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Multi-dimensional Backward Stochastic Differential Equations of Diagonally Quadratic Generators with a Special Structure*

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Abstract The present paper is devoted to the well-posedness of a type of multi-dimensional backward stochastic differential equations (BSDE for short) with a diagonally quadratic generator. The author gives a new priori estimate, and prove that the BSDE admits a unique solution on a given interval when the generator has a sufficiently small growth of the off-diagonal elements (i.e., for each i, the i-th component of the generator has a small growth of the j-th row z^j of the variable z for each $j \neq i$). Finally, a solvability result is given when the diagonally quadratic generator is triangular.

Keywords Multi-dimensional BSDE, Diagonally quadratic generator, BMO martingale
2000 MR Subject Classification 60H10

1 Introduction

Bismut [2] first introduced Backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T], \tag{1.1}$$

where $(W_t)_{t\in[0,T]}$ is a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented natural filtration generated by the standard Brownian motion W. The terminal value ξ is an $\mathcal{F}_T$-measurable n-dimensional random vector, the generator function $f(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ is $(\mathcal{F}_t)$-progressively measurable for each pair (y, z), and the solution $(Y_t, Z_t)_{t\in[0,T]}$ is a pair of $(\mathcal{F}_t)$-progressively measurable processes with values in $\mathbb{R}^n \times \mathbb{R}^{n\times d}$ which almost surely verifies BSDE (1.1). In 1990, Pardoux and Peng [18] established the existence and uniqueness result for B-SDE with an L^2-terminal value and a generator satisfying a uniformly Lipschitz continuous condition. When the generators have a quadratic growth in the state variable z, the situation is more complicated. In the one-dimensional case, Kobylanski [16] established the first existence and uniqueness result for quadratic BSDE with bounded terminal values, Tevzadze [19] gave a fixed-point argument, Briand and Elie [3] provided a constructive approach to quadratic BSDE

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with and without delay. Briand and Hu [4–5], Delbaen et al. [7–8], Barrieu and El Karoui [1] and Fan et al. [9] considered the unbounded terminal value case.

For multi-dimensional quadratic BSDE, when the terminal value is small enough in the supremum norm, Tevzadze [19] proved a general existence and uniqueness result for multi-dimensional quadratic BSDE. Frei and Dos Reis [12] provided a counterexample which shows that multi-dimensional quadratic BSDE with a bounded terminal value may fail to have a global solution. Frei [11] introduced the notion of split solution and studied the existence of solution by considering a special kind of terminal value. Cheridito and Nam [6] and Xing and Žitković [20] obtained the solvability for multi-dimensional quadratic BSDE in the Markovian setting. Jamneshan et al. [14] provided solutions for multi-dimensional quadratic BSDE with separated generators. Cheridito and Nam [6], Hu and Tang [13] and Luo [17] obtained local solvability of systems of BSDE with sub-quadratic, diagonally quadratic and triangularly quadratic generators respectively, which under additional assumptions on the generator can be extended to global solutions. When the terminal value is unbounded, Jamneshan et al. [14] provided solutions when the terminal value is small in the BMO-sense, and Fan et al. [10] obtained global solutions when the generator is convex or concave.

As a continuation of Hu and Tang [13] and Fan et al. [10], we are devoted to the solvability of multi-dimensional diagonally quadratic BSDE when the generator has a small growth of the off-diagonal elements. The local solution is constructed directly by [13, Theorem 2.2]. Together with the new priori estimate we build and a special kind of "intermediate value" property of the $\mathcal{S}^\infty$-norm of the local solution, we are able to stitch local solutions to get the global solution. In contrast to [13, Theorem 2.3] and [10, Theorem 2.4], we allow the generator to have a small growth of the off-diagonal elements. In contrast to [10, Theorem 2.5] and [17], we do not assume that the generator is strictly quadratic. Finally, assuming that for each $i = 1, \cdots, n$, the i-th component f^i of the generator f is diagonally quadratic, depends only on the first i components of the state variable y and the first i rows of the state variable z, we prove existence and uniqueness of the global solution to the multi-dimensional diagonally quadratic BSDE with a bounded terminal value.

The rest of the paper is organized as follows. In Section 2, we prepare some notations and state the main results of this paper. In Section 3, we give an estimate and prove our main results. In Section 4, we prove a global solvability result for triangular and diagonally quadratic BSDE.

2 Preliminaries and Statement of Main Results

2.1 Notations

Let $W = (W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t\geq 0}$ be the augmented natural filtration generated by W. Throughout this paper, we fix a $T \in (0, \infty)$. We endow $\Omega \times [0, T]$ with the predictable σ-algebra $\mathcal{P}$ and $\mathbb{R}^n$ with its Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$. All the processes are assumed to be $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable, and all equalities and inequalities between random variables and processes are understood in the sense of $\mathbb{P}$-a.s. and $\mathrm{d}\mathbb{P} \times \mathrm{d}t$-a.e., respectively. The Euclidean norm is always denoted by $|\cdot|$, and $\|\cdot\|_\infty$ denotes the L^∞-norm for one-dimensional or multi-dimensional random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We define the following four Banach spaces of stochastic processes. By $\mathcal{S}^p(\mathbb{R}^n)$ for $p \geq 1$, we denote the set of all $\mathbb{R}^n$-valued continuous adapted processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \Big(\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^p]\Big)^{\frac{1}{p}} < +\infty.$$

By $\mathcal{S}^\infty(\mathbb{R}^n)$, we denote the set of all $\mathbb{R}^n$-valued continuous adapted processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^\infty} := \Big\| \sup_{t\in[0,T]} |Y_t| \Big\|_\infty < +\infty.$$

By $\mathcal{H}^p(\mathbb{R}^{n\times d})$ for $p \geq 1$, we denote the set of all $\mathbb{R}^{n\times d}$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-progressively measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathcal{H}^p} := \Big\{\mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big\}^{\frac{1}{p}} < +\infty.$$

By $\mathrm{BMO}(\mathbb{R}^{n\times d})$, we denote the set of all $Z \in \mathcal{H}^2(\mathbb{R}^{n\times d})$ such that

$$\|Z\|_{\mathrm{BMO}} := \sup_\tau \Big\|\mathbb{E}_\tau\Big[\int_\tau^T |Z_s|^2 \mathrm{d}s\Big]\Big\|_\infty^{\frac{1}{2}} < +\infty.$$

Here and hereafter the supremum is taken over all $(\mathcal{F}_t)$-stopping times τ with values in $[0, T]$, and $\mathbb{E}_\tau$ denotes the conditional expectation with respect to $\mathcal{F}_\tau$.

The spaces $\mathcal{S}^p_{[a,b]}(\mathbb{R}^n)$, $\mathcal{S}^\infty_{[a,b]}(\mathbb{R}^n)$, $\mathcal{H}^p_{[a,b]}(\mathbb{R}^{n\times d})$, and $\mathrm{BMO}_{[a,b]}(\mathbb{R}^{n\times d})$ are identically defined for stochastic processes over the time interval $[a, b]$. We note that for $Z \in \mathrm{BMO}(\mathbb{R}^{n\times d})$, the process $\int_0^t Z_s \mathrm{d}B_s$, $t \in [0, T]$, is an n-dimensional BMO martingale. For the theory of BMO martingales, we refer the reader to Kazamaki [15].

For $i = 1, \cdots, n$, denote by z^i, y^i and f^i the i-th row of matrix $z \in \mathbb{R}^{n\times d}$, the i-th component of the vector $y \in \mathbb{R}^n$ and the generator f, respectively.

2.2 Statement of the main results

The main result of this paper concerns global solutions for bounded terminal value case. Consider the multi-dimensional BSDE (1.1) of the following structured quadratic generator:

$$f^i(t, y, z) = g^i(t, z^i) + h^i(t, y, z), \quad i = 1, \cdots, n. \tag{2.1}$$

We need the following assumptions.

(H1) There exist two positive real constants γ and C and a real constant $\delta \in [0, 1)$, such that for $i = 1, \cdots, n$, $g^i : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $h^i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ satisfy the following inequalities:

$|g^i(t, z)| \leq \frac{\gamma}{2}|z|^2, \quad \forall z \in \mathbb{R}^d;$

$|g^i(t, z_1) - g^i(t, z_2)| \leq C(1 + |z_1| + |z_2|)|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^d;$

$|h^i(t, 0, 0)| \leq C;$

$|h^i(t, y_1, z_1) - h^i(t, y_2, z_2)| \leq C|y_1 - y_2| + C(1 + |z_1|^\delta + |z_2|^\delta)|z_1 - z_2|, \quad \forall y_1, y_2 \in \mathbb{R}^n,$
$z_1, z_2 \in \mathbb{R}^{n\times d}.$

(H2) There exists a three-dimensional non-negative deterministic vector function $(\alpha_t, \beta_t, \eta_t)_{t\in[0,T]}$ and a positive constant $r \in (0, 1+\delta]$ such that for $i = 1, \cdots, n$, h^i satisfies

$$\mathrm{sgn}(y^i)h^i(t, y, z) \le \alpha_t + \beta_t|y| + \eta_t|z|^r, \quad \forall y \in \mathbb{R}^n,\ z \in \mathbb{R}^{n\times d}.$$

(H3) There exist four non-negative constants C_0, C_1, C_2 and C_3 such that

$$\|\xi\|_\infty \le C_0, \quad \int_0^T \alpha_t \mathrm{d}t \le C_1, \quad \int_0^T \beta_t \mathrm{d}t \le C_2, \quad \int_0^T (\eta_t + \eta_t^{\frac{2}{1-\delta}})\mathrm{d}t \le C_3.$$

Our main result ensures existence and uniqueness for the diagonally quadratic BSDE (1.1).

Theorem 2.1 *There exists a constant $r_0 > 0$ (depending only on the vector of parameters $(n, \gamma, \delta, C_0, C_1, C_2, C_3)$) such that if (H1)–(H3) holds for $r \in (0, r_0)$, then BSDE (1.1) has a unique solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$ on $[0, T]$.*

The proof is given in Section 3.

Example 2.1 Assuming that $T = 1$, then the following generator f satisfies (H1)–(H3) with $(\alpha_t, \beta_t, \eta_t) = (2, 1, 1)$ and $(\gamma, \delta, C_1, C_2, C_3) = (2, 0.5, 2, 1, 2)$ when $r \in (0, 1.5]$:

$$f^i(t, y, z) = |z^i|^2 + |y| + \sin(|z|^{\frac{3}{2}}) + |z|^r \mathbf{1}_{\{|z|>1\}} + |z|\mathbf{1}_{\{|z|\le 1\}}, \quad i = 1, \cdots, n.$$

The second result of this paper concerns a special type of diagonally quadratic BSDE as follows:

$$Y_t^i = \xi^i + \int_t^T k^i(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}W_s, \quad 0 \le t \le T,\ 1 \le i \le n. \tag{2.2}$$

For each $i = 1, \cdots, n$, $H \in \mathbb{R}^{n\times d}$, $z \in \mathbb{R}^{1\times d}$, $Y \in \mathbb{R}^n$ and $y \in \mathbb{R}$, define by $H(z; i)$ the matrix in $\mathbb{R}^{n\times d}$ whose i-th row is z and whose j-th row is H^j for any $j \ne i$, define by $Y(y; i)$ the vector in $\mathbb{R}^n$ whose i-th component is y and whose j-th component is Y^j for any $j \ne i$. We make the following assumptions.

(A1) There exists a constant $\alpha \in (-1, 1)$ and a positive constant K_1 such that for $i = 1, \cdots, n$, the function $k^i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ depends only on the first i components of y and the first i rows of z, and

$$|k^i(t, y, z)| \le K_1\Big(1 + \sum_{j=1}^{i} |y^j| + \sum_{j=1}^{i-1} |z^j|^{1+\alpha} + |z^i|^2\Big), \quad \forall y \in \mathbb{R}^n,\ z \in \mathbb{R}^{n\times d}.$$

(A2) There exists a non-negative constant β and a positive constant K_2 such that for $i = 1, \cdots, n$ and each $(Y, Z, y_1, y_2, z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}^{1\times d}$, the function k^i satisfies:

$$|k^i(t, Y(y_1; i), Z(z_1; i)) - k^i(t, Y(y_2; i), Z(z_2; i))| \le \beta|y_1 - y_2| + K_2(1 + |z_1| + |z_2|)|z_1 - z_2|.$$

(A3) There exists a non-negative constant K_3 such that $\xi = (\xi^1, \cdots, \xi^n)^*$ satisfies

$$\|\xi\|_\infty \le K_3.$$

We have the following result.

Theorem 2.2 *Let* (A1)–(A3) *be satisfied. Then BSDE* (2.2) *has a unique solution* $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$ *on* $[0, T]$.

The proof is given in Section 4.

Remark 2.1 In Theorem 2.2, we do not require (H1)–(H3).

Example 2.2 The following generator k satisfies (A1)–(A2) in Theorem 2.2:

$$k^1(t, y, z) = 1 + y^1 + \sin(y^1) + |z^1|^2;$$
$$k^i(t, y, z) = 1 + \sum_{j=1}^{i} y^j + \sin(y^{i-1})y^i + \sum_{j=1}^{i-1} |z^j|^{1+\alpha} + \cos(|z^{i-1}|)|z^i|^2, \quad i = 2, \cdots, n.$$

3 Diagonally Quadratic BSDE

We first give an estimate.

Lemma 3.1 *Let* (H1)–(H3) *hold,* $(Y, Z) \in \mathcal{S}^\infty_{[t_0,T]}(\mathbb{R}^n) \times \mathcal{H}^2{}_{[t_0,T]}(\mathbb{R}^{n\times d})$ *is a solution of BSDE* (1.1) *on* $[t_0, T]$*, then there exist two positive constants* C_4, C_5 (*depending on the vector of parameters* $(n, \gamma, \delta, C_0, C_1, C_2, C_3)$) *such that*

$$\|Y\|_{\mathcal{S}^\infty_{[t_0,T]}} \le C_4 + C_5 \exp\Big(\frac{r\gamma}{1-\delta}\|Y\|_{\mathcal{S}^\infty_{[t_0,T]}}\Big). \tag{3.1}$$

Proof Define

$$u(x) = \frac{\exp(\gamma|x|) - \gamma|x| - 1}{\gamma^2}, \quad x \in \mathbb{R}.$$

Then we have that for $x \in \mathbb{R}$,

$$u'(x) = \frac{\exp(\gamma|x|) - 1}{\gamma}\mathrm{sgn}(x), \quad u''(x) = \exp(\gamma|x|), \quad u''(x) - \gamma|u'(x)| = 1.$$

Using Itô's formula to compute $u(Y_t^i)$ and using the assumption (H2), we have

$$\begin{aligned}
u(Y_t^i) &= u(\xi^i) + \int_t^T \Big[u'(Y_s^i)\big(g^i(s, Z_s^i) + h^i(s, Y_s, Z_s)\big) - \frac{1}{2}u''(Y_s^i)|Z_s^i|^2\Big]\mathrm{d}s - \int_t^T u'(Y_s^i)Z_s^i\mathrm{d}W_s \\
&\le u(\xi^i) - \int_t^T u'(Y_s^i)Z_s^i\mathrm{d}W_s \\
&\quad + \int_t^T \Big[\frac{\exp(\gamma|Y_s^i|) - 1}{\gamma}\Big(\frac{\gamma}{2}|Z_s^i|^2 + \alpha_s + \beta_s|Y_s| + \eta_s|Z_s|^r\Big) - \frac{1}{2}\exp(\gamma|Y_s^i|)|Z_s^i|^2\Big]\mathrm{d}s \\
&= u(\xi^i) - \int_t^T u'(Y_s^i)Z_s^i\mathrm{d}W_s \\
&\quad + \int_t^T \Big[-\frac{1}{2}|Z_s^i|^2 + \frac{\exp(\gamma|Y_s^i|) - 1}{\gamma}(\alpha_s + \beta_s|Y_s| + \eta_s|Z_s|^r)\Big]\mathrm{d}s
\end{aligned} \tag{3.2}$$

Using Hölder's inequality, we get

$$\eta_s|Z_s|^r = \varepsilon^{\frac{r}{2}}|Z_s|^r \cdot \varepsilon^{-\frac{r}{2}}\eta_s \le \frac{r}{2}\varepsilon|Z_s|^2 + \frac{2-r}{2}(\varepsilon^{-\frac{r}{2}}\eta_s)^{\frac{2}{2-r}}. \tag{3.3}$$

Taking

$$\varepsilon = \frac{\gamma}{nr}\exp(-\gamma\|Y\|_{\mathcal{S}^\infty_{[s,T]}}),$$

we have

$$\eta_s|Z_s|^r \le \frac{\gamma}{2n}\exp(-\gamma\|Y\|_{\mathcal{S}^\infty_{[s,T]}})|Z_s|^2 + \frac{2-r}{2}\eta_s^{\frac{2}{2-r}}\Big(\frac{\gamma}{nr}\exp(-\gamma\|Y\|_{\mathcal{S}^\infty_{[s,T]}})\Big)^{-\frac{r}{2-r}}. \tag{3.4}$$

From $0 < r \le 1+\delta < 2$, we have

$$\frac{2-r}{2} \le 1, \quad 1 \le \frac{2}{2-r} \le \frac{2}{1-\delta}, \quad \frac{r}{2-r} \le \frac{1+\delta}{1-\delta}.$$

Therefore

$$\eta_s^{\frac{2}{2-r}} \le \eta_s^{\frac{2}{1-\delta}} + \eta_s, \qquad \Big(\frac{nr}{\gamma}\Big)^{\frac{r}{2-r}} \le \Big(\frac{2n}{\gamma}\Big)^{\frac{r}{2-r}} \le \Big(\frac{2n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1.$$

From (3.4), we deduce

$$\eta_s|Z_s|^r \le \frac{\gamma}{2n}\exp(-\gamma\|Y\|_{\mathcal{S}^\infty_{[s,T]}})|Z_s|^2 + (\eta_s^{\frac{2}{1-\delta}} + \eta_s)\Big(\Big(\frac{2n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1\Big)\exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big). \tag{3.5}$$

Let

$$k_s = (\eta_s^{\frac{2}{1-\delta}} + \eta_s)\Big(\Big(\frac{2n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1\Big).$$

From (3.2) and (3.5), we have

$$\begin{aligned} u(Y_t^i) \le\ & u(\xi^i) - \int_t^T u'(Y_s^i)Z_s^i \mathrm{d}W_s + \int_t^T \Big[-\frac{1}{2}|Z_s^i|^2 + \frac{1}{2n}|Z_s|^2\Big]\mathrm{d}s \\ & + \int_t^T \frac{\exp(\gamma|Y_s^i|)}{\gamma}\Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + k_s\exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\mathrm{d}s. \end{aligned} \tag{3.6}$$

Hence it holds that

$$\begin{aligned} \sum_{i=1}^n u(Y_t^i) \le\ & \sum_{i=1}^n u(\xi^i) - \int_t^T \sum_{i=1}^n u'(Y_s^i)Z_s^i \mathrm{d}W_s \\ & + \frac{1}{\gamma}\int_t^T \Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + k_s\exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\sum_{i=1}^n \exp(\gamma|Y_s^i|)\mathrm{d}s. \end{aligned} \tag{3.7}$$

Noting that

$$\frac{\exp(\gamma|x|) - 2}{2\gamma^2} \le u(x) \le \frac{\exp(\gamma|x|)}{\gamma^2},$$

we have

$$\begin{aligned} & \sum_{i=1}^n \frac{\exp(\gamma|Y_t^i|) - 2}{2\gamma^2} \\ & \le \frac{n\exp(\gamma\|\xi\|_\infty)}{\gamma^2} - \int_t^T \sum_{i=1}^n u'(Y_s^i)Z_s^i \mathrm{d}W_s \\ & \quad + \frac{1}{\gamma}\int_t^T \Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + k_s\exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\sum_{i=1}^n \exp(\gamma|Y_s^i|)\mathrm{d}s. \end{aligned} \tag{3.8}$$

Taking conditional expectation with respect to $\mathcal{F}_\tau$ for $\tau \in [t_0, t]$, we show

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^n \exp(\gamma|Y_t^i|)|\mathcal{F}_\tau\Big] \\
&\le 2n(\exp(\gamma\|\xi\|_\infty)+1) + \int_t^T 2\gamma\Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} \\
&\quad + k_s \exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\mathbb{E}\Big[\sum_{i=1}^n \exp(\gamma|Y_s^i|)|\mathcal{F}_\tau\Big]\mathrm{d}s. \qquad (3.9)
\end{aligned}$$

Using Gronwall's inequality, we get

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^n \exp(\gamma|Y_t^i|)|\mathcal{F}_\tau\Big] \\
&\le 2n(\exp(\gamma\|\xi\|_\infty)+1)\cdot \exp\Big(\int_t^T 2\gamma\Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + k_s \exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\mathrm{d}s\Big). \qquad (3.10)
\end{aligned}$$

Setting $\tau = t$ and noting $r \le 1+\delta$, we have

$$\begin{aligned}
&\sum_{i=1}^n \exp(\gamma|Y_t^i|) \\
&\le 2n(\exp(\gamma\|\xi\|_\infty)+1)\exp\Big(\int_t^T 2\gamma\Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + k_s \exp\Big(\frac{r\gamma}{1-\delta}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\mathrm{d}s\Big).
\end{aligned}$$

Using Jensen's inequality, we obtain

$$\sum_{i=1}^n \exp(\gamma|Y_t^i|) \ge n\exp\Big(\frac{\sum_{i=1}^n \gamma|Y_t^i|}{n}\Big) \ge n\exp\Big(\frac{\gamma|Y_t|}{n}\Big).$$

Combining the preceding inequalities and the assumption (H3), we have

$$\begin{aligned}
|Y_t| &\le \frac{n}{\gamma}\log(2\exp(\gamma\|\xi\|_\infty)+2) + \int_t^T 2n\Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + k_s\exp\Big(\frac{r\gamma}{1-\delta}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\mathrm{d}s \\
&\le \frac{n}{\gamma}\log(2\exp(\gamma C_0)+2) + 2nC_1 + 2n\int_{t_0}^T k_s\exp\Big(\frac{r\gamma}{1-\delta}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\mathrm{d}s \\
&\quad + \int_t^T 2n\beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\mathrm{d}s. \qquad (3.11)
\end{aligned}$$

Let

$$K_0 := \frac{n}{\gamma}\log(2\exp(\gamma C_0)+2) + 2nC_1 + 2n\int_{t_0}^T k_s\exp\Big(\frac{r\gamma}{1-\delta}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\mathrm{d}s.$$

We have

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \le K_0 + \int_t^T 2n\beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\mathrm{d}s, \quad \forall t \in [t_0, T]. \qquad (3.12)$$

Using Gronwall's inequality and the assumption (H3), we have

$$
\begin{aligned}
\|Y\|_{\mathcal{S}^\infty_{[t_0,T]}} &\le K_0 \exp\Big(\int_{t_0}^T 2n\beta_s \mathrm{d}s\Big) \\
&\le \exp(2nC_2)\Big(\frac{n}{\gamma}\log(2\exp(\gamma C_0)+2)+2nC_1 \\
&\quad + 2n\int_{t_0}^T k_s \exp\Big(\frac{r\gamma}{1-\delta}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\mathrm{d}s\Big).
\end{aligned}
\tag{3.13}
$$

Let

$$
C_4 = \exp(2nC_2)\Big(\frac{n}{\gamma}\log(2\exp(\gamma C_0)+2)+2nC_1\Big), \quad C_5 = 2nC_3\exp(2nC_2)\Big(\Big(\frac{2n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}}+1\Big).
$$

From the definition of k_s and the assumption (H3), we get (3.1). The proof is complete.

From Lemma 3.1, we get the following proposition.

Proposition 3.1 *There exists a constant $r_0 > 0$ (depending on the vector of parameters $(n, \gamma, \delta, C_0, C_1, C_2, C_3)$) such that if* (H1)–(H3) *holds for $r \in (0, r_0)$, and $(Y, Z) \in \mathcal{S}^\infty_{[t_0,T]}(\mathbb{R}^n) \times \mathcal{H}^2{}_{[t_0,T]}(\mathbb{R}^{n\times d})$ is a solution of BSDE* (1.1) *on $[t_0, T]$, then*

$$
\|Y\|_{\mathcal{S}^\infty_{[t_0,T]}} \le C_4 + 2C_5, \tag{3.14}
$$

where C_4 and C_5 are given by (3.1).

Proof Define

$$
F(x) = C_4 + C_5 \exp\Big(\frac{r\gamma x}{1-\delta}\Big) - x, \quad x \ge 0.
$$

Then we have

$$
F'(x) = \frac{r\gamma C_5}{1-\delta}\exp\Big(\frac{r\gamma x}{1-\delta}\Big) - 1, \quad F''(x) = \frac{r^2\gamma^2 C_5}{(1-\delta)^2}\exp\Big(\frac{r\gamma x}{1-\delta}\Big) > 0.
$$

Let

$$
r_0 = \frac{(1-\delta)\log 2}{\gamma(C_4+2C_5)}.
$$

For a given $r \in (0, r_0)$, let

$$
x_0 = \frac{1-\delta}{r\gamma}\log\frac{1-\delta}{C_5 r\gamma}.
$$

Then we have

$$
r < r_0 < \frac{1-\delta}{\gamma C_5}, \quad x_0 > 0, \quad F'(x_0) = 0.
$$

Hence, F is decreasing on $[0, x_0]$ and increasing on $[x_0, +\infty)$, and

$$
\begin{aligned}
F(C_4+2C_5) &= C_4 + C_5\exp\Big(\frac{r\gamma(C_4+2C_5)}{1-\delta}\Big) - C_4 - 2C_5 \\
&= C_5\Big(\exp\Big(\frac{r\gamma(C_4+2C_5)}{1-\delta}\Big) - 2\Big) \\
&< C_5\Big(\exp\Big(\frac{r_0\gamma(C_4+2C_5)}{1-\delta}\Big) - 2\Big) = 0.
\end{aligned}
$$

Then $F(x) = 0$ has two zeros x_1, x_2 and they satisfy

$$C_0 < C_4 < x_1 < C_4 + 2C_5 < x_2, \quad \{x : F(x) \geq 0\} = [0, x_1] \cup [x_2, +\infty). \tag{3.15}$$

From Lemma 3.1, we obtain that

$$F(\|Y\|_{\mathcal{S}^\infty_{[t,T]}}) \geq 0, \quad \forall t \in [t_0, T].$$

Hence

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \in [0, x_1] \cup [x_2, +\infty), \quad \forall t \in [t_0, T].$$

Define

$$\widehat{t} = \inf\{t \in [t_0, T] : \|Y\|_{\mathcal{S}^\infty_{[t,T]}} \leq x_1\}.$$

Notice that

$$\|Y\|_{\mathcal{S}^\infty_{[T,T]}} = \|\xi\|_\infty \leq C_0 < x_1.$$

$\widehat{t}$ is well defined. $\|Y\|_{\mathcal{S}^\infty_{[t,T]}}$ is decreasing and right-continuous about t, so we have $\|Y\|_{\mathcal{S}^\infty_{[\widehat{t},T]}} \leq x_1$. If $\widehat{t} > t_0$, then

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} > x_1, \quad \forall t \in [t_0, \widehat{t}).$$

Therefore

$$\|Y\|_{\mathcal{S}^\infty_{[t,T]}} \geq x_2, \quad \forall t \in [t_0, \widehat{t}).$$

From (3.13), we deduce that

$$\begin{aligned}
x_2 &\leq \limsup_{t \to \widehat{t}^-} \|Y\|_{\mathcal{S}^\infty_{[t,T]}} \\
&\leq \limsup_{t \to \widehat{t}^-} \Big[C_4 + 2n \exp(2nC_2) \int_t^T k_s \exp\Big(\frac{r\gamma}{1-\delta} \|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big) \mathrm{d}s\Big] \\
&= C_4 + 2n \exp(2nC_2) \int_{\widehat{t}}^T k_s \exp\Big(\frac{r\gamma}{1-\delta} \|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big) \mathrm{d}s \\
&\leq C_4 + 2n \exp(2nC_2) \Big(\Big(\frac{2n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1\Big) C_3 \exp\Big(\frac{r\gamma}{1-\delta} \|Y\|_{\mathcal{S}^\infty_{[\widehat{t},T]}}\Big) \\
&\leq C_4 + C_5 \exp\Big(\frac{r\gamma x_1}{1-\delta}\Big) = F(x_1) + x_1 = x_1.
\end{aligned} \tag{3.16}$$

This is a contradiction. Hence $\widehat{t} = t_0$, and

$$\|Y\|_{\mathcal{S}^\infty_{[t_0,T]}} \leq x_1 < C_4 + 2C_5.$$

The proof is complete.

Proof of Theorem 2.1 For the number r_0 given in Proposition 3.1 and a given $r \in (0, r_0)$, define

$$\lambda := C_4 + 2C_5,$$

where C_4 and C_5 are the same as in Lemma 3.1. From (3.15), we have

$$\|\xi\|_\infty \leq C_0 \leq C_4 \leq \lambda.$$

From [13, Theorem 2.2, p. 1072], there exists $t_\lambda > 0$ which depends on constants $(n, C, \gamma, \delta, \lambda)$, such that BSDE (1.1) has a local solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$ on $[T - t_\lambda, T]$. From Proposition 3.1, we obtain

$$\|Y_{T-t_\lambda}\|_\infty \le \|Y\|_{\mathcal{S}^\infty_{[T-t_\lambda,T]}} \le \lambda.$$

Taking $T - t_\lambda$ as the terminal time and Y_{T-t_λ} as the terminal value, BSDE (1.1) has a local solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^{n\times d})$ on $[T - 2t_\lambda, T - t_\lambda]$. Stitching the solutions we have a solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$ on $[T - 2t_\lambda, T]$ and $\|Y_{T-2t_\lambda}\|_\infty \le \lambda$. Repeating the preceding process, we can extend the pair (Y, Z) to the whole interval $[0, T]$ within finite steps such that Y is uniformly bounded by λ and $Z \in \mathcal{H}^2(\mathbb{R}^{n\times d})$. We now show that $Z \in \mathrm{BMO}(\mathbb{R}^{n\times d})$. Identical to the proofs of inequalities (3.5)–(3.6), we have

$$\eta_s|Z_s|^r \le \frac{\gamma}{4n}\exp(-\gamma\|Y\|_{\mathcal{S}^\infty_{[s,T]}})|Z_s|^2 + (\eta_s^{\frac{2}{1-\delta}} + \eta_s)\Big(\Big(\frac{4n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1\Big)\exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big) \tag{3.17}$$

and

$$\begin{aligned} u(Y_t^i) \le u(\xi^i) &- \int_t^T u'(Y_s^i)Z_s^i \mathrm{d}W_s + \int_t^T \Big[-\frac{1}{2}|Z_s^i|^2 + \frac{1}{4n}|Z_s|^2\Big]\mathrm{d}s \\ &+ \int_t^T \frac{\exp(\gamma|Y_s^i|)}{\gamma}\Big(\alpha_s + \beta_s\|Y\|_{\mathcal{S}^\infty_{[s,T]}} + \widehat{k}_s\exp\Big(\frac{r\gamma}{2-r}\|Y\|_{\mathcal{S}^\infty_{[s,T]}}\Big)\Big)\mathrm{d}s, \end{aligned} \tag{3.18}$$

where

$$\widehat{k}_s = (\eta_s^{\frac{2}{1-\delta}} + \eta_s)\Big(\Big(\frac{4n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1\Big).$$

Summing i from 1 to n and taking conditional expectation with respect to $\mathcal{F}_t$, we have

$$\frac{1}{4}\mathbb{E}\Big[\int_t^T |Z_s|^2\mathrm{d}s|\mathcal{F}_t\Big] \le \frac{n\exp(\gamma C_0)}{\gamma^2} + \frac{n\exp(\gamma\lambda)}{\gamma}\Big(C_1 + C_2\lambda + C_3\Big(\Big(\frac{4n}{\gamma}\Big)^{\frac{1+\delta}{1-\delta}} + 1\Big)\exp\Big(\frac{r\gamma\lambda}{2-r}\Big)\Big).$$

Hence $Z \in \mathrm{BMO}(\mathbb{R}^{n\times d})$. Finally, the uniqueness on the given interval $[0, T]$ is a consequence of [13, Theorem 2.2, p. 1072] via a pasting technique.

Remark 3.1 Assumptions (H1)–(H3) of Theorem 2.1 are different from those of [13, Theorem 2.3, p. 1072] and [10, Theorem 2.4]. We allow the generator to have a small growth of the off-diagonal elements. They are different from those of [10, Theorem 2.5] in that the generator is not required to be strictly quadratic. For example, the following generator f satisfies Theorem 2.1 rather than the others when r is sufficiently small:

$$f^i(t, y, z) = |z^i|^2\sin\big(\log(|z^i| + 1)\big) + |y| + \sin(|z|^{1+\delta}) + |z|^r\mathbf{1}_{\{|z|>1\}} + |z|\mathbf{1}_{\{|z|\le 1\}}, \quad i = 1, \cdots, n.$$

Remark 3.2 When C_3 is sufficiently small such that $C_5 < \exp\big(-\frac{\gamma(1+\delta)(1+C_4)}{1-\delta}\big)$, taking $r_0 = 1 + \delta$, then for $r \in (0, r_0]$, we have

$$F(C_4 + 1) = C_5\exp\Big(\frac{\gamma(1+\delta)(1+C_4)}{1-\delta}\Big) - 1 < 0.$$

In a similar way we have Theorem 2.1. In particular, when $C_3 = 0$ we have $C_5 = 0$, then we have $\|Y\|_{\mathcal{S}^\infty_{[t_0,T]}} \le C_4$ by (3.1) and thus Theorem 2.1 holds, which is the case of [10, Theorem 2.4].

Remark 3.3 From [10, Theorem 2.1], (H1) and (H2) can be replaced with the following in Theorem 2.1.

(H1′) There exist a deterministic scalar-valued positive function $(\alpha_t)_{t\in[0,T]}$, a deterministic nondecreasing continuous function $\phi(\cdot):[0,+\infty)\to[0,+\infty)$ with $\phi(0)=0$ and several real constants $\gamma>0$, $C\geq 0$, $\delta\in[0,1)$ such that for $i=1,\cdots,n$ and each $(y,\overline{y},z,\overline{z})\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^{n\times d}$, f^i satisfies the following inequalities:

$$\begin{aligned}&|f^i(t,y,z)|\leq\alpha_t+\phi(|y|)+\frac{\gamma}{2}|z^i|^2+C\sum_{j\neq i}|z^j|^{1+\delta},\\&|f^i(t,y,z)-f^i(t,\overline{y},\overline{z})|\\\leq{}&\phi(|y|\vee|\overline{y}|)\Big[(1+|z|+|\overline{z}|)(|y-\overline{y}|+|z^i-\overline{z}^i|)+(1+|z|^\delta+|\overline{z}|^\delta)\sum_{j\neq i}|z^j-\overline{z}^j|\Big].\end{aligned}$$

(H2′) There exist a two-dimensional non-negative deterministic vector function $(\beta_t,\eta_t)_{t\in[0,T]}$ and a positive constant $r\in(0,1+\delta]$ such that for $i=1,\cdots,n$ and $(y,z)\in\mathbb{R}^n\times\mathbb{R}^{n\times d}$, the function f^i satisfies:

$$\mathrm{sgn}(y^i)f^i(t,y,z)\leq\alpha_t+\beta_t|y|+\eta_t|z|^r+\frac{\gamma}{2}|z^i|^2.$$

4 Diagonally Quadratic and Triangular BSDE

To prove Theorem 2.2, we need the following lemma.

Lemma 4.1 *We consider the following one-dimensional BSDE*:

$$Y_t=\eta+\int_t^T l(s,Y_s,Z_s)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s,\quad 0\leq t\leq T.\tag{4.1}$$

The terminal value η *and the generator* l *satisfy the following assumptions*:

(B1) $\forall K>0$, *the function* $l:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\to\mathbb{R}$ *satisfies*:

$$\sup_{t\in[0,T]}\Big\|\mathbb{E}\Big[\exp\Big(K\int_t^T|l(s,0,0)|\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]\Big\|_\infty<+\infty.$$

(B2) *There exist a non-negative constant* β *and a positive constant* C *such that for each* $(y,\overline{y},z,\overline{z})\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}^{1\times d}$, *the function* l *satisfies*:

$$|l(t,y,z)-l(t,\overline{y},\overline{z})|\leq\beta|y-\overline{y}|+C(1+|z|+|\overline{z}|)|z-\overline{z}|.$$

(B3) *There exists a non-negative constant* C_1 *such that* η *satisfies*:

$$\|\eta\|_\infty\leq C_1.$$

Then BSDE (4.1) *has a unique solution* $(Y,Z)\in\mathcal{S}^\infty(\mathbb{R})\times\mathrm{BMO}(\mathbb{R}^{1\times d})$ *on* $[0,T]$.

Proof When $\beta=0$, l is independent of y. From [13, Lemma 2.1], we know that the result holds. When $\beta>0$, for $y\in\mathcal{S}^\infty(\mathbb{R})$, we define a map $\varphi(y)=Y$, where Y is given by

$$Y_t=\eta+\int_t^T l(s,y_s,Z_s)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s,\quad 0\leq t\leq T.$$

From the preceding result, we know that φ is well-defined and maps $\mathcal{S}^\infty(\mathbb{R})$ to itself. For $y,\overline{y}\in\mathcal{S}^\infty(\mathbb{R})$, let $Y=\varphi(y),\overline{Y}=\varphi(\overline{y})$. Denote $\delta Y=Y-\overline{Y},\ \delta y=y-\overline{y},\ \delta Z=Z-\overline{Z}$. We have

$$\begin{aligned}\delta Y_t&=\int_t^T[l(s,y_s,Z_s)-l(s,\overline{y}_s,\overline{Z}_s)]\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}W_s\\&=\int_t^T(\beta_s\delta y_s+\delta Z_s\alpha_s)\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}W_s.\end{aligned}$$

Here $|\beta_s|\le\beta,|\alpha_s|\le C(1+|Z_s|+|\overline{Z}_s|)$, therefore $\alpha\cdot W$ is a BMO martingale. Define

$$\widetilde{W}_t:=W_t-\int_0^t\alpha_s\,\mathrm{d}s,\quad t\in[0,T];\quad \mathrm{d}\widetilde{\mathbb{P}}:=\mathscr{E}(\alpha\cdot W)_0^T\mathrm{d}\mathbb{P}.$$

Then, $\widetilde{\mathbb{P}}$ is a new probability equivalent to $\mathbb{P}$, and $\widetilde{W}$ is a Brownian motion with respect to $\widetilde{\mathbb{P}}$. We have

$$\delta Y_t=\int_t^T\beta_s\delta y_s\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}\widetilde{W}_s.$$

Taking the conditional expectation with respect to $\widetilde{\mathbb{P}}$, we have

$$\|\delta Y\|_{\mathcal{S}^\infty}\le\int_t^T\beta\|\delta y\|_{\mathcal{S}^\infty}\mathrm{d}s\le\beta T\|\delta y\|_{\mathcal{S}^\infty}.$$

When $T\le\frac{1}{2\beta}$, φ is a contraction map and and the statement follows from the Banach fixed point theorem. For general T, we can repeat the preceding process and get the result within finite steps. The proof is complete.

Proof of Theorem 2.2 We will solve BSDE (2.2) in order. For the first equation, noting that $\|\xi^1\|_\infty\le K_3$, $|k^1(s,0,0)|\le K_1$, from [16] we know that it has a unique solution $(Y^1,Z^1)\in\mathcal{S}^\infty(\mathbb{R})\times\mathrm{BMO}(\mathbb{R}^{1\times d})$. Suppose that we already solve the first $(i-1)$ equations with $(Y^j,Z^j)\in\mathcal{S}^\infty(\mathbb{R})\times\mathrm{BMO}(\mathbb{R}^{1\times d}),\ j=1,\cdots,i-1$. For the i-th equation, we have that $\forall K>0$,

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(K\int_t^T|k^i(s,Y_s(0;i),Z_s(0;i))|\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]\\&\le\exp(KTK_1)\cdot\mathbb{E}\Big[\exp\Big(KK_1\int_t^T\Big(\sum_{j=1}^{i-1}|Y_s^j|+\sum_{j=1}^{i-1}|Z_s^j|^{1+\alpha}\Big)\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]\\&\le\exp\Big(KTK_1+KTK_1\sum_{j=1}^{i-1}\|Y^j\|_{\mathcal{S}^\infty}\Big)\cdot\mathbb{E}\Big[\exp\Big(KK_1\int_t^T\sum_{j=1}^{i-1}|Z_s^j|^{1+\alpha}\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big].\end{aligned}\tag{4.2}$$

By Hölder's inequality and Young's inequality we get

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(KK_1\int_t^T\sum_{j=1}^{i-1}|Z_s^j|^{1+\alpha}\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]\\&\le\Big(\prod_{j=1}^{i-1}\mathbb{E}\Big[\exp\Big(KK_1(i-1)\int_t^T|Z_s^j|^{1+\alpha}\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]\Big)^{\frac{1}{i-1}}\end{aligned}\tag{4.3}$$

and

$$L|Z_s^j|^{1+\alpha}=\varepsilon^{\frac{1+\alpha}{2}}|Z_s^j|^{1+\alpha}\cdot L\varepsilon^{-\frac{1+\alpha}{2}}\le\frac{1+\alpha}{2}\varepsilon|Z_s^j|^2+\frac{1-\alpha}{2}L^{\frac{2}{1-\alpha}}\varepsilon^{-\frac{1+\alpha}{1-\alpha}},\quad\forall L>0.$$

Let ε be sufficiently small such that $\frac{1+\alpha}{2}\varepsilon\|Z^j\|^2_{\mathrm{BMO}} \leq \frac{1}{2}$. From John-Nirenberg inequality (see [15, Theorem 2.2]), we have that for $1 \leq j \leq i-1$,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(L\int_t^T |Z_s^j|^{1+\alpha}\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big] &\leq \frac{1}{1-\frac{1+\alpha}{2}\varepsilon\|Z^j\|^2_{\mathrm{BMO}}}\exp\Big(\frac{1-\alpha}{2}L^{\frac{2}{1-\alpha}}\varepsilon^{-\frac{1+\alpha}{1-\alpha}}T\Big) \\ &\leq 2\exp\Big(\frac{1-\alpha}{2}L^{\frac{2}{1-\alpha}}\varepsilon^{-\frac{1+\alpha}{1-\alpha}}T\Big), \quad \forall t\in[0,T]. \end{aligned} \tag{4.4}$$

Combining (4.2)–(4.4), we obtain that $\forall K > 0$,

$$\sup_{t\in[0,T]}\Big\|\mathbb{E}\Big[\exp\Big(K\int_t^T \big|k^i(s, Y_s(0;i), Z_s(0;i))\big|\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big]\Big\|_\infty < +\infty. \tag{4.5}$$

Therefore, we can apply Lemma 4.1 to see the i-th equation admits a unique solution $(Y^i, Z^i) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathrm{BMO}(\mathbb{R}^{1\times d})$ on $[0,T]$. The proof is complete.

5 Conclusion Remark

We study the well-posedness of the multi-dimensional BSDE (1.1) with a diagonally quadratic generator. When the generator has a small growth of the off-diagonal elements, we build a new priori estimate and get the existence and uniqueness of the global solution, which generalizes the results in Hu and Tang [13] and Fan et al. [10]. Besides, when the generator is diagonally quadratic and triangular, we get the global solvability of the multi-dimensional BSDE (2.2) without the small growth condition. Finally, when the generator is non-triangular and has a general sub-quadratic growth of the off-diagonal elements, the existence and uniqueness of the global solutions are interesting and challenging, which remains to be studied in the future.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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