Chin. Ann. Math. Ser. B
45(5), 2024, 805–822
DOI: 10.1007/s11401-024-0040-6

Chinese Annals of Mathematics, Series B

Exact Convergence Rate of the Local Limit Theorem for a Branching Random Walk in $\mathbb{Z}^d$ with a Random Environment in Time*

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Abstract Consider a branching random walk with a random environment in time in the d-dimensional integer lattice. The branching mechanism is governed by a supercritical branching process, and the particles perform a lazy random walk with an independent, non-identical increment distribution. For $A \subset \mathbb{Z}^d$, let $\mathbb{Z}_n(A)$ be the number of offsprings of generation n located in A. The exact convergence rate of the local limit theorem for the counting measure $Z_n(\cdot)$ is obtained. This partially extends the previous results for a simple branching random walk derived by Gao (2017, Stoch. Process Appl.).

Keywords Branching random walk, Random environment, Local limit theorems, Exact convergence rate
2000 MR Subject Classification 60F05, 60J10, 60G50, 60J80

1 Introduction

Branching random walk is a classical model in probability. The model gives a description of the evolution of a population of particles where spatial motion is present, hence generalizes the classical Galton-Watson branching processes. Although the model has a long history, it is still quite central in pure and applied probability. It serves as a popular model for describing and analyzing phenomena in various applied disciplines, such as biology, population dynamics and computer science. Meanwhile, because the model captures the fundamental nature of the stochastic dynamics, it is frequently found in many other random models (e.g. multiplicative cascades, infinite particle systems, random fractals). The reader may refer to [1, 34] for the classical results on BRW.

Since Harris [21, Chapter III §16] raised the conjecture of central limit theorem for branching random walk, plenty of works were devoted to this topic. See e.g. [2, 6, 10, 18, 20, 23, 25–28, 32, 36, 38] and references therein.

In this article, we aim to derive the exact convergence rate of the local limit theorem (abbreviated as LLT) for a branching random walk in $\mathbb{Z}^d$ with a time-varying random environment. In the literature, Révész [34] initiated the study of the convergence speed in local limit theorems for a simple branching random walk on $\mathbb{Z}^d$ (without random environment setting), and Chen [9]

Manuscript received September 22, 2018. Revised October 30, 2022 .
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*This work was supported by the National Natural Science Foundation of China (No. 11971063).

derived the exact convergence rate of local limit theorems in this case under a second moment condition for the branching mechanism. Later, Gao [15] improved Chen's result by weakening the moment condition therein. The main objective of this article is to extend the result in [15] to the case of branching random walk with a time-varying random environment setting. Due to the appearance of the random environment, the model may be more suitable to modelling real world applications but the analysis is much more difficult and awkward.

When the walks are governed by non-lattice random distributions in $\mathbb{R}$, some closely related results on the central limit theorems for branching random walk with a time-varying random environment have been obtained by Gao and Liu in [16–17]. This kind of branching random walk in time-varying random environment appeared firstly in [7] and then some other related limit theorems were surveyed in [23] and [31]. Different kinds of random environments have been considered in the context of branching random walks in the literature (see [5, 8, 11–13, 19, 22, 33, 39]). The readers may refer to these articles and references therein for more information.

1.1 Description of the model and the notations

A random environment $\xi = (\xi_n)$ is defined by a sequence of independent and identically distributed random variables in a some abstract space Θ. Each realization of ξ_n determines the probability distribution $p(\xi_n) = \{p_k(\xi_n) : k \in \mathbb{N}\}$ with $\mathbb{N} = \{0, 1, 2, \cdots\}$, and a real number $r_n = r(\xi_n) \in (0, 1)$, with which the associated motion law $G_{\xi_n}(\cdot)$ is define by

$$G_{\xi_n}(\mathbf{0}) = r_n, \quad G_{\xi_n}(\mathbf{e}_v) = G_{\xi_n}(-\mathbf{e}_v) = \frac{1 - r_n}{2d}, \quad v = 1, 2, \cdots, d, \tag{1.1}$$

where $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{Z}^d$, $\mathbf{e}_v (1 \le v \le d)$ are the orthogonal unit vectors in $\mathbb{Z}^d$.

Without loss of generality, we can take ξ_n as coordinate functions defined on the product space $(\Theta^{\mathbb{N}}, \mathscr{F}^{\bigotimes \mathbb{N}})$ equipped with the product law τ of some probability law τ_0 on $(\Theta, \mathscr{F})$, which is invariant and ergodic under the usual shift transformation T on $\Theta^{\mathbb{N}}$: $T(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$.

Given the environment $\xi = (\xi_n)$, the process can be described as follows. It begins at time 0 with one initial particle $\varnothing$ of generation $\mathbf{0}$ located at $S_\varnothing = \mathbf{0} \in \mathbb{Z}^d$; at time 1, it is replaced by $N = N_\varnothing$ new particle $\varnothing i = i \, (1 \le i \le N)$ of generation 1, located at $S_i = L_i \in \mathbb{Z}^d$, where $N, L_1, L_2, \cdots$ are mutually independent, N has the law $p(\xi_0)$, and each L_i has the law $G(\xi_0)$. In general, each particle $u = u_1 \cdots u_n$ of generation n is replaced at time $n + 1$ by N_u new particles $ui \, (1 \le i \le N_u)$ of generation $n+1$, with displacements L_{ui}, so that the ith child ui is located at $S_{ui} = S_u + L_{ui}$, where $N_u, L_{u1}, L_{u2}, \cdots$ are mutually independent, N_u has the law $p(\xi_n)$, and each L_{ui} has the same law G_{ξ_n}. By definition, given the environment ξ, the random variables N_u and L_u, indexed by all the finite sequences u of positive integers, are independent of each other.

For each realization $\xi \in \Theta^{\mathbb{N}}$ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ be the probability space under which the process is defined (when the environment ξ is fixed to the given realization). The probability $\mathbb{P}_\xi$ is usually called quenched law. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \varepsilon^{\mathbb{N}} \bigotimes \mathcal{G}, \mathbb{P})$, where $\mathbb{P} = \mathbb{E}(\delta_\xi \bigotimes \mathbb{P}_\xi)$ with δ_ξ the Dirac measure at ξ and $\mathbb{E}$ the expectation with respect to the random variable ξ, so that for all measurable and positive g defined on $\Theta^{\mathbb{N}} \times \Gamma$, we have

$$\int_{\Theta^{\mathbb{N}} \times \Gamma} g(x, y) \mathrm{d}\mathbb{P}(x, y) = \mathbb{E} \int_\Gamma g(\xi, y) \mathrm{d}\mathbb{P}_\xi(y).$$

The total probability $\mathbb{P}$ is usually called annealed law. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}$ will still be denoted by $\mathbb{E}$; there will be no confusion for reason of consistence. The expectation with respect to $\mathbb{P}_\xi$ will be denoted by $\mathbb{E}_\xi$.

Let $\mathbb{T}$ be the genealogical tree with $\{N_u\}$ as defining elements. By definition, we have (a) $\varnothing \in \mathbb{T}$; (b) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$; (c) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \le i \le N_u$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n, where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs.

1.2 The main result

Define $Z_n(\cdot)$ as the counting measure of particles of generation n: For $B \subset \mathbb{Z}^d$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_B(S_u).$$

In particular, we will frequently write $Z_n(z)$ instead of $Z_n(\{z\})$, which is the number of the nth generation individuals located at $z \in \mathbb{Z}^d$ by definition.

Then $\{Z_n(\mathbb{Z}^d)\}$ constitutes a branching process in random environment (see e.g. [3–4, 35]). For $n \ge 0$, let $\widehat{N}_n$ (resp. $\widehat{L}_n$) be a random variable with the law $p(\xi_n)$ (resp. G_{ξ_n}) under the law $\mathbb{P}_\xi$, and define

$$m_n = \mathbb{E}_\xi \widehat{N}_n, \quad \Pi_n = m_0 \cdots m_{n-1}, \quad \Pi_0 = 1.$$

It is well known that the normalized sequence

$$W_n = \frac{1}{\Pi_n} Z_n(\mathbb{Z}^d), \quad n \ge 1 \tag{1.2}$$

constitutes a martingale with respect to the filtration $(\mathcal{F}_n)$ defined by

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma(\xi, N_u : |u| < n) \quad \text{for } n \ge 1.$$

Throughout the paper, we shall always assume the following conditions:

$$\mathbb{E} \ln m_0 > 0 \quad \text{and} \quad \mathbb{E}\Big[\frac{1}{m_0} \widehat{N}_0 (\ln^+ \widehat{N}_0)^{\lambda+1}\Big] < \infty, \tag{1.3}$$

where the value of $\lambda > 0$ is to be specified in the hypothesis of the theorems. Under these conditions, the underlying branching process $\big\{Z_n(\mathbb{Z}^d)\big\}$ is supercritical, which means that $Z_n\big(\mathbb{Z}^d\big) \to \infty$ with positive probability, and the limit $W = \lim\limits_n W_n$ verifies $\mathbb{E}W = 1$ and $W > 0$ almost surely on the explosion event $\{Z_n \to \infty\}$ (see e.g. [4, 37]) .

For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, define

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d, \quad \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

Set

$$s_n^2 = \sum_{i=0}^{n-1} (1 - r_i), \quad u_n = \sum_{i=0}^{n-1} (1 - r_i)^2.$$

By the assumptions on ξ, we deduce from the law of large number that

$$\frac{s_n^2}{n} \xrightarrow[a.s.]{n\to\infty} 1-\mathbb{E}r_0, \quad \frac{u_n}{n} \xrightarrow[a.s.]{n\to\infty} \mathbb{E}(1-r_0)^2. \tag{1.4}$$

Then our main results can be stated as follows.

Theorem 1.1 *Assume* (1.3) *for some* $\lambda > 4(d+3)$, $\mathbb{E}m_0^{-\iota} < \infty$ *for some* $\iota > 0$. *Then for each* $x = (x_1, \cdots, x_d) \in \mathbb{Z}^d$,

$$\begin{aligned} & n\Big[\Big(\frac{2\pi s_n^2}{d}\Big)^{\frac{d}{2}}\frac{Z_n(x)}{\Pi_n} - W\exp\Big(-\frac{d}{2s_n^2}\|x\|^2\Big)\Big] \\ & \xrightarrow[a.s.]{n\to\infty} \frac{d}{1-\mathbb{E}r_0}\Big[-\frac{1}{2}V_2 + \langle x, V_1\rangle + \frac{1}{8}\Big(d-(d+2)\frac{\mathbb{E}(1-r_0)^2}{1-\mathbb{E}r_0}\Big)W\Big], \end{aligned} \tag{1.5}$$

where respectively $V_1 \in \mathbb{R}^d$ *and* $V_2 \in \mathbb{R}$ *are the almost sure limit of the sequences* $\{N_{v,n}\}_n(v = 1, 2)$, *defined by*

$$N_{1,n} = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n} S_u \quad \textit{and} \quad N_{2,n} = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n}(\|S_u\|^2 - s_n^2). \tag{1.6}$$

Remark 1.1 This theorem generalizes [15, Theorem 1.1(I)].

The rest of the paper is organized as follows. In Section 2, we define two martingales associated with the branching random walk and prove their a.s. convergence. To prove the main results, in Section 3, we consider a random walk with the independent, non-identical increment distribution on $\mathbb{Z}^d$, and derive a first order correction of the local limit theorem with asymptotical infinitesimal error terms. The proofs of the main theorem will be given in Section 4.

2 Two Martingales and Their Convergences

In this section, we prove the a.s. convergence of those two sequences defined by (1.6). In fact, they are martingales with respect to the filtration

$$\mathcal{D}_0 = \{\emptyset, \Omega\}, \quad \mathcal{D}_n = \sigma(N_u, L_{ui}: \ i \ge 1, |u| < n) \quad \text{for } n \ge 1.$$

Proposition 2.1 *The sequences* $\{(N_{1,n}, \mathcal{D}_n)\}$ *and* $\{(N_{2,n}, \mathcal{D}_n)\}$ *are both martingales. Moreover when* (1.3) *and* $\mathbb{E}(\ln^- m_0)^{1+\lambda} < \infty$ *for some* $\lambda > 2$, *these two martingales converge a.s. to* $V_1 \in \mathbb{R}^d$ *and* $V_2 \in \mathbb{R}$, *respectively*,

$$V_1 = \lim_{n\to\infty} N_{1,n} \quad a.s. \quad V_2 = \lim_{n\to\infty} N_{2,n} \quad a.s.$$

Proof The fact that $\{(N_{1,n}, \mathcal{D}_n)\}$ is a martingale can be easily shown

$$\begin{aligned} \mathbb{E}_{\xi,n}N_{1,n+1} &= \mathbb{E}_{\xi,n}\Big(\frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_{n+1}} S_u\Big) = \frac{1}{\Pi_{n+1}}\mathbb{E}_{\xi,n}\Big(\sum_{u\in\mathbb{T}_n}\sum_{i=1}^{N_u}(S_u + L_{ui})\Big) \\ &= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u}(S_u + L_{ui})\Big) \end{aligned}$$

$$= \frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_n} m_n S_u = N_{1,n}.$$

To see that $\{(N_{2,n+1}, \mathcal{D}_n)\}$ is a martingale, it suffices to notice that

$$\begin{aligned}
\mathbb{E}_{\xi,n} N_{2,n+1} &= \mathbb{E}_{\xi,n}\Big(\frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_{n+1}} (\|S_u\|^2 - s_{n+1}^2)\Big) \\
&= \frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u} (\|S_u + L_{ui}\|^2 - s_{n+1}^2)\Big) \\
&= \frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_n} \mathbb{E}_{\xi,n}\Big(\sum_{i=1}^{N_u} \mathbb{E}_{\xi,n}((\|S_u\|^2 + 2\langle S_s, L_{ui}\rangle + \|L_{ui}\|^2 - s_{n+1}^2) \mid N_u)\Big) \\
&= \frac{1}{\Pi_{n+1}} \sum_{u \in \mathbb{T}_n} m_n(\|S_u\|^2 + (1 - r_n) - s_{n+1}^2) \\
&= \frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} (\|S_u\|^2 - s_n^2) = N_{2,n}.
\end{aligned}$$

We shall prove the a.s. convergence of the martingale $\{N_{1,n}\}$ by showing that the series

$$\sum_{n=1}^{\infty} I_n \quad \text{with } I_n = N_{1,n+1} - N_{1,n} \tag{2.1}$$

converges a.s. For $n \geq 1$ and $|u| = n$, set

$$X_u = S_u\Big(\frac{N_u}{m_{|u|}} - 1\Big) + \sum_{i=1}^{N_u} \frac{L_{ui}}{m_{|u|}}. \tag{2.2}$$

It is plain to see

$$\|X_u\| \leq (n+1)\Big(\frac{N_u}{m_{|u|}} + 1\Big) \tag{2.3}$$

and

$$I_n = N_{1,n+1} - N_{1,n} = \frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} X_u.$$

Let $\widehat{X}_n$ be a generic random variable of X_u, i.e., $\widehat{X}_n$ has the same distribution with X_u (for $|u| = n$). Recall that $\widehat{N}_n$ has the same distribution as $N_u, |u| = n$.

We shall use a truncating argument to prove the convergence. Put

$$X'_u = X_u \mathbf{1}_{\{\frac{N_u}{m_n} \leq \Pi_{|u|}\}} \quad \text{and} \quad I'_n = \sum_{u \in \mathbb{T}_n} X'_u.$$

The following decomposition will play an important role:

$$\sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} (I_n - I'_n) + \sum_{n=0}^{\infty} (I'_n - \mathbb{E}_{\xi,n} I'_n) + \sum_{n=0}^{\infty} \mathbb{E}_{\xi,n} I'_n. \tag{2.4}$$

We shall prove that each of the three series on the right hand side converges a.s.

Since $\lim\limits_{n\to\infty} \ln \frac{\Pi_n}{n} = \mathbb{E} \ln m_0 > 0$ a.s., for a given constant $0 < c_\xi < \mathbb{E} \ln m_0$ and for n large enough, we have

$$\ln \Pi_n > c_\xi n. \tag{2.5}$$

For the first series $\sum_{n=0}^{\infty}(I_n - I'_n)$ in (2.4),

$$
\begin{aligned}
\mathbb{E}_\xi \|I_n - I'_n\| &= \mathbb{E}_\xi \Big\| \frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} X_u \mathbf{1}_{\{\frac{N_u}{m_n} > \Pi_n\}} \Big\| \\
&\le \mathbb{E}_\xi \Big(\frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} \mathbb{E}_{\xi,n}(\|X_u\| \mathbf{1}_{\{\frac{N_u}{m_n} > \Pi_n\}}) \Big) \\
&= \mathbb{E}_\xi (\|\widehat{X}_n\| \mathbf{1}_{\{\frac{\widehat{N}_n}{m_n} > \Pi_n\}}) \\
&\le \frac{1}{(\ln \Pi_n)^{1+\lambda}} \mathbb{E}_\xi \|\widehat{X}_n\| \Big(\ln^+ \Big(\frac{\widehat{N}_n}{m_n} \Big) \Big)^{1+\lambda} \\
&\le K_\xi n^{-\lambda} \mathbb{E}_\xi \Big(\frac{\widehat{N}_n}{m_n} + 1 \Big) (\ln^+ \widehat{N}_n)^{1+\lambda} + K_\xi n^{-\lambda} (\ln^- m_n)^{1+\lambda} \quad \text{a.s.}
\end{aligned}
$$

We see that

$$
\begin{aligned}
&\mathbb{E} \sum_{n=1}^{\infty} n^{-\lambda} \Big[\mathbb{E}_\xi \Big(\frac{\widehat{N}_n}{m_n} + 1 \Big) (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \Big] \\
=&\Big(\sum_{n=1}^{\infty} n^{-\lambda} \Big) \Big[\mathbb{E} \Big(\frac{\widehat{N}_n}{m_0} + 1 \Big) (\ln^+ \widehat{N}_0)^{1+\lambda} + \mathbb{E}(\ln^- m_0)^{1+\lambda} \Big] < \infty,
\end{aligned}
$$

which implies that

$$
\sum_{n=1}^{\infty} n^{-\lambda} \Big[\mathbb{E}_\xi \Big(\frac{\widehat{N}_n}{m_n} + 1 \Big) (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \Big] < \infty \quad \text{a.s.} \tag{2.6}
$$

Hence

$$
\begin{aligned}
&\mathbb{E}_\xi \Big\| \sum_{n=1}^{\infty} (\mathrm{I}_n - \mathrm{I}'_n) \Big\| \le \sum_{n=1}^{\infty} \mathbb{E}_\xi \|\mathrm{I}_n - \mathrm{I}'_n\| < \infty, \\
&\mathbb{E}_\xi \Big\| \sum_{n=1}^{\infty} \mathbb{E}_{\xi,n} \mathrm{I}'_n \Big\| = \mathbb{E}_\xi \Big\| \sum_{n=1}^{\infty} \mathbb{E}_{\xi,n} (\mathrm{I}_n - \mathrm{I}'_n) \Big\| \le \sum_{n=1}^{\infty} \mathbb{E}_\xi \|\mathrm{I}_n - \mathrm{I}'_n\| < \infty.
\end{aligned}
$$

It follows that the series $\sum_{n=1}^{\infty}(\mathrm{I}_n - \mathrm{I}'_n)$ and $\sum_{n=1}^{\infty}\mathbb{E}_{\xi,n}\mathrm{I}'_n$ converge a.s.

Thus, it suffices to prove that the second series

$$
\sum_{n=0}^{\infty} (I'_n - \mathbb{E}_{\xi,n} I'_n) \quad \text{converges a.s.} \tag{2.7}
$$

By using the fact that $\sum_{k=1}^{n}(I'_n - \mathbb{E}_{\xi,k} I_k)$ is a martingale w.r.t $\{\mathcal{D}_{n+1}\}$ and by the a.s. convergence of an L^2 bounded martingale (see e.g. [14, P.251, Example. 4.9]), we only need to show that the series $\sum_{n=0}^{\infty}\mathbb{E}_\xi\|I'_n - \mathbb{E}_{\xi,n} I'_n\|^2$ converges a.s.

First observe that

$$
\mathbb{E}_\xi \big\{ \|X_u\|^2 \big| N_u \big\} \le 2(\mathbb{E}_\xi \|S_u\|^2) \Big(\frac{N_u}{m_n} + 1 \Big)^2 + 2 \Big(\frac{N_u}{m_n} \Big)^2 \le 2(n+1) \Big(\frac{N_u}{m_n} + 1 \Big)^2.
$$

Notice

$$\begin{aligned}
&\mathbb{E}_\xi \|I'_n - \mathbb{E}_{\xi,n} I'_n\|^2 \\
&= \mathbb{E}_\xi \Big(\frac{1}{\Pi_n^2} \sum_{u\in\mathbb{T}_n} \mathbb{E}_{\xi,n} \|X'_u - \mathbb{E}_{\xi,n} X'_u\|^2 \Big) \\
&\le \frac{1}{\Pi_n^2} \mathbb{E}_\xi \sum_{u\in\mathbb{T}_n} \mathbb{E}_{\xi,n} \|X'_u\|^2 \\
&\le \frac{2}{\Pi_n} \mathbb{E}_\xi (n+1) \Big(\frac{\widehat{N}_n}{m_n} + 1 \Big)^2 (\mathbf{1}_{\{\frac{\widehat{N}_n}{m_n} \le \Pi_n\}} \mathbf{1}_{\{\frac{\widehat{N}_n}{m_n} \le \mathrm{e}^{2\lambda}\}} + \mathbf{1}_{\{\frac{\widehat{N}_n}{m_n} \le \Pi_n\}} \mathbf{1}_{\{\frac{\widehat{N}_n}{m_n} > \mathrm{e}^{2\lambda}\}}) \\
&\le \frac{2(n+1)(\mathrm{e}^{4\lambda}+3)}{\Pi_n} + \frac{2(n+1)}{\Pi_n} \mathbb{E}_\xi \frac{(\frac{\widehat{N}_n}{m_n})^2 \Pi_n (\ln \Pi_n)^{-1-\lambda}}{(\frac{\widehat{N}_n}{m_n})(\ln^+(\frac{\widehat{N}_n}{m_n}))^{-1-\lambda}} \\
&\quad (\text{because } \ x(\ln x)^{-1-\lambda} \ \text{ is increasing for } \ x > \mathrm{e}^{2\lambda}) \\
&\le \frac{2(n+1)(\mathrm{e}^{4\lambda}+3)}{\Pi_n} + 4\Big(\frac{2}{c_\xi}\Big)^{\lambda+1} n^{-\lambda} \Big(\mathbb{E}_\xi \frac{\widehat{N}_n}{m_n} (\ln^+ \widehat{N}_n)^{1+\lambda} + (\ln^- m_n)^{1+\lambda} \Big).
\end{aligned}$$

By using (2.6), we can obtain $\sum_{n=1}^{\infty} \mathbb{E}_\xi \|I_n - \mathbb{E}_{\xi,n} I'_n\|^2 < \infty$ a.s., which implies (2.7). Combining the above arguments, we see that the series $\sum I_n$ converges a.s. Therefore, $N_{1,n}$ converges a.s. to

$$V_1 = \sum_{n=1}^{\infty} (N_{1,n+1} - N_{1,n}) + N_{1,n}.$$

By mimicking the above proofs in the case of $\{(N_{1,n}, \mathcal{D}_n)\}$, we can prove the a.s. convergence of the martingale $\{(N_{2,n}, \mathcal{D}_n)\}$ by showing that

$$\sum_{n=1}^{\infty} (N_{2,n+1} - N_{2,n}) \quad \text{converges a.s.}$$

For $n \ge 1$ and $|u| = n$, we still use the nation X_u and I_n, which are defined respectively by

$$X_u = \Big(\frac{N_u}{m_n} - 1 \Big) (\|S_u\|^2 - s_n^2) + \frac{N_u}{m_n} r_n + \frac{2}{m_n} \Big\langle S_u, \sum_{i=1}^{N_u} L_{ui} \Big\rangle \tag{2.8}$$

and

$$I_n = N_{2,n+1} - N_{2,n} = \frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} X_u. \tag{2.9}$$

It is easy to see

$$\|X_u\| \le 2(n+1)^2 \Big(\frac{N_u}{m_{|u|}} + 1 \Big). \tag{2.10}$$

Using the decomposition (2.4), with X_u and I_n defined respectively by (2.8) and (2.9), and the following almost the same lines as above, we can prove that the series $\sum_{n=1}^{\infty} (N_{2,n+1} - N_{2,n})$ converges a.s. Then $\{N_{2,n}\}$ converges a.s. to

$$V_2 = \sum_{n=1}^{\infty} (N_{2,n+1} - N_{2,n}) + N_{2,1}.$$

The proposition is proved.

3 First Order Correction of the Local Limit Theorem for a Random Walk in a Time-Inhomogeneous Environment

In this section, we derive the first order correction in the local limit theorem for a time-inhomogeneous random walk in $\mathbb{Z}^d$, whose step size distributions vary in time.

Theorem 3.1 *Assume that* $\beta \in \left(\frac{1}{12}, \frac{1}{4}\right)$ *and* $\overline{r} = \{\overline{r}_n \in (0,1) : n \in \mathbb{N}\}$ *is a sequence of real numbers. Let* $S_n = L_1 + L_2 + \cdots + L_n$*, where*

$$\mathbb{P}(L_n = \mathbf{0}) = \overline{r}_n, \quad \mathbb{P}(L_n = \mathbf{e}_v) = \mathbb{P}(L_n = -\mathbf{e}_v) = \frac{1-\overline{r}_n}{2d}, \quad v = 1, \cdots, d, \quad n \in \mathbb{N}. \tag{3.1}$$

Set $\delta_c = \frac{1-\cos(d^{\frac{-1}{2}})}{d}$, $M_n = \sum_{i=1}^{n} \min\{2\overline{r}_i, \delta_c(1-\overline{r}_i)\}$, $\overline{s}_n^2 = \sum_{i=1}^{n}(1-\overline{r}_i), \overline{u}_n = \sum_{i=1}^{n}(1-\overline{r}_i)^2$. *If the following conditions*:

$$\frac{M_n}{n} \xrightarrow{n\to\infty} M \in (0,\infty), \qquad \frac{\overline{s}_n^2}{n} \xrightarrow{n\to\infty} \overline{s} > 0, \qquad \frac{\overline{u}_n}{n} \xrightarrow{n\to\infty} \overline{u} > 0 \tag{3.2}$$

hold, then we have

$$\mathbb{P}(S_n = x) = \left(\frac{d}{2\pi\overline{s}_n^2}\right)^{\frac{d}{2}} \left[1 + \frac{d}{\overline{s}_n^2}\left(-\frac{1}{2}\|x\|^2 + \frac{1}{8}d - \frac{1}{8}(d+2)\frac{\overline{u}_n}{\overline{s}_n^2}\right)\right] + \frac{1}{n^{1+\frac{d}{2}}} R_n(\overline{r}, x), \tag{3.3}$$

where $R_n(\overline{r}, x)$ *is asymptotically uniform bounded for* $\|x\| \le n^\beta$ *as* $n \to \infty$,

$$\begin{aligned}\sup_{\|x\|\le n^\beta} |R(\overline{r}, x)| \le & C_d\left(\frac{\overline{s}_n^2}{n}\right)^{-\frac{d}{2}} n^{-\frac{1}{2}+2\beta} + 2^d n^{\frac{d}{2}} \exp\left\{-\frac{11}{24} n^{\frac{1}{4}-\beta}\left(\frac{\overline{s}_n^2}{n}\right)\right\} \\ & + n^{(1+\frac{d}{2})} \mathrm{e}^{-M_n} + C_d\left(\frac{\overline{s}_n^2}{n}\right)^{-d} n^{\frac{1}{2}+\frac{d}{8}} \exp\left\{-\frac{1}{2d}\left(\frac{\overline{s}_n^2}{n}\right) n^{\frac{1}{4}-\beta}\right\}.\end{aligned} \tag{3.4}$$

Corollary 3.1 *Under the hypothesis of Theorem* 3.1,

$$\begin{aligned} & n^{1+\frac{d}{2}}\left[\mathbb{P}(S_n = x) - \left(\frac{d}{2\pi\overline{s}_n^2}\right)^{\frac{d}{2}} \exp\left(-\frac{d\|x\|^2}{2\overline{s}_n^2}\right)\right] \\ & \xrightarrow{n\to\infty} 2\pi\left(\frac{d}{2\pi\overline{s}}\right)^{\frac{d}{2}+1}\left(\frac{1}{8}d - \frac{1}{8}(d+2)\frac{\overline{u}}{\overline{s}}\right).\end{aligned} \tag{3.5}$$

Proof of Theorem 3.1 We start by the following simple fact:

$$\psi_{L_i}(\lambda) = \mathbb{E}\mathrm{e}^{i\langle\lambda, L_i\rangle} = \overline{r}_i + \frac{1-\overline{r}_i}{d}\sum_{j=1}^{d}\cos\lambda_j, \quad \lambda = (\lambda_1, \cdots, \lambda_d) \in [-\pi, \pi]^d.$$

Because $\{L_n\}$ are independent, we have

$$\psi_{S_n}(\lambda) = \mathbb{E}\mathrm{e}^{i\langle\lambda, \sum_{i=1}^{n} L_i\rangle} = \prod_{i=1}^{n}\mathbb{E}\mathrm{e}^{i\langle\lambda, L_i\rangle} = \prod_{i=1}^{n}\psi_{L_i}(\lambda).$$

Then, by [29, Proposition 2.2.2], we see that

$$\mathbb{P}(S_n = x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{e}^{-i\langle\lambda, x\rangle}\psi_{S_n}(\lambda)\mathrm{d}\lambda$$

$$= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{e}^{-i\langle \lambda, x\rangle} \prod_{i=1}^n \psi_{L_i}(\lambda) \mathrm{d}\lambda$$

$$= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \cos\langle \lambda, x\rangle \prod_{i=1}^n \psi_{L_i}(\lambda) \mathrm{d}\lambda. \tag{3.6}$$

It can be easily seen that

$$2\overline{r}_i - 1 \le \psi_{L_i}(\lambda) \le 1 \quad \text{for } \overline{r}_i \in (0,1), \quad \lambda \in [-\pi,\pi]^d$$

and

$$\psi_{L_i}(\lambda) = 1 \Leftrightarrow \lambda = \mathbf{0}.$$

When $\|\lambda\| < 1$, by Taylor's expansion, we have

$$\psi_{L_i}(\lambda) = \overline{r}_i + \frac{1-\overline{r}_i}{d} \sum_{j=1}^d \Big(1 - \frac{1}{2}\lambda_j^2 + \frac{\cos\eta_j}{24}\lambda_j^4\Big),$$

hence

$$\|\psi_{L_i}(\lambda)\| \le 1 - \frac{11(1-\overline{r}_i)}{24}\|\lambda\|^2 \le \mathrm{e}^{-\frac{11}{24}(1-\overline{r}_i)\|\lambda\|^2}. \tag{3.7}$$

While when $\lambda \in [-\pi,\pi]^d$ and $\|\lambda\| \ge 1$,

$$-d \le \sum_{j=1}^d \cos\lambda_j \le d - 1 + \cos(d^{-\frac{1}{2}}) < d.$$

Therefore, we have

$$\begin{aligned} |\psi_{L_i}(\lambda)| &\le \max\{|2\overline{r}_i - 1|, \overline{r}_i + (1-\delta_c)(1-\overline{r}_i)\} \\ &\le \exp\{-(1 - \max\{|2\overline{r}_i - 1|, \overline{r}_i + (1-\delta_c)(1-\overline{r}_i)\})\} \\ &= \mathrm{e}^{-\min\{2r_i, \delta_c(1-r_i)\}}. \end{aligned} \tag{3.8}$$

On the basis of the above discussions, we write

$$\mathbb{P}(S_n = x) = \mathrm{I}_n(x) + \mathrm{J}_n(x)$$

with

$$\mathrm{J}_n(x) := \frac{1}{(2\pi)^d} \int_{\lambda\in[-\pi,\pi]^d, \|\lambda\|\ge 1} \cos\langle \lambda, x\rangle \prod_{i=1}^n \psi_{L_i}(\lambda) \mathrm{d}\lambda,$$

$$\mathrm{I}_n(x) := \frac{1}{(2\pi)^d} \int_{\|\lambda\|<1} \cos\langle \lambda, x\rangle \prod_{i=1}^n \psi_{L_i}(\lambda) \mathrm{d}\lambda.$$

With the help of (3.8) and the definition of M_n, we see that

$$|J_n(x)| \le \prod_{i=1}^n |\psi_{L_i}(\lambda)| \le \mathrm{e}^{-M_n} = o(n^{-1-\frac{d}{2}}). \tag{3.9}$$

We next estimate $I_n(x)$. Let $\alpha = \sqrt{n}\lambda$. Then,

$$\mathrm{I}_n(x) = \frac{1}{(2\pi\sqrt{n})^d} \int_{\|\alpha\|<\sqrt{n}} \cos\frac{\langle x, \alpha\rangle}{\sqrt{n}} \prod_{i=1}^n \psi_{L_i}\Big(\frac{\alpha}{\sqrt{n}}\Big) \mathrm{d}\alpha.$$

To estimate this integral, we put $\delta = \frac{1}{8} - \frac{1}{2}\beta$ and use the following decomposition:

$$(2\pi\sqrt{n})^d \mathrm{I}_n = \int_{n^\delta \le \|\alpha\| < \sqrt{n}} \cos\frac{\langle x,\alpha\rangle}{\sqrt{n}} \prod_{i=1}^n \psi_{L_i}\Big(\frac{\alpha}{\sqrt{n}}\Big)\mathrm{d}\alpha + \int_{\|\alpha\|<n^\delta} \cos\frac{\langle x,\alpha\rangle}{\sqrt{n}} \prod_{i=1}^n \psi_{L_i}\Big(\frac{\alpha}{\sqrt{n}}\Big)\mathrm{d}\alpha$$
$$=: \mathrm{I}_{1n}(x) + \mathrm{I}_{2n}(x).$$

By (3.7),

$$|\mathrm{I}_{1n}(x)| \le 2^d n^{\frac{d}{2}} \exp\Big\{-\frac{11}{24} n^{\frac{1}{4}-\beta}\Big(\frac{\overline{s}_n^2}{n}\Big)\Big\}. \tag{3.10}$$

Next we turn to the estimate of $\mathrm{I}_{2n}(x)$. By Taylor's expansion, as n tends to infinity, for $\|\alpha\| < n^\delta$, $\|x\| < n^\beta$, we have

$$\Big|\cos\frac{\langle x,\alpha\rangle}{\sqrt{n}} - 1 + \frac{\langle x,\alpha\rangle^2}{2n}\Big| \le \frac{|\langle x,\alpha\rangle|^4}{4!n^2} < \frac{1}{24} n^{-2+4(\delta+\beta)}.$$

Then we can write

$$\cos\frac{\langle x,\alpha\rangle}{\sqrt{n}} = 1 - \frac{\langle x,\alpha\rangle^2}{2n} + \frac{\langle x,\alpha\rangle^4}{n^2} M_{1n}(x,\alpha), \tag{3.11}$$

where $\sup\limits_{\|\alpha\|<n^\delta,\|x\|<n^\beta} \|M_{1n}(x,\alpha)\| < \frac{1}{24}$.

On the basis of (3.11), we have

$$\prod_{i=1}^n \psi_{L_i}\Big(\frac{\alpha}{\sqrt{n}}\Big) = \prod_{i=1}^n \Big(\overline{r}_i + \frac{1-\overline{r}_i}{d}\sum_{j=1}^d \cos\frac{\alpha_j}{\sqrt{n}}\Big)$$
$$= \exp\Big\{\sum_{i=1}^n \log\Big(1 - (1-\overline{r}_i)\frac{\|\alpha\|^2}{2nd} + (1-\overline{r}_i)\frac{\sum_{j=1}^d \alpha_j^4}{24n^2 d} + (1-\overline{r}_i)\frac{\|\alpha\|^6}{n^3} M_{2n}(\alpha)\Big)\Big\}$$
$$= \exp\Big\{-\overline{s}_n^2 \frac{\|\alpha\|^2}{2nd} + \overline{s}_n^2 \frac{\sum_{j=1}^d \alpha_j^4}{24n^2 d} - \overline{u}_n \frac{\|\alpha\|^4}{8n^2 d^2} + \overline{s}_n^2 \frac{\|\alpha\|^6}{n^3} M_{3n}(\overline{r},\alpha)\Big\}$$
$$= \mathrm{e}^{-\overline{s}_n^2 \frac{\|\alpha\|^2}{2nd}} \Big\{1 + \overline{s}_n^2 \frac{\sum_{j=1}^d \alpha_j^4}{24n^2 d} - \overline{u}_n \frac{\|\alpha\|^4}{8n^2 d^2} + \overline{s}_n^2 \frac{\|\alpha\|^6}{n^3} M_{3n}(\overline{r},\alpha) + \overline{s}_n^4 \frac{\|\alpha\|^8}{n^4} M_{4n}(\overline{r},\alpha)\Big\},$$

where, just as in (3.11), $M_{in}(\cdot)$ is a continuous function uniformly bounded by a constant, $i = 2,3,4$.

Now, combining the above expansions with (3.11), we obtain

$$\cos\frac{\langle x,\alpha\rangle}{\sqrt{n}} \prod_{i=1}^n \psi_{L_i}\Big(\frac{\alpha}{\sqrt{n}}\Big)$$
$$= \mathrm{e}^{-\overline{s}_n^2 \frac{\|\alpha\|^2}{2nd}} \Big(1 - \frac{\langle x,\alpha\rangle^2}{2n} + \overline{s}_n^2 \frac{\sum_{j=1}^d \alpha_j^4}{24n^2 d} - \overline{u}_n \frac{\|\alpha\|^4}{8n^2 d^2} + \frac{1}{n}\varepsilon_n(x,\overline{r},\alpha)\Big), \tag{3.12}$$

where the term $\varepsilon_n(x,\overline{r},\alpha)$ satisfies

$$\sup_{\|\alpha\|<n^\delta,\|x\|<n^\beta} |\varepsilon_n(x,\overline{r},\alpha)| \le C_d \max\Big\{\frac{1}{n^{1-4(\delta+\beta)}}, \frac{1}{n^{1-8\delta}}\Big\}$$

$$= C_d n^{-\min(\frac{1}{2}-2\beta,4\beta)} = C_d n^{-\frac{1}{2}+2\beta} \tag{3.13}$$

with C_d an absolute constant only depending on d. Thus

$$\begin{aligned}\sup_{\|x\|\le n^\beta}\Big|\int_{\|\alpha\|<n^\delta} \mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\varepsilon_n(x,\overline{r},\alpha)\mathrm{d}\alpha\Big| &\le C_d n^{-\frac{1}{2}+2\beta}\int_{\mathbb{R}^d}\mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\mathrm{d}\alpha\\ &\le C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{-\frac{d}{2}}n^{-\frac{1}{2}+2\beta}. \end{aligned}\tag{3.14}$$

Observe that

$$\begin{aligned}\int_{\|\alpha\|\ge n^\delta}\mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\mathrm{d}\alpha &= 2\pi^{\frac{d}{2}}/\Gamma\Big(\frac{d}{2}\Big)\int_{n^\delta}^\infty \mathrm{e}^{-\frac{\overline{s}_n^2}{2nd}r^2}r^{d-1}\mathrm{d}r\\ &\le C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{-d}n^{d\delta}\mathrm{e}^{-\frac{\overline{s}_n^2}{2nd}n^{2\delta}}\\ &\le C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{-d}n^{\frac{1}{2}+\frac{d}{8}}\exp\Big\{-\frac{1}{2d}\Big(\frac{\overline{s}_n^2}{n}\Big)n^{\frac{1}{4}-\beta}\Big\}. \end{aligned}\tag{3.15}$$

By elementary calculus, we evaluate

$$\begin{aligned}&\int_{\|\alpha\|<n^\delta}\mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\mathrm{d}\alpha = \Big(\frac{2\pi nd}{\overline{s}_n^2}\Big)^{\frac{d}{2}} + \gamma_n^{\mathrm{e}}(\overline{s}_n^2),\\ &\int_{\|\alpha\|<n^\delta}\mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\frac{1}{d}\sum_{j=1}^d\alpha_j^4\mathrm{d}\alpha = 3(2\pi)^{\frac{d}{2}}\Big(\frac{d}{(\frac{\overline{s}_n^2}{n})}\Big)^{\frac{d}{2}+2} + \gamma_n^{\mathrm{e}}(\overline{s}_n^2),\\ &\int_{\|\alpha\|<n^\delta}\mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\|\alpha\|^4\mathrm{d}\alpha = (2\pi)^{\frac{d}{2}}d(d+2)\Big(\frac{d}{(\frac{\overline{s}_n^2}{n})}\Big)^{\frac{d}{2}+2} + \gamma_n^{\mathrm{e}}(\overline{s}_n^2),\\ &\int_{\|\alpha\|<n^\delta}\mathrm{e}^{-\frac{\overline{s}_n^2\|\alpha\|^2}{2nd}}\langle\alpha,x\rangle^2\mathrm{d}\alpha = (2\pi)^{\frac{d}{2}}\Big(\frac{d}{(\frac{\overline{s}_n^2}{n})}\Big)^{\frac{d}{2}+1}\|x\|^2 + \sigma_n^{\mathrm{e}}(\overline{s}_n^2,x). \end{aligned}$$

In the above estimates, similar to (3.15), all the infinitesimals $\gamma_n^{\mathrm{e}}(\overline{s}_n^2)$ and $\sup_{\|x\|\le n^\beta}\sigma_n^{\mathrm{e}}(\overline{s}_n^2,x)$ are bounded by

$$C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{-d}n^{\frac{1}{2}+\frac{d}{8}}\exp\Big\{-\frac{1}{2d}\Big(\frac{\overline{s}_n^2}{n}\Big)n^{\frac{1}{4}-\beta}\Big\}.$$

Taking these evaluations and the expansion (3.12) into account, we obtain

$$\mathrm{I}_{2n} = \Big(\frac{2\pi d}{\frac{\overline{s}_n^2}{n}}\Big)^{\frac{d}{2}}\Big[1+\frac{d}{\overline{s}_n^2}\Big(-\frac{1}{2}\|x\|^2+\frac{1}{8}d\Big(1-\frac{\overline{u}_n}{\overline{s}_n^2}\Big)-\frac{1}{4}\frac{\overline{u}_n}{\overline{s}_n^2}\Big)\Big]+\frac{1}{n}\sigma_n(\overline{s}_n^2,x),$$

where $\sigma_n(\overline{s}_n^2,x)$ satisfies

$$\begin{aligned}\sup_{\|x\|\le n^\beta}\big\|\sigma_n(\overline{s}_n^2,x)\big\| \le& C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{\frac{-d}{2}}n^{\frac{-1}{2}+2\beta}\\ &+C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{-d}n^{\frac{1}{2}+\frac{d}{8}}\exp\Big\{-\frac{1}{2d}\Big(\frac{\overline{s}_n^2}{n}\Big)n^{\frac{1}{4}-\beta}\Big\}. \end{aligned}\tag{3.16}$$

Combining this with (3.9)–(3.10), we obtain the desired

$$\mathbb{P}(S_n=x) = \Big(\frac{d}{2\pi\overline{s}_n^2}\Big)^{\frac{d}{2}}\Big[1+\frac{d}{\overline{s}_n^2}\Big(-\frac{1}{2}\|x\|^2+\frac{1}{8}d-\frac{1}{8}(d+2)\frac{\overline{u}_n}{\overline{s}_n^2}\Big)\Big]+\frac{1}{n^{1+\frac{d}{2}}}R_n(\overline{r},x), \tag{3.17}$$

where $R_n(\overline{r},x)$ is asymptotically uniform bounded satisfying (3.4), i.e., for $\|x\|\le n^{\beta}$ as $n\to\infty$,

$$\sup_{\|x\|\le n^{\beta}}|R(\overline{r},x)|\le C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{\frac{-d}{2}}n^{-\frac{1}{2}+2\beta}+2^dn^{\frac{d}{2}}\exp\Big\{-\frac{11}{24}n^{\frac{1}{4}-\beta}\frac{\overline{s}_n^2}{n}\Big\}$$
$$+n^{(1+\frac{d}{2})}\mathrm{e}^{-M_n}+C_d\Big(\frac{\overline{s}_n^2}{n}\Big)^{-d}n^{\frac{1}{2}+\frac{d}{8}}\exp\Big\{-\frac{1}{2d}\frac{\overline{s}_n^2}{n}n^{\frac{1}{4}-\beta}\Big\}.$$

The lemma has been proved.

4 Proof of Theorem 1.1

In the proof of the main results, a key decomposition plays an important role. It goes back at least as far as [28] and is widely used in later literature (see e.g. [9, 16, 18, 34]). To introduce the decomposition, we need some notations.

Let $\mathbb{T}(u)$ be the shifted tree of $\mathbb{T}$ at u with defining elements $\{N_{uv}\}$ satisfying (1) $\varnothing\in\mathbb{T}(u)$, (2) $vi\in\mathbb{T}(u)\Rightarrow v\in\mathbb{T}(u)$ and (3) if $v\in\mathbb{T}(u)$, then $vi\in\mathbb{T}(u)$ if and only if $1\le i\le N_{uv}$. Set $\mathbb{T}_n(u)=\{v\in\mathbb{T}(u):|v|=n\}$ and denote by $|T_n(u)|$ the cardinality of $\mathbb{T}_n(u)$ (i.e., the number of descendants of nth generation of u).

For $u\in(\mathbb{N}^*)^k(k\ge 0)$ and $n\ge 1$, let $Z_n(u,x)$ denote the number of descendants of nth generation of u located at $x+S_u\in\mathbb{Z}^d$. More precisely,

$$Z_n(u,x)=\sum_{v\in\mathbb{T}_n(u)}\delta_x(S_{uv}-S_u),$$

where

$$\delta_x(y)=\begin{cases}1, & x=y,\\ 0, & x\ne y.\end{cases}$$

Let β be a real number satisfying $\frac{d+3}{\lambda}<\beta<\frac{1}{4}$ and set $k_n=\lfloor n^{\beta}\rfloor$, the largest integer not bigger than n^{β}. On the basis of the additivity property of the branching process, we have the following decomposition:

$$Z_n(x)=\sum_{u\in\mathbb{T}_{k_n}}Z_{n-k_n}(u,x-S_u). \tag{4.1}$$

By definition, for $u\in\mathbb{T}_{k_n}$, we have

$$Z_{n-k_n}(u,x-S_u)=\sum_{v_1\cdots v_{n-k_n}\in\mathbb{T}_{n-k_n}(u)}\delta_x(S_{uv_1\cdots v_{n-k_n}}).$$

In addition, we also need the following $\sigma-$fields:

$$\mathcal{I}_0=\{\emptyset,\Omega\},\quad \mathcal{I}_n=\sigma(\xi_k,N_u,L_{ui}:k<n,i\ge 1,|u|<n)\quad\text{for } n\ge 1.$$

For conditional probabilities and expectations, we write

$$\mathbb{P}_{\xi,n}(\cdot)=\mathbb{P}_{\xi}(\cdot|\mathcal{D}_n),\quad \mathbb{E}_{\xi,n}(\cdot)=\mathbb{E}_{\xi}(\cdot|\mathcal{D}_n),\quad \mathbb{P}_n(\cdot)=\mathbb{P}(\cdot|\mathcal{I}_n),\quad \mathbb{E}_n(\cdot)=\mathbb{E}(\cdot|\mathcal{I}_n).$$

Also we write $\widehat{S}_n=S_{1_n}$ for $n\ge 0$, with $1_n=\underbrace{1\cdots 1}_{n}$ for $n>1$ and $1_0=\varnothing$. Then $\widehat{S}_n$ is a random walk with the random environment ξ in time .

As the environment sequence is independent, identically distributed, the distribution of $Z_{n-k_n}(u,x)$ under $\mathbb{P}_\xi$ coincides with that of $Z_{n-k_n}(x)$ under $\mathbb{P}_{T^{k_n}\xi}$. Then for $u\in\mathbb{T}_{k_n}$, we have

$$\mathbb{E}_\xi\Big(\frac{Z_{n-k_n}(u,x)}{m_{k_n}\cdots m_{n-1}}\Big)=\mathbb{P}_{T^{k_n}\xi}(\widehat{S}_{n-k_n}=x). \tag{4.2}$$

Therefore, we provide the following key decomposition:

$$\begin{aligned}\frac{1}{\Pi_n}Z_n(x)&=\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\Big[\frac{Z_{n-k_n}(u,x-S_u)}{m_{k_n}\cdots m_{n-1}}-\mathbb{P}_{T^{k_n}\xi}(\widehat{S}_{n-k_n}=x-y)|_{y=S_u}\Big]\\&\quad+\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{P}_{T^{k_n}\xi}(\widehat{S}_{n-k_n}=x-y)|_{y=S_u}\\&=:A_n+B_n.\end{aligned} \tag{4.3}$$

On the basis of (4.3), we divide the proof of Theorem 1.1 into two lemmas.

Lemma 4.1 *Assume the conditions of Theorem* 1.1. *Then*

$$n^{1+\frac{d}{2}}A_n\xrightarrow[a.s.]{n\to\infty}0. \tag{4.4}$$

Lemma 4.2 *Assume the conditions of Theorem* 1.1. *Then*

$$\begin{aligned}&n\Big[\Big(\frac{2\pi s_n^2}{d}\Big)^{\frac{d}{2}}B_n-W\exp\Big(-\frac{d\|x\|^2}{2s_n^2}\Big)\Big]\\&\xrightarrow[a.s.]{n\to\infty}\frac{d}{1-\mathbb{E}r_0}\Big[-\frac{1}{2}V_2+\langle x,V_1\rangle+\Big(\frac{1}{8}d-\frac{1}{8}(d+2)\frac{\mathbb{E}(1-r_0)^2}{1-\mathbb{E}r_0}\Big)W\Big].\end{aligned} \tag{4.5}$$

Proof of Lemma 4.1 Firstly we give some notations. For $u\in\mathbb{T}_{k_n}$, define

$$\begin{aligned}X_{n,u}&=\frac{Z_{n-k_n}(u,x-S_u)}{m_{k_n}\cdots m_{n-1}}-\mathbb{P}_{T^{k_n}\xi}(\widehat{S}_{n-k_n}=x-y)|_{y=S_u},\\\overline{X}_{n,u}&=X_{n,u}\mathbf{1}_{\{|X_{n,u}|\le\Pi_{k_n}\}},\\\overline{A}_n&=\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\overline{X}_{n,u}.\end{aligned}$$

Then we have the following fact:

$$|X_{n,u}|\le W_{n-k_n}(u)+1\quad\text{with}\quad W_{n-k_n}(u)=:\frac{\mathbb{T}_{n-k_n}(u)}{m_{k_n}\cdots m_{n-1}}.$$

We remind that $\{W_{n-k_n}(u):u\in\mathbb{T}_{k_n}\}$ are mutually independent and identically distributed as W_{n-k_n} under the conditional probability $\mathbb{P}_{\xi,k_n}$. We can obtain the required result once we prove $\forall\varepsilon>0$,

$$\sum_{n=1}^\infty\mathbb{P}_{k_n}(|n^{1+\frac{d}{2}}A_n|>2\varepsilon)<\infty. \tag{4.6}$$

Notice

$$\begin{aligned}\mathbb{P}_{k_n}\Big(|A_n|>\frac{2\varepsilon}{n^{1+\frac{d}{2}}}\Big)&\le\mathbb{P}_{k_n}(A_n\ne\overline{A}_n)+\mathbb{P}_{k_n}\Big(|\overline{A}_n-\mathbb{E}_{\xi,k_n}\overline{A}_n|>\frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big)\\&\quad+\mathbb{P}_{k_n}\Big(|\mathbb{E}_{\xi,k_n}\overline{A}_n|>\frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big).\end{aligned} \tag{4.7}$$

Then we divide the proof into 3 steps.

Step 1 Prove

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(A_n \neq \overline{A}_n) < \infty. \tag{4.8}$$

To this end, we need the following result with $W^* = \sup_n W_n$.

Lemma 4.3 (see [30, Theorem 1.2]) *Assume* (1.3) *for some* $\lambda > 0$ *and* $\mathbb{E} m_0^{-\iota} < \infty$ *for some* $\iota > 0$. *Then*

$$\mathbb{E}(W^* + 1)(\ln(W * + 1))^{\lambda} < \infty. \tag{4.9}$$

By virtue of this result, we have

$$\begin{aligned}
\mathbb{P}_{k_n}(A_n \neq \overline{A}_n) &\le \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(X_{n,u} \neq \overline{X}_{n,u}) = \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(|X_{n,u}| \ge \Pi_{k_n}) \\
&\le \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(W_{n-k_n}(u) + 1 \ge \Pi_{k_n}) \\
&= W_{k_n}[\pi_n \mathbb{P}(W_{n-k_n} + 1 \ge \pi_n)]_{\pi_n = \Pi_{k_n}} \\
&\le W_{k_n}[\mathbb{E}((W_{n-k_n} + 1)\mathbf{1}_{\{W_{n-k_n}+1 \ge \pi_n\}})]_{\pi_n = \Pi_{k_n}} \\
&\le W_{k_n}[\mathbb{E}((W^* + 1)\mathbf{1}_{\{W^*+1 \ge \pi_n\}})]_{\pi_n = \Pi_{k_n}} \\
&\le W^*(\ln \Pi_{k_n})^{-\lambda} \mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} \\
&\le K_{\xi} W^* n^{-\lambda\beta} \mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda},
\end{aligned}$$

where K_{ξ} is constant depending on ξ. And the last inequality holds because of the conditions $k_n \sim n^{\beta}$ and

$$\frac{1}{n} \ln \Pi_n \to \mathbb{E} \ln m_0 > 0 \quad \text{a.s.} \tag{4.10}$$

By the choice of β and using Lemma 4.3, we achieve the first inequality (4.8).

Step 2 We next prove the following inequality for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}\Big(|\overline{A}_n - \mathbb{E}_{\xi,k_n} \overline{A}_n| > \frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big) < \infty. \tag{4.11}$$

Take a constant $b \in (1, \mathrm{e}^{\mathbb{E} \ln m_0})$. For all $u \in \mathbb{T}_{k_n}$ and $n \ge 1$, we can see

$$\begin{aligned}
\mathbb{E}_{k_n} \overline{X}_{n,u}^2 &= \int_0^{\infty} 2x \mathbb{P}_{k_n}(|\overline{X}_{n,u}| > x) \mathrm{d}x = 2 \int_0^{\infty} x \mathbb{P}_{k_n}(|X|_{n,u} \mathbf{1}_{\{|X_{n,u}| < \Pi_{k_n}\}}) \mathrm{d}x \\
&\le 2 \int_0^{\Pi_{k_n}} x \mathbb{P}_{k_n}(|W_{n-k_n}(u) + 1| > x) \mathrm{d}x = 2 \int_0^{\Pi_{k_n}} x \mathbb{P}_{k_n}(|W_{n-k_n} + 1| > x) \mathrm{d}x \\
&\le 2 \int_0^{\Pi_{k_n}} x \mathbb{P}(|W^* + 1| > x) \mathrm{d}x \\
&\le 2 \int_{\mathrm{e}}^{\Pi_{k_n}} (\ln x)^{-\lambda} \mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} \mathrm{d}x + 9 \\
&\le 2\mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} \Big(\int_{\mathrm{e}}^{b^{k_n}} (\ln x)^{-\lambda} \mathrm{d}x + \int_{b^{k_n}}^{\Pi_{k_n}} (\ln x)^{-\lambda} \Big) + 9 \\
&\le 2\mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} (b^{k_n} + (\Pi_{k_n} - b^{k_n})(k_n \ln b)^{-\lambda}) + 9.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}_{k_n}\Big(|\overline{A}_n-\mathbb{E}_{\xi,k_n}\overline{A}_n|>\frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big)\\
&=\sum_{n=1}^{\infty}\mathbb{E}_{k_n}\mathbb{P}_{\xi,k_n}\Big(|\overline{A}_n-\mathbb{E}_{\xi,k_n}\overline{A}_n|>\frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big)\\
&\le \varepsilon^{-2}\sum_{n=1}^{\infty}n^{2+d}\mathbb{E}_{k_n}\Big(\Pi_{k_n}^{-2}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{\xi,k_n}\overline{X}_{n,u}^2\Big)=\varepsilon^{-2}\sum_{n=1}^{\infty}n^{2+d}\Big(\Pi_{k_n}^{-2}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{k_n}\overline{X}_{n,u}^2\Big)\\
&\le \varepsilon^{-2}\sum_{n=1}^{\infty}\frac{n^{2+d}W_{k_n}}{\Pi_{k_n}}[(W^*+1)(\ln(W^*+1)^{\lambda})(b^{k_n}+(\Pi_{k_n}-b^{k_n})(k_n\ln b)^{-\lambda})+9]\\
&\le 2\varepsilon^{-2}W^*\mathbb{E}(W^*+1)(\ln(W^*+1)^{\lambda})\Big(\sum_{n=1}^{\infty}\frac{n^{2+d}}{\Pi_{k_n}}b^{k_n}+\sum_{n=1}^{\infty}n^{2+d}(k_n\ln b)^{-\lambda}\Big)\\
&\quad+9\varepsilon^{-2}W^*\sum_{n=1}^{\infty}\frac{n^{2+d}}{\Pi_{k_n}}.
\end{aligned}$$

Using the condition (4.10) and $\lambda\beta>3+d$, the last three series in the above inequality converge and the second inequality (4.11) follows.

Step 3 As $\mathbb{E}_{\xi,k_n}X_{n,u}=0$, we have

$$\begin{aligned}
\mathbb{P}_{k_n}\Big(|\mathbb{E}_{\xi,k_n}\overline{A}_n|>\frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big)&\le\frac{n^{1+\frac{d}{2}}}{\varepsilon}\mathbb{E}_{k_n}|\mathbb{E}_{\xi,k_n}\overline{A}_n|=\frac{n^{1+\frac{d}{2}}}{\varepsilon}\mathbb{E}_{k_n}\Big|\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{\xi,k_n}\overline{X}_{n,u}\Big|\\
&=\frac{n^{1+\frac{d}{2}}}{\varepsilon}\mathbb{E}_{k_n}\Big|\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}(-\mathbb{E}_{\xi,k_n}X_{n,u}\mathbf{1}_{\{|X_{n,u}|\ge\Pi_{k_n}\}})\Big|\\
&\le\frac{n^{1+\frac{d}{2}}}{\varepsilon}\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{k_n}(W_{n-k_n}(u)+1)\mathbf{1}_{\{W_{n-k_n}(u)+1\ge\Pi_{k_n}\}}\\
&=\frac{n^{1+\frac{d}{2}}}{\varepsilon}W_{k_n}[\mathbb{E}(W^*+1)\mathbf{1}_{\{W_{n-k_n}+1\ge\pi_n\}}]_{\pi_n=\Pi_{k_n}}\\
&\le\frac{W^*}{\varepsilon}n^{1+\frac{d}{2}}[\mathbb{E}(W^*+1)\mathbf{1}_{\{W^*+1\ge\pi_n\}}]_{\pi_n=\Pi_{k_n}}\\
&\le\frac{W^*}{\varepsilon}\frac{n^{1+\frac{d}{2}}}{(\ln\Pi_{k_n})^{\lambda}}\mathbb{E}(W^*+1)\ln^{\lambda}(W^*+1)\\
&\le\frac{W^*}{\varepsilon}K_{\xi}n^{1+\frac{d}{2}-\lambda\beta}\mathbb{E}(W^*+1)\ln^{\lambda}(W^*+1).
\end{aligned}$$

Then by the condition (4.10) and $\lambda\beta>3+d$, we obtain

$$\sum_{n=1}^{\infty}\mathbb{P}_{k_n}\Big(|\mathbb{E}_{\xi,k_n}\overline{A}_n|>\frac{\varepsilon}{n^{1+\frac{d}{2}}}\Big)<\infty.$$

Through Steps 1–3, We prove (4.6). Hence Lemma 4.1 has been proved.

Proof of Lemma 4.2 Just to keep the notations consistent with Theorem 3.1, we introduce

$$\widehat{s}_{n-k_n}^2:=\sum_{i=0}^{n-k_n-1}(1-r_{k_n+i}),\quad \widehat{u}_{n-k_n}:=\sum_{i=0}^{n-k_n-1}(1-r_{k_n+i})^2,$$

$$\widehat{M}_{n-k_n} := \sum_{i=0}^{n-k_n-1} \min\{2r_{k_n+i}, \delta_c(1-r_{k_n+i})\}.$$

By the law of large numbers, we get

$$\begin{aligned}
&\frac{\widehat{s}^2_{n-k_n}}{n-k_n} = \frac{s_n^2 - s_{k_n}^2}{n-k_n} \to 1 - \mathbb{E}r_0,\\
&\frac{\widehat{u}_{n-k_n}}{n-k_n} = \frac{u_n - u_{k_n}}{n-k_n} \to \mathbb{E}(1-r_0)^2,\\
&\frac{\widehat{M}_{n-k_n}}{n-k_n} \to \mathbb{E}\min\{2r_0, \delta_c(1-r_0)\} > 0.
\end{aligned} \tag{4.12}$$

Using Theorem 3.1, we have

$$\begin{aligned}
&\mathbb{P}_{T^{k_n}\xi}(\widehat{S}_{n-k_n} = x-y)|_{y=S_u}\\
=&\Big(\frac{d}{2\pi \widehat{s}^2_{n-k_n}}\Big)^{\frac{d}{2}}\Big[1 + \frac{d}{\widehat{s}^2_{n-k_n}}\Big(-\frac{1}{2}\|x-S_u\|^2 + \frac{1}{8}d - \frac{1}{8}(d+2)\frac{\widehat{u}_{n-k_n}}{\widehat{s}^2_{n-k_n}}\Big)\Big]\\
&+\frac{1}{(n-k_n)^{1+\frac{d}{2}}}R_{n-k_n}(\xi, x-S_u),
\end{aligned} \tag{4.13}$$

where the remainder terms $R_{n-k_n}\big(\xi, x-S_u\big)(u \in \mathbb{T}_{k_n})$ are asymptotically uniform infinitesimals.

Observe $\|S_u\| \leq k_n$ for $u \in \mathbb{T}_{k_n}$ and $n \gg k_n$. Then

$$\begin{aligned}
&\|x-S_u\|^2 = \|x\| - 2\langle x, S_u\rangle + \|S_u\|^2,\\
&(s_n^2 - s_{k_n}^2)^{\frac{-d}{2}} = (s_n^2)^{\frac{-d}{2}}\Big(1 + \frac{ds^2_{k_n}}{2s_n^2}\Big) + (s_n^2)^{-\frac{d}{2}}o\Big(\frac{s^2_{k_n}}{s_n^2}\Big),\\
&(s_n^2 - s_{k_n}^2)^{-1} = (s_n^2)^{-1}\Big(1 + \frac{s^2_{k_n}}{s_n^2}\Big) + (s_n^2)^{-1}o\Big(\frac{s^2_{k_n}}{s_n^2}\Big),\\
&(n-k_n)^{\frac{-d}{2}} = n^{\frac{-d}{2}}\Big(1 + \frac{dk_n}{2n}\Big) + o\Big(\frac{1}{n^{1+\frac{d}{2}}}\Big).
\end{aligned}$$

Plugging these into (4.3), we obtain

$$\begin{aligned}
&\mathbb{P}_{T^{k_n}\xi}(\widehat{S}_{n-k_n} = x-y)|_{y=S_u}\\
=&\Big(\frac{d}{2\pi s_n^2}\Big)^{\frac{d}{2}}\Big[1 + \frac{d}{s_n^2}\langle x, S_u\rangle + \frac{d}{s_n^2}\Big(-\frac{1}{2}(\|S_u\|^2 - s^2_{k_n})\Big)\\
&+\frac{d}{s_n^2}\Big(-\frac{1}{2}\|x\|^2 + \frac{1}{8}d - \frac{1}{8}(d+2)\frac{\widehat{u}_{n-k_n}}{\widehat{s}^2_{n-k_n}}\Big)\Big]\\
&+\frac{1}{n^{\frac{d}{2}+1}}\varepsilon_{u,n},
\end{aligned} \tag{4.14}$$

where $\varepsilon_{u,n}(u \in \mathbb{T}_{k_n})$ denote a family of infinitesimals dominated by an infinitesimal τ_n, i.e.,

$$\sup\{|\varepsilon_{u,n}| : \ u \in \mathbb{T}_{k_n}\} \leq \tau_n \to 0.$$

Hence, combining (4.14) with (4.3), we conclude

$$\Big(\frac{2\pi s_n^2}{d}\Big)^{\frac{d}{2}}B_n = W_{k_n} + \frac{d}{s_n^2}\Big[-\frac{1}{2}N_{2,k_n} + \langle x, N_{1,k_n}\rangle$$

$$+\Big(-\frac{1}{2}\|x\|^2+\frac{1}{8}d-\frac{1}{8}(d+2)\frac{\widehat{u}_{n-k_n}}{\widehat{s}^2_{n-k_n}}\Big)W_{k_n}\Big]+\frac{1}{n}\Big(\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\varepsilon_{u,n}\Big).$$

Then we obtain

$$\Big(\frac{2\pi s_n^2}{d}\Big)^{\frac{d}{2}}B_n-W_{k_n}\exp\Big(-\frac{d\|x\|^2}{2s_n^2}\Big)$$
$$=\frac{d}{s_n^2}\Big[-\frac{1}{2}N_{2,k_n}+\langle x,N_{1,k_n}\rangle+\Big(\frac{1}{8}d-\frac{1}{8}(d+2)\frac{\widehat{u}_{n-k_n}}{\widehat{s}^2_{n-k_n}}\Big)W_{k_n}\Big]+o\Big(\frac{1}{n}\Big)W_{k_n}. \tag{4.15}$$

Under the condition (1.3), applying the result given in [24, Theorem 1.2], we obtain

$$W-W_n=o(n^{-\lambda}),\quad \text{a.s.}$$

By the choice of β and k_n, we see that

$$W-W_{k_n}=o(n^{-\lambda\beta})=o(n^{-1}). \tag{4.16}$$

So we can deduce the desired (4.5) from the results (4.12), (4.15)–(4.16) and Proposition 2.1.

Proof of Theorem 1.1 Theorem 1.1 follows directly from Lemmas 4.1–4.2.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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