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Persistence Approximation Property for L^p Operator Algebras*

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Abstract In this paper, the authors study the persistence approximation property for quantitative K-theory of filtered L^p operator algebras. Moreover, they define quantitative assembly maps for L^p operator algebras when $p \in [1, \infty)$. Finally, in the case of L^p crossed products and L^p Roe algebras, sufficient conditions for the persistence approximation property are found. This allows to give some applications involving the L^p (coarse) Baum-Connes conjecture.

Keywords L^p operator algebra, Quantitative assembly map, Persistence approximation property, L^p Baum-Connes conjecture

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1 Introduction

Quantitative operator K-theory was primarily developed first by Yu [20] on the Novikov conjecture for groups with finite asymptotic dimension, and then by Oyono-Oyono and Yu in [13] to study a general quantitative K-theory for filtered C^*-algebras. Based on their work, Chung later extended the framework of quantitative K-theory to the class of algebras of bounded linear operators on subquotients of L^p spaces for $p \in [1, \infty)$ (i.e., SQ_p algebras) in [2]. Since an L^p operator algebra is obviously an SQ_p algebra, we can derive a framework of quantitative K-theory for L^p operator algebras by applying Chung's work to the L^p operator algebras. For a filtered L^p operator algebra A, the K-theory of A can be approximated by the quantitative K-theory group $K_*^{\varepsilon,r,N}(A)$ as r and N tend to infinity, i.e., $\lim\limits_{r,N\to\infty} K_*^{\varepsilon,r,N}(A) = K_*(A)$. Compared to the usual K-theory of a complex Banach algebra, quantitative K-theory is more computable and more flexible by using quasi-idempotents and quasi-invertibles instead of idempotents and invertibles, respectively.

To explore a way of approximating K-theory with quantitative K-theory, Oyono-Oyono

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and Yu studied the persistence approximation property for quantitative K-theory of filtered C^*-algebras in [14]. Subsequently, Wang and Wang investigated the persistence approximation property for maximal Roe algebras, and proved that if X is a coarsely uniformly contractible discrete metric space with bounded geometry, and it admits a fibred coarse embedding into Hilbert space, then the maximal Roe algebra for X satisfies the persistence approximation property in [17]. Motivated by these successful researches on the persistence approximation property for the quantitative K-theory, we will in this paper extend these methods and results for C^*-algebras to L^p operator algebras.

Recently, the research on L^p operator algebras has been revived. In the work of [15], Phillips introduced full and reduced L^p crossed products and proved that the K-theory of L^p analogs of Cuntz algebras is the same as that of C^*-algebras. This work has inspired mathematicians to study L^p operator algebras that behave like C^*-algebras, including group L^p operator algebras (see [7, 9–11, 15]) and groupoid L^p operator algebras (see [8]). There are also related works on ℓ^p uniform Roe algebras in comparison with classical uniform Roe algebras, such as [4–5, 12]. These researches provide sufficient methods and techniques for dealing with the problem of the L^p operator algebras in this paper.

In order to investigate an L^p version of persistence approximation property, we have to give a definition of the quantitative L^p assembly map. In this important article [3], Chung defined the L^p assembly map, and showed that a certain L^p assembly map is an isomorphism if the action $\Gamma \curvearrowright X$ has finite dynamical complexity. Moreover, Zhang and Zhou in [21] studied L^p localization algebras and L^p Roe algebras, which are basic ingredients for defining quantitative L^p assembly maps.

The main aim of this paper is to define the L^p analog of the quantitative assembly map to study the persistence approximation property for the quantitative K-theory of filtered L^p operator algebras. More precisely, we say that a filtered L^p operator algebra A has the persistence approximation property if for any ε in $\left(0, \frac{1}{20}\right)$, any $r > 0$ and any $N \geq 1$, there exist $\varepsilon' \in \left[\varepsilon, \frac{1}{20}\right)$, $r' \geq r$ and $N' \geq N$ such that the following statement $\mathcal{PA}_*(A, \varepsilon, \varepsilon', r, r', N, N')$ is satisfied: An element from $K_*^{\varepsilon,r,N}(A)$ is zero in $K_*(A)$ implies that it is zero in $K_*^{\varepsilon',r',N'}(A)$. For the case of a crossed product of an L^p operator algebra by a finitely generated group, we obtain the main theorem.

Theorem 1.1 (see Theorem 4.1) *Let Γ be a finitely generated group, and let A be a Γ-L^p operator algebra. Assume that*

(1) *Γ admits a cocompact universal example for proper actions;*

(2) *for any positive integer $\mathscr{N}$, there exists a non-decreasing function $\omega : [1, \infty) \to [1, \infty)$ such that the $\mathscr{N}$-L^p Baum-Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)$$

is ω-surjective;

(3) *the L^p Baum-Connes assembly map for Γ with coefficients in A is injective.*
Then for any $N \geq 1$, there exists a universal constant $\lambda_{PA} \geq 1$ such that for any ε in $\left(0, \frac{1}{20\lambda_{PA}}\right)$ and any $r > 0$, there exist $r' \geq r$ and $N' \geq N$ such that $\mathcal{PA}_*(A \rtimes \Gamma, \varepsilon, \lambda_{PA}\varepsilon, r, r', N, N')$ holds.

This theorem is a generalization of Oyono-Oyono and Yu's work on persistence approximation property for C^* crossed products (see [14]). We call it the L^p version of persistence approximation property. To demonstrate this result, we define a quantitative L^p assembly map by using the L^p localization algebra and the L^p Roe algebra. Moreover, we carefully estimate the changing parameters of (ε, r, N)-idempotent and (ε, r, N)-invertible elements in the proof of the theorem to present a cleaner result.

Parallel to the main theorem, we obtain a similar result for the L^p Roe algebra for a discrete metric space X with bounded geometry. Replacing the assumption that the group admits a cocompact universal example for proper actions by that X is coarsely uniformly contractible, we have the following theorem.

Theorem 1.2 (see Theorem 5.1) *Let X be a discrete metric space with bounded geometry, and let A be an L^p operator algebra. Assume that*

(1) *X is coarsely uniformly contractible*;

(2) *for any positive integer $\mathscr{N}$, there exists a non-decreasing function $\omega : [1, \infty) \to [1, \infty)$ such that the $\mathscr{N}$-L^p coarse Baum-Connes assembly map for X with coefficients in*

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)$$

is ω-surjective;

(3) *the L^p coarse Baum-Connes assembly map for X with coefficients in A is injective.*

Then for any $N \geq 1$, there exists a universal constant $\lambda_{PA} \geq 1$ such that for any ε in $\left(0, \frac{1}{20\lambda_{PA}}\right)$ and any $r > 0$, there exist $r' \geq r$ and $N' \geq N$ such that $\mathcal{PA}_*(B^p(X, A), \varepsilon, \lambda_{PA}\varepsilon, r, r', N, N')$ holds.

As a corollary of this theorem, we prove that any L^p Roe algebra for a discrete Gromov hyperbolic metric space satisfies the persistence approximation property.

The outline of this paper is as follows: In Section 2, we recall the main results of quantitative K-theory for filtered L^p operator algebras. In Section 3, we define a quantitative L^p assembly map and show the connection between the quantitative statements and the L^p Baum-Connes conjecture. In Section 4, for the case of L^p crossed products, we find a sufficient condition for the persistence approximation property. Finally, in Section 5, we show that if X is a coarsely uniformly contractible discrete metric space with bounded geometry and finite asymptotic dimension, then the L^p Roe algebra for X has the persistence approximation property.

2 Quantitative K-Theory for L^p Operator Algebras

The ordinary K-theory of Banach algebras developed in [1] focuses on idempotents or invertibles. In comparison, quantitative K-theory for Banach algebras studied in [2] focuses on quasi-idempotents or quasi-invertibles. In this section, we recall some basic definitions and theorems of quantitative K-theory for filtered SQ_p algebras from [2]. Moreover, by applying these conclusions to filtered L^p operator algebras, we can obtain some basic concepts and main results of quantitative K-theory for filtered L^p operator algebras.

Definition 2.1 (see [7]) *Let A be a Banach algebra. For $p \in [1, \infty)$, we say that A is an L^p operator algebra if there exist an L^p space E and an isometric homomorphism $A \to \mathcal{B}(E)$.*

Remark 2.1 The L^p operator algebra was initially defined by Phillips in [15], and the above definition is compatible with the original one.

Definition 2.2 (see [2]) *A filtered L^p operator algebra is an L^p operator algebra A with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive real numbers $r \in (0, \infty)$ such that*

(1) $A_r \subset A_{r'}$ *if* $r \leq r'$;

(2) $A_r A_{r'} \subset A_{r+r'}$ *for all* $r, r' > 0$;

(3) *the subalgebra* $\bigcup\limits_{r>0} A_r$ *is dense in* A.

If A is unital with identity 1_A, we require $1_A \in A_r$. For any $r > 0$, we call the family $(A_r)_{r>0}$ a filtration of A. We say that a has propagation r if $a \in A_r$.

If A is not unital, we write the unitization of A as

$$A^+ = \{(a, z) : a \in A, z \in \mathbb{C}\}$$

with multiplication given by $(a, z)(a', z') = (aa' + za' + z'a, zz')$. We use $\widetilde{A}$ to represent A^+ if A is non-unital or to represent A if A is unital.

In order to control the matrix norm in quantitative K-theory of Banach algebras, we need to establish the matrix norm structure.

Definition 2.3 (see [6]) *For $p \in [1, +\infty)$, an abstract p-operator space is a Banach space X together with a family of norms $\|\cdot\|_n$ on $M_n(X)$ satisfying*:

(1) $\mathcal{D}_\infty$: *For $u \in M_n(X)$ and $v \in M_m(X)$, we have*

$$\left\| \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right\|_{n+m} = \max(\|u\|_n, \|v\|_m).$$

(2) $\mathcal{M}_p$: *For $u \in M_m(X)$, $\alpha \in M_{n,m}(\mathbb{C})$ and $\beta \in M_{m,n}(\mathbb{C})$, we have*

$$\|\alpha u \beta\|_n \leq \|\alpha\|_{B(\ell^p_m, \ell^p_n)} \|u\|_m \|\beta\|_{B(\ell^p_n, \ell^p_m)},$$

where ℓ^p_n denotes $\mathbb{C}^n$ with the ℓ^p norm.

Clearly, an L^p operator algebra is an abstract p-operator space.

Definition 2.4 (see [16]) *Let X and Y be p-operator spaces, and let $\phi : X \to Y$ be a bounded linear map. For each $n \in \mathbb{N}$, let $\phi_n : M_n(X) \to M_n(Y)$ be the induced map given by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$. We say that ϕ is p-completely bounded if $\sup_n \|\phi_n\| < \infty$. In this case, we let $\|\phi\|_{pcb} = \sup_n \|\phi_n\|$.*

We say that ϕ is p-completely contractive if $\|\phi\|_{pcb} \leq 1$ and ϕ is p-completely isometric if $\|\phi\|_{pcb} = 1$.

Definition 2.5 (see [2]) *Let A and B be filtered L^p operator algebras with filtrations $(A_r)_{r>0}$ and $(B_r)_{r>0}$, respectively. A filtered homomorphism $\phi : A \to B$ is an algebra homomorphism such that*

(1) *ϕ is p-completely bounded ;*

(2) *$\phi(A_r) \subset B_r$ for all $r > 0$.*

If $\phi : A \to B$ is a filtered homomorphism, then it induces a filtered homomorphism $\phi^+ : A^+ \to B^+$ given by $\phi^+(a, z) = (\phi(a), z)$.

Definition 2.6 (see [2]) *Let A be a unital filtered L^p operator algebra. For $0 < \varepsilon < \frac{1}{20}$, $r > 0$ and $N \geq 1$,*

(1) *an element $e \in A$ is called an (ε, r, N)-idempotent if $\|e^2 - e\| < \varepsilon$, $e \in A_r$ and $\max(\|e\|, \|1_{\widetilde{A}} - e\|) \leq N$;*

(2) *if A is unital, an element $u \in A$ is called an (ε, r, N)-invertible if $u \in A_r$, $\|u\| \leq N$, and there exists $v \in A_r$ with $\|v\| \leq N$ such that $\max(\|uv - 1\|, \|vu - 1\|) < \varepsilon$.*

We call v an (ε, r, N)-inverse for u and we call (u, v) an (ε, r, N)-inverse pair. In addition, ε is called the control and r is called the propagation of the (ε, r, N)-idempotent or of the (ε, r, N)-invertible.

Next, we recall the definitions of quantitative K-theory for filtered L^p operator algebras. Given a filtered L^p operator algebra A,

(1) we let $\mathrm{Idem}^{\varepsilon,r,N}(A) := \{e \in A \mid e \text{ is an } (\varepsilon, r, N)\text{-idempotent}\}$;

(2) we set $\mathrm{Idem}_n^{\varepsilon,r,N}(A) := \mathrm{Idem}^{\varepsilon,r,N}(M_n(A))$ for each $n \in \mathbb{N}$;

(3) we have inclusions $\mathrm{Idem}_n^{\varepsilon,r,N}(A) \hookrightarrow \mathrm{Idem}_{n+1}^{\varepsilon,r,N}(A)$, $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$;

(4) we set $\mathrm{Idem}_\infty^{\varepsilon,r,N}(A) := \bigcup_{n\in\mathbb{N}} \mathrm{Idem}_n^{\varepsilon,r,N}(A)$;

(5) we define the equivalence relation $\sim$ on $\mathrm{Idem}_\infty^{\varepsilon,r,N}(A)$ as follows: $e \sim f$ if and only if e and f are $(4\varepsilon, r, 4N)$-homotopic in $\mathrm{Idem}_\infty^{4\varepsilon,r,4N}(A)$;

(6) we denote $[e] := \{f \in \mathrm{Idem}_\infty^{\varepsilon,r,N}(A) \mid f \sim e \text{ in } \mathrm{Idem}_\infty^{\varepsilon,r,N}(A)\}$;

(7) $\mathrm{Idem}_\infty^{\varepsilon,r,N}(A)/\sim := \{[e] \mid e \in \mathrm{Idem}_\infty^{\varepsilon,r,N}(A)\}$ and $[e] + [f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right]$;

(8) $\mathrm{Idem}_\infty^{\varepsilon,r,N}(A)/\sim$ is an abelian semigroup with identity $[0]$.

If we want to keep track of changes of parameters, we write $[e]_{\varepsilon,r,N}$ instead of $[e]$.

Definition 2.7 (see [2]) *Let A be a filtered L^p operator algebra. For $0 < \varepsilon < \frac{1}{20}$, $r > 0$*

and $N \geq 1$*,*

(1) *if* A *is unital, define* $K_0^{\varepsilon,r,N}(A)$ *to be the Grothendieck group of* $\mathrm{Idem}_\infty^{\varepsilon,r,N}(A)/\sim$;

(2) *if* A *is non-unital, define* $K_0^{\varepsilon,r,N}(A) := \ker(\pi_* : K_0^{\varepsilon,r,N}(A^+) \to K_0^{\varepsilon,r,N}(\mathbb{C}))$*, where* $\pi : A^+ \to \mathbb{C}$ *is the usual quotient homomorphism, which is p-completely contractive.*

If $[e]-[f] \in K_0^{\varepsilon,r,N}(A)$, where $e, f \in M_k(\widetilde{A})$, then $[e]-[f] = [e']-[I_k]$ in $K_0^{\varepsilon,r,N}(A)$ for some $e' \in M_{2k}(\widetilde{A})$. Therefore, if we relax control, we can write elements in $K_0^{\varepsilon,r,N}(A)$ in the form $[e]-[I_k]$ with $\pi(e) = \mathrm{diag}(I_k, 0)$.

Given a unital filtered L^p operator algebra A,

(1) we let $GL^{\varepsilon,r,N}(A) := \{u \in A \mid u \text{ is an } (\varepsilon,r,N)\text{-invertible}\}$;

(2) we set $GL_n^{\varepsilon,r,N}(A) := GL^{\varepsilon,r,N}(M_n(A))$ for each positive integer n;

(3) we have inclusions $GL_n^{\varepsilon,r,N}(A) \hookrightarrow GL_{n+1}^{\varepsilon,r,N}(A)$, $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$;

(4) we set $GL_\infty^{\varepsilon,r,N}(A) := \bigcup\limits_{n\in\mathbb{N}} GL_n^{\varepsilon,r,N}(A)$;

(5) we define the equivalence relation $\sim$ on $GL_\infty^{\varepsilon,r,N}(A)$ as follows: $u \sim v$ if and only if u and v are $(4\varepsilon, 2r, 4N)$-homotopic in $GL_\infty^{4\varepsilon,2r,4N}(A)$;

(6) we denote $[u] := \{v \in GL_\infty^{\varepsilon,r,N}(A) \mid v \sim u \text{ in } GL_\infty^{\varepsilon,r,N}(A)\}$;

(7) $GL_\infty^{\varepsilon,r,N}(A)/\sim := \{[u] \mid u \in GL_\infty^{\varepsilon,r,N}(A)\}$ and $[u]+[v] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]$;

(8) $GL_\infty^{\varepsilon,r,N}(A)/\sim$ is an abelian group with identity $[1]$.

If we want to take into account parameter changes, we usually write $[u]_{\varepsilon,r,N}$ instead of $[u]$.

Definition 2.8 (see [2]) *Let* A *be a unital filtered* L^p *operator algebra. For* $0 < \varepsilon < \frac{1}{20}$*,* $r > 0$ *and* $N \geq 1$*,*

(1) *if* A *is unital, define* $K_1^{\varepsilon,r,N}(A) := GL_\infty^{\varepsilon,r,N}(A)/\sim$;

(2) *if* A *is non-unital, define* $K_1^{\varepsilon,r,N}(A) := \ker(\pi_* : K_1^{\varepsilon,r,N}(A^+) \to K_1^{\varepsilon,r,N}(\mathbb{C}))$.

Remark 2.2 (see [2]) If e is an (ε,r,N)-idempotent in A, we can choose a function κ_0 that is holomorphic on a neighborhood of $Sp(e)$, and

$$\kappa_0(z) = \begin{cases} 0, & z \in \overline{B}_{\sqrt{\varepsilon}}(0), \\ 1, & z \in \overline{B}_{\sqrt{\varepsilon}}(1), \end{cases}$$

then we apply holomorphic functional calculus to get an idempotent

$$\kappa_0(e) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \kappa_0(z)(z-e)^{-1} \mathrm{d}z \in A,$$

where γ is the contour $\{z \in \mathbb{C} : |z| = \sqrt{\varepsilon}\} \cup \{z \in \mathbb{C} : |z-1| = \sqrt{\varepsilon}\}$, and

$$\|\kappa_0(e)\| < \frac{N+1}{1-2\sqrt{\varepsilon}},$$

which implies that $\|\kappa_0(e)\| < 2(N+1)$. Since each (ε,r,N)-invertible is invertible, we can define a function κ_1 such that $\kappa_1(u) = u$, thus $\|\kappa_1(u)\| \leq N$.

Definition 2.9 *For any filtered L^p operator algebra A and any positive numbers $r, r', \varepsilon, \varepsilon'$ and $N, N' \geq 1$ with $\varepsilon \leq \varepsilon' < \frac{1}{20}$, $r \leq r'$ and $N \leq N'$, we have natural group homomorphisms:*

(1) $\iota_0 : K_0^{\varepsilon,r,N}(A) \to K_0(A)$, $[e]_{\varepsilon,r,N} \mapsto [\kappa_0(e)]$;

(2) $\iota_1 : K_1^{\varepsilon,r,N}(A) \to K_1(A)$, $[u]_{\varepsilon,r,N} \mapsto [\kappa_1(u)] = [u]$;

(3) $\iota_* = \iota_0 \oplus \iota_1$;

(4) $\iota_0^{\varepsilon',r',N'} : K_0^{\varepsilon,r,N}(A) \to K_0^{\varepsilon',r',N'}(A)$, $[e]_{\varepsilon,r,N} \mapsto [e]_{\varepsilon',r',N'}$;

(5) $\iota_1^{\varepsilon',r',N'} : K_1^{\varepsilon,r,N}(A) \to K_1^{\varepsilon',r',N'}(A)$, $[u]_{\varepsilon,r,N} \mapsto [u]_{\varepsilon',r',N'}$;

(6) $\iota_*^{\varepsilon',r',N'} = \iota_0^{\varepsilon',r',N'} \oplus \iota_1^{\varepsilon',r',N'}$.

Remark 2.3 We sometimes refer to these natural homomorphisms as relaxation of control maps. In addition, from the above definition, we know that the origin of variable parameters of quasi-idempotents or quasi-invertibles, thus we only mark the destination of the parameters to reduce to three superscripts.

Proposition 2.1 (see [2]) *There exists a polynomial $\rho \geq 1$ with positive coefficients such that for any filtered L^p operator algebra A, any $\varepsilon \in \big(0, \frac{1}{20\rho(N)}\big)$, any $r > 0$ and any $N \geq 1$, the following holds:*

Let $[x], [x']$ be in $K_*^{\varepsilon,r,N}(A)$ such that $\iota_*([x]) = \iota_*([x'])$ in $K_*(A)$, there exist $r' \geq r$ and $N' \geq N$ such that

$$\iota_*^{\rho(N)\varepsilon,r',N'}([x]) = \iota_*^{\rho(N)\varepsilon,r',N'}([x']) \quad \text{in } K_*^{\rho(N)\varepsilon,r',N'}(A).$$

Remark 2.4 From the proof of [2, Proposition 3.21], we know that the choice of N' depends on the norm of the homotopy path of the idempotents or invertibles, and we can choose

$$\rho(N) = \begin{cases} 1 + \dfrac{9}{20}(N+1)^2, & * = 0, \\ 1, & * = 1. \end{cases}$$

The item (ii) of the next proposition is a consequence of the preceding proposition.

Proposition 2.2 (see [2]) *Let A be an L^p operator algebra filtered by $(A_r)_{r>0}$.*

(i) *For any $\varepsilon \in \big(0, \frac{1}{20}\big)$ and any $[y] \in K_*(A)$, there exist $r > 0$, $N \geq 1$ and $[x] \in K_*^{\varepsilon,r,N}(A)$ such that $\iota_*([x]) = [y]$.*

(ii) *There exists a polynomial $\rho \geq 1$ with positive coefficients such that the following is satisfied: For $\varepsilon \in \big(0, \frac{1}{20\rho(N)}\big)$, $r > 0$ and $N \geq 1$, let $[x]$ be an element of $K_*^{\varepsilon,r,N}(A)$ such that $\iota_*([x]) = 0$ in $K_*(A)$. Then there exist $r' \geq r$ and $N' \geq N$ such that*

$$\iota_*^{\rho(N)\varepsilon,r',N'}([x]) = 0 \quad \text{in } K_*^{\rho(N)\varepsilon,r',N'}(A).$$

Remark 2.5 From the proof of [2, Proposition 3.20], we may put

$$N = \begin{cases} \|y\| + 1, & * = 0, \\ \|y\| + \|y^{-1}\| + 1, & * = 1 \end{cases}$$

in the item (i) of the above proposition.

Definition 2.10 (see [2]) *A control pair is a pair (λ, h) such that*

(1) $\lambda : [1, \infty) \to [1, \infty)$ *is a non-decreasing function;*

(2) $h : \big(0, \frac{1}{20}\big) \times [1, \infty) \to [1, \infty)$ *is a function such that $h(\cdot, N)$ is non-increasing for fixed N.*

We will write λ_N for $\lambda(N)$ and $h_{\varepsilon,N}$ for $h(\varepsilon, N)$. Given two control pairs (λ, h) and (λ', h'), we say that $(\lambda, h) \le (\lambda', h')$ if $\lambda_N \le \lambda'_N$ and $h_{\varepsilon,N} \le h'_{\varepsilon,N}$ for all $\varepsilon \in \big(0, \frac{1}{20}\big)$ and $N \ge 1$.

Given a filtered L^p operator algebra A, we write the families

$$\mathcal{K}_i(A) = (K_i^{\varepsilon,r,N}(A))_{0<\varepsilon<\frac{1}{20}, r>0, N\ge 1}, \quad \text{where } i \in \{0, 1\}.$$

Definition 2.11 (see [2]) *Let A and B be filtered L^p operator algebras, and let (λ, h) be a control pair. A (λ, h)-controlled morphism $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$, where $i, j \in \{0, 1\}$, is a family*

$$\mathcal{F} = (F^{\varepsilon,r,N})_{0<\varepsilon<\frac{1}{20\lambda_N}, r>0, N\ge 1}$$

of group homomorphisms

$$F^{\varepsilon,r,N} : K_i^{\varepsilon,r,N}(A) \to K_j^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(B)$$

such that whenever $0 < \varepsilon \le \varepsilon' < \frac{1}{20\lambda_{N'}}$, $h_{\varepsilon,N}r \le h_{\varepsilon',N'}r'$ and $N \le N'$, we have the following commutative diagram

$$\begin{array}{ccc}
K_i^{\varepsilon,r,N}(A) & \xrightarrow{\iota_i} & K_i^{\varepsilon',r',N'}(A) \\
{\scriptstyle F^{\varepsilon,r,N}}\downarrow & & \downarrow{\scriptstyle F^{\varepsilon',r',N'}} \\
K_j^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(B) & \xrightarrow{\iota_j} & K_j^{\lambda_{N'}\varepsilon', h_{\varepsilon',N'}r', \lambda_{N'}}(B).
\end{array}$$

We write ι_i for $\iota_i^{\varepsilon',r',N'}$ and ι_j for $\iota_j^{\lambda_{N'}\varepsilon', h_{\varepsilon',N'}r', \lambda_{N'}}$. We say that $\mathcal{F}$ is a controlled morphism if it is a (λ, h)-controlled morphism for some control pair (λ, h).

Definition 2.12 (see [2]) *Let A and B be filtered L^p operator algebras. Let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ and $\mathcal{G} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled and $(\lambda^{\mathcal{G}}, h^{\mathcal{G}})$-controlled morphisms, respectively. Let (λ, h) be a control pair. We write $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ if $(\lambda^{\mathcal{F}}, h^{\mathcal{F}}) \le (\lambda, h)$, $(\lambda^{\mathcal{G}}, h^{\mathcal{G}}) \le (\lambda, h)$, and the following diagram commutes whenever $0 < \varepsilon < \frac{1}{20\lambda_N}$, $r > 0$ and $N \ge 1$:*

$$\begin{array}{ccccc}
 & & K_j^{\lambda^{\mathcal{F}}_N\varepsilon, h^{\mathcal{F}}_{\varepsilon,N}r, \lambda^{\mathcal{F}}_N}(B) & & \\
 & \overset{F^{\varepsilon,r,N}}{\nearrow} & & \overset{\iota_j}{\searrow} & \\
K_i^{\varepsilon,r,N}(A) & & & & K_j^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(B) \\
 & \underset{G^{\varepsilon,r,N}}{\searrow} & & \underset{\iota_j}{\nearrow} & \\
 & & K_j^{\lambda^{\mathcal{G}}_N\varepsilon, h^{\mathcal{G}}_{\varepsilon,N}r, \lambda^{\mathcal{G}}_N}(B). & &
\end{array}$$

Observe that if $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ for some control pair (λ, h), then $\mathcal{F}$ and $\mathcal{G}$ induce the same homomorphism in K-theory.

Definition 2.13 (see [2]) *Let A and B be filtered L^p operator algebras. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled morphism with $(\lambda^{\mathcal{F}}, h^{\mathcal{F}}) \leq (\lambda, h)$.*

(1) *We say that $\mathcal{F}$ is left (resp. right) (λ, h)-invertible if there exists a controlled morphism $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$ such that $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_i(A)}$ (resp. $\mathcal{F} \circ \mathcal{G} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_j(B)}$). In this case, we call $\mathcal{G}$ a left (resp. right) (λ, h)-inverse for $\mathcal{F}$.*

(2) *We say that $\mathcal{F}$ is (λ, h)-invertible or a (λ, h)-isomorphism if there exists a controlled morphism $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$ that is both a left (λ, h)-inverse and a right (λ, h)-inverse for $\mathcal{F}$. In this case, we call $\mathcal{G}$ a (λ, h)-inverse for $\mathcal{F}$.*

We say that $\mathcal{F}$ is a controlled isomorphism if it is a (λ, h)-isomorphism for some control pair (λ, h).

Definition 2.14 (see [2]) *Let A and B be filtered L^p operator algebras. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled morphism with $(\lambda^{\mathcal{F}}, h^{\mathcal{F}}) \leq (\lambda, h)$.*

(1) *We say that $\mathcal{F}$ is (λ, h)-injective if for any $0 < \varepsilon < \frac{1}{20\lambda_N}$, $r > 0$, $N \geq 1$ and $[x] \in K_i^{\varepsilon,r,N}(A)$, if $F^{\varepsilon,r,N}([x]) = 0$ in $K_j^{\lambda_N^F \varepsilon, h^{\mathcal{F}}_{\varepsilon,N} r, \lambda_N^F}(B)$, then $\iota_i^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}([x]) = 0$ in $K_i^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}(A)$.*

(2) *We say that $\mathcal{F}$ is (λ, h)-surjective if for any $0 < \varepsilon < \frac{1}{20(\lambda_F \cdot \lambda)_N}$, $r > 0$, $N \geq 1$ and $[y] \in K_j^{\varepsilon,r,N}(B)$, there exists $[x] \in K_i^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}(A)$ such that*

$$F^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}([x]) = \iota_j^{(\lambda^{\mathcal{F}} \cdot \lambda)_N \varepsilon, (h^{\mathcal{F}} \cdot h)_{\varepsilon,N} r, (\lambda^{\mathcal{F}} \cdot \lambda)_N}([y]) \text{ in } K_j^{(\lambda^{\mathcal{F}} \cdot \lambda)_N \varepsilon, (h^{\mathcal{F}} \cdot h)_{\varepsilon,N} r, (\lambda^{\mathcal{F}} \cdot \lambda)_N}(B).$$

Proposition 2.3 (see [2]) *Let A be a unital filtered L^p operator algebra.*

(i) *If e and f are homotopic as (ε, r, N)-idempotents in A, then there exist $\alpha_N > 0$, an integer k and an α_N-Lipschitz homotopy of $\left(2\varepsilon, r, \frac{5}{2}N\right)$ -idempotents between* $\operatorname{diag}(e, I_k, 0_k)$ *and* $\operatorname{diag}(f, I_k, 0_k)$.

(ii) *If u and v are homotopic as (ε, r, N)-invertibles in A, then there exist $\beta_N > 0$, an integer k and a β_N-Lipschitz homotopy of $((4N^2 + 2)\varepsilon, 2r, 2(N + \varepsilon))$-invertibles between* $\operatorname{diag}(u, I_k)$ *and* $\operatorname{diag}(v, I_k)$.

Remark 2.6 In fact, the proof of item (ii) is similar to that of item (i) (see [2, Lemma 2.29]).

Remark 2.7 Let A be an L^p operator algebra, and let $\otimes$ denote the spatial L^p operator tensor product. $M_n(A)$ can be regarded as $M_n(C) \otimes A$ when $M_n(\mathbb{C})$ is viewed as $B(\ell^p_n)$. Recall from [14, Proposition 1.8, Example 1.10] we see that $\overline{M^p_\infty} = \mathscr{K}(\ell^p)$ for $p \in (1, \infty)$ when $\overline{M^p_\infty}$ denotes $\overline{\bigcup_{n\in\mathbb{N}} M_n(\mathbb{C})}^{\|\cdot\|_{\ell^p}}$. However, when $p = 1$, there is a rank one operator on ℓ^1 that is not in $\overline{M^1_\infty}$.

Now we collect some concepts of [15] concerning L^p operator tensor products. For $p \in [1, \infty)$ and for measure spaces (X, μ) and (Y, ν), there is an L^p tensor product such that we have a canonical isometric isomorphism $L^p(X,\mu)\otimes L^p(Y,\nu)\cong L^p(X\times Y,\mu\times\nu)$ via $(x,y)\mapsto \xi(x)\eta(y)$ for any $\xi\in L^p(X,\mu)$, $\eta\in L^p(Y,\nu)$, this tensor product has the following properties:

(1) Under the previous isomorphism, the linear span of all $\xi\otimes\eta$ is dense in $L^p(X\times Y,\mu\times\nu)$.

(2) $\|\xi\otimes\eta\|_p=\|\xi\|_p\|\eta\|_p$ for all $\xi\in L^p(X,\mu)$ and $\eta\in L^p(Y,\nu)$.

(3) The tensor product is commutative and associative.

(4) If $a\in B(L^p(X_1,\mu_1),L^p(X_2,\mu_2))$ and $b\in B(L^p(Y_1,\nu_1),L^p(Y_2,\nu_2))$, then there exists a unique

$$c\in B(L^p(X_1\times Y_1,\mu_1\times\nu_1),L^p(X_2\times Y_2,\mu_2\times\nu_2))$$

such that $c(\xi\otimes\eta)=a(\xi)\otimes b(\eta)$ for all $\xi\in L^p(X_1,\mu_1)$ and $\eta\in L^p(Y_1,\nu_1)$. We will denote this operator by $a\otimes b$, thus $\|a\otimes b\|=\|a\|\|b\|$.

(5) The tensor product of operators is associative, bilinear and satisfies $(a_1\otimes b_1)(a_2\otimes b_2)=a_1a_2\otimes b_1b_2$.

If $A\subset B(L^p(X,\mu))$ and $B\subset B(L^p(Y,\nu))$ are norm-closed subalgebras, we can define $A\otimes B\subset B(L^p(X\times Y,\mu\times\nu))$ to be the closed linear span of all elements of the form $a\otimes b$ with $a\in A$ and $b\in B$.

Proposition 2.4 (see [2]) *If A is a filtered L^p operator algebra for some $p\in(1,\infty)$, then the homomorphism*

$$A\to\mathscr{K}(\ell^p)\otimes A,\quad a\mapsto\begin{pmatrix} a & & \\ & 0 & \\ & & \ddots\end{pmatrix}$$

induces a group isomorphism (the Morita equivalence)

$$K_*^{\varepsilon,r,N}(A)\to K_*^{\varepsilon,r,N}(\mathscr{K}(\ell^p)\otimes A).$$

For $p=1$, we denote $\mathscr{K}(\ell^1)$ by $\overline{\bigcup_{n\in\mathbb{N}}M_n(\mathbb{C})}^{\|\cdot\|_{\ell^1}}$, then we still have the Morita equivalance.

Proposition 2.5 (see [2]) *If A is a filtered L^1 operator algebra, then we have a group isomorphism*

$$K_*^{\varepsilon,r,N}(\mathscr{K}(\ell^1)\otimes A)\cong K_*^{\varepsilon,r,N}(A).$$

Remark 2.8 For any $r>0$, the L^p operator tensor product $\mathscr{K}(\ell^p)\otimes A$ has a filtration $(\mathscr{K}(\ell^p)\otimes A_r)_{r>0}$.

If $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ is any family of filtered L^p operator algebras. For any $r>0$, we set

$$\mathcal{A}_{c,r}^{\infty}=\prod_{i\in\mathbb{N}}\mathscr{K}(\ell^p)\otimes A_{i,r},$$

and we define the L^p operator algebra $\mathcal{A}_c^{\infty}$ as the closure of $\bigcup_{r>0}\mathcal{A}_{c,r}^{\infty}$ in $\prod_{i\in\mathbb{N}}\mathscr{K}(\ell^p)\otimes A_i$.

Lemma 2.1 *Let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of filtered L^p operator algebras. There exist a control pair (λ, h) independent of the family $\mathcal{A}$ and a (λ, h)-isomorphism*

$$\mathcal{F} = (F^{\varepsilon,r,N})_{0<\varepsilon<\frac{1}{20},r>0,N\geq 1} : \mathcal{K}_*(\mathcal{A}_c^\infty) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(A_i),$$

where

$$F^{\varepsilon,r,N} : K_*^{\varepsilon,r,N}(\mathcal{A}_c^\infty) \to \prod_{i\in\mathbb{N}} K_*^{\varepsilon,r,N}(A_i)$$

is induced on the j-th factor by the projection $\prod\limits_{i\in\mathbb{N}} \mathscr{K}(\ell^p) \otimes A_i \to \mathscr{K}(\ell^p) \otimes A_j$ and up to the Morita equivalence restricted to $\mathcal{A}_c^\infty$.

Remark 2.9 If A_i is unital, then the above Lemma 2.1 is a consequence of Proposition 2.3. In this case, we let $\lambda_N = \frac{5}{2}N$, $h(\cdot, N) = 2$. If A_i is not unital for some i, the proof is similar to that of [14, Lemma 2.14].

3 Quantitative L^p Assembly Maps

In this section, we will introduce L^p localization algebras, L^p Roe algebras and reduced L^p crossed products to define quantitative L^p assembly maps, and establish the connection between the L^p Baum-Connes conjecture and the quantitative L^p Baum-Connes conjecture.

3.1 L^p Roe algebras and L^p localization algebras

In this section, we consider the case of finitely generated groups. Let Γ be a finitely generated group with a length function $\ell : \Gamma \to \mathbb{R}^+$ such that

(1) $\ell(\gamma) = 0$ if and only if $\gamma = e$, where e is the identity element of Γ;

(2) $\ell(\gamma\gamma') \leq \ell(\gamma) + \ell(\gamma')$ for all $\gamma, \gamma' \in \Gamma$;

(3) $\ell(\gamma) = \ell(\gamma^{-1})$ for all $\gamma \in \Gamma$.

We assume that ℓ is the word length

$$\ell(\gamma) = \inf\{d \mid \gamma = \gamma_1 \cdots \gamma_d \text{ with } \gamma_1, \cdots, \gamma_d \in S\},$$

where S is a finite symmetric generating set. Let the ball of radius $r \in (0, \infty)$ around the identity of Γ be

$$B(e, r) = \{\gamma \in \Gamma \mid \ell(\gamma) \leq r\}.$$

Definition 3.1 (see [19]) *Let Γ be a finitely generated group and let $d \geq 0$. The spherical Rips complex of Γ at scale d, denoted by $S_d(\Gamma)$, consists as a set of all formal sums*

$$x = \sum_{\gamma\in\Gamma} t_\gamma \gamma$$

such that each $t_\gamma \in [0, 1]$ with $\sum\limits_{\gamma\in\Gamma} t_\gamma = 1$ and such that the support of x defined by

$$\operatorname{supp}(x) := \{\gamma \in \Gamma \mid t_\gamma \neq 0\}$$

has diameter at most d.

Definition 3.2 (see [19]) *Let Γ be a finitely generated group, and let $S_d(\Gamma)$ be the associated spherical Rips complex at scale d. A semi-simplicial path δ between points x and y in $S_d(\Gamma)$ consists of a sequence of the form*

$$x = x_0, y_0, x_1, y_1, x_2, y_2, \cdots, x_n, y_n = y,$$

where each of $x_1, \cdots, x_n$ and each of $y_0, \cdots, y_{n-1}$ are in Γ. The length of such a path is

$$l(\delta) := \sum_{i=0}^{n} d_{S_d}(x_i, y_i) + \sum_{i=0}^{n-1} d_\Gamma(y_i, x_{i+1}).$$

We define the semi-spherical distance on $S_d(\Gamma)$ by

$$d_{P_d}(x, y) := \inf\{l(\gamma) \mid \gamma \text{ is a semi-simplicial path between } x \text{ and } y\}$$

(*note that a semi-simplicial path between two points always exists*).

The Rips complex of Γ is defined to be the space $P_d(\Gamma)$ equipped with the metric d_{P_d} above.

Remark 3.1 $P_d(\Gamma)$ is a locally finite simplicial complex and is locally compact when endowed with the simplicial topology, and it is endowed with a proper and cocompact action of Γ by left translation.

Definition 3.3 *For $d \geq 0$, we define*

$$Q_d := \Big\{ \sum_{\gamma\in\Gamma} t_\gamma \gamma \in P_d(\Gamma) \mid t_\gamma \in \mathbb{Q} \text{ for all } \gamma \in \Gamma \Big\}.$$

Then Q_d is a Γ-invariant, countable, dense subset of $P_d(\Gamma)$.

Definition 3.4 *Let Γ be a discrete group, and let A be an L^p operator algebra. We say that A is a Γ-L^p operator algebra if $\alpha : \Gamma \to \mathrm{Aut}(A)$ is an action by isometric automorphisms.*

Definition 3.5 (see [15]) *Let (Γ, A, α) be a Γ-L^p operator algebra, and let $(X, \mathcal{B}, \mu)$ be a measure space. Then a covariant representation of (Γ, A, α) on $L^p(X, \mu)$ is a pair (v, π) consisting of a representation $\gamma \mapsto v_\gamma$ from Γ to the invertible operators on $L^p(X, \mu)$ such that $\gamma \mapsto v_\gamma \xi$ is continuous for all $\xi \in L^p(X, \mu)$, and a representation $\pi : A \to B(L^p(X, \mu))$ such that the following covariance condition is satisfied*: $\pi(\alpha_\gamma(a)) = v_\gamma \pi(a) v_\gamma^{-1}$ *for all $\gamma \in \Gamma$ and $a \in A$.*

We say that a covariant representation is isometric if π is isometric.

Definition 3.6 *Let A be a Γ-L^p operator algebra, and let E be a covariant represented L^p space of A. An L^p-module is defined to be an L^p space*

$$L_d = \ell^p(Q_d) \otimes E \otimes \ell^p \otimes \ell^p(\Gamma) \cong \ell^p(Q_d, E \otimes \ell^p \otimes \ell^p(\Gamma))$$

equipped with an isometric Γ-action given by

$$u_\gamma \cdot (\delta_x \otimes e \otimes \eta \otimes \delta_{\gamma'}) = \delta_{x\gamma^{-1}} \otimes \gamma e \otimes \eta \otimes \delta_{\gamma\gamma'}$$

for $x \in Q_d, e \in E, \eta \in \ell^p$ and $\gamma, \gamma' \in \Gamma$.

Remark 3.2 For each $d \geq d_0 \geq 0$, the canonical inclusion $i_{d_0,d} : P_{d_0}(\Gamma) \hookrightarrow P_d(\Gamma)$ is a homeomorphism on its image and a coarse equivalence, and $Q_{d_0} \subset Q_d$. Hence, we have an equivariant isometric inclusion $L_{d_0} \subset L_d$.

Remark 3.3 Let $\mathscr{K}_\Gamma$ be the algebra of compact operators on $\ell^p \otimes \ell^p(\Gamma) \cong \ell^p(\mathbb{N} \times \Gamma)$ equipped with the Γ-action induced by the tensor product of the trivial action on ℓ^p and the left regular representation on $\ell^p(\Gamma)$. Also, we equip the algebra $A \otimes \mathscr{K}_\Gamma$ with the diagonal action of Γ. We say that the representation of $A \otimes \mathscr{K}_\Gamma$ on $E \otimes \ell^p \otimes \ell^p(\Gamma)$ is faithful and covariant if this representation is obtained by tensoring the natural action on E, trivial on ℓ^p and regular on $\ell^p(\Gamma)$.

Next, we will define equivariant L^p Roe algebras and equivariant L^p localization algebras.

Definition 3.7 *Let L_d be the L^p-module as in Definition 3.6, and let T be a bounded linear operator on L_d, which we regard as a $(Q_d \times Q_d)$-indexed matrix $T = (T_{y,z})$ with*

$$T_{y,z} \in B(E \otimes \ell^p \otimes \ell^p(\Gamma))$$

for all $y, z \in Q_d$.

(1) *T is Γ-invariant if $u_\gamma T u_\gamma^{-1} = T$ for all $\gamma \in \Gamma$, i.e., $T_{y,z} = \gamma \cdot T_{y\gamma, z\gamma}$ for all $\gamma \in \Gamma$.*

(2) *The propagation of T is defined to be*

$$\mathrm{prop}(T) := \sup\{d_{P_d(\Gamma)}(y, z) : T_{y,z} \neq 0\}.$$

(3) *T is E-locally compact if $T_{y,z} \in A \otimes \mathcal{K}_\Gamma$ for all $y, z \in Q_d$, and if for each compact subset $G \subset P_d(\Gamma)$, the set*

$$\{(y, z) \in (G \times G) \cap (Q_d \times Q_d) : T_{y,z} \neq 0\}$$

is finite.

Definition 3.8 *Let L_d be the L^p-module, and let $\mathbb{C}[L_d, A]^\Gamma$ denote the algebra of all Γ-invariant, E-locally compact operators on L_d with finite propagation. The equivariant L^p Roe algebra with coefficients in A, denoted by $B^p(P_d(\Gamma), A)^\Gamma$, is defined to be closure of $\mathbb{C}[L_d, A]^\Gamma$ in the operator norm on $B(L_d)$.*

Definition 3.9 *Let L_d be the L^p-module, and let $\mathbb{C}_L[L_d, A]^\Gamma$ denote the algebra of all bounded, uniformly continuous functions $f : [0, \infty) \to \mathbb{C}[L_d, A]^\Gamma$ such that*

$$\mathrm{prop}(f(t)) \to 0 \quad \text{as } t \to \infty.$$

The equivariant L^p localization algebra with coefficients in A, denoted by $B^p_L(P_d(\Gamma), A)^\Gamma$, is the completion of $\mathbb{C}_L[L_d, A]^\Gamma$ with respect to the norm

$$\|f\| := \sup_{t \in [0,\infty)} \|f(t)\|_{B(L_d)}.$$

3.2 The quantitative L^p assembly maps

For $p \in [1, \infty)$, to give a definition of a quantitative L^p assembly map, we replace the equivariant KK-theory by the equivariant K-theory of the L^p localization algebra on the left-hand side of the map and replace the reduced C^* crossed product by the reduced L^p crossed product on the right-hand side of the map. In the setting of L^p operator algebras, we need to study reduced L^p crossed products and L^p Baum-Connes assembly maps.

Definition 3.10 *Let A be a Γ-L^p operator algebra, and let E be an L^p representation space of A. The reduced L^p crossed product $A \rtimes_{\alpha,\lambda} \Gamma$ is the completion of $C_c(\Gamma, A, \alpha)$ in the operator norm on $B(E \otimes \ell^p(\Gamma))$.*

Remark 3.4 If A is a matrix algebra $M_n(\mathbb{C})$ or a commutative algebra $C(X)$ for some compact space X, then the above definition is identical with Phillips's reduced L^p crossed products (see [15, Definition 3.3]) since it is independent of the representation of A (see [18, Lemma 2.6]).

Remark 3.5 In the following, we will write $A \rtimes \Gamma$ for $A \rtimes_{\alpha,\lambda} \Gamma$. Note that the identification between $A \rtimes \Gamma$ and $B^p(P_d(\Gamma), A)^\Gamma$ is derived from the Morita equivalence between $C_c(\Gamma, A, \alpha)$ and $\mathbb{C}[L_d, A]^\Gamma$. In addition, for $r > 0$, the reduced L^p crossed product $A \rtimes \Gamma$ has a filtration

$$(A \rtimes \Gamma)_r := \{f \in C_c(\Gamma, A) \text{ with } \mathrm{supp}(f) \in B(e, r)\}.$$

Definition 3.11 *Let A be an L^p operator algebra. For $N \geq 1$,*

(1) *an element $z \in A$ is called an N-idempotent if $z^2 = z$ and $\|z\| \leq N$;*

(2) *if A is unital, an element $w \in A$ is called an N-invertible if w is invertible and $\max\{\|w\|, \|w^{-1}\|\} \leq N$.*

Then we will define a variant of K-theory of L^p operator algebras, which is labeled by the norm of the element and the norm of the homotopy path.

Given an L^p operator algebra A, for $N \geq 1$,

(1) we set $\mathrm{Idem}^N(A) := \{z \in A \mid z \text{ is an } N\text{-idempotent}\}$;

(2) we let $\mathrm{Idem}^N_m(A) = \mathrm{Idem}^N(M_m(A))$ for each $m \in \mathbb{N}$;

(3) we have inclusions $\mathrm{Idem}^N_m(A) \hookrightarrow \mathrm{Idem}^N_{m+1}(A)$, $z \mapsto \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$;

(4) we put $\mathrm{Idem}^N_\infty(A) := \bigcup_{m \in \mathbb{N}} \mathrm{Idem}^N_m(A)$;

(5) we define the equivalence relation $\sim$ on $\mathrm{Idem}^N_\infty(A)$ as follows: $z \sim z'$ if z and z' are homotopic in $\mathrm{Idem}^{4N}_\infty(A)$;

(6) we denote by $[z]$ the equivalence class of $z \in \mathrm{Idem}^N_\infty(A)$;

(7) we equip $\mathrm{Idem}^N_\infty(A)/\sim$ with the addition given by $[z] + [z'] = [\mathrm{diag}(z, z')]$;

(8) $\mathrm{Idem}^N_\infty(A)/\sim$ is an abelian semigroup with identity $[0]$.

If we wish to keep track of changes in the norm, we write $[z]_N$ instead of $[z]$.

Definition 3.12 *Let A be an L^p operator algebra. For $N \geq 1$,*

(1) *if A is unital, define $K_0^N(A)$ to be the Grothendieck group of* $\mathrm{Idem}_\infty^N(A)/\sim$;

(2) *if A is non-unital, define*

$$K_0^N(A) := \ker(\pi_* : K_0^N(A^+) \to \mathbb{Z}).$$

If $[z] - [z'] \in K_0^N(A)$, where $z, z' \in M_k(\widetilde{A})$, then $[z] - [z'] = [z''] - [I_k]$ in $K_0^N(A)$ for some $z'' \in M_{2k}(\widetilde{A})$. Hence, each element of $K_0^N(A)$ can be written by $[z]-[I_k]$ with $\pi(z) = \mathrm{diag}(I_k, 0)$.

Given a unital L^p operator algebra A, for $N \geq 1$,

(1) we set $GL^N(A) := \{w \in A \mid w \text{ is an } N\text{-invertible}\}$;

(2) we let $GL_m^N(A) = GL^N(M_m(A))$ for each $m \in \mathbb{N}$;

(3) we have inclusions $GL_m^N(A) \hookrightarrow GL_{m+1}^N(A)$, $w \mapsto \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$;

(4) we put $GL_\infty^N(A) := \bigcup\limits_{m \in \mathbb{N}} GL_m^N(A)$;

(5) we define the equivalence relation $\sim$ on $GL_\infty^N(A)$ as follows: $w \sim w'$ if w and w' are homotopic in $GL_\infty^{4N}(A)$;

(6) we denote by $[w]$ the equivalence class of $w \in GL_\infty^N(A)$;

(7) we equip $GL_\infty^N(A)/\sim$ with the addition defined by $[w] + [w'] = [\mathrm{diag}(w, w')]$;

(8) $GL_\infty^N(A)/\sim$ is an abelian group with identity $[1]$.

If we wish to keep track of changes in norm, we write $[w]_N$ instead of $[w]$.

Definition 3.13 *Let A be an L^p operator algebra. For $N \geq 1$,*

(1) *if A is unital, define $K_1^N(A) := GL_\infty^N(A)/\sim$;*

(2) *if A is non-unital, define $K_1^N(A) := K_1^N(A^+)$.*

In the odd case, each element of $K_1^N(A)$ can be written as $[w]$ satisfying $\pi(w) = I_k$. Observe that there is a natural map $K_*^N(A) \to K_*^{N'}(A)$ if $N \leq N'$ and $K_*(A) = \lim\limits_{N\to\infty} K_*^N(A)$.

The evaluation-at-zero homomorphism

$$ev_0 : B_L^p(P_d(\Gamma), A)^\Gamma \to B^p(P_d(\Gamma), A)^\Gamma$$

induces a homomorphism on K-theory

$$ev_* : K_*(B_L^p(P_d(\Gamma), A)^\Gamma) \to K_*(B^p(P_d(\Gamma), A)^\Gamma).$$

Definition 3.14 (see [3]) *Let A be a Γ-L^p operator algebra. We define an L^p assembly map*

$$\mu_{A,*}^d : K_*(B_L^p(P_d(\Gamma), A)^\Gamma) \xrightarrow{ev_*} K_*(B^p(P_d(\Gamma), A)^\Gamma) \cong K_*(A \rtimes \Gamma),$$

which gives rise to a homomorphism

$$\mu_{A,*} : \lim_{d>0} K_*(B_L^p(P_d(\Gamma), A)^\Gamma) \to K_*(A \rtimes \Gamma)$$

called the L^p Baum-Connes assembly map. Moreover, the L^p Baum-Connes conjecture for Γ predicts that the L^p Baum-Connes assembly map $\mu_{A,*}$ is an isomorphism.

Subsequently, we will give a definition of a quantitative L^p assembly map. Let us do some preparation. Considering the even case, the odd case is similar. Let $[z]$ be in $K_0^N(B_L^p(P_d(\Gamma),A)^\Gamma)$ with $z\in \mathrm{Idem}_m^N(\widetilde{B_L^p}(P_d(\Gamma),A)^\Gamma)$ for some m. Then for any $0<\varepsilon<\frac{1}{20}$, there exist $r'>0$, $\widetilde{z}\in \mathrm{Idem}_m(\mathbb{C}_L\widetilde{[L_d,A]}^\Gamma_{r'})$ such that $\|z-\widetilde{z}\|<\frac{\varepsilon}{6N(N+1)^2}$, then $\widetilde{z}$ is an $(\varepsilon,r',2N)$-idempotent in $M_m(\mathbb{C}_L\widetilde{[L_d,A]}^\Gamma)$ and $\iota_0([\widetilde{z}]_{\varepsilon,r',2N})=[z]$ (see [2, Proposition 3.20]). Observe that the propagation of $\widetilde{z}$ tends to zero when t goes to infinity. Hence, for $r>0$, we can choose $t\in[0,\infty)$ such that the prop$(\widetilde{z}_t)\le r$. Since $\|z_t-\widetilde{z}_t\|\le\|z-\widetilde{z}\|<\frac{\varepsilon}{6N(N+1)^2}$, we get that $\widetilde{z}_t$ is an $(\varepsilon,r,2N)$-idempotent in $M_m(\mathbb{C}\widetilde{[L_d,A]}^\Gamma)$ and $\iota_0([\widetilde{z}_t]_{\varepsilon,r,2N})=[z_t]$ by applying [2, Proposition 3.20].

Definition 3.15 *Let A be a Γ-L^p operator algebra. For $0<\varepsilon<\frac{1}{20}$, $r>0$, $N\ge 1$ and $d>0$, we define a quantitative L^p assembly map*

$$\mu_{A,*}^{\varepsilon,r,N,d}:K_*^N(B_L^p(P_d(\Gamma),A)^\Gamma)\to K_*^{\varepsilon,r,9N}(B^p(P_d(\Gamma),A)^\Gamma)\cong K_*^{\varepsilon,r,9N}(A\rtimes\Gamma),\quad [z]\mapsto[\widetilde{z}_t]_{\varepsilon,r,9N}$$

for some $t\in[0,\infty)$ satisfying

$$\iota_*([\widetilde{z}_t]_{\varepsilon,r,9N})=[z_t]\quad \text{in } K_*(A\rtimes\Gamma).$$

Remark 3.6 Put $B=B_L^p(P_d(\Gamma),A)^\Gamma$. In the even case, If $[z]=[z']\in K_0^N(B)$, then $[z]+[g]=[z']+[g]$ in $\mathrm{Idem}_\infty^N(\widetilde{B})/\sim$ for some g in $\mathrm{Idem}_k^N(\widetilde{B})$, thus $\mathrm{diag}(z,g)$ and $\mathrm{diag}(z',g)$ are homotopic in $\mathrm{Idem}_\infty^{4N}(\widetilde{B})$. Let $(Z^s)_{s\in[0,1]}$ be a homotopy of $4N$-idempotents between $\mathrm{diag}(z,g)$ and $\mathrm{diag}(z',g)$, and let $0=s_0<s_1<\cdots<s_k=1$ be such that

$$\|Z^{s_i}-Z^{s_{i-1}}\|<\frac{\varepsilon}{6(10N+1)}\qquad\text{for } i=1,\cdots,k.$$

For each i, there exist $r_i>0$, $\widetilde{Z^{s_i}}\in M_m(\widetilde{B_{r_i}})$ such that $\|Z^{s_i}-\widetilde{Z^{s_i}}\|<\frac{\varepsilon}{30N(5N+1)^2}$. Then $\widetilde{Z^{s_i}}$ is an $(\varepsilon,r_i,5N)$-idempotent in $M_m(\widetilde{B})$ and $\iota_0([\widetilde{Z^{s_i}}])=[Z^{s_i}]$ in $K_0(B)$ (see [2, Proposition 3.20]). For $r>0$, by the definition of the localization algebra, we can choose an appropriate t_i in $[0,\infty)$ such that $Z_{t_i}^{s_i}$ is in $M_m(\widetilde{A\rtimes\Gamma})$ and the propagation of $Z_{t_i}^{s_i}$ is no more than r. Let $t=\max\limits_{0\le i\le k}t_i$, and define $\widetilde{Z_t^l}=\frac{l-s_{i-1}}{s_i-s_{i-1}}\widetilde{Z_t^{s_i}}+\frac{s_i-l}{s_i-s_{i-1}}\widetilde{Z_t^{s_{i-1}}}$ for $l\in[s_{i-1},s_i]$. Then $\widetilde{Z_t^l}$ is a homotopy of $(\varepsilon,r,5N)$-idempotent in $M_m(\widetilde{A\rtimes\Gamma})$ between $\widetilde{Z_t^0}$ and $\widetilde{Z_t^1}$. The odd case is similar: We can also construct a homotopy of $(\varepsilon,r,9N)$-invertible in $M_m(\widetilde{A\rtimes\Gamma})$. Note that $\max\{5N,9N\}=9N$. Hence, for any $[z]\in K_*^N(B)$, there exists a unique element $[\widetilde{z}_t]_{\varepsilon,r,9N}\in K_*^{\varepsilon,r,9N}(A\rtimes\Gamma)$ such that $\iota_*([\widetilde{z}_t]_{\varepsilon,r,9N})=[z_t]$ for some $t\in[0,\infty)$ in $K_*(A\rtimes\Gamma)$. Therefore, the quantitative L^p assembly map $\mu_{A,*}^{\varepsilon,r,N,d}$ is well-defined.

Moreover, the quantitative L^p assembly maps are compatible with the usual ones, namely, if $[z]$ is an element of $K_*^N(B_L^p(P_d(\Gamma),A)^\Gamma)$, then

$$\mu_{A,*}^d([z])=\iota_*\circ\mu_{A,*}^{\varepsilon,r,N,d}([z]_N)\quad\text{in } K_*(A\rtimes\Gamma).\tag{3.1}$$

For any positive numbers d, d' such that $d \leq d'$, we denote by

$$i^N_{d,d',*} : K^N_*(B^p_L(P_d(\Gamma), A)^\Gamma) \to K^N_*(B^p_L(P_{d'}(\Gamma), A)^\Gamma),$$

the homomorphism induced by the canonical inclusion $i_{d,d'} : P_d(\Gamma) \hookrightarrow P_{d'}(\Gamma)$, then

$$\mu^{\varepsilon,r,N,d}_{A,*} = \mu^{\varepsilon,r,N,d'}_{A,*} \circ i^N_{d,d',*},$$

which implies that $\mu^d_{A,*} = \mu^{d'}_{A,*} \circ i_{d,d',*}$. Moreover, for $0 < \varepsilon \leq \varepsilon' < \frac{1}{20}$, $0 < r \leq r'$ and $1 \leq N \leq N'$, we have

$$\iota^{\varepsilon',r',9N'}_* \circ \mu^{\varepsilon,r,N,d}_{A,*} = \mu^{\varepsilon',r',N',d}_{A,*}. \tag{3.2}$$

For $N \geq 1$, the evaluation-at-zero homomorphism

$$ev_0 : B^p_L(P_d(\Gamma), A)^\Gamma \to B^p(P_d(\Gamma), A)^\Gamma$$

induces a homomorphism on a variant of K-theory

$$ev^N_* : K^N_*(B^p_L(P_d(\Gamma), A)^\Gamma) \to K^N_*(B^p(P_d(\Gamma), A)^\Gamma).$$

Definition 3.16 *Let A be a Γ-L^p operator algebra. For $N \geq 1$, we define an N-L^p assembly map*

$$\mu^{N,d}_{A,*} : K^N_*(B^p_L(P_d(\Gamma), A)^\Gamma) \xrightarrow{ev^N_*} K^N_*(B^p(P_d(\Gamma), A)^\Gamma) \cong K^N_*(A \rtimes \Gamma),$$

which gives rise to a homomorphism

$$\mu^N_{A,*} : \lim_{d>0} K^N_*(B^p_L(P_d(\Gamma), A)^\Gamma) \to K^N_*(A \rtimes \Gamma)$$

called the N-L^p Baum-Connes assembly map.

Remark 3.7 When A is a C^*-algebra, the N-L^p Baum-Connes assembly map is indeed the Baum-Connes assembly map. In fact, in the context of C^*-algebras in [1], idempotents are homotopic to projections and invertibles are homotopic to unitaries. And the norm of the projection or the unitary is no more than 1.

Definition 3.17 *Let A and B be L^p operator algebras, and let $\omega : [1, \infty) \to [1, \infty)$ be a non-decreasing function. We say that $F^N : K^N_i(A) \to K^N_j(B)$ is ω-surjective if for any integer $N \geq 1$ and $[y] \in K^N_j(B)$, there exists $[x] \in K^{\omega(N)}_i(A)$ such that*

$$F^{\omega(N)}([x]) = [y] \text{ in } K^{\omega(N)\cdot N}_j(B).$$

Remark 3.8 By the proof of [3, Theorem 5.17], we know that if $\Gamma \curvearrowright X$ has finite dynamical complexity, then the N-L^p Baum-Connes assembly map for $\Gamma \curvearrowright X$ is ω-surjective, and the function ω depends on the dynamic asymptotic dimension m and Mayer-Vietoris control pair (λ, h). In addition, we may use the term controlled-surjective when we do not want to emphasize the function ω.

Definition 3.18 *Let A be a filtered L^p operator algebra. For $0<\varepsilon<\frac{1}{20}$, $r>0$ and $N\geq 1$, we have a canonical group homomorphism*

$$\iota_*^N : K_*^{\varepsilon,r,N}(A)\to K_*^{4N}(A),\quad [z]_{\varepsilon,r,N}\mapsto [\kappa_*(z)]_{4N}.$$

Furthermore, the quantitative L^p assembly maps are compatible with the N-L^p assembly maps, namely, if $[z]$ is the element of $K_*^N(B_L^p(P_d(\Gamma),A)^\Gamma)$, then

$$\mu_{A,*}^{36N,d}([z]_{36N})=\iota_*^{9N}\circ\mu_{A,*}^{\varepsilon,r,N,d}([z]_N)\quad \text{in } K_*^{36N}(A\rtimes\Gamma).$$

Proposition 3.1 *There exists a polynomial $\rho\geq 1$ with positive coefficients such that for any filtered L^p operator algebra A, any $\varepsilon\in(0,\frac{1}{20\rho(N)})$, any $r>0$ and any $N\geq 1$, the following holds: Let $[x],[x']$ be in $K_*^{\varepsilon,r,N}(A)$ such that $\iota_*^N([x])=\iota_*^N([x'])$ in $K_*^{4N}(A)$, there exists $r'\geq r$ such that*

$$[x]_{\rho(N)\varepsilon,r',33N}=[x']_{\rho(N)\varepsilon,r',33N}\quad \textit{in } K_*^{\rho(N)\varepsilon,r',33N}(A).$$

Proof (i) In the even case, let $(g_t)_{t\in[0,1]}$ be a homotopy of $16N$-idempotents in $M_n(\widetilde{A})$ between $\kappa_0(x)$ and $\kappa_0(x')$. Then $G:=(g_t)$ is a $16N$-idempotent in $C([0,1],M_n(\widetilde{A}))$. There exist $r'\geq r$ and $H:=(h_t)\in C([0,1],M_n(\widetilde{A_{r'}}))$ such that $\|H-G\|<\frac{\varepsilon}{68N}$. In particular, we have $\|h_0-\kappa_0(x)\|<\frac{\varepsilon}{68N}$ and $\|h_1-\kappa_0(x')\|<\frac{\varepsilon}{68N}$. Then h_t is an $(\varepsilon,r',17N)$-idempotent in $M_n(\widetilde{A})$ for each $t\in[0,1]$. Also

$$\begin{aligned}\|h_0-x\|&<\|h_0-\kappa_0(x)\|+\|\kappa_0(x)-x\|\\&<\frac{\varepsilon}{68N}+\frac{2(N+1)\varepsilon}{(1-\sqrt{\varepsilon})(1-2\sqrt{\varepsilon})}\\&<6(N+1)\varepsilon\end{aligned}$$

and similarly $\|h_1-x'\|<6(N+1)\varepsilon$. Then h_0 and x are $(\varepsilon',r',17N)$-homotopic, where $\varepsilon'=\varepsilon+\frac{1}{4}(6N+6)^2\varepsilon^2$, and similarly for h_1 and x'. Hence $[x]_{\varepsilon',r',17N}=[x']_{\varepsilon',r',17N}$.

(ii) In the odd case, let $(f_t)_{t\in[0,1]}$ be a homotopy of $16N$-invertibles in $M_n(\widetilde{A})$ between x and x'. The path $F:=(f_t)$ can be regarded as an invertible element in $C([0,1],M_n(\widetilde{A}))$. Then there exist $r'\geq r$ and $W\in C([0,1],M_n(\widetilde{A_{r'}}))$ such that

$$\|W-F\|<\frac{1}{33N}(\varepsilon-\max\{\|xy-1\|,\|yx-1\|,\|x'y'-1\|,\|y'x'-1\|\}),$$

where y is an (ε,r,N)-inverse for x and y' is an (ε,r,N)-inverse for x'. Then W is an $(\varepsilon,r',33N)$-invertible in $C([0,1],M_n(\widetilde{A}))$, and we have a homotopy of $(\varepsilon,r',33N)$-invertibles $x\sim W_0\sim W_1\sim x'$.

3.3 Quantitative statements

Oyono-Oyono established the connection between the Baum-Connes conjecture and the quantitative Baum-Connes conjecture in [13]. In parallel, we will give the connection between the L^p Baum-Connes conjecture and the quantitative L^p Baum-Connes conjecture.

For a Γ-L^p operator algebra A and positive numbers $d, d', r, r', \varepsilon, \varepsilon', N, N'$ with $d \le d'$, $\varepsilon \le \varepsilon' < \frac{1}{20}$, $r \le r'$ and $1 \le N \le N'$, let us consider the following statements:

(1) $QI_{A,*}(d, d', \varepsilon, r, N)$: For every $[x] \in K_*^N(B_L^p(P_d(\Gamma), A)^\Gamma)$, then

$$\mu_{A,*}^{\varepsilon,r,N,d}([x]) = 0 \quad \text{in } K_*^{\varepsilon,r,9N}(A \rtimes \Gamma)$$

implies that $i_{d,d',*}([x]) = 0$ in $K_*(B_L^p(P_{d'}(\Gamma), A)^\Gamma)$.

(2) $QS_{A,*}(d, \varepsilon, \varepsilon', r, r', N, N')$: For every $[y] \in K_*^{\varepsilon,r,N}(A \rtimes \Gamma)$, there exists an element $[x] \in K_*^{N'}(B_L^p(P_d(\Gamma), A)^\Gamma)$ such that

$$\mu_{A,*}^{\varepsilon',r',N',d}([x]) = \iota_*^{\varepsilon',r',9N'}([y]) \quad \text{in } K_*^{\varepsilon',r',9N'}(A \rtimes \Gamma).$$

Using equation (3.1) and Proposition 2.1, we get the following proposition.

Proposition 3.2 *Let Γ be a finitely generated group, and let A be a Γ-L^p operator algebra. For a positive number ε with $\varepsilon < \frac{1}{20}$:*

(i) *Assume that for any $r > 0, N \ge 1$ and $d > 0$, there exists $d' \ge d$ such that $QI_{A,*}(d, d', \varepsilon, r, N)$ is satisfied. Then $\mu_{A,*}$ is injective.*

(ii) *Assume that for any $r > 0$ and $N \ge 1$, there exist positive numbers ε', d, r' and N' with $\varepsilon \le \varepsilon' < \frac{1}{20}$, $r \le r'$, $N \le N'$ and $d > 0$ such that $QS_{A,*}(d, \varepsilon, \varepsilon', r, r', N, N')$ is true. Then $\mu_{A,*}$ is surjective.*

The following results construct the connection between quantitative injectivity (resp. surjectivity) and injectivity (resp. surjectivity) of the L^p Baum-Connes assembly map.

Theorem 3.1 *Let Γ be a discrete group, and let A be a Γ-L^p operator algebra. Then the following two statements are equivalent*:

(i) *$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}$ is injective.*

(ii) *For $0 < \varepsilon < \frac{1}{20}$, $r > 0$, $N \ge 1$ and $d > 0$, there exists $d' \ge d$ such that $QI_{A,*}(d, d', \varepsilon, r, N)$ holds.*

Proof The proof relies on Proposition 3.3, which will be proved later. Suppose (ii) holds. Let $[x]$ be in $K_*(B_L^p(P_d(\Gamma), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))^\Gamma)$ for some $d > 0$ such that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^d([x]) = 0 \quad \text{in } K_*(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma).$$

Then there exists $N' \ge 1$ such that $[z] \in K_*^{N'}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$, thus $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\varepsilon',r',N',d}([x])$ is an element of $K_*^{\varepsilon',r',9N'}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$. By equation (3.1), we obtain that

$$\iota_*(\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\varepsilon',r',N',d}([x])) = 0 \quad \text{for any } \varepsilon \in \Big(0, \frac{1}{20}\Big).$$

Hence, by Proposition 2.2 (ii) and equation (3.2), there exist $\varepsilon \ge \varepsilon'$, $r \ge r'$ and $N \ge N'$ such that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\varepsilon,r,N,d}([x]) = 0 \quad \text{in } K_*^{\varepsilon,r,9N}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma).$$

According to Proposition 3.3, we have an isomorphism

$$K_*(B_L^p(P_d(\Gamma),\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A))^\Gamma)\xrightarrow{\cong}K_*(B_L^p(P_d(\Gamma),A)^\Gamma)^{\mathbb{N}} \tag{3.3}$$

induced on the j-th factor by the projection $\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A)\to\mathscr{K}(\ell^p)\otimes A$ and up to the Morita equivalence

$$K_*(B_L^p(P_d(\Gamma),A)^\Gamma)\cong K_*(B_L^p(P_d(\Gamma),\mathscr{K}(\ell^p)\otimes A)^\Gamma). \tag{3.4}$$

Assume that $([x_m])_{m\in\mathbb{N}}$ is the element in $K_*(B_L^p(P_d(\Gamma),A)^\Gamma)^{\mathbb{N}}$ corresponding to $[x]$ under this identification, and let $d'\geq d$ be a positive number such that $QI_{A,*}(d,d',\varepsilon,r,N)$ holds. By naturality of the quantitative L^p assembly maps, we get that

$$\mu_{A,*}^{\varepsilon,r,N,d}([x_m])=0\quad\text{in } K_*^{\varepsilon,r,9N}(B_L^p(P_d(\Gamma),A)^\Gamma),$$

which implies that $i_{d,d',*}([x_m])=0$ in $K_*(B_L^p(P_{d'}(\Gamma),A)^\Gamma)$ for each integer m. Finally, using equation (3.3), we obtain that

$$i_{d,d',*}([x])=0\quad\text{in } K_*(B_L^p(P_{d'}(\Gamma),\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A))^\Gamma).$$

Hence $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}$ is injective. Thus (ii) implies (i).

Suppose (ii) is false. In the even case, there exist ε in $\left(0,\frac{1}{20}\right)$, $r>0$, $N\geq 1$ and $d>0$ such that for all $d'\geq d$, the statement $QI_{A,0}(d,d',\varepsilon,r,N)$ does not hold. So it suffices to prove that $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),0}$ is not injective. Let $(d_m)_{m\in\mathbb{N}}$ be an increasing and unbounded sequence of positive numbers such that $d_m\geq d$ for all $m\in\mathbb{N}$. For each positive integer m, let $[x_m]$ be in $K_0^N(B_L^p(P_d(\Gamma),A)^\Gamma)$ such that

$$\mu_{A,0}^{\varepsilon,r,N,d}([x_m])=0\quad\text{in } K_0^{\varepsilon,r,9N}(A\rtimes\Gamma)$$

but

$$i_{d,d_i,0}([x_m])\neq 0\quad\text{in } K_0(B_L^p(P_{d_m}(\Gamma),A)^\Gamma).$$

Assume that $[x]$ is the element in $K_0^N(B_L^p(P_d(\Gamma),\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A))^\Gamma)$ corresponding to $([x_m])_{m\in\mathbb{N}}$ under the identification of equation (3.3). Let $(e_m)_{m\in\mathbb{N}}$ be a family of $(\varepsilon,r,9N)$-idempotents with e_m in $M_{n_k}(\widetilde{A\rtimes\Gamma})$ for some n_k such that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),0}^{\varepsilon,r,N,d}([x])=[(e_m)_{m\in\mathbb{N}}]_{\varepsilon,r,9N}\quad\text{in } K_0^{\varepsilon,r,9N}(\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A)\rtimes\Gamma).$$

By naturality of $\mu_{A,0}^{\varepsilon,r,N,d}$, we know that $[e_m]_{\varepsilon,r,9N}=0$ in $K_0^{\varepsilon,r,N}(A\rtimes\Gamma)$ for all integers m, hence

$$\iota_0([(e_m)_{m\in\mathbb{N}}]_{\varepsilon,r,9N})=0\quad\text{in } K_0(\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A)\rtimes\Gamma).$$

This gives $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),0}^d([x])=\iota_0\circ\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),0}^{\varepsilon,r,N,d}([x])=0$. For each positive integer m, $i_{d,d_m,0}([x_m])\neq 0$ implies $i_{d,d_m,0}([x])\neq 0$, thus we see that $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),0}$ is not injective, hence (i) is false. In the odd case, we have a similar proof.

Theorem 3.2 *Let Γ be a discrete group. Assume that for any Γ-L^p operator algebra A, there exists a polynomial $\rho \geq 1$ with positive coefficients such that for any ε in $\left(0, \frac{1}{20\rho(N)}\right)$, $r > 0$ and $N \geq 1$, there exist $r' \geq r$, $N' \geq N$ and $d > 0$ such that $QS_{A,*}(d, \varepsilon, \rho(N)\varepsilon, r, r', N, N')$ holds. Then $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}$ is surjective.*

Proof The proof relies on Proposition 3.3, which will be proved later. Let ρ be as in Proposition 2.1. Suppose the statement $QS_{A,*}(d, \varepsilon, \rho(N)\varepsilon, r, r', N, N')$ holds. Let $[z]$ be the element in $K_*(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$ and let $[y]$ be in $K_*^{\varepsilon,r',N'}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$ such that $\iota_*([y]) = [z]$ with $\varepsilon \in \left(0, \frac{1}{20\rho(N)}\right)$, $r > 0$ and $N \geq 1$. Let $[y_i]$ be the image of $[y]$ under the composition

$$K_*^{\varepsilon,r,N}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma) \to K_*^{\varepsilon,r,N}(\mathscr{K}(\ell^p) \otimes A \rtimes \Gamma) \xrightarrow{\cong} K_*^{\varepsilon,r,N}(A \rtimes \Gamma), \tag{3.5}$$

where the first map is induced on the j-th factor by the projection

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \to \mathscr{K}(\ell^p) \otimes A$$

and the second map is the Morita equivalence of Propositions 2.4–2.5. Let d, r' and N' be positive numbers with $r' \geq r$ and $N' \geq N$ such that $QS_{A,*}(d, \varepsilon, \rho(N)\varepsilon, r, r', N, N')$ holds. Then for each positive integer m, there exists $[x_m]$ in $K_*^{N'}(B_L^p(P_d(\Gamma), A)^\Gamma)$ such that

$$\mu_{A,*}^{\rho(N)\varepsilon,r',N',d}([x_m]) = \iota_*^{\rho(N)\varepsilon,r',9N'}([y_m]) \quad \text{in } K_*^{\rho(N)\varepsilon,r',9N'}(A \rtimes \Gamma).$$

Let $[x]$ be the element of $K_*^{N'}(B_L^p(P_d(\Gamma), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))^\Gamma)$ corresponding to $([x_m])_{m\in\mathbb{N}}$ under the identification of equation (3.3). By naturality of the quantitative L^p assembly maps, we get that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\rho(N)\varepsilon,r',N',d}([x]) = \iota_*^{\rho(N)\varepsilon,r',9N'}([y])$$

in $K_*^{\rho(N)\varepsilon,r',9N'}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$. Hence, we conclude that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{d}([x]) = \iota_*([y]) = [z],$$

and therefore $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}$ is surjective.

The next theorem relates controlled-surjectivity of the $\mathscr{N}$-L^p Baum-Connes assembly map and quantitative surjectivity.

Theorem 3.3 *Let Γ be a discrete group. Assume that for any Γ-L^p operator algebra A and any positive integer $\mathscr{N}$, there exists a non-decreasing function $\omega : [1, \infty) \to [1, \infty)$ such that $\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\mathscr{N}}$ is ω-surjective. Then for some polynomial $\rho \geq 1$ with positive coefficients and for any ε in $\left(0, \frac{1}{20\rho(9N\omega(4N))}\right)$, $r > 0$ and $N \geq 1$, there exist $r' \geq r$, $N' \geq N$ and $d > 0$ such that $QS_{A,*}(d, \varepsilon, \rho(9N\omega(4N))\varepsilon, r, r', N, N')$ holds.*

Proof Assume that this statement does not hold. Then there exist

(1) ε in $\left(0, \frac{1}{20\rho(N)}\right)$, $r > 0$ and $N \geq 1$,

(2) an unbounded increasing sequence $(r_m)_{m\in\mathbb{N}}$ with $r_m \geq r$,

(3) an unbounded increasing sequence $(N_m)_{m\in\mathbb{N}}$ with $N_m \geq N$,

(4) an unbounded increasing sequence $(d_m)_{m\in\mathbb{N}}$ with $d_m > 0$,

(5) an element $[y_m]$ in $K_*^{\varepsilon,r,N}(A \rtimes \Gamma)$,

such that for each $m \in \mathbb{N}$ and any $[x_m]$ in $K_*^{N_m}(B_L^p(P_{d_m}(\Gamma), A)^\Gamma)$,

$$\iota_*^{\rho(9N\omega(4N))\varepsilon,r_m,9N_m}([y_m]) \neq \mu_{A,*}^{\rho(9N\omega(4N))\varepsilon,r_m,N_m,d_m}([x_m])$$

in $K_*^{\rho(9N\omega(4N))\varepsilon,r_m,9N_m}(A \rtimes \Gamma)$. According to equation (3.5), there exists

$$[y] \in K_*^{\varepsilon,r,N}(\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$$

such that for every positive integer m, the image of $[y]$ is $[y_m]$. Since $\mu^{\mathscr{N}}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}$ is ω-surjective, then for some $d' > 0$ there exists $[x]$ in $K_*^{\omega(4N)}(B_L^p(P_{d'}(\Gamma), \ell^\infty(\mathbb{N},\mathscr{K}(\ell^p) \otimes A))^\Gamma)$ such that

$$\iota_*^N([y]) = \mu^{\omega(4N),d'}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}([x]) \quad \text{in } K_*^{\omega(4N)\cdot 4N}(\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma).$$

Since the quantitative L^p assembly maps are compatible with the $\omega(4N)$-L^p assembly maps, we get that

$$\mu^{4N_1,d'}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}([x]_{4N_1}) = \iota_*^{N_1} \circ \mu^{\varepsilon,r,\omega(4N),d'}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}([x]_{\omega(4N)}),$$

where $N_1 = \max\{\omega(4N)\cdot N, 9\omega(4N)\}$. We now apply Proposition 3.1 and conclude that there exists $r' \geq r$ such that

$$\iota_*^{\rho(9N\omega(4N))\varepsilon,r',33N_1} \circ \mu^{\varepsilon,r,\omega(4N),d'}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}([x]) = \iota_*^{\rho(9N\omega(4N))\varepsilon,r',33N_1}([y]).$$

However, if we choose m such that $r_m \geq r'$, $N_m \geq 33N_1$ and $d_m \geq d'$, using naturality of the L^p assembly map and equation (3.2), we obtain that

$$\iota_*^{\rho(9N\omega(4N))\varepsilon,r_m,9N_m}([y_m]) = \mu_{A,*}^{\rho(9N\omega(4N))\varepsilon,r_m,N_m,d_m}([x_m]),$$

which contradicts our assumption.

In the proof of (i) implying (ii) of Theorems 3.1 and 3.3, replacing the algebra $\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A)$ by $\prod_{i\in\mathbb{N}}(\mathscr{K}(\ell^p)\otimes A_i)$ for a family of Γ-L^p operator algebras $(A_i)_{i\in\mathbb{N}}$, we can obtain the following theorem.

Theorem 3.4 *Let Γ be a discrete group.*

(i) *Assume that for any Γ-L^p operator algebra A, the L^p Baum-Connes assembly map $\mu_{A,*}$ is injective. Then for $0 < \varepsilon < \frac{1}{20}$, $r > 0$, $N \geq 1$ and $d > 0$, there exists $d' \geq d$ such that $QI_{A,*}(d,d',\varepsilon,r,N)$ holds.*

(ii) *Assume that for any Γ-L^p operator algebra A and for any integer $\mathscr{N}$, there exists a non-decreasing function $\omega : [1,\infty) \to [1,\infty)$ such that the $\mathscr{N}$-L^p Baum-Connes assembly map $\mu^{\mathscr{N}}_{A,*}$ is ω-surjective. Then for some polynomial $\rho \geq 1$ with positive coefficients and for any ε in $\left(0, \frac{1}{20\rho(9N\omega(4N))}\right)$, $r > 0$ and $N \geq 1$, there exist $d > 0$, $r' \geq r$ and $N' \geq N$ such that $QS_{A,*}(d,\varepsilon,\rho(9N\omega(4N))\varepsilon,r,r',N,N')$ holds.*

Remark 3.9 To complete the proof of Theorems 3.1 and 3.3, we need Proposition 3.3 which is based on a couple of lemmas.

Lemma 3.1 *Let A be a unital L^p operator algebra. There exists a map $\varphi:(0,\infty)\to(0,\infty)$ such that*:

(1) *If e and f are homotopic idempotents in $M_n(A)$, then there exist $k,N\in\mathbb{N}$ with $n+k\le N$, and a homotopy of idempotents $(E_t)_{t\in[0,1]}$ in $M_N(A)$ between* $\mathrm{diag}(e,I_k,0)$ *and* $\mathrm{diag}(f,I_k,0)$ *such that $\|E_t-E_s\|\le\varepsilon$ when $|s-t|\le\varphi(\varepsilon)$ for any $\varepsilon>0$ and any $s,t\in[0,1]$.*

(2) *If u and v are homotopic invertibles in $GL_n(A)$, then there exist an integer k and a homotopy $(U_t)_{t\in[0,1]}$ in $GL_{n+k}(A)$ between* $\mathrm{diag}(u,I_k)$ *and* $\mathrm{diag}(v,I_k)$ *such that $\|U_s-U_t\|\le\varepsilon$ when $|s-t|\le\phi(\varepsilon)$ for any $\varepsilon>0$ and any $s,t\in[0,1]$.*

Proof Let us prove the property in the case of idempotents, the case of invertibles being similar. Without loss of generality, we suppose $n=1$.

(i) Recall from [1, Propositions 4.3.3 and 3.4.3] that if e and f are idempotents in A, and there exists $0<\delta<\frac{1}{\|2e-1\|}$ such that $\|e-f\|\le\delta$, then $f=z^{-1}ez$ for some invertible z in A with $\|z-1\|<1$. Hence there exists $a\in A$ with $\|a\|<\log 2$ such that $z=\exp(a)$. Considering the homotopy $(e_t)_{t\in[0,1]}=(\exp(ta)\cdot e\cdot\exp(-ta))_{t\in[0,1]}$ between e and f, we see that there exists a map $\varphi_1:(0,\infty)\to(0,\infty)$ such that $\|e_s-e_t\|\le\varepsilon$ when $|s-t|\le\varphi_1(\varepsilon)$ for any $\varepsilon>0$ and any $s,t\in[0,1]$.

(ii) For $t\in[0,1]$, let $c_t=\cos\frac{\pi t}{2}$ and $s_t=\sin\frac{\pi t}{2}$. Define

$$E_t=\begin{pmatrix}e&0\\0&0\end{pmatrix}+\begin{pmatrix}c_t&-s_t\\s_t&c_t\end{pmatrix}\begin{pmatrix}1-e&0\\0&0\end{pmatrix}\begin{pmatrix}c_t&s_t\\-s_t&c_t\end{pmatrix}$$

in $M_2(A)$. Then we know that $(E_t)_{t\in[0,1]}$ is a homotopy of idempotents between $\mathrm{diag}(1,0)$ and $\mathrm{diag}(e,1-e)$. Also, there exists a map $\varphi_2:(0,\infty)\to(0,\infty)$ such that $\|E_s-E_t\|\le\varepsilon$ when $|s-t|\le\varphi_2(\varepsilon)$ for any $\varepsilon>0$ and any $s,t\in[0,1]$.

(iii) In the general case, let $(e_t)_{t\in[0,1]}$ be a homotopy of idempotents between e and f, and let $0=t_0<t_1<\cdots<t_k=1$ be such that

$$\|e_{t_i}-e_{t_{i-1}}\|\le\delta\quad\text{ for } i=1,\cdots,k.$$

Then we have the following sequence of homotopies of idempotents in $M_{2k+1}(A)$ in which the first and last homotopies are conjugated by some permutation matrices:

$h_0\overset{h_t^0}{\sim}h_1\overset{h_t^1}{\sim}h_2\overset{h_t^2}{\sim}h_3\overset{h_t^3}{\sim}h_4\overset{h_t^4}{\sim}h_5$, where

$h_0=\mathrm{diag}(e_{t_0},I_k,0_k)$,

$h_1=\mathrm{diag}(e_{t_0},1,0,\cdots,1,0)$,

$h_2=\mathrm{diag}(e_{t_0},1-e_{t_1},e_{t_1},\cdots,1-e_{t_k},e_{t_k})$,

$h_3=\mathrm{diag}(e_{t_0},1-e_{t_0},e_{t_1},1-e_{t_1},\cdots,e_{t_{k-1}},1-e_{t_{k-1}},e_{t_k})$,

$h_4=\mathrm{diag}(1,0,\cdots,1,0,e_{t_k})$,

$h_5=\mathrm{diag}(e_{t_k},I_k,0_k)$.

If we let $\varphi = \min\{\varphi_1, \varphi_2\}$, then the result is obtained from cases (i) and (ii). Indeed, the fact that $\|h_3 - h_2\| \leq \delta$ implies that for every $m \in \{0, 4\}$, there are homotopies $(h_t^m)_{t\in[0,1]}$ between h_m and h_{m+1} such that $\|h_s^m - h_t^m\| \leq \varepsilon$ when $|s-t| \leq \varphi(\varepsilon)$ for any $\varepsilon > 0$ and any $s, t \in [0, 1]$.

In the next lemma, the injectivity of $\Phi_*^{\mathcal{A}}$ follows immediately from Lemma 3.1, and $\Phi_*^{\mathcal{A}}$ is clearly surjective. Hence the following result is obtained.

Lemma 3.2 *Let $\mathcal{A} = (A_i)_{i\in I}$ be a family of unital L^p operator algebras. Let*

$$\Phi_*^{\mathcal{A}} : K_*\Big(\prod_{i\in I}(\mathscr{K}(\ell^p) \otimes A_i)\Big) \to \prod_{i\in I} K_*(\mathscr{K}(\ell^p) \otimes A_i) \cong \prod_{i\in I} K_*(A_i)$$

be the homomorphism induced on the j-th factor by the projection

$$\prod_{i\in I}(\mathscr{K}(\ell^p) \otimes A_i) \to \mathscr{K}(\ell^p) \otimes A_j.$$

Then $\Phi_*^{\mathcal{A}}$ is an isomorphism.

Remark 3.10 Observe that $\mathscr{K}(\ell^p) \otimes \mathscr{K}(\ell^p) \otimes A_i$ is isometrically isomorphic to $\mathscr{K}(\ell^p) \otimes A_i$ for each $i \in \mathbb{N}$, thus $\Phi_*^{\mathcal{A}}$ is an isometric isomorphism.

As a consequence of this lemma, we have the following important proposition.

Proposition 3.3 *Let Γ be a discrete group and let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of Γ-L^p operator algebras. Suppose $A_i \otimes \mathscr{K}(\ell^p)$ is equipped with the diagonal action, the action of Γ on $\mathscr{K}(\ell^p)$ is trivial. Let*

$$\begin{aligned}\Phi_*^{\Gamma,\mathcal{A}} : K_*\Big(B_L^p\Big(P_d(\Gamma), \prod_{i\in I}(\mathscr{K}(\ell^p) \otimes A_i)\Big)^\Gamma\Big) &\to \prod_{i\in I} K_*(B_L^p(P_d(\Gamma), \mathscr{K}(\ell^p) \otimes A_i)^\Gamma) \\ &\cong \prod_{i\in I} K_*(B_L^p(P_d(\Gamma), A_i)^\Gamma)\end{aligned}$$

be the homomorphism induced on the j-th factor by the projection

$$\prod_{i\in I}(\mathscr{K}(\ell^p) \otimes A_i) \to \mathscr{K}(\ell^p) \otimes A_j.$$

Then $\Phi_*^{\Gamma,\mathcal{A}}$ is an isomorphism.

Proof Put $B_i = \mathscr{K}(\ell^p) \otimes A_i$, $i \in I$. For any locally compact space X equipped with an action of Γ, we define

$$\Phi_*^X : K_*\Big(B_L^p\Big(X, \prod_{i\in I} B_i\Big)^\Gamma\Big) \to \prod_{i\in I} K_*(B_L^p(X, B_i)^\Gamma).$$

The homomorphism induced by the projection on the j-th factor is

$$\Phi_{j,*}^X : K_*\Big(B_L^p\Big(X, \prod_{i\in I} B_i\Big)^\Gamma\Big) \to K_*(B_L^p(X, B_j)^\Gamma).$$

Let $Z_0, \cdots, Z_n$ be the skeleton decomposition of $P_d(\Gamma)$, then Z_j is a locally finite simplicial complex of dimension j, and endowed with a proper, cocompact and type preserving action of Γ.

Next, we prove that $\Phi_*^{Z_j}$ is an isomorphism by induction on j.

(i) For $j = 0$, the 0-skeleton Z_0 is a finite union of orbits, thus it suffices to prove that $\Phi_*^{\Gamma/F}$ is an isomorphism when F is a finite subgroup of Γ. For any Γ-L^p operator algebra B, let χ_0 be the charateristic map of F in Γ/F, and let π be a representation of $C_0(\Gamma/F)$ in E_d. Then $E_{d_0} = \pi(\chi_0) \cdot E_d$ is stable under the action of group F and under the endmorphism of a bounded linear operator T. The element restricted to E_{d_0} defines an element of $K_*(B_L^p(\mathbb{C}, B)^F)$ and there is a natural restriction isomorphism

$$R_{F,\Gamma}^B : K_*(B_L^p(\Gamma/F, B)^\Gamma) \to K_*(B_L^p(\mathbb{C}, B)^F) \cong K_*(B \rtimes F).$$

By naturality, we obtain the following commutative diagram:

$$\begin{array}{ccc} K_*\Big(B_L^p\Big(\Gamma/F, \prod\limits_{i\in I} B_i\Big)^\Gamma\Big) & \xrightarrow{\Phi_{j,*}^{\Gamma/F}} & K_*(B_L^p(\Gamma/F, B_j)^\Gamma) \\ {\scriptstyle R_{F,\Gamma}^{\prod_{i\in I} B_i}}\Big\downarrow & & \Big\downarrow {\scriptstyle R_{F,\Gamma}^{B_j}} \\ K_*\Big(\prod\limits_{i\in I} B_i \rtimes F\Big) & \longrightarrow & K_*(B_j \rtimes F), \end{array}$$

where the bottom row is induced by the homomorphism

$$\prod_{i\in I} B_i \rtimes F \to B_j \rtimes F$$

determined by the projection on the j-th factor $\prod\limits_{i\in I} B_i \to B_j$. Since F is finite, we see that $\prod\limits_{i\in I} B_i \rtimes F \cong \big(\prod\limits_{i\in I} B_i\big) \rtimes F$. Applying Lemma 3.2, we have an isomorphism

$$K_*\Big(\Big(\prod_{i\in I} B_i\Big) \rtimes F\Big) \cong K_*\Big(\prod_{i\in I} B_i \rtimes F\Big) \to \prod_{i\in I} K_*(B_i \rtimes F).$$

Hence $\Phi_*^{\Gamma/F}$ is an isomorphism.

(ii) Suppose $\Phi_*^{Z_{j-1}}$ is an isomorphism, and it remains to prove that $\Phi_*^{Z_j}$ is an isomorphism. The short exact sequence

$$0 \to C_0(Z_j \backslash Z_{j-1}) \to C_0(Z_j) \to C_0(Z_{j-1}) \to 0$$

induces a natural long exact sequence

$$\to K_*(B_L^p(Z_{j-1}, \cdot)^\Gamma) \to K_*(B_L^p(Z_j, \cdot)^\Gamma) \to K_*(B_L^p(Z_j \backslash Z_{j-1}, \cdot)^\Gamma) \to K_{*+1}(B_L^p(Z_{j-1}, \cdot)^\Gamma) \to$$

and hence by naturality, we obtain a commutative diagram

$$\begin{array}{ccccccc}
K_*(B_L^p(Z_{j-1},B)^\Gamma) & \to & K_*(B_L^p(Z_j,B)^\Gamma) & \to & K_*(B_L^p(Z_j\backslash Z_{j-1},B)^\Gamma) & \to & K_{*+1}(B_L^p(Z_{j-1},B)^\Gamma) \\
\downarrow{\scriptstyle \Phi_*^{Z_{j-1}}} & & \downarrow{\scriptstyle \Phi_*^{Z_j}} & & \downarrow{\scriptstyle \Phi_*^{Z_j\backslash Z_{j-1}}} & & \downarrow{\scriptstyle \Phi_{*+1}^{Z_{j-1}}} \\
\prod\limits_{i\in I} K_*(B_L^p(Z_{j-1})^\Gamma) & \to & \prod\limits_{i\in I} K_*(B_L^p(Z_j)^\Gamma) & \to & \prod\limits_{i\in I} K_*(B_L^p(Z_j\backslash Z_{j-1})^\Gamma) & \to & \prod\limits_{i\in I} K_{*+1}(B_L^p(Z_{j-1})^\Gamma),
\end{array}$$

where $\prod\limits_{i\in I} B_i$ and $\prod\limits_{i\in I} K_*(B_L^p(Z_j,B_i)^\Gamma)$ are denoted by B and $\prod\limits_{i\in I} K_*(B_L^p(Z_j)^\Gamma)$ respectively. We denote by I_j the interior of the standard j-simplex. Since the action of Γ is type preserving, then

$$Z_j\backslash Z_{j-1} \cong I_j \times C_j,$$

where C_j is the set of center of j-simplices of Z_j, Γ acts trivially on I_j. Together with the Bott periodicity, we have a commutative diagram

$$\begin{array}{ccc}
K_*\Big(B_L^p\Big(Z_j\backslash Z_{j-1}, \prod\limits_{i\in I} B_i\Big)^\Gamma\Big) & \longrightarrow & K_{*+1}\Big(B_L^p\Big(C_j, \prod\limits_{i\in I} B_i\Big)^\Gamma\Big) \\
\downarrow{\scriptstyle \Phi_*^{Z_j\backslash Z_{j-1}}} & & \downarrow{\scriptstyle \Phi_{*+1}^{C_j}} \\
\prod\limits_{i\in I} K_*(B_L^p(Z_j\backslash Z_{j-1},B_i)^\Gamma) & \longrightarrow & \prod\limits_{i\in I} K_{*+1}(B_L^p(C_j,B_i)^\Gamma).
\end{array}$$

Finally, $\Phi_*^{C_j}$ is an isomorphism obtained from case (i), and thus $\Phi_*^{Z_j\backslash Z_{j-1}}$ is an isomorphism. According to the induction hypothesis and the five lemma, we know that $\Phi_*^{Z_j}$ is an isomorphism.

4 Persistence Approximation Property

In this section, we introduce the persistence approximation property for filtered L^p operator algebras. In the case of a reduced crossed product of an L^p operator algebra by a finitely generated group, we find a sufficient condition for the persistence approximation property.

Let A be a filtered L^p operator algebra. Applying Proposition 2.2 (i), we see that for any $\varepsilon \in \big(0, \frac{1}{20}\big)$ and any $N \geq 1$, there exists a surjective map

$$\lim_{r>0} K_*^{\varepsilon,r,N}(A) \to K_*^N(A)$$

induced by a family of relaxation of control maps $(\iota_*)_{r>0}$. Moreover, if $\varepsilon > 0$ is small enough, then for any $r > 0$, any $N \geq 1$ and any $[x] \in K_*^{\varepsilon,r,N}(A)$, there exist positive numbers $\varepsilon' \in \big[\varepsilon, \frac{1}{20}\big)$ independent of x and A, $r' \geq r$ and $N' \geq N$ such that

$$\iota_*([x]) = 0 \text{ in } K_*(A) \Rightarrow \iota_*^{\varepsilon',r',N'}([x]) = 0 \ \text{ in } K_*^{\varepsilon',r',N'}(A).$$

However, we may wonder whether this r' depends on x, in other words whether the family $(K_*^{\varepsilon,r,N}(A))_{0<\varepsilon<\frac{1}{20},r>0,N\geq 1}$ has a persistence approximation for $K_*(A)$ in the following sense:

For any sufficiently small $\varepsilon \in \left(0, \frac{1}{20}\right)$, any $r > 0$ and any $N \geq 1$, there exist $\varepsilon' \in \left[\varepsilon, \frac{1}{20}\right)$, $r' \geq r$ and $N' \geq N$ such that for any $[x] \in K_*^{\varepsilon,r,N}(A)$, we have

$$\iota_*^{\varepsilon',r',N'}([x]) \neq 0 \quad \text{in } K_*^{\varepsilon',r',N'}(A) \Rightarrow \iota_*([x]) \neq 0 \quad \text{in } K_*(A).$$

Therefore, we consider the following statement: For a filtered L^p operator algebra A and positive numbers ε, r and $N \geq 1$, there exist ε' in $\left[\varepsilon, \frac{1}{20}\right)$, $r' \geq r$ and $N' \geq N$:

$\mathcal{PA}_*(A, \varepsilon, \varepsilon', r, r', N, N')$: For any $[x] \in K_*^{\varepsilon,r,N}(A)$,

$$\iota_*([x]) = 0 \quad \text{in } K_*(A) \Rightarrow \iota_*^{\varepsilon',r',N'}([x]) = 0 \quad \text{in } K_*^{\varepsilon',r',N'}(A).$$

4.1 The case of crossed products

Theorem 4.1 *Let Γ be a finitely generated group, and let A be a Γ-L^p operator algebra. Assume that*

(1) *Γ admits a cocompact universal example for proper actions.*

(2) *For any positive integer $\mathscr{N}$, there exists a non-decreasing function $\omega : [1, \infty) \to [1, \infty)$ such that the $\mathscr{N}$-L^p Baum-Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)$$

is ω-surjective.

(3) *The L^p Baum-Connes assembly map for Γ with coefficients in A is injective.*

Then for any $N \geq 1$, there exists a universal constant $\lambda_{PA} \geq 1$ such that for any ε in $\left(0, \frac{1}{20\lambda_{PA}}\right)$ and any $r > 0$, there exist $r' \geq r$ and $N' \geq N$ such that $\mathcal{PA}_*(A \rtimes \Gamma, \varepsilon, \lambda_{PA}\varepsilon, r, r', N, N')$ holds.

Remark 4.1 Here, the constant λ_{PA} does not depend on r, but on the positive integer N.

Proof Let A be a Γ-L^p operator algebra, and let Γ admit a cocompact universal example for proper actions. Assume that for every positive integer $\mathscr{N}$, there exists a non-decreasing function ω such that the $\mathscr{N}$-L^p Baum-Connes assembly map with coefficients in $\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)$ is ω-surjective and the L^p Baum-Connes assembly map with coefficients in A is injective, then there exist positive numbers d and d' with $d \leq d'$ such that the following two conditions are satisfied:

(1) For every $\mathscr{N} \in \mathbb{N}$ and any $[z]$ in $K_*^{\mathscr{N}}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$, there exists $[x]$ in $K_*^{\omega(\mathscr{N})}(B_L^p(P_d(\Gamma), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))^\Gamma)$ such that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\omega(\mathscr{N}),d}([x]) = [z] \quad \text{in } K_*^{\omega(\mathscr{N})\cdot\mathscr{N}}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma).$$

(2) For any $[x]$ in $K_*(B_L^p(P_d(\Gamma), A)^\Gamma)$ such that $\mu_{A,*}^d([x]) = 0$, we have

$$i_{d,d',*}([x]) = 0 \quad \text{in } K_*(B_L^p(P_{d'}(\Gamma), A)^\Gamma),$$

where $i_{d,d',*} : K_*(B_L^p(P_d(\Gamma), A)^\Gamma) \to K_*(B_L^p(P_{d'}(\Gamma), A)^\Gamma)$ is induced by the inclusion $P_d(\Gamma) \hookrightarrow P_{d'}(\Gamma)$.

Fix such d and d', and let ρ be as in Proposition 3.1, pick (λ, h) as in Lemma 2.1 and put $\lambda_{PA} = \rho(9\lambda_N \omega(4\lambda_N))$. Assume that there exists $N \geq 1$ such that this statement does not hold. Then there exist

(1) $\varepsilon \in \left(0, \frac{1}{20\lambda_{PA}}\right)$ and $r > 0$,

(2) an unbounded increasing sequence $(r_i)_{i\in\mathbb{N}}$ with $r_i \geq r$,

(3) an unbounded increasing sequence $(N_i)_{i\in\mathbb{N}}$ with $N_i \geq N$,

(4) a sequence of elements $([x_i])_{i\in\mathbb{N}}$ with $[x_i] \in K_*^{\varepsilon,r,N}(A \rtimes \Gamma)$,

such that, for each $i \in \mathbb{N}$,

$$\iota_*([x_i]) = 0 \quad \text{in } K_*(A \rtimes \Gamma)$$

and

$$\iota_*^{\lambda_{PA}\varepsilon,r_i,N_i}([x_i]) \neq 0 \quad \text{in } K_*^{\lambda_{PA}\varepsilon,r_i,N_i}(A \rtimes \Gamma).$$

Since

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma_{h_{\varepsilon,N}r} = \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A \rtimes \Gamma_{h_{\varepsilon,N}r})$$

and according to Lemma 2.1, there exists an element

$$[x] \in K_*^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$$

that maps to $\iota_*^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}([x_i])$, for all integers i under the composition

$$K_*^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p)\otimes A)\rtimes\Gamma) \to K_*^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(\mathscr{K}(\ell^p)\otimes A\rtimes\Gamma) \xrightarrow{\cong} K_*^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(A\rtimes\Gamma),$$

where the first map is induced by the j-th projection

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \to \mathscr{K}(\ell^p) \otimes A \tag{4.1}$$

and the isomorphism is the Morita equivalence of Propositions 2.4–2.5. Note that $\iota_*^{\lambda_N}([x])$ is in $K_*^{4\lambda_N}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma)$. Let

$$[z] \in K_*^{\omega(4\lambda_N)}(B_L^p(P_d(\Gamma), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))^\Gamma)$$

such that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\omega(4\lambda_N),d}([z]) = \iota_*^{\lambda_N}([x]) \quad \text{in } K_*^{\omega(4\lambda_N)\cdot 4\lambda_N}(\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A) \rtimes \Gamma).$$

Since the quantitative L^p assembly maps are compatible with the $\omega(4\lambda_N)$-L^p assembly maps, we obtain that

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{4N_1,d}([z]_{4N_1}) = \iota_*^{N_1} \circ \mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\omega(4\lambda_N),d}([z]_{\omega(4\lambda_N)}),$$

where $N_1 = \max\{\omega(4\lambda_N)\cdot\lambda_N, 9\omega(4\lambda_N)\}$. However, according to Proposition 3.1, there exists $R \geq h_{\varepsilon,N}r$ such that

$$\begin{aligned}\iota_*^{\lambda_{PA}\varepsilon,R,33N_1}([x]) &= \iota_*^{\lambda_{PA}\varepsilon,R,33N_1}\circ\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\omega(4\lambda_N),d}([z]_{\omega(4\lambda_N)})\\ &= \mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\lambda_{PA}\varepsilon,R,33N_1,d}([z]_{33N_1}).\end{aligned}$$

By Proposition 3.3, we have an isomorphism

$$K_*(B_L^p(P_d(\Gamma),\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A))^\Gamma)\xrightarrow{\cong}\prod_{j\in\mathbb{N}}K_*(B_L^p(P_d(\Gamma),A)^\Gamma) \tag{4.2}$$

induced by the j-th projection in equation (4.1). Let $([z_j])_{j\in\mathbb{N}}$ be the element of

$$\prod_{j\in\mathbb{N}}K_*(B_L^p(P_d(\Gamma),A)^\Gamma)$$

corresponding to $[z]$ under this identification. Using the compatibility of the quantitative L^p assembly maps with the usual ones, we obtain by naturality that $\mu_{A_i,*}^d([z_i]) = 0$, for every $i\in\mathbb{N}$ and hence

$$i_{d,d',*}([z_i]) = 0 \quad \text{in } K_*(B_L^p(P_{d'}(\Gamma),A)^\Gamma).$$

Using once more equation (4.2), we deduce that

$$i_{d,d',*}([z]) = 0 \quad \text{in } K_*(B_L^p(P_{d'}(\Gamma),\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A))^\Gamma).$$

Let $(p_t)_{t\in[0,1]}$ be a homotopy of idempotents (resp. invertibles) in $M_n(\widetilde{B})$ between $i_{d,d',*}([z])$ and 0, then $P := (p_t)$ is an idempotent (resp. invertible) element in $C([0,1],M_n(\widetilde{B}))$, where $B = B_L^p(P_{d'}(\Gamma),\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A))^\Gamma$. Put $N' = \max\{33N_1,\|P\|\}$. Since

$$\mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\lambda_{PA}\varepsilon,R,N',d}([z]) = \mu_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}^{\lambda_{PA}\varepsilon,R,N',d'}\circ i_{d,d',*}([z]),$$

then

$$\iota_*^{\lambda_{PA}\varepsilon,R,N'}([x]) = 0 \quad \text{in } K_*^{\lambda_{PA}\varepsilon,R,N'}(\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A)\rtimes\Gamma).$$

By naturality, we see that $\iota_*^{\lambda_{PA}\varepsilon,R,N'}([x_i]) = 0$ in $K_*^{\lambda_{PA}\varepsilon,R,N'}(A\rtimes\Gamma)$ for all integers i. Picking an integer i such that $r_i \geq R$ and $N_i \geq N'$, we have

$$\iota_*^{\lambda_{PA}\varepsilon,r_i,N_i}([x_i]) = \iota_*^{\lambda_{PA}\varepsilon,r_i,N_i}\circ\iota_*^{\lambda_{PA}\varepsilon,R,N'}([x_i]) = 0,$$

which contradicts our assumption.

For any L^p operator algebra A, the L^p Baum-Connes assembly map for Γ with coefficients in $C_0(\Gamma,A)$ is an isomorphism and $C_0(\Gamma,A)\rtimes\Gamma\cong A\otimes\mathscr{K}(\ell^p(\Gamma))$, hence by Theorem 4.1, we immediately obtain the following corollary.

Corollary 4.1 *Let Γ be a finitely generated group, and let A be an L^p operator algebra. Assume that*

(1) Γ *admits a cocompact universal example for proper actions*;

(2) *for any positive integer $\mathcal{N}$, there exists a non-decreasing function $\omega : [1,\infty) \to [1,\infty)$ such that the $\mathcal{N}$-L^p Baum-Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N}, C_0(\Gamma, \mathscr{K}(\ell^p) \otimes A))$$

is ω-surjective.

Then for any $N \geq 1$, there exists a universal constant $\lambda_{PA} \geq 1$ such that for any ε in $\big(0, \frac{1}{20\lambda_{PA}}\big)$ and any $r > 0$, there exist $r' \geq r$ and $N' \geq N$ such that $\mathcal{PA}_*(A \otimes \mathcal{K}(\ell^p(\Gamma)), \varepsilon, \lambda_{PA}\varepsilon, r, r', N, N')$ holds.

In particular, if we put $A = \mathbb{C}$, we have the following conclusion.

Proposition 4.1 *Let Γ be a finitely generated group. Assume that*

(1) Γ *admits a cocompact universal example for proper actions*;

(2) *for any positive integer $\mathcal{N}$, there exists a non-decreasing function $\omega : [1,\infty) \to [1,\infty)$ such that the $\mathcal{N}$-L^p Baum-Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N}, C_0(\Gamma, \mathscr{K}(\ell^p)))$$

is ω-surjective.

Then for any $N \geq 1$, there exists a universal constant $\lambda \geq 1$ such that for any $\varepsilon \in \big(0, \frac{1}{20\lambda}\big)$ and any $r > 0$, there exist $R \geq r$ and $N' \geq N$ such that the following holds:

(1) *If u is an (ε, r, N)-invertible of $\mathcal{K}(\ell^p(\Gamma) \otimes \ell^p) + \mathbb{C}Id_{\ell^p(\Gamma)\otimes\ell^p}$, then u is connected to $Id_{\ell^p(\Gamma)\otimes\ell^p}$ by a homotopy of $(\lambda\varepsilon, R, N')$-invertibles.*

(2) *If e and f are (ε, r, N)-idempotents of $\mathcal{K}(\ell^p(\Gamma) \otimes \ell^p)$ such that*

$$\operatorname{rank} \kappa_0(e) = \operatorname{rank} \kappa_0(f),$$

then e and f are connected by a homotopy of $(\lambda\varepsilon, R, N')$-idempotents.

5 Applications Involving L^p Coarse Baum-Connes Conjecture

In this section, X will be a discrete metric space with bounded geometry and A will be an L^p operator algebra. We will present a result on the persistence approximation property of the L^p Roe algebra for X. This result is applied to show that if any such space is coarsely uniformly contractible and satisfies controlled-surjectivity of the $\mathcal{N}$-L^p coarse Baum-Connes assembly map and injectivity of the L^p coarse Baum-Connes assembly map, then the L^p Roe algebra $B^p(X, A)$ has the persistence approximation property.

Assume that $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is any family of filtered L^p operator algebras. For each $i \in \mathbb{N}$, there is a representation of A_i on an L^p space E_i. We define $E := \bigoplus_{i\in\mathbb{N}} E_i = \{(e_i)_{i\in\mathbb{N}} \mid e_i \in E_i\}$

with the norm $\|(e_1, e_2, \cdots)\| = \Big\{\sum_{i\in\mathbb{N}} |e_i|^p\Big\}^{\frac{1}{p}}$. Clearly, E is an L^p space. Let $L'_d = \ell^p(Q_d)\otimes E\otimes \ell^p$ be a certain L^p-X-module defined in [21], and let $\mathbb{C}[L'_d, A_i]$ denote the algebra of all E-locally compact operators on L'_d with finite propagation. For any $r > 0$, we set

$$\mathcal{A}^{\infty}_{d,r} = \prod_{i\in\mathbb{N}} \mathbb{C}[L'_d, A_i]_r,$$

and we define the L^p operator algebra $\mathcal{A}^{\infty}_d$ as the closure of $\bigcup_{r>0} \mathcal{A}^{\infty}_{d,r}$ in $\prod_{i\in\mathbb{N}} B^p(P_d(X), A_i)$.

Lemma 5.1 *Let X be a discrete metric space with bounded geometry, and let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of filtered L^p operator algebras. Then there exist a control pair (λ, h) independent of the family $\mathcal{A}$ and a (λ, h)-isomorphism*

$$\mathcal{G} = (G^{\varepsilon,r,N})_{0<\varepsilon<\frac{1}{20},r>0,N\geq 1} : \mathcal{K}_*(\mathcal{A}^{\infty}_d) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(B^p(P_d(X), A_i)),$$

where

$$G^{\varepsilon,r,N} : K^{\varepsilon,r,N}_*(\mathcal{A}^{\infty}_d) \to \prod_{i\in\mathbb{N}} K^{\varepsilon,r,N}_*(B^p(P_d(X), A_i))$$

is induced on the j-th factor by the projection $\prod_{i\in\mathbb{N}} B^p(P_d(X), A_i) \to B^p(P_d(X), A_j)$.

Proof Let us first consider the even case. For $0 < \varepsilon < \frac{1}{20}$, $r > 0$ and $N \geq 1$, there exist a control pair (λ, h) and a (λ, h)-controlled morphism

$$G^{\varepsilon,r,N} : K^{\varepsilon,r,N}_*(\mathcal{A}^{\infty}_d) \to \prod_{i\in\mathbb{N}} K^{\varepsilon,r,N}_*(B^p(P_d(X), A_i))$$

induced on the j-th factor by the projection $\prod_{i\in\mathbb{N}} B^p(P_d(X), A_i) \to B^p(P_d(X), A_j)$. For any positive integer i and n, we know that

$$M_n(\ell^{\infty}(X, A_i \otimes \mathscr{K}(\ell^p))) \subset \ell^{\infty}(X, A_i) \otimes \mathscr{K}(\ell^p).$$

Hence, $M_n(B^p(P_d(X), A_i)) \subset B^p(P_d(X), A_i)$. Assume that x is in $\prod_{i\in\mathbb{N}} K^{\varepsilon,r,N}_0(B^p(P_d(X), A_i))$, then we can write $[x] = ([x_i])_{i\in\mathbb{N}}$ for $[x_i] \in K^{\varepsilon,r,N}_0(B^p(\widetilde{P_d(X)}, A_i))$. Let $(e_i)_{i\in\mathbb{N}}$ be a family of (ε, r, N)-idempotents with e_i in some $M_n(B^p(P_d(X), A_i))$ such that $[x]_{\varepsilon,r,N} = [(e_i)_{i\in\mathbb{N}}]_{\varepsilon,r,N}$, then $G^{\varepsilon,r,N}$ is (λ, h)-surjective.

According to the item (i) of Proposition 2.3, we construct the Lipschitz homotopy of (ε, r, N)-idempotents in larger matrix size, thus we can prove that $G^{\varepsilon,r,N}$ is (λ, h)-injective. In the odd case, we have a similar proof.

Lemma 5.2 *Let X be a discrete metric space with bounded geometry, and let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of filtered L^p operator algebras, then we have a filtered isomorphism*

$$\phi : B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big) \to \mathcal{A}^{\infty}_d.$$

Proof By the universal property of $B^p\big(P_d(X), \prod_{i\in\mathbb{N}} A_i\big)$, there exists a filtered homomorphism

$$B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big) \to \mathcal{A}_d^\infty.$$

Note that the filtered homomorphism ϕ maps the dense subalgebra $\mathbb{C}\big[L'_d, \prod_{i\in\mathbb{N}} A_i\big]$ to a dense subalgebra of $\mathcal{A}_d^\infty$, thus we can easily get that ϕ is surjective. It thus suffices to show that ϕ is injective. For every positive integer i, we have the inclusion $A_i \to \prod_{i\in\mathbb{N}} A_i$. Hence, we have a filtered homomorphism

$$B^p(P_d(X), A_i) \to B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big),$$

which induces a filtered homomorphism

$$\psi : A_d^\infty \to B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big)$$

such that the composition

$$B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big) \xrightarrow{\phi} A_d^\infty \xrightarrow{\psi} B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big)$$

is an identity map. Let x be in $B^p\big(P_d(X), \prod_{i\in\mathbb{N}} A_i\big)$ such that $\phi(x) = 0$ in A_d^∞, then $x = \psi(\phi(x)) = 0$, thus ϕ is injective. This implies that ϕ is a filtered isomorphism.

The preceding Lemma 5.2 yields the following.

Corollary 5.1 *Let X be a discrete metric space with bounded geometry, and let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of filtered L^p operator algebras, then there exist a control pair (λ, h) and a (λ, h)-isomorphism*

$$\mathcal{K}_*\Big(B^p\Big(P_d(X), \prod_{i\in\mathbb{N}} A_i\Big)\Big) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(B^p(P_d(X), A_i)).$$

Moreover, passing to the limit we obtain

$$\mathcal{K}_*\Big(B^p\Big(X, \prod_{i\in\mathbb{N}} A_i\Big)\Big) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(B^p(X, A_i)).$$

Definition 5.1 (see [14]) *A discrete metric space X is coarsely uniformly contractible, if for each $d > 0$, there exists $d' > d$ such that any compact subset of $P_d(X)$ lies in a contractible compact subset of $P_{d'}(X)$.*

Example 5.1 (see [14]) Any discrete Gromov hyperbolic metric space is coarsely uniformly contractible.

Definition 5.2 *Let A be an L^p operator algebra. The evaluation-at-zero homomorphism*

$$ev_0 : B_L^p(P_d(X), A) \to B^p(P_d(X), A)$$

induces a homomorphism on K-theory

$$\mu^d_{A,*} = ev_* : K_*(B^p_L(P_d(X), A)) \to K_*(B^p(P_d(X), A)) \cong K_*(B^p(X, A)),$$

called an L^p coarse assembly map.

The family of L^p coarse assembly maps $(\mu^d_{A,*})_{d>0}$ gives rise to a homomorphism

$$\mu_{A,*} : \lim_{d>0} K_*(B^p_L(P_d(X), A)) \to K_*(B^p(X, A)),$$

called the L^p coarse Baum-Connes assembly map. Moreover, the L^p coarse Baum-Connes conjecture for X posits that this map $\mu_{A,*}$ is an isomorphism.

Definition 5.3 *Let A be an L^p operator algebra. For $N \geq 1$, we define an N-L^p coarse assembly map*

$$\mu^{N,d}_{A,*} : K^N_*(B^p_L(P_d(X), A)) \to K^N_*(B^p(P_d(X), A)) \cong K^N_*(B^p(X, A))$$

induced by the evaluation-at-zero homomorphism

$$ev_0 : B^p_L(P_d(X), A) \to B^p(P_d(X), A).$$

The family of N-L^p coarse assembly maps $(\mu^{N,d}_{A,*})_{d>0}$ gives rise to a homomorphism

$$\mu^N_{A,*} : \lim_{d>0} K^N_*(B^p_L(P_d(X), A)) \to K^N_*(B^p(X, A)),$$

called the N-L^p coarse Baum-Connes assembly map.

Remark 5.1 From the proof of [21, Theorem 4.6], we see that if X is a proper metric space with finite asymptotic dimension, then the N-L^p coarse Baum-Connes assembly map for X is ω-surjective, and the function ω depends on the asymptotic dimension m, strong Lipschitz constant C and Mayer-Vietoris control pair (λ, h).

The following result gives a sufficient condition for persistence approximation property to be satisfied for a class of L^p operator algebras.

Theorem 5.1 *Let X be a discrete metric space with bounded geometry, and let A be an L^p operator algebra. Assume that*

(1) *X is coarsely uniformly contractible*;

(1) *for any positive integer $\mathcal{N}$, there exists a non-decreasing function $\omega : [1, \infty) \to [1, \infty)$ such that the $\mathcal{N}$-L^p coarse Baum-Connes assembly map for X with coefficients in*

$$\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)$$

is ω-surjective;

(3) *the L^p coarse Baum-Connes assembly map for X with coefficients in A is injective.*

Then for any $N \geq 1$, there exists a universal constant $\lambda_{PA} \geq 1$ such that for any ε in $\big(0, \frac{1}{20\lambda_{PA}}\big)$ and any $r > 0$, there exist $r' \geq r$ and $N' \geq N$ such that $\mathcal{PA}_*(B^p(X, A), \varepsilon, \lambda_{PA}\varepsilon, r, r', N, N')$ holds.

Proof Let ρ be as in Proposition 3.1, pick (λ, h) as in Corollary 5.1 and put $\lambda_{PA} = \rho(9\lambda_N \omega(4\lambda_N))$. Assume that there exists $N \geq 1$ such that this statement does not hold. Then there exist

(1) $\varepsilon \in \left(0, \frac{1}{20\lambda_{PA}}\right)$ and $r > 0$,

(2) an unbounded increasing sequence $(r_i)_{i \in \mathbb{N}}$ bounded below by r,

(3) an unbounded increasing sequence $(N_i)_{i \in \mathbb{N}}$ bounded below by N,

(4)a sequence of elements $([x_i])_{i \in \mathbb{N}}$ with $[x_i] \in K_*^{\varepsilon, r, N}(B^p(X, A))$, such that, for each $i \in \mathbb{N}$,

$$\iota_*([x_i]) = 0 \quad \text{in } K_*(B^p(X, A))$$

and

$$\iota_*^{\lambda_{PA}\varepsilon, r_i, N_i}([x_i]) \neq 0 \quad \text{in } K_*^{\lambda_{PA}\varepsilon, r_i, N_i}(B^p(X, A)).$$

Let $[x]$ be an element of $K_*^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}(B^p(X, \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)))$ corresponding to $([x_i])_{i \in \mathbb{N}}$ in $\prod_{i \in \mathbb{N}} K_*^{\varepsilon, r, N}(B^p(X, A))$ under the (λ, h)-isomorphism of Corollary 5.1. Observe that $\iota_*^{\lambda_N}([x])$ is the element of $K_*^{4\lambda_N}(B^p(X, \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)))$. Then there exist $d > 0$ and

$$[z] \in K_*^{\omega(4\lambda_N)}(B_L^p(P_d(X), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A)))$$

such that

$$\mu_{\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A), *}^{\omega(4\lambda_N), d}([z]) = \iota_*^{\lambda_N}([x]) \quad \text{in } K_*^{\omega(4\lambda_N) \cdot 4\lambda_N}(B^p(X, \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))).$$

Since the quantitative L^p coarse assembly maps are compatible with the $\omega(4\lambda_N)$-L^p coarse assembly maps, we obtain that

$$\mu_{\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A), *}^{4N_1, d}([z]_{4N_1}) = \iota_*^{N_1} \circ \mu_{\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A), *}^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \omega(4\lambda_N), d}([z]_{\omega(4\lambda_N)}),$$

where $N_1 = \max\{\omega(4\lambda_N) \cdot \lambda_N, 9\omega(4\lambda_N)\}$. However, according to Proposition 3.1, there exists $R \geq h_{\varepsilon,N} r$ such that

$$\iota_*^{\lambda_{PA}\varepsilon, R, 33N_1}([x]) = \mu_{\ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A), *}^{\lambda_{PA}\varepsilon, R, 33N_1, d}([z]_{33N_1}).$$

By Proposition 3.3, we have an isomorphism

$$K_*(B_L^p(P_d(X), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))) \cong \prod_{i \in \mathbb{N}} K_*(B_L^p(P_d(X), A)).$$

Let $([z_i])_{i \in \mathbb{N}}$ be the element of $\prod_{i \in \mathbb{N}} K_*(B_L^p(P_d(X), A))$ corresponding to $[z]$ under this identification. Using the compatibility of the quantitative L^p assembly maps with the usual ones, we obtain by naturality that $\mu_{A,*}^d([z_i]) = 0$ for each $i \in \mathbb{N}$. Since X is coarsely uniformly contractible and $\mu_{A,*}$ is injective, we deduce that there exists $d' \geq d$ such that

$$i_{d,d',*}([z]) = 0 \quad \text{in } K_*(B_L^p(P_{d'}(X), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p) \otimes A))).$$

Let $(p_t)_{t\in[0,1]}$ be a homotopy of idempotents (resp. invertibles) in $M_n(\widetilde{B})$ between $i_{d,d',*}([z])$ and 0, then $P := (p_t)$ is an idempotent (resp. invertible) element in $C([0,1], M_n(\widetilde{B}))$, where $B = B^p_L(P_{d'}(X), \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p)\otimes A))$. Put $N' = \max\{33N_1, \|P\|\}$. Since

$$\mu^{\lambda_{PA}\varepsilon,R,N',d}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*}([z]) = \mu^{\lambda_{PA}\varepsilon,R,N',d'}_{\ell^\infty(\mathbb{N},\mathscr{K}(\ell^p)\otimes A),*} \circ i_{d,d',*}([z]),$$

then

$$\iota_*^{\lambda_{PA}\varepsilon,R,N'}([x]) = 0 \quad \text{in } K_*^{\lambda_{PA}\varepsilon,R,N'}(B^p(X, \ell^\infty(\mathbb{N}, \mathscr{K}(\ell^p)\otimes A))).$$

By naturality, we see that $\iota_*^{\lambda_{PA}\varepsilon,R,N'}([x_i]) = 0$ in $K_*^{\lambda_{PA}\varepsilon,R,N'}(B^p(X,A))$ for all integers i. Picking an integer i such that $r_i \geq R$ and $N_i \geq N'$, we have

$$\iota_*^{\lambda_{PA}\varepsilon,r_i,N_i}([x_i]) = 0,$$

which contradicts our assumption.

Theorem 5.2 (see [21]) *For any $p \in [1,\infty)$, the L^p coarse Baum-Connes conjecture holds for proper metric spaces with finite asymptotic dimension.*

Since hyperbolic metric spaces have finite asymptotic dimension, and combining this with Remark 5.1 and Theorems 5.1–5.2, we have the following result.

Corollary 5.2 *For any $N \geq 1$, there exists a universal constant $\lambda_{PA} \geq 1$ such that for any discrete Gromov hyperbolic metric space X, the following holds: For any ε in $\left(0, \frac{1}{20\lambda_{PA}}\right)$ and any $r > 0$, there exist $r' \geq r$ and $N' \geq N$ such that $\mathcal{PA}_*(B^p(X,A), \varepsilon, \lambda_{PA}\varepsilon, r, r', N, N')$ holds for any L^p operator algebra A.*

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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