $\gamma = \left(\frac{2}{3}-\frac{4}{f}\right)\pi$. By α_7 and Parity Lemma, the degree 5 vertex $\Vert\gamma_1\vdots\delta_6\Vert\cdots = \beta^3\gamma\delta, \beta\gamma^3\delta$ or $\beta\gamma\delta^3$. All have angle sum $> 2\pi$, a contradiction.

Therefore, $\vdots\delta_1\Vert\delta_4\vdots\cdots = \delta^4$. Then $\delta = \frac{\pi}{2}$. By α_7, we get $E_{78} = b$ or c. This determines T_7 in the fifth and sixth pictures.

In the fifth picture, by the edge length consideration, the degree 5 vertex $\beta_7\gamma_1\gamma_6\cdots = \alpha\beta^2\gamma^2$ or $\beta\gamma^3\delta$. By $\beta^2\gamma^2$, we get $\beta_7\gamma_1\gamma_6\cdots = \beta\gamma^3\delta$, which determines T_8, T_9. By $\alpha = \frac{2\pi}{3}, \beta+\gamma=\pi, \delta=\frac{\pi}{2}$ and $\beta\gamma^3\delta$, we get $\beta = \frac{3\pi}{4}, \gamma = \frac{\pi}{4}$. Then $\beta_5\beta_6\cdots = \beta_5\beta_6\gamma_{10}\gamma$ and $\alpha_6\alpha_9\cdots = \alpha_6\alpha_9\alpha_{10}$ determine T_{10}. Then by $\beta+\delta > \pi, \beta > \frac{\pi}{2}$ and Lemma 2.12, $\Vert\beta\vert\beta\Vert\cdots$ is not a vertex, contradicting $\Vert\beta_9\vert\beta_{10}\Vert\cdots$.

In the sixth picture, by the edge length consideration, the degree 5 vertex $\gamma_1\gamma_6\gamma_7\cdots = \alpha\gamma^4$ or $\beta\gamma^3\delta$. By $\alpha = \frac{2\pi}{3}, \beta+\gamma=\pi, \delta=\frac{\pi}{2}$, we get $\beta = \frac{2\pi}{3}$, $\gamma = \frac{\pi}{3}$ for $\alpha\gamma^4$, or $\beta = \frac{3\pi}{4}$, $\gamma = \frac{\pi}{4}$ for $\beta\gamma^3\delta$. Both imply $\beta+\delta > \pi, \beta > \frac{\pi}{2}$. By Lemma 2.12, $\Vert\beta\vert\beta\Vert\cdots$ is not a vertex, contradicting

$\Vert\beta_2|\beta_7\Vert\cdots$.

Proposition 5.7 *There is no tiling with the* 5*th special tile in* Figure 30.

Proof Let the fifth of Figure 30 be the center tile T_1 in the partial neighborhood in Figure 36. By Lemma 2.9, the degree 3 vertex $\beta_1\cdots=\alpha_3\beta_1\beta_2$ determines T_2. By $\alpha_3\beta_1\beta_2$ and Lemma 2.10, the degree 4 vertex $\alpha_1\cdots=\alpha^2\gamma^2$. This determines T_3, T_4. By Parity Lemma and Lemma 2.13′, the degree 5 vertex $\text{¦}\delta_1\Vert\delta_2\text{¦}\cdots$ has no α and must be odd. Then $\text{¦}\delta_1\Vert\delta_2\text{¦}\cdots=\beta\gamma\delta^3$. By $\alpha\beta^2, \alpha^2\gamma^2$ and $\beta\gamma\delta^3$, we get $3(\alpha+\beta+\gamma+\delta)=(\alpha+2\beta)+(2\alpha+2\gamma)+(\beta+\gamma+3\delta)=6\pi$, contradicting Lemma 2.2.

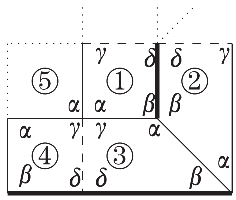


Figure 36 Partial neighborhood of the 5th special tile.

Proposition 5.8 *There is no tiling with the* 6*th special tile in* Figure 30*, and without* 3344*-tile.*

Proof Let the sixth of Figure 30 be the center tile T_1 in the partial neighborhoods in Figure 37. By Lemma 2.9, the degree 3 vertex $\beta_1\cdots=\alpha_3\beta_1\beta_2$ determines T_2. By $\alpha_3\beta_1\beta_2$ and Lemma 2.10, the degree 4 vertex $\alpha_1\cdots=\alpha^2\gamma^2$. This determines T_3, T_4. Then by Lemma 2.10, the degree 4 vertex $\text{¦}\delta_1\Vert\delta_2\text{¦}\cdots=\gamma^2\delta^2$ or δ^4. This determines T_5, T_6 in Figure 37.

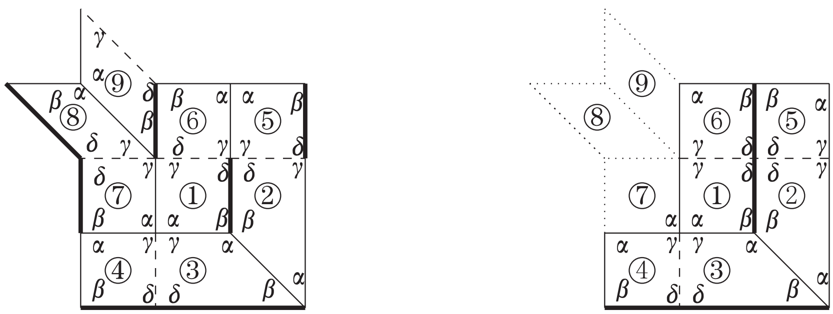


Figure 37 Partial neighborhoods of the 6th special tile.

In the first picture, by $\alpha\beta^2$, $\alpha^2\gamma^2$, $\gamma^2\delta^2$, we get $\alpha=\delta=\frac{8\pi}{f}$, $\beta=\left(1-\frac{4}{f}\right)\pi$, $\gamma=\left(1-\frac{8}{f}\right)\pi$. By $\gamma>0$, we have $f>8$. By α_7 and the edge length consideration, the degree 5 vertex $\gamma_1\delta_6\cdots=\beta^3\gamma\delta, \beta\gamma^3\delta$ or $\beta\gamma\delta^3$. However $3\beta+\gamma+\delta>2\pi$, $\beta+\gamma+3\delta>2\pi$. Then $\gamma_1\delta_6\cdots=\beta\gamma^3\delta$, which determines T_7, T_8, T_9 and implies $\alpha=\delta=\frac{4\pi}{5}$, $\beta=\frac{3\pi}{5}$, $\gamma=\frac{\pi}{5}$. Then by $\beta+\delta>\pi, \delta>\frac{\pi}{2}$ and Lemma 2.12, $\Vert\delta\text{¦}\delta\Vert\cdots$ is not a vertex, contradicting $\Vert\delta_3\text{¦}\delta_4\Vert\cdots$.

In the second picture, by α_7, the degree 5 vertex $\gamma_1\gamma_6\cdots=\alpha\beta^2\gamma^2$, $\alpha\gamma^4$ or $\beta\gamma^3\delta$. By $\alpha\beta^2$, $\alpha\beta^2\gamma^2$ is not a vertex. If $\gamma_1\gamma_6\cdots=\alpha\gamma^4$, by $\alpha\beta^2,\alpha^2\gamma^2,\delta^4,\alpha\gamma^4$, we get $\alpha=\beta=\frac{2\pi}{3}$, $\gamma=\frac{\pi}{3}$, $\delta=\frac{\pi}{2}$. If $\gamma_1\gamma_6\cdots=\beta\gamma^3\delta$, we get $\alpha=\frac{5\pi}{7}$, $\beta=\frac{9\pi}{14}$, $\gamma=\frac{2\pi}{7}$, $\delta=\frac{\pi}{2}$. Both imply $\delta_3\delta_4\cdots=\delta^4$ and $\alpha_2\beta_3\cdots=\alpha\beta^2$. Then T_3 is a special 3344-tile, which has been discussed in Section 4.

Proposition 5.9 *There is no tiling with the* 7*th special tile in Figure* 30*, and without* 3345-*tile.*

Proof Let the seventh of Figure 30 be the center tile T_1 in the partial neighborhoods in Figure 38. By Lemma 2.9, the degree 3 vertex $\beta_1\cdots=\alpha_7\beta_1\beta_2$, which determines T_2. Then by Lemma 2.10, the degree 4 vertex $\vdots\delta_1\|\delta_2\vdots\cdots=\gamma^2\delta^2$ or δ^4. This determines T_3,T_4 in Figure 38.

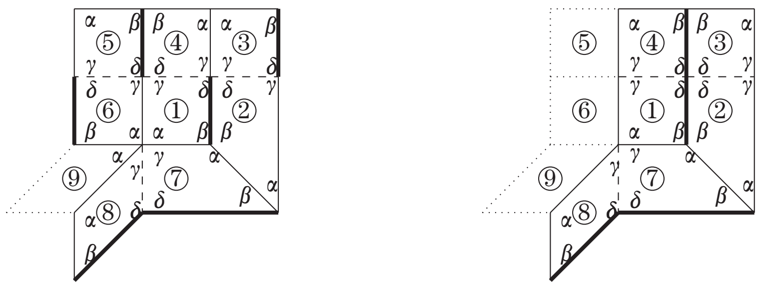


Figure 38 Partial neighborhoods of the 7th special tile.

In the first picture, by Lemma 2.10, the degree 4 vertex $\gamma_1\delta_4\cdots=\gamma^2\delta^2$ determines T_5,T_6. By $\alpha\beta^2$ and Lemma 2.13$'$, and α_7, the degree 5 vertex $\alpha_1\alpha_6\cdots=\alpha^3\gamma^2$, which determines T_7,T_8. By $\alpha\beta^2,\gamma^2\delta^2,\alpha^3\gamma^2$, we get $\alpha=\frac{8\pi}{f}$, $\beta=\left(1-\frac{4}{f}\right)\pi$, $\gamma=\left(1-\frac{12}{f}\right)\pi$, $\delta=\frac{12\pi}{f}$. If $\delta>\frac{\pi}{2}$, then by $\beta+\delta>\pi$ and Lemma 2.12, $\|\delta\vdots\delta\|\cdots$ is not a vertex, contradicting $\|\delta_7\vdots\delta_8\|\cdots$. Therefore, $\delta\le\frac{\pi}{2}$. Then we have $f\ge 24$ and $\alpha\le\frac{\pi}{3},\beta\ge\frac{5\pi}{6},\gamma\ge\frac{\pi}{2}$. By $\beta>\frac{\pi}{2}$, $\beta+\delta>\pi$ and Lemma 2.12, $\|\beta|\beta\|\cdots$ is not a vertex. Then by α_9, we get $\|\beta_6|\gamma_9\vdots\cdots=\theta\|\beta_6|\gamma_9\vdots\rho\cdots$, where $\theta=\beta$ or δ and $\rho=\gamma$ or δ, and θ,ρ are not the same angle. However we always have $\theta+\beta>\pi$ and $\rho+\gamma\ge\pi$, a contradiction.

In the second picture, by $\alpha\beta^2$ and Lemma 2.13$'$, and α_7, the degree 5 vertex $\alpha_1\cdots=\alpha\gamma^4$ or $\alpha^3\gamma^2$, which determines T_7,T_8.

If $\alpha_1\cdots=\alpha\gamma^4$, we get $E_{56}=a$ and the degree 4 vertex $|\gamma_1\vdots\gamma_4|\cdots=\alpha^2\gamma^2$. By $\alpha\beta^2$, $\alpha^2\gamma^2$, $\alpha\gamma^4$, δ^4, we get $\alpha=\beta=\frac{2\pi}{3}$, $\gamma=\frac{\pi}{3}$, $\delta=\frac{\pi}{2}$.

If $\alpha_1\cdots=\alpha^3\gamma^2$, then by $\alpha\beta^2$, δ^4 and $\alpha^3\gamma^2$, we get $\alpha=\left(\frac{1}{2}-\frac{4}{f}\right)\pi$, $\beta=\left(\frac{3}{4}+\frac{2}{f}\right)\pi$, $\gamma=\left(\frac{1}{4}+\frac{6}{f}\right)\pi$. By $\alpha^3\gamma^2$, the degree 4 vertex $|\gamma_1\vdots\gamma_4|\cdots=\beta^2\gamma^2$ or γ^4. However $\beta^2\gamma^2$ contradicts $\beta+\gamma>\pi$. Therefore, $\gamma_1\gamma_4\cdots=\gamma^4$. Then $f=24$ and $\alpha=\frac{\pi}{3}$, $\beta=\frac{5\pi}{6}$, $\gamma=\delta=\frac{\pi}{2}$.

Both possibilities imply $\alpha_2\beta_7\cdots=\alpha\beta^2$ and $\|\delta_7\vdots\delta_8\|\cdots=\delta^4$. Then T_7 is a special 3345-tile, which has been discussed in Section 4.

Finally, we describe the tilings with more than one types of special tiles. Except quadrilateral subdivisions of the octahedron, where every tile is 3444-tile, all other a^2bc-tilings with a 344d-tile also have a $334d$-tile:

• The flip of a special quadrilateral subdivision of the octahedron with $f = 24$ has a 3444-tile and a 3344-tile: $T(2\alpha^3, 6\alpha\gamma^2, 6\alpha^2\beta^2, 6\beta^2\gamma^2, 6\delta^4)$;

• the 3-layer earth map tiling with $f = 16$ has a 3444-tile and a 3344-tile: $T(8\alpha\gamma^2, 4\alpha^2\beta^2, 2\beta^4, 4\delta^4)$;

• the 1st flip of the 3-layer earth map tiling with $f = 24$ has a 3445-tile, a 3344-tile and a 3345-tile: $T(12\alpha\gamma^2, 4\alpha^2\beta^2, 6\delta^4, 4\alpha\beta^4)$;

• the 2nd flip of the 3-layer earth map tiling with $f = 24$ has a 3444-tile, a 3445-tile, a 3344-tile and a 3345-tile: $T(10\alpha\gamma^2, 6\alpha^2\beta^2, 2\beta^2\gamma^2, 6\delta^4, 2\alpha\beta^4)$.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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