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A Study on the Second Order Tangent Bundles over Bi-Kählerian Manifolds

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Abstract This paper aims to study the Berger type deformed Sasaki metric g_{BS} on the second order tangent bundle T^2M over a bi-Kählerian manifold M. The authors firstly find the Levi-Civita connection of the Berger type deformed Sasaki metric g_{BS} and calculate all forms of Riemannian curvature tensors of this metric. Also, they study geodesics on the second order tangent bundle T^2M and bi-unit second order tangent bundle $T^2_{1,1}M$, and characterize a geodesic of the bi-unit second order tangent bundle in terms of geodesic curvatures of its projection to the base. Finally, they present some conditions for a section $\sigma : M \to T^2M$ to be harmonic and study the harmonicity of the different canonical projections and inclusions of (T^2M, g_{BS}). Moreover, they search the harmonicity of the Berger type deformed Sasaki metric g_{BS} and the Sasaki metric g_{S} with respect to each other.

Keywords Berger type deformed Sasaki metric, Bi-Kählerian structure, Geodesics, Harmonicity, Riemannian curvature tensor, Second order tangent bundle
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1 Introduction

The geometry of the second order tangent bundle T^2M over an n-dimensional manifold M which is the equivalent classes of curves with the same aceleration vector fields on M was studied in [14–15, 25, 32–33] . Dodson and Radivoiovici proved that a second-order tangent bundle T^2M of finite n-dimensional M becomes a vector bundle over M if and only if M has a linear connection in [15]. The lifts of geometric objects on M to its second order tangent bundle T^2M were developed in [34]. In [22], Ishikawa defined a Sasaki-type metric on the second order tangent bundle T^2M of a Riemanian manifold and searched some of its properties. With this in hand Gezer and Magden [20] studied the geometry of a second order tangent bundle with a Sasaki-type metric. The Levi-Civita connection and all forms of Riemannian curvature tensor of Sasaki-type metric on T^2M were computed and the relations between the geometric properties M and T^2M were derived in [20] and [24]. From a different perspective, in [2, 10, 14], the sections on the second-order tangent bundle T^2M (bundle of accelerations on a smooth manifold M),

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locally, described in detail the second order ordinary differential equations on M.

These equations have received renewed geometric attention in recent years from interactions with jet fields, linear and nonlinear connections, Lagrangians, Finsler structures and the theory of timedependent Lagrangian particle systems (see [5–6, 28, 30–31]).

Next, we assume that M is a $4k$-dimensional complex manifold and φ_i ($\varphi_i^2 = -I$) for $i = 1, 2$, are two independent compatible integrable almost complex structures. Here $\varphi_1(x) \neq \varphi_2(x)$ for a point x in M. Also, a pseudo-Riemannian metric g is a Hermitian metric with respect to both complex structures φ_1 and φ_2 , i.e., $g(\varphi_1 X, \varphi_1 Y) = g(X, Y)$ and $g(\varphi_2 X, \varphi_2 Y) = g(X, Y)$. In this case, the quartet $(M_{4k}, g, \varphi_1, \varphi_2)$ is called a bi-Hermitian manifold. If $\varphi_1(x) \neq \varphi_2(x)$ everywhere on M, a bi-Hermitian structure $(g, \varphi_1, \varphi_2)$ is called strongly bi-Hermitian. The real function p is defined by $p = -\frac{1}{4k}\mathrm{trace}(\varphi_1 \circ \varphi_2)$ or equivalently $\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -2pI$, where p is the angle function of a bi-Hermitian structure and where I is the field of identity endomorphisms.

As is known, an almost Hermitian structure on a manifold M consists of a nondegenerate 2-form ω, an almost complex structure φ and a Riemannian metric g satisfying the compatibility condition $\omega(X, Y) = g(\varphi X, Y)$. If the 2-form ω is closed, i.e., $d\omega = 0$, the triple (g, φ, ω) is called an almost Kählerian structure. Also, the triple (g, φ, ω) is called Kählerian structure if the almost complex structure φ is integrable.

We can define bi-Kählerian manifolds by following analogue of Kählerian geometry. Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Hermitian manifold. For such a structure we define 2-forms ω_i setting $\omega_i(X, Y) = g(\varphi_i X, Y)$, $i = 1, 2$. If the 2-forms ω_i are closed ($d\omega_i = 0$), the bi-Hermitian structure $(g, \varphi_1, \varphi_2)$ is called bi-Kähler. Such bi-Hermitian structures have been studied by many authors (see [7, 9, 29, 36]). The bi-Hermitian geometry is known in the physics literature: Gates et al. showed in [19] that upon imposing $N = (2, 2)$ supersymmetry, the geometry induced on the target of a 2-dimensional sigma model is precisely this one.

The existence of bi-Kählerian structures on the base manifolds offers the possibility to construct the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$ on the second order tangent bundle T^2M over a $4k$-dimensional bi-Kählerian manifold M in the sense of Yampolsky . In this note, we define the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$ on the second-order tangent bundle T^2M over a bi-Kählerian manifold $(M_{4k}, g, \varphi_1, \varphi_2)$ as a natural metric and firstly obtain the Levi-Civita connection of this metric. Secondly, we calculate all forms of the Riemannian curvature tensors of this metric and present some results concerning with them. Thirdly, we study the geodesics and geodesic curvatures of projections to the base of geodesics on the second-order tangent bundle T^2M and bi-unit second order tangent bundle $T^2_{1,1}M$. Finally, we present some conditions for a section $\sigma : M \to T^2M$ to be harmonic and study the harmonicity of the different canonical projections and inclusions of $(T^2M, \mathrm{g_{BS}})$. We ends the paper with harmonicity of the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$ and the Sasaki metric $\mathrm{g_S}$ with respect to each other.

2 The Berger Type Deformed Sasaki Metric on T^2M over Bi-Kählerian Manifolds

2.1 Tangent bundle TM

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle denoted by $\pi : TM \to M$. A system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i)$, $\bar{i} = n + i = n + 1, \cdots, 2n$, where (u^i) is the cartesian coordinates in each tangent space T_pM at $p \in M$ with respect to the natural base $\{\frac{\partial}{\partial x^i}|_p\}$, p being an arbitrary point in U whose coordinates are (x^i).

Given a vector field $X = X^i\frac{\partial}{\partial x^i}$ on M, the vertical lift ${}^VX$ and the horizontal lift ${}^HX$ of X are given, respectively, with respect to the induced coordinates, by

$$X^V = X^i\partial_{\bar{i}}, \tag{2.1}$$

$$X^H = X^i\partial_i - u^s\Gamma^i_{sk}X^k\partial_{\bar{i}}, \tag{2.2}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial u^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g (see [34]).

In particular, we have the vertical spray ${}^Vu$ and the horizontal spray ${}^Hu$ on TM defined by

$${}^Vu = u^{i\,V}(\partial_i) = u^i\partial_{\bar{i}}, \quad {}^Hu = u^{i\,H}(\partial_i) = u^i\delta_i,$$

respectively, where $\delta_i = \partial_i - u^j\Gamma^s_{ji}\partial_{\bar{s}}$. ${}^Vu$ is also called the canonical or Liouville vector field on TM.

Let f be a smooth function of M to $\mathbb{R}$ and X, Y, Z be any vector fields on M. We have (see [34])

$$\begin{aligned} X^H(f^V) &= X(f), \\ X^V(f^V) &= 0, \\ X^H((g(Y,Z))^V) &= X(g(Y,Z)), \\ X^V((g(Y,Z))^V) &= 0. \end{aligned}$$

The bracket operations of vertical and horizontal vector fields are given by the formulas (see [16, 34])

$$\begin{cases} [X^H, Y^H] = [X,Y]^H - (R(X,Y)u)^V, \\ [X^H, Y^V] = (\nabla_X Y)^V, \\ [X^V, Y^V] = 0 \end{cases} \tag{2.3}$$

for all X, $Y \in \Gamma(TM)$, where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Definition 2.1 (see [3, 33]) *Let (M_{2k}, g, φ) be a (anti-para) Kählerian manifold. On the tangent bundle TM, a fiber-wise Berger type deformation of the Sasaki metric on TM (called also (φ, δ)-metric) is defined by*

(1) $g_{\varphi,\delta}(X^H, Y^H) = g(X, Y) \circ \pi$,

(2) $g_{\varphi,\delta}(X^H, Y^V) = 0$,

(3) $g_{\varphi,\delta}|_{(x,u)}(X^V, Y^V) = g_x(X, Y) + \delta^2 g_x(X, \varphi(u)) g_x(Y, \varphi(u))$,

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$, and δ is some constant.

The Berger type deformed Sasaki metric on the tangent bundle over a Kählerian manifold firstly introduced and studied by Yampolsky [33]. Then, in [3–4], Altunbas and his coauthors defined the Berger type deformed Sasaki metric on the tangent bundle over an anti-paraKählerian manifold and they studied its curvature properties and some harmonic problems in this setting.

2.2 Whitney tangent fiber bundle $TM \oplus TM$

Let $\pi : TM \to M$ be a canonical projection. The Whitney tangent fiber bundle $TM \oplus TM$ is defined by

$$TM \oplus TM = \{(u, w) \in TM \times TM;\ \pi(u) = \pi(w)\} = \bigcup_{x \in M} T_xM \times T_xM,$$

where π is denoted by

$$\begin{aligned} \pi : TM \oplus TM &\to M \\ (u, \omega) &\mapsto \pi(u, \omega) = \pi(u) = \pi(\omega). \end{aligned}$$

A local chart $(U, \varphi) = (U, x^i)$ on M induces a chart $(\pi^{-1}(U), \widetilde{\varphi}) = (\pi^{-1}(U), x^i, u^i)$ on TM and $(\pi^{-1}(U), \overline{\varphi}) = (\pi^{-1}(U), x^i, u^i, z^i)$ on $TM \oplus TM$ such

$$\overline{\varphi}(x, u, \omega) = (\varphi(x), \widetilde{\varphi}_x(u), \widetilde{\varphi}_x(\omega)) = (\varphi(x), u, z).$$

Let $\widetilde{X}, \widetilde{Y} \in \mathcal{H}(TM)$ then $(\widetilde{X}, \widetilde{Y}) \in \mathcal{H}(TM \oplus TM)$ if and only if

$$\mathrm{d}\pi(\widetilde{X}) = \mathrm{d}\pi(\widetilde{Y}).$$

Relatively to the chart $(\pi^{-1}(U), \overline{\varphi}) = (\pi^{-1}(U), x^i, u^i, z^i)$, the local frame vector fields are given by

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\Big), \\ \frac{\partial}{\partial u^i} &= \Big(\frac{\partial}{\partial u^i}, 0\Big), \\ \frac{\partial}{\partial z^i} &= \Big(0, \frac{\partial}{\partial z^i}\Big). \end{aligned}$$

For $X \in \Gamma(TM)$ and $f \in C^\infty(M)$, then we have

$$(X^V, 0) = X^i \frac{\partial}{\partial u^i}, \quad (0, X^V) = X^i \frac{\partial}{\partial z^i},$$

$$(X^H, X^H) = X^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij} X^i u^j \frac{\partial}{\partial u^k} - \Gamma^k_{ij} X^i z^j \frac{\partial}{\partial z^k},$$
$$(X^V, 0)(f \circ \pi) = (0, X^V)(f \circ \pi) = 0,$$
$$(X^H, X^H)(f \circ \pi) = X(f) \circ \pi.$$

Let (M, g) be a Riemannian manifold, ∇ be its Levi-Civita connection and $\gamma_1, \gamma_2 : 0 \in I \subset \mathbb{R} \to M$ be a smooth curve. Then we have

$$[\gamma_1 \sim \gamma_2] \Leftrightarrow \Big[\gamma_1(0) = \gamma_2(0),\ \frac{\mathrm{d}\gamma_1}{\mathrm{d}t}(0) = \frac{\mathrm{d}\gamma_2}{\mathrm{d}t}(0) \text{ and } \frac{\mathrm{d}^2\gamma_1}{\mathrm{d}t^2}(0) = \frac{\mathrm{d}^2\gamma_2}{\mathrm{d}t^2}(0)\Big],$$
$$j_0^2\gamma = \big\{\overline{\gamma};\quad \overline{\gamma} \sim \gamma\big\}.$$

The second-order tangent bundle is the natural bundle of 2-jets of differentiable curves, defined by

$$T^2M = \{j_0^2\gamma;\ \gamma : \mathbb{R}_0 \to M, \text{ is a smooth curve at } 0 \in \mathbb{R}\}.$$

The canonical projection P on T^2M is given by

$$P : T^2M \to M,$$
$$j_0^2\gamma \mapsto \gamma(0).$$

A local chart (U, φ) induces a chart $(P^{-1}(U), \phi)$ on T^2M given by

$$\phi(j_0^2\gamma) = \Big(\varphi(\gamma(0)), \frac{\mathrm{d}\varphi\circ\gamma}{\mathrm{d}t}(0), \frac{\mathrm{d}^2\varphi\circ\gamma}{\mathrm{d}t^2}(0)\Big).$$

Theorem 2.1 (see [10]) *If $TM \oplus TM$ denotes the Whitney sum, then*

$$S : T^2M \to TM \oplus TM,$$
$$j_0^2\gamma \mapsto (\dot\gamma(0), (\nabla_{\dot\gamma(0)}\dot\gamma)(0))$$

is a diffeomorphism of natural bundles.

In the induced coordinate, we have

$$S : (x^i, u^i, z^i) \mapsto (x^i, u^i, z^i + u^j u^k \Gamma^i_{jk}). \tag{2.4}$$

Definition 2.2 (see [12]) *Let T^2M be a second-order tangent bundle endowed with the vectorial structure induced by the diffeomorphism S. For any section $\sigma \in \Gamma(T^2M)$, we define two vector fields on M by*

$$X_\sigma = P_1 \circ S \circ \sigma,$$
$$Y_\sigma = P_2 \circ S \circ \sigma,$$

where P_1 and P_2 denote the first and the second projection from $TM \oplus TM$ onto TM.

2.3 λ-Lifts on T^2M

Let (U,φ) be a local chart on M, then the diffeomorphism S induces a local chart $((\pi_\oplus \circ S)^{-1}(U), \overline{\varphi}\circ S)$ on T^2M such as

$$\frac{\partial}{\partial x^i} = S_*^{-1}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\Big), \tag{2.5}$$
$$\frac{\partial}{\partial u^i} = S_*^{-1}\Big(\frac{\partial}{\partial u^i}, 0\Big), \tag{2.6}$$
$$\frac{\partial}{\partial z^i} = S_*^{-1}\Big(0, \frac{\partial}{\partial z^i}\Big), \tag{2.7}$$

where $\pi_\oplus : (u,\omega) \in TM \oplus TM \mapsto \pi(u) = \pi(\omega) = x$.

Definition 2.3 (see [10]) *Let (M,g) be a Riemannian manifold and $X \in \Gamma(TM)$ be a vector field on M. For $\lambda = 0,1,2$, the λ-lift of X to T^2M is defined by respectively*

$$X^{(0)} = S_*^{-1}(X^H, X^H),$$
$$X^{(1)} = S_*^{-1}(X^V, 0),$$
$$X^{(2)} = S_*^{-1}(0, X^V).$$

From formulae (2.5)–(2.7) and Definition 2.3, we obtain the following lemma.

Lemma 2.1 *For any $X \in \mathcal{H}(M)$ and any smooth function $f \in C^\infty(M)$, we have*

$$X^{(1)} = X^i\frac{\partial}{\partial u^i},$$
$$X^{(2)} = X^i\frac{\partial}{\partial z^i},$$
$$X^{(0)} = X^i\frac{\partial}{\partial x^i} - \Gamma_{ij}^k X^i u^j \frac{\partial}{\partial u^k} - \Gamma_{ij}^k X^i z^j \frac{\partial}{\partial z^k},$$
$$X^{(1)}(f\circ\pi) = X^{(2)}(f\circ\pi) = 0,$$
$$X^{(0)}(f\circ\pi) = X(f)\circ\pi.$$

From Definition 2.3 and (2.3), we obtain the following theorem.

Theorem 2.2 (see [12, 20, 24]) *Let (M,g) be a Riemannian manifold. If R denotes the Riemannian curvature tensor of (M,g), then for all vector fields $X,Y \in \Gamma(TM)$ and $p \in T^2M$ we have*

(1) $[X^{(0)}, Y^{(0)}]_p = [X,Y]_p^{(0)} - (R_x(X,Y)u)^{(1)} - (R_x(X,Y)w)^{(2)}$,

(2) $[X^{(0)}, Y^{(i)}] = (\nabla_X Y)^{(i)}$,

(3) $[X^{(i)}, Y^{(j)}] = 0$,

where $(x,u,w) = S(p)$ and $i,j = 1,2$.

Lemma 2.2 *Let (M_{2k}, g, φ) be a Kählerian manifold. For all $x \in M$, $u = u^i\frac{\partial}{\partial x^i}, \omega = \omega^i\frac{\partial}{\partial x^i} \in T_xM$ and any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have the followings*

(1) $X^{(0)}(g(Y,u))_p = g_x(\nabla_X Y, u)$,

(2) $X^{(0)}(g(Y,\omega))_p = g_x(\nabla_X Y, \omega)$,
(3) $X^{(0)}(g(Y,\varphi(u)))_p = g_x((\nabla_X Y), \varphi(u))$,
(4) $X^{(0)}(g(Y,\varphi(\omega)))_p = g_x((\nabla_X Y), \varphi(\omega))$,
(5) $X^{(0)}(f(r_1^2))_p = X^{(0)}(f(r_2^2))_p = 0 = X^{(0)}(g(u,u))_p = X^{(0)}(g(\omega,\omega))_p$,
(6) $X^{(1)}(g(u,u))_p = 2g_x(X,u)$,
(7) $X^{(1)}(g(\omega,\omega))_p = 0 = X^{(2)}(g(u,u))_p$,
(8) $X^{(2)}(g(\omega,\omega))_p = 2g_x(X,\omega)$,
(9) $X^{(1)}(g(Y,u))_p = g_x(X,Y) = X^{(2)}(g(Y,\omega))_p$,
(10) $X^{(1)}(g(Y,\omega))_p = 0 = X^{(2)}(g(Y,u))_p$,
(11) $X^{(1)}(g(Y,\varphi(u)))_p = g_x(Y,\varphi(X)) = X^{(2)}(g(Y,\varphi(\omega)))_p$,
(12) $X^{(1)}(g(Y,\varphi(\omega)))_p = 0 = X^{(2)}(g(Y,\varphi(u)))_p$,
(13) $X^{(1)}(f(r_1^2))_p = 2f'(r_1^2)g_x(X,u)$,
(14) $X^{(1)}(f(r_2^2))_p = 0 = X^{(2)}(f(r_1^2))_p$,
(15) $X^{(2)}(f(r_2^2))_p = 2f'(r_2^2)g_x(X,\omega)$,

where $p = S^{-1}(x,u,\omega)$, $r_1^2 = g(u,u) = |u|^2$, $r_2^2 = g(\omega,\omega) = |\omega|^2$.

2.4 The Berger type deformed sasaki metric

Definition 2.4 *Let* $(M_{4k}, g, \varphi_1, \varphi_2)$ *be a bi-Kählerian manifold. We define a Berger type deformed Sasaki metric* $\mathrm{g_{BS}}$ *on the second order tangent bundle* T^2M *by*

$$\mathrm{g_{BS}} = S_*^{-1}(g_{\varphi_1,\delta} \oplus g_{\varphi_2,\eta}). \tag{2.8}$$

From Definition 2.4, we obtain the following proposition.

Proposition 2.1 *Let* $(M_{4k}, g, \varphi_1, \varphi_2)$ *be a bi-Kählerian manifold. If* $p \in T^2M$, *then for all* $X, Y \in \Gamma(TM)$ *and* $i, j \in \{0,1,2\}$ $(i \neq j)$, *we obtain*

(1) $\mathrm{g_{BS}}(X^{(0)}, Y^{(0)})_p = g(X,Y)_x$,
(2) $\mathrm{g_{BS}}(X^{(i)}, Y^{(j)})_p = 0$,
(3) $\mathrm{g_{BS}}(X^{(1)}, Y^{(1)})_p = g(X,Y) + \delta^2 g(X,\varphi_1(u))g(Y,\varphi_1(u))_x$,
(4) $\mathrm{g_{BS}}(X^{(2)}, Y^{(2)})_p = g(X,Y) + \eta^2 g(X,\varphi_2(\omega))g(Y,\varphi_2(\omega))_x$,

where $S(p) = (x,u,w) \in T_xM \oplus T_xM$.

From Lemma 2.2 and Proposition 2.1, standard calculations give the following lemma.

Lemma 2.3 *Let* $(M_{4k}, g, \varphi_1, \varphi_2)$ *be a bi-Kählerian manifold and* T^2M *be its second-order tangent bundle with the Berger type deformed Sasaki metric* $\mathrm{g_{BS}}$. *Then*

$$X^{(0)}(\mathrm{g_{BS}}(Y^{(0)}, Z^{(0)}))_p = X(g(Y,Z)),$$
$$X^{(0)}(\mathrm{g_{BS}}(Y^{(1)}, Z^{(1)}))_p = \mathrm{g_{BS}}((\nabla_X Y)^{(1)}, Z^{(1)}) + \mathrm{g_{BS}}(Y^{(1)}, (\nabla_X Z)^{(1)}),$$
$$X^{(0)}(\mathrm{g_{BS}}(Y^{(2)}, Z^{(2)}))_p = \mathrm{g_{BS}}((\nabla_X Y)^{(2)}, Z^{(2)}) + \mathrm{g_{BS}}(Y^{(2)}, (\nabla_X Z)^{(2)}),$$
$$X^{(1)}(\mathrm{g_{BS}}(Y^{(0)}, Z^{(0)}))_p = 0 = X^{(2)}(\mathrm{g_{BS}}(Y^{(0)}, Z^{(0)}))_p,$$

$$X^{(1)}(\mathrm{g_{BS}}(Y^{(1)},Z^{(1)}))_p=\delta^2 g(\varphi_1(X),Y)g(Z,\varphi_1(u))+\delta^2 g(Y,\varphi_1(u))g(Z,\varphi_1(X)),$$
$$X^{(2)}(\mathrm{g_{BS}}(Y^{(2)},Z^{(2)}))_p=\delta^2 g(\varphi_2(X),Y)g(Z,\varphi_2(\omega))+\delta^2 g(Y,\varphi_2(\omega))g(Z,\varphi_2(X)),$$
$$X^{(1)}(\mathrm{g_{BS}}(Y^{(2)},Z^{(2)}))_p=0=X^{(2)}(\mathrm{g_{BS}}(Y^{(1)},Z^{(1)}))_p$$

for all $X,Y,Z\in\Gamma(TM)$ and $p\in T^2M$.

Lemma 2.4 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. Then*

$$g(Z,\varphi_1 u)=\frac{1}{\lambda}\mathrm{g_{BS}}(Z^{(1)},(\varphi_1 u)^{(1)}),$$
$$g(Z,\varphi_2 w)=\frac{1}{\beta}\mathrm{g_{BS}}(Z^{(2)},(\varphi_2 \omega)^{(2)}),$$
$$g(Z,\varphi_1 X)=\mathrm{g_{BS}}\Big((\varphi_1 X)^{(1)}-\frac{\delta^2}{\lambda}g(X,u)(\varphi_1 u)^{(1)},Z^{(1)}\Big),$$
$$g(Z,\varphi_2 X)=\mathrm{g_{BS}}\Big((\varphi_2 X)^{(2)}-\frac{\eta^2}{\beta}g(X,\omega)(\varphi_2 \omega)^{(2)},Z^{(2)}\Big),$$

where $\lambda=1+\delta^2|u|^2$, $\beta=1+\eta^2|\omega|^2$, X,Z are vector fields and $u\in TM$.

Theorem 2.3 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If $\widetilde\nabla$ denote the Levi-Civita connection of T^2M, then for $p\in T^2M$ and $X,Y\in\Gamma(TM)$ we have*

$$(1)\quad (\widetilde\nabla_{X^{(0)}}Y^{(0)})_p=(\nabla_X Y)^{(0)}-\frac{1}{2}(R(X,Y)u)^{(1)}-\frac{1}{2}(R(X,Y)\omega)^{(2)},$$
$$(2)\quad (\widetilde\nabla_{X^{(0)}}Y^{(1)})_p=(\nabla_X Y)^{(1)}+\frac{1}{2}[R(u,Y)X+\delta^2 g(Y,\varphi_1(u))R(u,\varphi_1(u))X]^{(0)},$$
$$(3)\quad (\widetilde\nabla_{X^{(0)}}Y^{(2)})_p=(\nabla_X Y)^{(2)}+\frac{1}{2}[R(\omega,Y)X+\eta^2 g(Y,\varphi_2(\omega))R(\omega,\varphi_2(\omega))X]^{(0)},$$
$$(4)\quad (\widetilde\nabla_{X^{(1)}}Y^{(0)})_p=\frac{1}{2}[R(u,X)Y+\delta^2 g(X,\varphi_1(u))R(u,\varphi_1(u))Y]^{(0)},$$
$$(5)\quad (\widetilde\nabla_{X^{(2)}}Y^{(0)})_p=\frac{1}{2}[R(\omega,X)Y+\eta^2 g(X,\varphi_2(\omega))R(\omega,\varphi_2(\omega))Y]^{(0)},$$
$$(6)\quad (\widetilde\nabla_{X^{(1)}}Y^{(1)})_p=\delta^2[g(Y,\varphi_1(u))\varphi_1 X+g(X,\varphi_1(u))\varphi_1 Y]^{(1)}$$
$$-\frac{\delta^4}{\lambda}[g(X,u)g(Y,\varphi_1 u)+g(Y,u)g(X,\varphi_1 u)](\varphi_1 u)^{(1)},$$
$$(7)\quad (\widetilde\nabla_{X^{(2)}}Y^{(2)})_p=\eta^2[g(Y,\varphi_2(\omega))\varphi_2 X+g(X,\varphi_2(\omega))\varphi_1 Y]^{(2)},$$
$$-\frac{\eta^4}{\beta}[g(X,\omega)g(Y,\varphi_2\omega)+g(Y,\omega)g(X,\varphi_2\omega)](\varphi_2\omega)^{(2)},$$
$$(8)\quad (\widetilde\nabla_{X^{(1)}}Y^{(2)})_p=(\widetilde\nabla_{X^{(2)}}Y^{(1)})_p=0,$$

where $S(p)=(x,u,w)$, $\lambda=1+\delta^2|u|^2$, $\beta=1+\eta^2|w|^2$, ∇ and R denote the Levi-Civita connection and the Riemannian curvature tensor of (M_{4k},g), respectively.

Proof Using Proposition 2.1, Lemmas 2.3–2.4 and Koszul formula, we have

$$2\mathrm{g_{BS}}(\widetilde\nabla_{\widetilde X}\widetilde Y,\widetilde Z)=\widetilde X(\mathrm{g_{BS}}(\widetilde Y,\widetilde Z))+\widetilde Y(\mathrm{g_{BS}}(\widetilde X,\widetilde Z))-\widetilde Z(\mathrm{g_{BS}}(\widetilde X,\widetilde Y))$$

$$- \mathrm{g}_{\mathrm{BS}}(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]) + \mathrm{g}_{\mathrm{BS}}(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) + \mathrm{g}_{\mathrm{BS}}(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]),$$

then Theorem 2.3 immediately follows.

If we denote the horizontal and vertical projections by $\mathcal{H}$, $\mathcal{V}^1$ and $\mathcal{V}^2$ respectively, then we can state the followings (see [3]):

(i) The vertical distribution V^1T^2M is totally geodesic in TT^2M if $\mathcal{H}\widetilde{\nabla}_{X^{(1)}}Y^{(1)} = 0$ and $\mathcal{V}^2\widetilde{\nabla}_{X^{(1)}}Y^{(1)} = 0$.

(ii) The vertical distribution V^2T^2M is totally geodesic in TT^2M if $\mathcal{H}\widetilde{\nabla}_{X^{(2)}}Y^{(2)} = 0$ and $\mathcal{V}^1\widetilde{\nabla}_{X^{(2)}}Y^{(2)} = 0$.

(iii)The horizontall distribution HT^2M is totally geodesic in TT^2M if $\mathcal{V}^1\widetilde{\nabla}_{X^{(0)}}Y^{(0)} = \mathcal{V}^2\widetilde{\nabla}_{X^{(0)}}Y^{(0)} = 0$.

As an application of the Levi-Civita connection $\widetilde{\nabla}$, we can state the following result.

Proposition 2.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. Then*

(i) *the vertical distributions V^1T^2M and V^2T^2M are totally geodesic in TT^2M;*

(ii) *the horizontal distribution HT^2M is totally geodesic in TT^2M if and only the base manifold M_{4k} is flat.*

Proof The results come immediately from (1), (6) and (7) of Theorem 2.3.

3 The Riemannian Curvature Tensors on T^2M

Let $F : TM \to TM$ be a smooth bundle endomorphism of TM. The horizontal and vertical vector fields $F^{(0)}, F^{(1)}, F^{(2)}$ are defined respectively on T^2M by

$$\begin{aligned} F^{(0)} &: TM \to T(T^2M), \\ &(x,u) \mapsto (F_x u)^{(0)}, \qquad (3.1)\\ F^{(1)} &: TM \to T(T^2M), \\ &(x,u) \mapsto (F_x u)^{(1)}, \qquad (3.2)\\ F^{(2)} &: TM \to T(T^2M), \\ &(x,\omega) \mapsto (F_x \omega)^{(2)}. \qquad (3.3) \end{aligned}$$

Locally, we have

$$\begin{aligned} F^{(0)}_{(x,u)} &= u^i (F(\partial_i))^{(0)}, \\ F^{(1)}_{(x,u)} &= u^i (F(\partial_i))^{(1)}, \\ F^{(2)}_{(x,\omega)} &= \omega^i (F(\partial_i))^{(2)}. \qquad (3.4) \end{aligned}$$

From Lemma 2.1 and formulas (3.4), we get the following proposition.

Proposition 3.1 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS} and $F : TM \to TM$ be a smooth bundle endomorphism of TM. Then we have the following formulas*

$$(1)\quad \widetilde{\nabla}_{X^{(0)}} F^{(1)}(x,u) = ((\nabla_X F)u)^{(1)} + \frac{1}{2}[R(u,Fu)X + \delta^2 g(Fu,\varphi_1 u)R(u,\varphi_1 u)X]^{(0)},$$

$$(2)\quad \widetilde{\nabla}_{X^{(0)}} F^{(2)}(x,\omega) = ((\nabla_X F)\omega)^{(2)} + \frac{1}{2}[R(\omega,F\omega)X + \eta^2 g(F\omega,\varphi_2 \omega)R(\omega,\varphi_2 \omega)X]^{(0)},$$

$$(3)\quad \widetilde{\nabla}_{X^{(1)}} F^{(1)}(x,u) = (FX)^{(1)} + \delta^2[g(Fu,\varphi_1 u)\varphi_1 X + g(X,\varphi_1 u)\varphi_1 Fu]^{(1)} - \frac{\delta^4}{\lambda}[g(X,u)g(Fu,\varphi_1 u) + g(Fu,u)g(X,\varphi_1 u)](\varphi_1 u)^{(1)},$$

$$(4)\quad \widetilde{\nabla}_{X^{(2)}} F^{(2)}(x,\omega) = (FX)^{(2)} + \eta^2[g(F\omega,\varphi_2 \omega)\varphi_2 X + g(X,\varphi_2 \omega)\varphi_2 F\omega]^{(2)} - \frac{\eta^4}{\beta}[g(X,\omega)g(F\omega,\varphi_2 \omega) + g(F\omega,\omega)g(X,\varphi_2 \omega)](\varphi_2 \omega)^{(2)},$$

$$(5)\quad \widetilde{\nabla}_{X^{(2)}} F^{(1)}(x,u) = \widetilde{\nabla}_{X^{(1)}} F^{(2)}(x,\omega) = 0.$$

Using Theorem 2.3, Proposition 3.1 and the second Bianchi identity, we obtain the following proposition.

Proposition 3.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold, (T^2M, g_{BS}) be its second order tangent bundle equipped with the Berger type deformed Sasaki metric and $\widetilde{R}$ be the curvature tensor of (T^2M, g_{BS}). Then we have the following formulas*

$$\begin{aligned}(1)\quad \widetilde{R}(X^{(0)},Y^{(0)})Y^{(0)} &= [R(X,Y)Y]^{(0)} + \frac{3\delta^2}{4}g(R(X,Y)u,\varphi_1 u)[R(u,\varphi_1 u)Y]^{(0)} \\ &+ \frac{3}{4}[R(u,R(X,Y)u)Y]^{(0)} + \frac{1}{2}[(\nabla_Y R)(X,Y)u]^{(1)} \\ &+ \frac{3\eta^2}{4}g(R(X,Y)\omega,\varphi_2 \omega)[R(\omega,\varphi_2 \omega)Y]^{(0)} \\ &+ \frac{3}{4}[R(u,R(X,Y)\omega)Y]^{(0)} + \frac{1}{2}[(\nabla_Y R)(X,Y)\omega]^{(2)},\end{aligned}$$

$$\begin{aligned}(2)\quad \widetilde{R}(X^{(0)},Y^{(1)})Y^{(1)} &= -\frac{\delta^2}{4}g(Y,\varphi_1 u)[R(u,\varphi_1 u)R(u,Y)X + R(u,Y)R(u,\varphi_1 u)X]^{(0)} \\ &- \frac{\delta^4}{4}g(Y,\varphi_1 u)^2[R(u,\varphi_1 u)R(u,\varphi_1 u)X]^{(0)} - \frac{1}{4}[R(u,Y)R(u,Y)X]^{(0)},\end{aligned}$$

$$\begin{aligned}(3)\quad \widetilde{R}(X^{(0)},Y^{(2)})Y^{(2)} &= -\frac{\eta^2}{4}g(Y,\varphi_2 \omega)[R(\omega,\varphi_2 \omega)R(\omega,Y)X + R(\omega,Y)R(\omega,\varphi_2 \omega)X]^{(0)} \\ &- \frac{\eta^4}{4}g(Y,\varphi_2 \omega)^2[R(\omega,\varphi_2 \omega)R(\omega,\varphi_2 \omega)X]^{(0)} - \frac{1}{4}[R(\omega,Y)R(\omega,Y)X]^{(0)},\end{aligned}$$

$$\begin{aligned}(4)\quad \widetilde{R}(X^{(1)},Y^{(1)})Y^{(1)} &= \delta^4 g(Y,\varphi_1 u)[g(Y,\varphi_1 u)X^{(1)} - g(X,\varphi_1 u)Y^{(1)}] \\ &+ \frac{\delta^6}{\lambda}g(Y,\varphi_1 u)[g(X,\varphi_1 u)g(Y,u) - g(Y,\varphi_1 u)g(X,u)][u]^{(1)} \\ &+ \Big[\frac{\delta^6}{\lambda^2}g(Y,u)(g(X,u)g(Y,\varphi_1 u) - g(Y,u)g(X,\varphi_1 u)) \\ &+ \frac{\delta^4}{\lambda}(g(X,\varphi_1 u)g(Y,Y) - g(Y,\varphi_1 u)g(X,Y)) \\ &+ 3\frac{\delta^4}{\lambda}g(X,\varphi_1 Y)g(Y,u)\Big][\varphi_1 u]^{(1)} - 3\delta^2 g(X,\varphi_1 Y)[\varphi_1 Y]^{(1)},\end{aligned}$$

$$
\begin{aligned}
(5)\quad \widetilde{R}(X^{(2)},Y^{(2)})Y^{(2)} &= \eta^4 g(Y,\varphi_2\omega)[g(Y,\varphi_2\omega)X^{(2)} - g(X,\varphi_2\omega)Y^{(2)}] \\
&\quad + \frac{\eta^6}{\beta} g(Y,\varphi_2\omega)[g(X,\varphi_2\omega)g(Y,\omega) - g(Y,\varphi_2 u)g(X,\omega)][\omega]^{(2)} \\
&\quad + \Big[\frac{\eta^6}{\beta^2} g(Y,\omega)(g(X,\omega)g(Y,\varphi_2\omega) - g(Y,\omega)g(X,\varphi_2\omega)) \\
&\quad + \frac{\eta^4}{\beta}(g(X,\varphi_2\omega)g(Y,Y) - g(Y,\varphi_2\omega)g(X,Y)) \\
&\quad + 3\frac{\eta^4}{\beta} g(X,\varphi_2 Y)g(Y,\omega)\Big][\varphi_2\omega]^{(2)} - 3\eta^2 g(X,\varphi_2 Y)[\varphi_2 Y]^{(2)},
\end{aligned}
$$

$$(6)\quad \widetilde{R}(X^{(1)},Y^{(2)})Y^{(2)} = 0.$$

Proposition 3.3 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$ and $\widetilde{R}$ be the curvature tensor of $(T^2M,\mathrm{g_{BS}})$. Then we have*

$$
\begin{aligned}
(1)\quad \mathrm{g_{BS}}(\widetilde{R}(X^{(0)},Y^{(0)})Y^{(0)},X^{0})_p &= g_x(R(X,Y)Y,X) - \frac{3}{4}|R(X,Y)u|^2 - \frac{3}{4}|R(X,Y)\omega|^2 \\
&\quad - \frac{3\delta^2}{4} g(R(X,Y)u,\varphi_1 u)^2 - \frac{3\eta^2}{4} g(R(X,Y)\omega,\varphi_2\omega)^2,
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad \mathrm{g_{BS}}(\widetilde{R}(X^{(0)},Y^{(1)})Y^{(1)},X^{(0)})_p &= \frac{1}{4}|R(u,Y)X|^2 + \frac{\delta^4}{4} g(Y,\varphi_1 u)^2 |R(u,\varphi_1 u)X|^2 \\
&\quad + \frac{\delta^2}{2} g(Y,\varphi_1 u) g(R(u,\varphi_1 u)X, R(u,Y)X),
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad \mathrm{g_{BS}}(\widetilde{R}(X^{(0)},Y^{(2)})Y^{(2)},X^{(0)})_p &= \frac{1}{4}|R(\omega,Y)X|^2 + \frac{\eta^4}{4} g(Y,\varphi_2\omega)^2 |R(\omega,\varphi_2\omega)X|^2 \\
&\quad + \frac{\eta^2}{2} g(Y,\varphi_2\omega) g(R(\omega,\varphi_2\omega)X, R(\omega,Y)X),
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad \mathrm{g_{BS}}(\widetilde{R}(X^{(1)},Y^{(1)})Y^{(1)},X^{(1)})_p &= \delta^4(g(X,\varphi_1 u)^2 + g(Y,\varphi_1 u)^2) - 3\delta^2 g(X,\varphi_1 Y)^2 \\
&\quad - \frac{\delta^6}{\lambda}(g(X,u)g(Y,\varphi_1 u) - g(X,\varphi_1 u)g(Y,u))^2,
\end{aligned}
$$

$$
\begin{aligned}
(5)\quad \mathrm{g_{BS}}(\widetilde{R}(X^{(2)},Y^{(2)})Y^{(2)},X^{(2)})_p &= \eta^4(g(X,\varphi_2\omega)^2 + g(Y,\varphi_2\omega)^2) - 3\eta^2 g(X,\varphi_2 Y)^2 \\
&\quad - \frac{\eta^6}{\beta}(g(X,\omega)g(Y,\varphi_2\omega) - g(X,\varphi_2\omega)g(Y,\omega))^2,
\end{aligned}
$$

$$(6)\quad \mathrm{g_{BS}}(\widetilde{R}(X^{(1)},Y^{(2)})Y^{(2)},X^{(1)})_p = 0,$$

where $p = S^{-1}(x,u,\omega)$.

Proposition 3.3 implies the following theorem.

Theorem 3.1 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$ and $\widetilde{K}$ be the sectional curvature of $(T^2M,\mathrm{g_{BS}})$. Then we have*

$$
\begin{aligned}
(1)\quad \widetilde{K}_p(X^{(0)},Y^{(0)}) &= K_x(X,Y) - \frac{3}{4}|R(X,Y)u|^2 - \frac{3}{4}|R(X,Y)\omega|^2 \\
&\quad - \frac{3\delta^2}{4} g(R(X,Y)u,\varphi_1 u)^2 - \frac{3\eta^2}{4} g(R(X,Y)\omega,\varphi_2\omega)^2,
\end{aligned}
$$

$$(2)\quad \widetilde{K}_p(X^{(0)},Y^{(1)}) = \frac{1}{1+\delta^2 g_x(Y,\varphi_1 u)^2}\Big[\frac{\delta^4}{4}g(Y,\varphi_1 u)^2|R(u,\varphi_1 u)X|^2 + \frac{\delta^2}{2}g(Y,\varphi_1 u)g(R(u,\varphi_1 u)X,R(u,Y)X) + \frac{1}{4}|R(u,Y)X|^2\Big],$$

$$(3)\quad \widetilde{K}_p(X^{(0)},Y^{(2)}) = \frac{1}{1+\delta^2 g_x(Y,\varphi_2 \omega)^2}\Big[\frac{\eta^4}{4}g(Y,\varphi_2 \omega)^2|R(\omega,\varphi_2 \omega)X|^2 + \frac{\eta^2}{2}g(Y,\varphi_2 \omega)g(R(\omega,\varphi_2 \omega)X,R(\omega,Y)X) + \frac{1}{4}|R(\omega,Y)X|^2\Big],$$

$$(4)\quad \widetilde{K}_p(X^{(1)},Y^{(1)}) = \frac{1}{1+\delta^2 (g_x(X,\varphi_1 u)^2+g_x(Y,\varphi_1 u)^2)}\Big[-3\delta^2 g(X,\varphi_1 Y)^2 + \delta^4(g(X,\varphi_1 u)^2+g(Y,\varphi_1 u)^2) - \frac{\delta^6}{\lambda}(g(X,u)g(Y,\varphi_1 u)-g(X,\varphi_1 u)g(Y,u))^2\Big],$$

$$(5)\quad \widetilde{K}_p(X^{(2)},Y^{(2)}) = \frac{1}{1+\eta^2 (g_x(X,\varphi_2 \omega)^2+g_x(Y,\varphi_2 \omega)^2)}\Big[-3\eta^2 g(X,\varphi_2 Y)^2 + \eta^4(g(X,\varphi_2 \omega)^2+g(Y,\varphi_2 \omega)^2) - \frac{\eta^6}{\beta}(g(X,\omega)g(Y,\varphi_2 \omega)-g(X,\varphi_2 \omega)g(Y,\omega))^2\Big],$$

$$(6)\quad \widetilde{K}_p(X^{(1)},Y^{(2)}) = 0,$$

where $p = S^{-1}(x,u,w)$, $X,Y \in \Gamma(TM)$ are orthonormal vector fields, and K is the sectional curvature of (M_{4k},g).

Let $m = 4k$, $p = S^{-1}(x,u,w) \in TM$ such as $u,w \in T_xM\backslash\{0\}$ and $\{E_i\}_{i=1,m}$ (resp. $\{\overline{E}_i\}_{i=1,m}$) be an orthonormal basis of the vector space T_xM, such that $E_1 = \frac{\varphi_1 u}{|u|}$ (resp. $\overline{E}_1 = \frac{\varphi_2 w}{|w|}$), then the orthonormal basis $\{F_i\}_{i=1,3m}$ of $T_p(T^2M)$ is given by

$$\begin{cases} F_i = E_i^{(0)}, \\ F_{m+1} = \dfrac{1}{\sqrt{\lambda}}(E_1)^{(1)}, \\ F_{m+j} = (E_j)^{(1)}, \\ F_{2m+1} = \dfrac{1}{\sqrt{\beta}}(\overline{E}_1)^{(2)}, \\ F_{2m+j} = (\overline{E}_j)^{(2)} \end{cases} \tag{3.5}$$

for $i = 1,m$ and $j = 2,m$.

From Theorem 3.1, we obtain the following lemma.

Lemma 3.1 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}, $p = S^{-1}(x,u,\omega) \in T^2M$ and $\{F_i\}_{i=1,3m}$ be an orthonormal basis of $T_p(T^2M)$ defined by formula (3.5). Then the sectional curvature $\widetilde{K}$ satisfies the following formulas*

$$\widetilde{K}_p(F_i,F_j) = K_x(E_i,E_j) - \frac{3}{4}|R(E_i,E_j)u|^2 - \frac{3}{4}|R(E_i,E_j)\omega|^2 - \frac{3\delta^2}{4}g(R(E_i,E_j)u,\varphi_1 u)^2 - \frac{3\eta^2}{4}g(R(E_i,E_j)\omega,\varphi_2\omega)^2,$$

$$\begin{aligned}
\widetilde{K}_p(F_i, F_{m+1}) &= \frac{\delta^2\lambda}{4(\lambda-1)}|R(u,\varphi_1 u)E_i|^2, \\
\widetilde{K}_p(F_i, F_{m+l}) &= \frac{1}{4}|R(u,E_l)E_i|^2, \\
\widetilde{K}_p(F_{m+t}, F_{m+1}) &= \frac{\delta^2(\lambda-1)}{\lambda} - \frac{\delta^4(\lambda^2+\lambda+1)}{\lambda^2(\lambda-1)}(g(E_t,u))^2, \\
\widetilde{K}_p(F_{m+t}, F_{m+l}) &= -3\delta^2 g(E_t,\varphi_1 E_l)^2, \\
\widetilde{K}_p(F_i, F_{2m+t}) &= \frac{1}{4}|R(\omega,\overline{E}_t)E_i|^2, \\
\widetilde{K}_p(F_i, F_{2m+1}) &= \frac{\eta^2\beta}{4(\beta-1)}|R(\omega,\varphi_2\omega)E_i|^2, \\
\widetilde{K}_p(F_{m+i}, F_{2m+j}) &= 0, \\
\widetilde{K}_p(F_{2m+t}, F_{2m+1}) &= \frac{\eta^2(\beta-1)}{\beta} - \frac{\eta^4(\beta^2+\beta+1)}{\beta^2(\beta-1)}(g(\overline{E}_t,\omega))^2, \\
\widetilde{K}_p(F_{2m+t}, F_{2m+l}) &= -3\eta^2 g(\overline{E}_t,\varphi_2\overline{E}_l)^2
\end{aligned}$$

for $i, j = 1, m$ and $t, l = 2, m$, where $m = 4k$.

Theorem 3.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. Then the scalar curvature is given by*

$$\begin{aligned}
\widetilde{\sigma}_p = \sigma_x &- \frac{1}{4}\sum_{i,j=1}^{m}|R(E_i,E_j)u|^2 + \frac{\delta^2(3-\lambda)}{4(\lambda-1)}\sum_{i=1}^{m}|R(u,\varphi_1 u)E_i|^2 \\
&- \frac{\delta^2}{\lambda}[(m-4)\lambda + 2(m-1)] - \frac{2\delta^2(\lambda^2+\lambda+1)}{\lambda^2} \\
&+ \frac{1}{2}\sum_{i=1,t=2}^{m}|R(\omega,\overline{E}_t)E_i|^2 - \frac{3}{4}\sum_{i,j=1}^{m}|R(E_i,E_j)\omega|^2 - \frac{2\eta^2(\beta^2+\beta+1)}{\beta^2} \\
&+ \frac{\eta^2(3-\beta)}{4(\beta-1)}|R(\omega,\varphi_2\omega),E_i|^2 - \frac{\eta^2}{\beta}[\beta(m-4) + 2(m-1)].
\end{aligned}$$

Proof From definition of scalar curvature (see [13]), we have

$$\begin{aligned}
\widetilde{\sigma}_p &= \sum_{j=1}^{3m}\widetilde{\mathrm{Ricci}}(F_j,F_j) \\
&= \sum_{i,j=1}^{3m}\mathrm{g_{BS}}(\widetilde{R}(F_j,F_i)F_i,F_j) \\
&= \sum_{i,j=1}^{3m}\widetilde{K}(F_i,F_j).
\end{aligned}$$

Using Lemma 3.1, Theorem 3.2 immediately follows.

Corollary 3.1 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and $(T^2M,)$ be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If (M_{4k}, g) is* *locally flat, then the scalar curvature is given by*

$$\widetilde{\sigma}_p = -\frac{\delta^2}{\lambda}[(m-4)\lambda + 2(m-1)] - \frac{2\delta^2(\lambda^2+\lambda+1)}{\lambda^2} - \frac{2\eta^2(\beta^2+\beta+1)}{\beta^2} - \frac{\eta^2}{\beta}[\beta(m-4)+2(m-1)],$$

where $m = 4k$.

Remark 3.1 In the case where (M_{4k}, g) is locally flat, the scalar curvature $\widetilde{\sigma}$ is negative.

4 Geodesics on T^2M

Lemma 4.1 (see [35]) *Let (M_{4k}, g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are vector fields and $(x,u) \in TM$ such that $X_x = u$, then we have*

$$\mathrm{d}_x X(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Lemma 4.2 *Let (M_{4k}, g) be a Riemannian manifold. If $Z \in \Gamma(TM)$ and $\sigma \in \Gamma(T^2M)$, then we have*

$$\mathrm{d}_x\sigma(Z_x) = Z^{(0)}_p + (\nabla_Z X_\sigma)^{(1)}_p + (\nabla_Z Y_\sigma)^{(2)}_p, \tag{4.1}$$

where $p = \sigma(x)$.

Proof Using Lemma 4.1, we obtain

$$\begin{aligned}\mathrm{d}_x\sigma(Z) &= \mathrm{d}S^{-1}(dX_\sigma(Z), \mathrm{d}Y_\sigma(Z))_{S(p)} \\ &= \mathrm{d}S^{-1}(Z^H, Z^H)_{S(p)} + \mathrm{d}S^{-1}((\nabla_Z X_\sigma)^V, (\nabla_Z Y_\sigma)^V)_{S(p)} \\ &= Z^{(0)}_p + (\nabla_Z X_\sigma)^{(1)}_p + (\nabla_Z Y_\sigma)^{(2)}_p.\end{aligned}$$

Lemma 4.3 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}, and $x : I \to M$ be a curve on M_{4k}. If $C : t \in I \to C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve in T^2M such that $y(t), z(t)$ are vector fields along $x(t)$(i.e., $y(t), z(t) \in T_{x(t)}M$), then*

$$\dot{C} = \dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}, \tag{4.2}$$

where $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\dot{C} = \frac{\mathrm{d}C}{\mathrm{d}t}$.

Proof If $Y, Z \in \Gamma(TM)$ are vector fields such that $Y(x(t)) = y(t)$ and $Z(x(t)) = z(t)$, then we have

$$\dot{C}(t) = \mathrm{d}C(t) = \mathrm{d}\sigma(\dot{x}(t)),$$

where $\sigma = S^{-1}((Y, Z))$. Using Lemma 4.2 we obtain

$$\dot{C}(t) = \mathrm{d}\sigma(x(t)) = \dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}. \tag{4.3}$$

In the following, we will denote $x' = \dot{x}$, $x'' = \nabla_{\dot{x}}\dot{x}$, $y' = \nabla_{\dot{x}}y$, $y'' = \nabla_{\dot{x}}\nabla_{\dot{x}}y$, $z' = \nabla_{\dot{x}}z$ and $z'' = \nabla_{\dot{x}}\nabla_{\dot{x}}z$.

Theorem 4.1 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then*

$$\begin{aligned}\widetilde{\nabla}_{\dot{C}}\dot{C} = {} & [x'' + R(y, y')x' + \delta^2 g(y', \varphi_1 y)R(y, \varphi_1 y)x' \\ & + R(z, z')x' + \eta^2 g(z', \varphi_2 z)R(z, \varphi_2 z)x']^{(0)} \\ & + \Big[y'' - 2\delta^2 g(y', \varphi_1 y)\Big[-\varphi_1(y') + \frac{\delta^2}{\lambda}g(y', y)\varphi_1(y)\Big]\Big]^{(1)} \\ & + \Big[z'' - 2\eta^2 g(z', \varphi_2 z)\Big[-\varphi_2(z') + \frac{\eta^2}{\beta}g(z', z)\varphi_2(z)\Big]\Big]^{(2)}.\end{aligned}$$

Proof From formula (4.3) and Theorem 2.3, we have

$$\begin{aligned}\widetilde{\nabla}_{\dot{C}}\dot{C} = {} & \widetilde{\nabla}_{[\dot{x}^{(0)} + (\nabla_{\dot{x}}y)^{(1)} + (\nabla_{\dot{x}}z)^{(2)}]}[\dot{x}^{(0)} + (\nabla_{\dot{x}}y)^{(1)} + (\nabla_{\dot{x}}z)^{(2)}] \\ = {} & \widetilde{\nabla}_{\dot{x}^{(0)}}\dot{x}^{(0)} + \widetilde{\nabla}_{\dot{x}^{(0)}}(\nabla_{\dot{x}}y)^{(1)} + \widetilde{\nabla}_{(\nabla_{\dot{x}}y)^{(1)}}\dot{x}^{(0)} + \widetilde{\nabla}_{(\nabla_{\dot{x}}y)^{(1)}}(\nabla_{\dot{x}}y)^{(1)} \\ & + \widetilde{\nabla}_{\dot{x}^{(0)}}(\nabla_{\dot{x}}z)^{(2)} + \widetilde{\nabla}_{(\nabla_{\dot{x}}z)^{(2)}}\dot{x}^{(0)} + \widetilde{\nabla}_{(\nabla_{\dot{x}}z)^{(2)}}(\nabla_{\dot{x}}z)^{(2)} \\ = {} & [x'' + R(y, y')x' + \delta^2 g(y', \varphi_1 y)R(y, \varphi_1 y)x' \\ & + R(z, z')x' + \eta^2 g(z', \varphi_2 z)R(z, \varphi_2 z)x']^{(0)} \\ & + \Big[y'' + 2\delta^2 g(y', \varphi_1 y)\varphi_1(y') - \frac{2\delta^4}{\lambda}g(y', \varphi_1 y)g(\varphi_1(y'), \varphi_1 y)\varphi_1(y)\Big]^{(1)} \\ & + \Big[z'' + 2\eta^2 g(z', \varphi_2 z)\varphi_2(z') - \frac{2\eta^4}{\beta}g(z', \varphi_2 z)g(\varphi_2(z'), \varphi_2 z)\varphi_2(z)\Big]^{(2)}.\end{aligned}$$

As a direct consequence of the theorem above we get the following theorem.

Theorem 4.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and $(T^2M, \mathrm{g_{BS}})$ be its second order tangent bundle equipped with the Berger type deformed Sasaki metric. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then C is a geodesic if and only if*

$$\begin{aligned}x'' = {} & -[R(y, y') + \delta^2 g(y', \varphi_1 y)R(y, \varphi_1 y) \\ & + R(z, z') + \eta^2 g(z', \varphi_2 z)R(z, \varphi_2 z)]x', \qquad (4.4)\end{aligned}$$

$$y'' = 2\delta^2 g(y', \varphi_1 y)\Big[-\varphi_1(y') + \frac{\delta^2}{\lambda}g(y', y)\varphi_1(y)\Big], \qquad (4.5)$$

$$z'' = 2\eta^2 g(z', \varphi_2 z)\Big[-\varphi_2(z') + \frac{\eta^2}{\beta}g(z', z)\varphi_2(z)\Big]. \qquad (4.6)$$

From Theorem 4.2, we obtain the following results.

Theorem 4.3 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally flat bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If*

$C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are a vector fields along $x(t)$, then $C(t)$ is a geodesic on T^2M if and only if $x(t)$ is a geodesic on $(M_{4k}, g, \varphi_1, \varphi_2)$ and

$$y'' = 2\delta^2 g(y', \varphi_1 y)\Big[-\varphi_1(y') + \frac{\delta^2}{\lambda} g(y', y)\varphi_1(y)\Big],$$
$$z'' = 2\eta^2 g(z', \varphi_2 z)\Big[-\varphi_2(z') + \frac{\eta^2}{\beta} g(z', z)\varphi_2(z)\Big].$$

Corollary 4.1 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally flat bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a horizontal lift of the curve $x(t)$(i.e., $y' = z' = 0$), then $C(t)$ is a geodesic on T^2M if and only if $x(t)$ is a geodesic on $(M_{4k}, g, \varphi_1, \varphi_2)$.*

Corollary 4.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally flat bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. The natural lift $C(t) = S^{-1}(x(t), \dot{x}(t), \dot{x}(t))$ of any geodesic $x(t)$ is a geodesic on T^2M.*

Theorem 4.4 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then $\mathcal{R}_1(y, y')$ and $\mathcal{R}_2(z, z')$ are parallel along $x(t)$ and we have the following formulae*

$$x^{(p+1)} = -[\mathcal{R}_1(y, y') + (\mathcal{R}_2(z, z')]x^{(p)}, \quad \forall p \geq 1, \tag{4.7}$$
$$|x^{(p)}| = \text{const.}, \quad \forall p \geq 1, \tag{4.8}$$

where

$$\mathcal{R}_1(y, y') = R(y, y') + \delta^2 g(y', \varphi_1 y)R(y, \varphi_1 y), \tag{4.9}$$
$$\mathcal{R}_2(z, z') = R(z, z') + \eta^2 g(z', \varphi_2 z)R(z, \varphi_2 z). \tag{4.10}$$

Proof Standard calculations give

$$\begin{aligned}
&\mathcal{R}_1'(y, y')\\
&= \nabla_{\dot{x}}[R(y, y') + \delta^2 g(y', \varphi_1(y))\, R(y, \varphi_1(y))]\\
&= R(y', y') + R(y, y'') + \delta^2 g(y'', \varphi_1(y))R(y, \varphi_1(y)) + \delta^2 g(y', \varphi_1(y'))\, R(y, \varphi_1(y))\\
&\quad + \delta^2 g(y', \varphi_1(y))\, R(y', \varphi_1(y)) + \delta^2 g(y', \varphi_1(y))\, R(y, \varphi_1(y'))\\
&= R(y, y'') + \delta^2 g(y'', \varphi_1(y))R(y, \varphi_1(y)) + \delta^2 g(y', \varphi_1(y))\, R(y', \varphi_1(y))\\
&\quad + \delta^2 g(y', \varphi_1(y))\, R(y, \varphi_1(y'))\\
&= R(y, y'') + \delta^2 g(y'', \varphi_1(y))R(y, \varphi_1(y)) + 2\delta^2 g(y', \varphi_1(y))\, R(y, \varphi_1(y'))\\
&= -2\delta^2 g(y', \varphi_1(y))R(y, \varphi_1(y') + \frac{2\delta^4}{\lambda} g(y', \varphi_1(y))g(\varphi_1(y'), \varphi_1(y))R(y, \varphi_1(y))\\
&\quad + \delta^2 g(y'', \varphi_1(y))R(y, \varphi_1(y)) + 2\delta^2 g(y', \varphi_1(y))\, R(y, \varphi_1(y'))\\
&= \frac{2\delta^4}{\lambda} g(y', \varphi_1(y))g(\varphi_1(y'), \varphi_1(y))R(y, \varphi_1(y)) + \delta^2 g(y'', \varphi_1(y))\, R(y, \varphi_1(y))
\end{aligned}$$

$$\begin{aligned}
&= \frac{2\delta^4}{\lambda} g(y', \varphi_1(y)) g(\varphi_1(y', \varphi_1(y)) R(y, \varphi_1(y)) - 2\delta^4 g(y', \varphi_1(y)) g(\varphi_1(y'), \varphi_1(y)) R(y, \varphi_1(y)) \\
&\quad + \frac{2\delta^6}{\lambda} g(y', \varphi_1(y)) g(\varphi_1(y'), \varphi_1(y)) g(\varphi_1(y), \varphi_1(y))\, R(y, \varphi_1(y)) \\
&= \Big[\frac{2\delta^4}{\lambda} - 2\delta^4 + \frac{2\delta^6}{\lambda} g(\varphi_1(y), \varphi_1(y))\Big] g(y', \varphi_1(y)) g(\varphi_1(y'), \varphi_1(y))\, R(y, \varphi_1(y)) \\
&= \frac{2\delta^4}{\lambda}[1 - \lambda + \delta^2 |\varphi_1(y)|^2] g(y', \varphi_1(y)) g(\varphi_1(y'), \varphi_1(y))\, R(y, \varphi_1(y)) \\
&= 0.
\end{aligned}$$

In the same way, we have $\mathcal{R}_2'(z, z') = 0$. Using (4.4), we obtain

$$\begin{aligned}
x^{(3)} &= -[\mathcal{R}_1'(y, y'') + \mathcal{R}_2'(z, z'')]x' - [\mathcal{R}_1(y, y') + \mathcal{R}_2(z, z')]x^{(2)} \\
&= -[\mathcal{R}_1(y, y') + \mathcal{R}_2(z, z')]x^{(2)}.
\end{aligned}$$

By induction on p the formula (4.7) follows.

On the other hand, we have

$$\begin{aligned}
(g(x^{(p)}, x^{(p)}))' &= 2g(x^{(p+1)}, x^{(p)}) \\
&= -2g(\mathcal{R}_1(y, y')x^{(p)}, x^{(p)}) - 2g(\mathcal{R}_2(z, z')x^{(p)}, x^{(p)}) \\
&= 0.
\end{aligned}$$

5 Geodesics of the Hypersurface $T^2_{1,1}M$

Let $T^2_{1,1}M$ be the hypersurface in T^2M defined by

$$T^2_{1,1}M = \{p = S^{-1}(x, u, w) \in T^2M,\ (|u|, |w|) = (1, 1)\}. \tag{5.1}$$

The unit normal vector fields to $T^2_{1,1}M$ are given by

$$\begin{aligned}
&\mathcal{U} : T^2M \to T(T^2M), \\
&p = S^{-1}(x, u, \omega) \mapsto \mathcal{U}_p = (u)^{(1)}, \qquad (5.2)\\
&\mathcal{W} : T^2M \to T(T^2M), \\
&p = S^{-1}(x, u, \omega) \mapsto \mathcal{W}_p = (\omega)^{(2)}. \qquad (5.3)
\end{aligned}$$

Indeed, for $p = S^{-1}(x, u, \omega) \in T^2_{1,1}M$, we have

$$\begin{aligned}
&g_{\mathrm{BS}}(\mathcal{U}, \mathcal{U})_p = g(u, u) + \delta^2 g(\varphi_1(u), u)^2 = g(u, u) = 1, \\
&g_{\mathrm{BS}}(\mathcal{W}, \mathcal{W})_p = g(\omega, \omega) + \eta^2 g(\varphi_2(\omega), \omega)^2 = g(\omega, \omega) = 1, \\
&g_{\mathrm{BS}}(\mathcal{U}, \mathcal{W})_p = 0.
\end{aligned}$$

On the other hand, if we set

$$F_1 : T^2M \to \mathbb{R},$$

$$\begin{aligned} p = S^{-1}(x,u,\omega) &\mapsto g(u,u),\\ F_2 : T^2M &\to \mathbb{R},\\ p = S^{-1}(x,u,\omega) &\mapsto g(\omega,\omega),\\ F : T^2M &\to \mathbb{R}^2,\\ p &\mapsto (F_1(p), F_2(p)), \end{aligned}$$

then the hypersurface $T^2_{1,1}M$ is given by

$$T^2_{1,1}M = \{p = S^{-1}(x,u,\omega) \in T^2M,\ (F_1(p), F_2(p)) = (1,1)\},$$

where $\mathrm{grad}_{\mathrm{g_{BS}}}(F_1)$ and $\mathrm{grad}_{\mathrm{g_{BS}}}(F_2)$ are vector fields normal to $T^2_{1,1}M$. From Lemma 2.2, for any vector field X on M, we get

$$\begin{aligned} \mathrm{g_{BS}}(X^{(0)}, \mathrm{grad}_{\mathrm{g_{BS}}}(F)) &= X^{(0)}(F) = X^{(0)}(g(u,u))\\ &= 0 = \mathrm{g_{BS}}(X^{(0)}, \mathcal{U}),\\ \mathrm{g_{BS}}(X^{(2)}, \mathrm{grad}_{\mathrm{g_{BS}}}(F)) &= X^{(2)}(F) = X^{(2)}(g(u,u))\\ &= 0 = 2\mathrm{g_{BS}}(X^{(2)}, \mathcal{U}),\\ \mathrm{g_{BS}}(X^{(1)}, \mathrm{grad}_{\mathrm{g_{BS}}}(F)) &= X^{(1)}(F) = X^{(1)}(g(u,u))\\ &= 2g(X,u) = 2\mathrm{g_{BS}}(X^{(1)}, \mathcal{U}). \end{aligned}$$

So, $\mathcal{U} = \frac{1}{2}\mathrm{grad}_{\mathrm{g_{BS}}}(F_1)$. By the same way, we obtain $\mathcal{W} = \frac{1}{2}\mathrm{grad}_{\mathrm{g_{BS}}}(F_2)$, therefore $\mathcal{U}$ and $\mathcal{W}$ are vector fields orthonormal to $T^2_{1,1}M$ and the second fundamental form is given by

$$B(\widetilde{X}, \widetilde{Y}) = \mathrm{g_{BS}}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \mathcal{U})\mathcal{U} + \mathrm{g_{BS}}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \mathcal{W})\mathcal{W} \tag{5.4}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathcal{H}(T^2_{1,1}M)$.

Lemma 5.1 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$, and $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ is a vector field along $x(t)$. Then, we have*

(1) $g(y,y) = 1 = g(z,z)$,
(2) $g(y',y) = 0 = g(z',z)$,
(3) $g(y'',y) = -|y'|^2 = -g(y',y')$,
(4) $g(z'',z) = -|z'|^2 = -g(z',z')$.

As $T^2_{1,1}M$ is a hypersurface in T^2M, the curve on $T^2_{1,1}M$ is a geodesic if and only if its second covariant derivative in T^2M is collinear to the unit normal vectors $(y)^{(1)}$ and $(z)^{(2)}$. From Theorem 4.1, formula (5.1) and Lemma 5.1, we obtain the following lemma.

Lemma 5.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$ and $C(t) =$*

$S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. Then, C is a geodesic on $T_{1,1}M$ if and only if

$$\begin{aligned} x'' = -[&R(y, y') + \delta^2 g(y', \varphi_1 y)R(y, \varphi_1 y) \\ &+ R(z, z') + \eta^2 g(z', \varphi_2 z)R(z, \varphi_2 z)]x', \end{aligned} \tag{5.5}$$

$$y'' = -2\delta^2 g(y', \varphi_1 y)\varphi_1(y') + \rho_1\, y, \tag{5.6}$$

$$z'' = -2\eta^2 g(z', \varphi_2 z)\varphi_1(z') + \rho_2\, z, \tag{5.7}$$

where ρ_1, ρ_2 are any function.

Lemma 5.3 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}, and $C(t) = S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. If we put $c_1 = |y'|$, $\mu_1 = g(y', \varphi_1 y)$, $c_2 = |z'|$, $\mu_2 = g(z', \varphi_1 z)$, then we have*

$$\rho_1 = -c_1^2 - 2\delta^2\mu_1^2, \tag{5.8}$$

$$\rho_2 = -c_2^2 - 2\eta^2\mu_2^2, \tag{5.9}$$

$$c_1' = 0 = c_2', \tag{5.10}$$

$$\mu_1' = 0 = \mu_2'. \tag{5.11}$$

Proof From formula (5.6), we obtain

$$\begin{aligned} y'' &= \rho_1\, y - 2\delta^2\mu_1\, \varphi_1(y'), \\ g(y'', y) &= -2\delta^2\mu_1\, g(\varphi_1(y'), y) + \rho_1\, g(y, y) \\ -|y'|^2 &= -2\delta^2\mu_1\, g(\varphi_1(y'), y) + \rho_1 \\ &= 2\delta^2\mu_1\, g(y', \varphi_1(y)) + \rho_1 \\ &= 2\delta^2\mu_1^2 + \rho_1, \\ \frac{1}{2}(c_1^2)' &= g(y'', y') \\ &= \rho_1\, g(y, y') - 2\delta^2\mu_1\, g(\varphi_1(y'), y') \\ &= \rho_1\, g(y, y') \\ &= 0, \quad (\text{from Lemma 5.1 (2)}) \\ \mu_1' &= g(y'', \varphi_1(y)) + g(y', \varphi_1(y')) \\ &= g(y'', \varphi_1(y)) + \rho_1\, g(y, \varphi_1(y)) - 2\delta^2\mu_1\, g(y', y) \\ &= 0. \end{aligned}$$

By the same way, we obtain the other formulae.

Using Lemmas 5.2–5.3, we obtain the following theorem.

Theorem 5.1 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}, and $C(t) =$*

$S^{-1}(x(t), y(t), z(t))$ be a curve on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. If we put $c_1 = |y'|$, $\mu_1 = g(y', \varphi_1 y)$, $c_2 = |z'|$, $\mu_2 = g(z', \varphi_1 z)$, then the curve $C(t) = S^{-1}(x(t), y(t), z(t))$ is a geodesic on $T^2_{1,1}M$ if and only if

$$c_1 = \text{const.}, \quad \mu_1 = \text{const.} \quad \textit{and} \quad \rho_1 = -c_1^2 - 2\delta^2\mu_1^2 = \text{const.}, \tag{5.12}$$

$$c_2 = \text{const.}, \quad \mu_2 = \text{const.} \quad \textit{and} \quad \rho_2 = -c_2^2 - 2\eta^2\mu_2^2 = \text{const.}, \tag{5.13}$$

$$\begin{aligned} x'' = &-[R(y, y') + \delta^2\mu_1 R(y, \varphi_1 y) \\ &+ R(z, z') + \eta^2 g(z', \varphi_2 z) R(z, \varphi_2 z)]x', \end{aligned} \tag{5.14}$$

$$y'' = -c_1^2 y - 2\delta^2\mu_1[\mu_1\, y + \varphi_1(y')], \tag{5.15}$$

$$z'' = -c_2^2 z - 2\eta^2\mu_2[\mu_2\, z + \varphi_2(z')]. \tag{5.16}$$

Theorem 5.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}, and $C(t) = S^{-1}(x(t), y(t), z(t))$ be a geodesic on $T^2_{1,1}M$ such that $y(t)$ and $z(t)$ are vector fields along $x(t)$. Then*

$$\mathcal{R}_1(y, y') = R(y, y') + \delta^2\mu_1\, R(y, \varphi_1(y)),$$
$$\mathcal{R}_2(z, z') = R(z, z') + \eta^2\mu_2\, R(z, \varphi_2(z))$$

are parallel along $x(t)$ for the case of $T^2_{1,1}M$.

Proof Using Theorem 5.1 and formula (5.15), we get

$$\begin{aligned} \mathcal{R}_1'(y, y') &= \nabla_{\dot{x}}\mathcal{R}_1(y, y') \\ &= (\nabla_{\dot{x}}R)(y, y') + R(y', y') + R(y, y'') + \delta^2\mu_1\, R(y', \varphi_1(y)) + \delta^2\mu_1\, R(y, \varphi_1(y')) \\ &= R(y, y'') + \delta^2\mu_1\, R(y', \varphi_1(y)) + \delta^2\mu_1\, R(y, \varphi_1(y')) \\ &= R(y, y'') - \delta^2\mu_1\, R(\varphi_1(y'), y) + \delta^2\mu_1\, R(y, \varphi_1(y')) \\ &= R(y, y'') + 2\delta^2\mu_1\, R(y, \varphi_1(y')) \\ &= R(y, -c_1^2 y - 2\delta^2\mu_1^2\, y) - 2\delta^2\mu_1\, R(y, \varphi_1(y') + 2\delta^2\mu_1\, R(y, \varphi_1(y')) \\ &= -(c_1^2 + 2\delta^2\mu_1^2)\, R(y, y)) \\ &= 0. \end{aligned}$$

By the same way, we obtain the other formula.

From Theorems 5.1–5.2, we obtain the following theorem.

Theorem 5.3 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a geodesic on $T^2_{1,1}M$ such that $y(t), z(t)$ are vector fields along $x(t)$, then we have*

$$x^{(p+1)} = -[\mathcal{R}_1(y, y') + \mathcal{R}_2(z, z')]x^{(p)}, \quad \forall\, p \geq 1, \tag{5.17}$$

$$|x^{(p)}| = \text{const.}, \quad \forall\, p \geq 1, \tag{5.18}$$

where

$$\mathcal{R}_1(y, y') = R(y, y') + \delta^2 \mu_1\, R(y, \varphi_1(y)),$$
$$\mathcal{R}_2(z, z') = R(z, z') + \eta^2 \mu_2\, R(z, \varphi_2(z)).$$

Theorem 5.4 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}, and $C(t) = S^{-1}(x(t), y(t), z(t))$ be a geodesic on $T^2_{1,1}M$ such that $y(t), z(t)$ are vector fields along $x(t)$, then all geodesic curvatures of $\gamma = x(t)$ are constants.*

Proof Denote by s an arc length parameter on $x(t)$. Then $x'_t = x'_s \frac{\mathrm{d}s}{\mathrm{d}t}$. Since C is a geodesic, $\|\dot{C}\| = \|\frac{\mathrm{d}}{\mathrm{d}t} C\| = K = \text{const.}$ and

$$\begin{aligned} K^2 = \|\dot{C}\|^2 &= \Big|\frac{\mathrm{d}s}{\mathrm{d}t}\Big|^2 + |y'|^2 + \delta^2 g(y', \varphi_1(y))^2 + |z'|^2 + \delta^2 g(z', \varphi_2(z))^2 \\ &= \Big|\frac{\mathrm{d}s}{\mathrm{d}t}\Big|^2 + c_1^2 + \delta^2 \mu_1^2 + c_2^2 + \eta^2 \mu_2^2. \end{aligned}$$

Hence

$$\Big|\frac{\mathrm{d}s}{\mathrm{d}t}\Big| = \sqrt{K^2 - (c_1^2 + \delta^2\mu_1^2 + c_2^2 + \eta^2\mu_2^2)} = \beta = \text{const.}, \tag{5.19}$$

where $\beta^2 = K^2 - (c_1^2 + \delta^2\mu_1^2 + c_2^2 + \eta^2\mu_2^2) = \text{const.}$

Denote by $\nu_1, \cdots, \nu_{2n-1}$ the Frenet frame along γ and by $k_1, \cdots, k_{2n-1}$ the geodesic curvatures of γ. From (5.19), we obtain

$$\begin{aligned} x' &= \beta \nu_1, \\ x'' &= \beta^2 k_1 \nu_2, \\ x^{(3)} &= \beta^3 k_1(-k_1\nu_1 + k_2\nu_3), \\ &\ \ \vdots \end{aligned}$$

Using (5.18) we deduce $k_1 = \text{const.}$, $k_2 = \text{const.}, \cdots, k_{2n-1} = \text{const.}$, which completes the proof.

6 Harmonicity

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. Then the second fundamental form of ϕ is defined by

$$B_\phi(X, Y) = (\nabla \mathrm{d}\phi)(X, Y) = \nabla^\phi_X \mathrm{d}\phi(Y) - \mathrm{d}\phi(\nabla_X Y). \tag{6.1}$$

Here ∇ is the Levi-Civita connection on M, ∇^ϕ is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi) = \mathrm{trace}_g \nabla \mathrm{d}\phi = \mathrm{trace}_g B_\phi \tag{6.2}$$

is the tension field of ϕ. A map ϕ is called to be harmonic if and only if $\tau(\phi)=0$.

If $\psi:(N^n,g)\to(\overline{N}^n,\overline{h})$ is a smooth map between two Riemannian manifolds, then we have

$$\tau(\psi\circ\phi)=\mathrm{d}\,\psi(\tau(\phi))+\mathrm{trace}_g\nabla\mathrm{d}\,\psi(\mathrm{d}\phi,\mathrm{d}\phi). \tag{6.3}$$

One can refer to [1, 4, 8, 11, 17–18, 21, 23, 26–27] for background on harmonic maps.

6.1 Harmonicity of section

Lemma 6.1 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If $\sigma\in\Gamma(T^2M)$, then the energy density associated to σ is given by*

$$\begin{aligned}e(\sigma)=&\frac{m}{2}+\frac{1}{2}\mathrm{trace}_g g(\nabla X_\sigma,\nabla X_\sigma)+\frac{\delta^2}{2}\mathrm{trace}_g g(\nabla X_\sigma,\varphi_1(X_\sigma))^2\\&+\frac{1}{2}\mathrm{trace}_g g(\nabla Y_\sigma,\nabla Y_\sigma)+\frac{\eta^2}{2}\mathrm{trace}_g g(\nabla Y_\sigma,\varphi_2(Y_\sigma))^2,\end{aligned}$$

where $m=4k$.

Proof Let $p=\sigma(x)=S^{-1}(x,u,w)\in T^2M$ and $(e_1,\cdots,e_m)$ be a local orthonormal frame on M_{4k} at x, then

$$2e(\sigma)_p=\sum_{i=1}^m \mathrm{g_{BS}}(\mathrm{d}\sigma(e_i),\mathrm{d}\sigma(e_i)).$$

Using formula (4.1), we obtain

$$\begin{aligned}2e(\sigma)_p=&\sum_{i=1}^m \mathrm{g_{BS}}(e_i^{(0)},e_i^{(0)})+\sum_{i=1}^m \mathrm{g_{BS}}((\nabla_{e_i}X_\sigma)^{(1)},(\nabla_{e_i}X_\sigma)^{(1)})\\&+\sum_{i=1}^m \mathrm{g_{BS}}((\nabla_{e_i}Y_\sigma)^{(2)},(\nabla_{e_i}Y_\sigma)^{(2)}),\end{aligned}$$

from which using Proposition 2.1, we deduce

$$\begin{aligned}2e(\sigma)=&m+\mathrm{trace}_g g(\nabla X_\sigma,\nabla X_\sigma)+\delta^2\mathrm{trace}_g g(\nabla X_\sigma,\varphi_1(X_\sigma))^2\\&+\mathrm{trace}_g g(\nabla Y_\sigma,\nabla Y_\sigma)+\eta^2\mathrm{trace}_g g(\nabla Y_\sigma,\varphi_2(Y_\sigma))^2.\end{aligned}$$

Theorem 6.1 *Let $(M_{4k},g,\varphi_1,\varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. Then the tension field associated with $\sigma\in\Gamma(T^2M)$ is given by*

$$\begin{aligned}\tau(\sigma)_p=&\mathrm{trace}_g[R(X_\sigma,\nabla_*X_\sigma)*+\delta^2g(\nabla_*X_\sigma,\varphi_1(X_\sigma))R(X_\sigma,\varphi_1(X_\sigma))*]^{(0)}\\&+\mathrm{trace}_g[R(Y_\sigma,\nabla_*Y_\sigma)*+\eta^2g(\nabla_*Y_\sigma,\varphi_2(Y_\sigma))R(Y_\sigma,\varphi_2(Y_\sigma))*]^{(0)}\\&+\mathrm{trace}_g[\nabla_*^2X_\sigma+2\delta^2g(\nabla_*X_\sigma,\varphi_1(X_\sigma))\varphi_1(\nabla_*X_\sigma)]^{(1)}\end{aligned}$$

$$
\begin{aligned}
&- \mathrm{trace}_g\Big[\frac{2\delta^4}{\lambda} g(\nabla_* X_\sigma, X_\sigma) g(\nabla_* X_\sigma, \varphi_1(X_\sigma))\varphi_1(X_\sigma)\Big]^{(1)} \\
&+ \mathrm{trace}_g[\nabla_*^2 Y_\sigma + 2\eta^2 g(\nabla_* Y_\sigma, \varphi_2(Y_\sigma))\varphi_2(\nabla_* Y_\sigma)]^{(2)} \\
&- \mathrm{trace}_g\Big[\frac{2\eta^4}{\beta} g(\nabla_* Y_\sigma, Y_\sigma) g(\nabla_* Y_\sigma, \varphi_2(Y_\sigma))\varphi_2(Y_\sigma)\Big]^{(2)},
\end{aligned}
$$

where $S(p) = (x, u, \omega) = (X_\sigma(x), Y_\sigma(x))$.

Proof Let $x \in M_{4k}$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame on T_xM, then by summing over i, we have

$$
\begin{aligned}
\tau(\sigma) &= \widetilde{\widetilde{\nabla}}_{\mathrm{d}\sigma(e_i)}\mathrm{d}\sigma(e_i) - \mathrm{d}\sigma(\nabla_{e_i} e_i) \\
&= \widetilde{\widetilde{\nabla}}_{e_i^{(0)} + (\nabla_{e_i} X_\sigma)^{(1)} + (\nabla_{e_i} Y_\sigma)^{(2)}} (e_i^{(0)} + (\nabla_{e_i} X_\sigma)^{(1)} + (\nabla_{e_i} Y_\sigma)^{(2)}) \\
&\quad - (\nabla_{e_i} e_i)^{(0)} - (\nabla_{\nabla_{e_i} e_i} X_\sigma)^{(1)} - (\nabla_{\nabla_{e_i} e_i} Y_\sigma)^{(2)} \\
&= \widetilde{\widetilde{\nabla}}_{e_i^{(0)}} e_i^{(0)} + \widetilde{\widetilde{\nabla}}_{e_i^{(0)}} (\nabla_{e_i} X_\sigma)^{(1)} + \widetilde{\widetilde{\nabla}}_{e_i^{(0)}} (\nabla_{e_i} Y_\sigma)^{(2)} + \widetilde{\widetilde{\nabla}}_{(\nabla_{e_i} X_\sigma)^1} e_i^{(0)} \\
&\quad + \widetilde{\widetilde{\nabla}}_{(\nabla_{e_i} Y_\sigma)^{(2)}} e_i^{(0)} + \widetilde{\widetilde{\nabla}}_{(\nabla_{e_i} X_\sigma)^{(1)}} (\nabla_{e_i} X_\sigma)^{(1)} + \widetilde{\widetilde{\nabla}}_{(\nabla_{e_i} Y_\sigma)^{(2)}} (\nabla_{e_i} Y_\sigma)^{(2)} \\
&\quad - (\nabla_{e_i} e_i)^{(0)} - (\nabla_{\nabla_{e_i} e_i} X_\sigma)^{(1)} - (\nabla_{\nabla_{e_i} e_i} Y_\sigma)^{(2)} \\
&= (\nabla_{e_i}\nabla_{e_i} X_\sigma)^{(1)} + (\nabla_{e_i}\nabla_{e_i} Y_\sigma)^{(2)} + [R(X_\sigma, \nabla_{e_i} X_\sigma)e_i + R(Y_\sigma, \nabla_{e_i} Y_\sigma)e_i]^{(0)} \\
&\quad + [\delta^2 g(\nabla_{e_i} X_\sigma, \varphi_1(X_\sigma)) R(X_\sigma, \varphi_1(X_\sigma))e_i + \eta^2 g(\nabla_{e_i} Y_\sigma, \varphi_2(Y_\sigma)) R(Y_\sigma, \varphi_2(Y_\sigma))e_i]^{(0)} \\
&\quad + \Big[2\delta^2 g(\nabla_{e_i} X_\sigma, \varphi_1(X_\sigma))\varphi_1(\nabla_{e_i} X_\sigma) - \frac{2\delta^4}{\lambda} g(\nabla_{e_i} X_\sigma, X_\sigma) g(\nabla_{e_i} X_\sigma, \varphi_1(X_\sigma))\varphi_1(X_\sigma)\Big]^{(1)} \\
&\quad + \Big[2\eta^2 g(\nabla_{e_i} Y_\sigma, \varphi_2(Y_\sigma))\varphi_2(\nabla_{e_i} Y_\sigma) - \frac{2\eta^4}{\beta} g(\nabla_{e_i} Y_\sigma, Y_\sigma) g(\nabla_{e_i} Y_\sigma, \varphi_2(Y_\sigma))\varphi_2(Y_\sigma)\Big]^{(2)} \\
&\quad - (\nabla_{\nabla_{e_i} e_i} X_\sigma)^{(1)} - (\nabla_{\nabla_{e_i} e_i} Y_\sigma)^{(2)}.
\end{aligned}
$$

Theorem 6.2 *Let* $(M_{4k}, g, \varphi_1, \varphi_2)$ *be a locally symmetric bi-Kählerian manifold and* T^2M *be its second order tangent bundle equipped with the Berger type deformed Sasaki metric* $\mathrm{g_{BS}}$*. A section* $\sigma : M_{4k} \to T^2M$ *is harmonic if and only if the following conditions are verified*

$$
\begin{aligned}
&\quad \mathrm{trace}_g[R(X_\sigma, \nabla_* X_\sigma) * + \delta^2 g(\nabla_* X_\sigma, \varphi_1(X_\sigma)) R(X_\sigma, \varphi_1(X_\sigma)) *] \\
&= -\mathrm{trace}_g[R(Y_\sigma, \nabla_* Y_\sigma) * + \eta^2 g(\nabla_* Y_\sigma, \varphi_2(Y_\sigma)) R(Y_\sigma, \varphi_2(Y_\sigma)) *], \\
&\quad \mathrm{trace}_g[\nabla_*^2 X_\sigma + 2\delta^2 g(\nabla_* X_\sigma, \varphi_1(X_\sigma))\varphi_1(\nabla_* X_\sigma)] \\
&= \mathrm{trace}_g\Big[\frac{2\delta^4}{\lambda} g(\nabla_* X_\sigma, X_\sigma) g(\nabla_* X_\sigma, \varphi_1(X_\sigma))\varphi_1(X_\sigma)\Big], \\
&\quad \mathrm{trace}_g[\nabla_*^2 Y_\sigma + 2\eta^2 g(\nabla_* Y_\sigma, \varphi_2(Y_\sigma))\varphi_2(\nabla_* Y_\sigma)] \\
&= \mathrm{trace}_g\Big[\frac{2\eta^4}{\beta} g(\nabla_* Y_\sigma, Y_\sigma) g(\nabla_* Y_\sigma, \varphi_2(Y_\sigma))\varphi_2(Y_\sigma)\Big].
\end{aligned}
$$

Corollary 6.1 *Let* $(M_{4k}, g, \varphi_1, \varphi_2)$ *be a locally symmetric bi-Kählerian manifold and* T^2M *be its second order tangent bundle equipped with the Berger type deformed Sasaki metric* $\mathrm{g_{BS}}$*. If* $\sigma : M_{4k} \to T^2M$ *is a section such that* X_σ *and* Y_σ *are parallel (i.e.,* $\nabla X_\sigma = \nabla Y_\sigma = 0$*), then* σ *is harmonic.*

Theorem 6.3 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If M_{4k} is a compact manifold, then $\sigma : M_{4k} \to T^2M$ is a harmonic section if and only if X_σ and Y_σ are parallel.*

Proof If σ is parallel, from Corollary 6.1, we deduce that σ is harmonic or vice versa. Let σ_t be a compactly supported variation of σ defined by $\sigma_t = (1+t)\sigma$. From Lemma 6.1 we have

$$\begin{aligned} e(\sigma_t) = \frac{m}{2} + \frac{(t+1)^2}{2}[&\mathrm{trace}_g g(\nabla X_\sigma, \nabla X_\sigma) + \delta^2 \mathrm{trace}_g g(\nabla X_\sigma, \varphi_1(X_\sigma))^2 \\ &+ \mathrm{trace}_g g(\nabla Y_\sigma, \nabla Y_\sigma) + \eta^2 \mathrm{trace}_g g(\nabla Y_\sigma, \varphi_2(Y_\sigma))^2]. \end{aligned}$$

If σ is a critical point of the energy functional, we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M e(\sigma_t)\mathrm{d}v_g|_{t=0} \\ &= \int_M [\mathrm{trace}_g g(\nabla X_\sigma, \nabla X_\sigma) + \delta^2 \mathrm{trace}_g g(\nabla X_\sigma, \varphi_1(X_\sigma))^2 \\ &\quad + \mathrm{trace}_g g(\nabla Y_\sigma, \nabla Y_\sigma) + \eta^2 \mathrm{trace}_g g(\nabla Y_\sigma, \varphi_2(Y_\sigma))^2]\mathrm{d}v_g. \end{aligned}$$

So

$$g(\nabla X_\sigma, \nabla X_\sigma) = g(\nabla Y_\sigma, \nabla Y_\sigma) = 0.$$

Theorem 6.4 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. $\sigma : (M_{4k}, g, \varphi_1, \varphi_2) \to (T^2M, \mathrm{g_{BS}})$ is an isometric immersion if and only if $\nabla X_\sigma = 0 = \nabla Y_\sigma$.*

Proof Let X, Y be vector fields. From Lemma 4.2 we have

$$\begin{aligned} &\mathrm{g_{BS}}(\mathrm{d}\sigma(X), \mathrm{d}\sigma(Y)) \\ &= \mathrm{g_{BS}}(X^{(0)} + (\nabla_X X_\sigma)^{(1)} + (\nabla_X Y_\sigma)^{(2)}, Y^{(0)} + (\nabla_Y X_\sigma)^{(1)} + (\nabla_Y Y_\sigma)^{(2)}) \\ &= g(X, Y) + g(\nabla_X X_\sigma, \nabla_Y X_\sigma) + \delta^2 g(\nabla_X X_\sigma, \varphi_1 X_\sigma) g(\nabla_Y X_\sigma, \varphi_1 X_\sigma) \\ &\quad + g(\nabla_X Y_\sigma, \nabla_Y Y_\sigma) + \eta^2 g(\nabla_X Y_\sigma, \varphi_2 Y_\sigma) g(\nabla_Y Y_\sigma, \varphi_2 Y_\sigma), \end{aligned}$$

from which it follows that

$$\mathrm{g_{BS}}(\mathrm{d}\sigma(X), \mathrm{d}\sigma(Y)) = g(X, Y).$$

Therefore, σ is an isometric immersion if and only if

$$\begin{aligned} 0 &= g(\nabla_X X_\sigma, \nabla_Y X_\sigma) + \delta^2 g(\nabla_X X_\sigma, \varphi_1 X_\sigma) g(\nabla_Y X_\sigma, \varphi_1 X_\sigma) \\ &\quad + g(\nabla_X Y_\sigma, \nabla_Y Y_\sigma) + \eta^2 g(\nabla_X Y_\sigma, \varphi_2 Y_\sigma) g(\nabla_Y Y_\sigma, \varphi_2 Y_\sigma), \end{aligned}$$

which is equivalent to $\nabla X_\sigma = 0$ and $\nabla Y_\sigma = 0$.

As a direct consequence of Theorems 6.3–6.4 , we obtain the following theorem.

Theorem 6.5 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a locally symmetric bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. If $\sigma : (M_{4k}, g, \varphi_1, \varphi_2) \to (T^2M, g_{\mathrm{BS}})$ is an isometric immersion, then σ is totally geodesic. Furthermore, σ is harmonic.*

6.2 Harmonicity conditions of inclusion

Theorem 6.6 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric g_{BS}. If g_{S} denotes the Sasaki metric on TM, then the inclusion*

$$\begin{aligned} I_2 : (TM, g_{\mathrm{S}}) &\to (T^2M, g_{\mathrm{BS}}), \\ (x, u) &\mapsto S^{-1}((x, u, u)) \end{aligned}$$

is a non-harmonic map and its tension field is given by

$$\tau(I_2)_{(x,u)} = -2\delta^2(u)^{(1)} - 2\eta^2(u)^{(2)}.$$

Proof Let $X \in \mathcal{H}(M_{4k})$, then we have

$$\begin{aligned} \mathrm{d}I_2(X^H) &= \mathrm{d}S^{-1}(X^H, X^H) = X^{(0)}, \\ \mathrm{d}I_2(X^V) &= \mathrm{d}S^{-1}(X^V, X^V) = X^{(1)} + X^{(2)}. \end{aligned}$$

Let $x \in M_{4k}$, $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M_{4k} and $\overline{\nabla}$ be the Levi-Civita connection associate with the Sasaki metric g_{S}. We have

$$\begin{aligned} B_{I_2}(e_i^H, e_i^H) &= \widetilde{\nabla}_{\mathrm{d}I_2(e_i^H)}\mathrm{d}I_2(e_i^H) - \mathrm{d}I_2(\overline{\nabla}_{e_i^H} e_i^H) = \widetilde{\nabla}_{e_i^0} e_i^0 - (\overline{\nabla}_{e_i} e_i)^0 = 0, \\ B_{I_2}(e_i^V, e_i^V) &= \widetilde{\nabla}_{\mathrm{d}I_2(e_i^V)}\mathrm{d}I_2(e_i^V) - \mathrm{d}I_2(\overline{\nabla}_{e_i^V} e_i^V) \\ &= \widetilde{\nabla}_{e_i^1 + e_i^2}(e_i^1 + e_i^2) = \widetilde{\nabla}_{e_i^1}(e_i^1) + \widetilde{\nabla}_{e_i^2}(e_i^2) \\ &= 2\delta^2\Big[g(e_i, \varphi_1(u))\varphi_1 e_i - \frac{\delta^2}{\lambda} g(e_i, u) g(e_i, \varphi_1 u)\varphi_1 u\Big]^{(1)} \\ &\quad + 2\eta^2\Big[g(e_i, \varphi_2(u))\varphi_2 e_i - \frac{\eta^2}{\beta} g(e_i, u) g(e_i, \varphi_2 u)\varphi_2 u\Big]^{(2)} \\ &= 2\delta^2[g(e_i, \varphi_1(u))\varphi_1 e_i]^{(1)} + 2\eta^2[g(e_i, \varphi_2(u))\varphi_2 e_i]^{(2)}. \end{aligned}$$

6.3 Harmonicity conditions of projections

Let $(E_1, \cdots, E_m)$ be the orthonormal vector fields on M_{4k}. The matrix of Berger type deformed Sasaki metric g_{BS} on T^2M with respect to $(E_1^{(0)}, \cdots, E_m^{(0)}, E_1^{(1)}, \cdots, E_m^{(1)}, E_1^{(2)}, \cdots, E_m^{(2)})$ is as follows

$$G_{\mathrm{BS}} = \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & aij & 0 \\ 0 & 0 & b_{ij} \end{pmatrix}, \tag{6.4}$$

$$G_{\mathrm{BS}}^{-1} = \begin{pmatrix} \delta^{ij} & 0 & 0 \\ 0 & a^{ij} & 0 \\ 0 & 0 & b^{ij} \end{pmatrix}, \tag{6.5}$$

where $a = (\delta_{ij} + \delta^2(\varphi_1 u)^i(\varphi_1 u)^j)_{i,j\le 4k}$ and $b = (\delta_{ij} + \delta^2(\varphi_2 w)_i(\varphi_2 w)_j)_{i,j\le 4k}$.

Using formula (6.1) and Theorem 2.3, we obtain the following lemma.

Lemma 6.2 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If $\pi : (T^2M, \mathrm{g_{BS}}) \to (M_{4k}, g)$ denotes the canonical projection, then we have*

$$\begin{aligned}
&B_\pi(E_i^0, E_j^0)_p = B_\pi(E_j^1, E_i^1) = B_\pi(E_j^2, E_i^2) = 0,\\
&B_\pi(E_i^0, E_j^1)_p = -\frac{1}{2}[R_x(u, E_j)E_i + \delta^2 g(E_j, \varphi_1 u)R(u, \varphi_1 u)E_i],\\
&B_\pi(E_i^0, E_j^2)_p = -\frac{1}{2}[R_x(w, E_j)E_i, +\eta^2 g(E_j, \varphi_2 w)R(w, \varphi_2 w)E_i],\\
&B_\pi(E_i^1, E_j^2)_p = 0.
\end{aligned}$$

Theorem 6.7 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. If ∇ is locally flat, then the canonical projection $\pi : (T^2M, \mathrm{g_{BS}}) \to (M_{4k}, g, \varphi_1, \varphi_2)$ is totally geodesic. Moreover, π is a harmonic map.*

Using formula (6.1) and Theorem 2.3, we obtain the following lemma.

Lemma 6.3 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kählerian manifold and T^2M be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$, and $(TM, \mathrm{g_S})$ be its tangent bundle equipped with the Sasaki metric $\mathrm{g_S}$. If π denotes the canonical projection, given by*

$$\begin{aligned}
\pi &: (T^2M, \mathrm{G_{BS}}) \to (TM, \mathrm{g_S}),\\
p &= S^{-1}(x, u, \omega) \mapsto (x, u),
\end{aligned}$$

then we have

$$\begin{aligned}
&\pi_*(X^0) = X^H, \quad \pi_*(X^1) = X^V, \quad \pi_*(X^2) = 0,\\
&B_\pi(E_i^0, E_j^0)_p = -\frac{1}{2}[R(E_i, E_j)u]^V,\\
&B_\pi(E_i^0, E_j^1)_p = -\frac{1}{2}[R(u, E_j)E_i + \delta^2 g(E_j, \varphi_1 u)R(u, \varphi_1 u)E_i]^H,\\
&B_\pi(E_i^0, E_j^2)_p = -\frac{1}{2}[R(\omega, E_j)E_i + \eta^2 g(E_j, \varphi_2 \omega)R(w, \varphi_2 \omega)E_i]^H,\\
&B_\pi(E_i^1, E_j^1)_p = 0 = B_\pi(E_i^2, E_j^2)_p = B_\pi(E_i^1, E_j^2)_p,
\end{aligned}$$

where $(E_1, \cdots, E_m)$ is a local orthonormal frame on M_{4k}.

From Lemma 6.2, we have the following theorem.

Theorem 6.8 *Let $(M_{4k}, g, \varphi_1, \varphi_2)$ be a bi-Kä hlerian manifold and $(T^2M, \mathrm{g_{BS}})$ be its second order tangent bundle equipped with the Berger type deformed Sasaki metric $\mathrm{g_{BS}}$. The canonical projection $\pi : (T^2M, G_{\mathrm{BS}}) \to (TM, \mathrm{g_S})$ is totally geodesic if and only if ∇ is locally flat. Moreover, π is a harmonic map.*

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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