Chin. Ann. Math. Ser. B
45(6), 2024, 905–926
DOI: 10.1007/s11401-024-0044-2

Chinese Annals of
Mathematics, Series B

Eigenvalues of Second-Order Left-Definite Linear Difference Operator with Spectral Parameters in Boundary Conditions*

Chenghua GAO[1] Jingjing WANG[2] Xiaobin YAO[3] Xueqin CAO[1]

Abstract In this paper, the authors consider the spectra of second-order left-definite difference operator with linear spectral parameters in two boundary conditions. First, they obtain the exact number of this kind of eigenvalue problem, and prove these eigenvalues are all real and simple. In details, they get that the number of the positive (negative) eigenvalues is related to not only the number of positive (negative) elements in the weight function, but also the parameters in the boundary conditions. Second, they obtain the interlacing properties of these eigenvalues and the sign-changing properties of the corresponding eigenfunctions according to the relations of the parameters in the boundary conditions.

Keywords Left-definite difference operator, Boundary conditions with spectral parameters, Interlacing properties, Oscillation properties
2000 MR Subject Classification 39A06, 39A12, 39A21, 39A70

1 Introduction

Let a and b be two integers with $a < b$. We use $[a,b]_{\mathbb{Z}}$ to denote the integer set $\{a, a+1, \cdots, b\}$. In this paper, we consider the spectra of the following second-order left-definite linear difference operator

$$-\nabla(p(t)\Delta y(t)) + q(t)y(t) = \lambda r(t)y(t), \quad t \in [1,T]_{\mathbb{Z}}, \tag{1.1}$$

$$(a_0 + b_0\lambda)y(0) = (c_0 + d_0\lambda)\Delta y(0), \tag{1.2}$$

$$(a_1 + b_1\lambda)y(T+1) = (c_1 + d_1\lambda)\nabla y(T+1), \tag{1.3}$$

where $T > 1$ is an integer, $[1,T]_{\mathbb{Z}} = \{1, \cdots, T\}$, $p : [0,T]_{\mathbb{Z}} \to (0,+\infty)$, $q : [1,T]_{\mathbb{Z}} \to [0,+\infty)$, $r(t) \neq 0$ and changes its sign on $[1,T]_{\mathbb{Z}}$.

Manuscript received September 28, 2022. Revised October 13, 2023.
[1]Department of Mathematics, Northwest Normal University, Lanzhou 730070, China.
E-mail: gaokuguo@163.com caoxqq@126.com
[2]School of Information Engineering, Lanzhou City University, Lanzhou 730070, China.
E-mail: WJJ950712@163.com
[3]Department of Mathematics, Qinghai Minzu University, Xining 810007, China.
E-mail: yaoxiaobin2008@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12461039, 12161071), the Doctoral Research Fund Project of Lanzhou City University (No. LZCU-BS2023-24), the Youth Fund Project of Lanzhou City University (No. LZCU-QN2023-09), Gansu Youth Science and Technology Fund Project (No. 24JRRA536) and the Discipline Construction Project of Lanzhou City University.

As an important problem in both mathematical subject and any other subject, Sturm-Liouville (S-L for short) problem has been discussed more than a hundred years. Many of the classical earliest results and references on the S-L problem could be found in Ince [34]. In details, for the following S-L problem

$$\begin{aligned}&-(p(t)y'(t))'+q(t)y(t)=\lambda m(t)y(t),\quad t\in(a,b),\\&a_0y(a)+c_0y'(a)=0,\quad a_1y(b)-b_1y'(b)=0,\end{aligned}$$

if the operator $Ly:=-(py')'+qy$ is right-definite, then this problem has a sequence of real and simple eigenvalues λ_k as

$$\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\to+\infty,$$

and the corresponding eigenfunction changes its sign exactly $k-1$ times. Meanwhile, if L is left-definite, the problem has two sequences of real and simple eigenvalues $\lambda_k^{\pm}$ as

$$\begin{aligned}&0\le\lambda_1^+<\lambda_2^+<\cdots<\lambda_k^+<\cdots\to+\infty,\\&0\ge\lambda_1^->\lambda_2^->\cdots>\lambda_k^->\cdots\to-\infty,\end{aligned}$$

and the corresponding eigenfunction to $\lambda_k^{\pm}$ changes its sign exactly $k-1$ times. After this, several authors paid attention to the right-definite and left-definite S-L problems for ordinary differential equations, such as [13–15, 47] and the references therein. Meanwhile, the discrete S-L problem attracted wide attention as well. In 1964, Atkinson [9] obtained the spectra of the discrete problem

$$c(t)y(t+1)=(\lambda a(t)+b(t))y(t)-c(t-1)y(t-1),\quad t\in[1,T]_{\mathbb{Z}},\tag{1.4}$$

$$y(0)=0,\quad y(T+1)+ly(T)=0.\tag{1.5}$$

Under the assumption that $a(t)>0$, this S-L problem has T real and simple eigenvalues. Later, Jirari [35] continued to consider the spectra of the same equation with a more general boundary condition and obtained similar results. Hartman [33] discussed the oscillation of the Sturm-Liouville difference equations including the possible sign-changing in the leading coeffcient $p(t)$, and the notion of a generalized zero was introduced to get the oscillation properties of the eigenfunctions. For the discrete right-definite periodic eigenvalue problems and the eigenvalue problems with coupled boundary conditions, the eigenvalue results have been obtained by Ma and Ma [43], Wang and Shi [46] and Sun and Shi [45]. Meanwhile, for the discrete left-definite S-L problems, Ma et al. [41] first considered the spectra of the discrete second-order left-definite S-L problem with Neumann boundary condition. They obtained this kind of problems has exact T real and simple eigenvalues. In details, the number of positive eigenvalues equals to the number of positive elements in the weight function, and the number of negative eigenvalues equals to the number of negative elements in the weight function. Meanwhile, they obtained the sign-changing time of the corresponding eigenvalues. Later, Ma et al. [40, 42] obtained the spectra of the discrete problem with Dirichlet boundary condition and the Sturm-Liouville boundary condition, respectively.

Now, let us recall the research history on the S-L problems with spectral parameters in the boundary conditions. As far as we know, the study of this kind of problems could be first found in 1820s, Poisson [44] deduced an ODE model with spectral parameter in the boundary conditions from a pendulum problem. After this, this kind of problems has received widespread attention (see [1–8, 10–12, 16–32, 36–39]) and it appeared in several practical problems, such as the vibrating string problems (see [1]), the acoustic wave problem (see [10]), the heat conduction problems (see [20]), the problem of vibrating beam (see [7–8]) and so on. In particular, Binding and Browne [11] considered the problem

$$-(py')' + qy = \lambda ry, \quad t \in [0, 1], \tag{1.6}$$

$$(a_j + b_j\lambda)y(j) = (c_j + d_j\lambda)(py')(j), \quad j = 0, 1. \tag{1.7}$$

Under the assumption $(-1)^i\sigma_i = (-1)^i(b_ic_i - a_id_i) < 0$ $(i = 1, 2)$ and the other assumptions which could guarantee that the operator L is a right-definite and self-adjoint opertor, the authors obtained (1.6)–(1.7) has a sequence of real and simple eigenvalues λ_k, which has only accumulation $+\infty$. Meanwhile, they obtained the oscillation properties of the corresponding eigenfunctions. The method they used is the Prüfer transformation. They also obtained the interlacing properties of the eigenvalues and the sign-changing time of the corresponding eigenfunctions. Later, the same results for (1.6)–(1.7) were obtained by Kerimov and Aliyev [36] and Kerimov and Poladov [38] by using a different method. Meanwhile, Aliyev [2] and Aliyev and Dun'yamalieva [3] continued to consider the spectra of (1.6)–(1.7) under the assumption $\sigma_0 < 0$ and $\sigma_1 < 0$ in a Pontryagin space and they obtained the basis properties of system of root functions of the S-L problem with spectral parameter in the boundary conditions. Moreover, Aliyev and Guliyeva [4], Aliyev and Kerimov [5], Aliyev and Namazov [6–8] also considered the spectral properties of the fourth-order S-L problems with spectral parameters in the boundary conditions. For the left-definite S-L problems with spectral parameters in the boundary conditions, Binding and Browne [12] considered the problem (1.6)–(1.7) under the left-definite assumptions. They obtained this eigenvalue problem has two sequences of real and simple eigenvalues λ_k, in which one is negative with only one accumulation $-\infty$ and the other one is positive with only one accumulation $+\infty$. Furthermore, the interlacing properties of the eigenvalues and the sign-changing time of the corresponding eigenvalues are obtained as well. Now, the question is what would happen for the discrete S-L problems. In 2002, Harmsen and Li [31] considered the spectra of the following right-definite discrete S-L problems: (1.1) with the boundary condition

$$y(0) = 0, \quad (a_1 + b_1\lambda)y(T + 1) = (c_1 + d_1\lambda)\nabla y(T + 1). \tag{1.8}$$

They obtained the problem (1.1), (1.8) has at most $T + 1$ real eigenvalues. Then they (see [32]) continued to consider the S-L problem, that is, (1.1) with the condition $y(0) = 0$ and a boundary condition with squared spectra parameter. They obtained this problem has at most $T + 2$ real eigenvalues. Later, Gao and Ma [25] considered the more general right-definite S-L problem (1.1)–(1.3) with $(-1)^i\sigma_i < 0$ and obtained this problem has exact $T+2$ real and simple eigenvalues (for (1.1), (1.8), it is exact $T + 1$ real and simple eigenvalues). They also obtained the interlacing properties of the eigenvalues and the oscillation properties of the corresponding eigenfunctions. Later, Gao et al. [23] obtained the spectral properties of (1.1)–(1.3) in a finite dimensional Pontryagin space under the assumptions: $r(t) > 0$, $\sigma_0 < 0$ and $\sigma_1 < 0$. Gao et al.

[28] considered the spectral properties of the right-definite S-L problems, (1.1) with nonlinear parameters in the boundary conditions, then obtained the exact number of the real and simple eigenvalues and the oscillation properties of the corresponding eigenfunctions. Meanwhile, Gao et al. [26] studied the spectra of second-order left-definite S-L problem (1.1) with the boundary condition

$$b_0 y(0) = d_0 \Delta y(0), \quad (a_1 + b_1\lambda) y(T+1) = (c_1 + d_1\lambda)\nabla y(T+1). \tag{1.9}$$

Under the following assumptions:

(A1) $p(t) > 0$ for $t \in [0,T]_{\mathbb{Z}}$, $q(t) \geq 0$ on $t \in [1,T]_{\mathbb{Z}}$.

(A2) $r(t)$ changes its sign on $t \in [1,T]_{\mathbb{Z}}$, i.e., there are m points in $t \in [1,T]_{\mathbb{Z}}$ such that $r(t) > 0$, while $r(t) < 0$ on other $T - m$ points in $t \in [1,T]_{\mathbb{Z}}$.

(H3) $b_0 + d_0 \neq 0, b_0 d_0 \geq 0$, and if $b_0 = 0$, then $q(t) \not\equiv 0$.

(A4) $-\sigma_1 M_1$ is positive definite, where $\sigma_1 = b_1 c_1 - a_1 d_1$ and

$$M_1 = \begin{pmatrix} a_1 b_1 & -a_1 d_1 \\ -a_1 d_1 & c_1 d_1 \end{pmatrix}.$$

They obtained the exact number of positive eigenvalues and negative eigenvalues in different cases. Furthermore, they obtained the interlacing properties of eigenvalues and oscillation properties of the corresponding eigenfunctions. However, for the problem (1.1)–(1.3), the fundamental functions defined in [26] is not useful for (1.1)–(1.3). Therefore, in this paper, we try to consider the spectral properties of (1.1)–(1.3). To get it, we have to give a new assumption on the first boundary condition.

(A3) $\sigma_0 M_0$ is positive definite, where $\sigma_0 = b_0 c_0 - a_0 d_0$ and

$$M_0 = \begin{pmatrix} a_0 b_0 & -a_0 d_0 \\ -a_0 d_0 & c_0 d_0 \end{pmatrix}.$$

Under the assumptions (A1)–(A4), we find that the S-L problem (1.1)–(1.3) is really a left-definite eigenvalue problem in a new Hilbert space, which will be shown in Section 2. Then, in Subsection 3.1, by defining and discussing the properties of the generalized Sturm's sequences, we obtain the eigenvalues of the Right Dirichlet Problem (RDP for short) and Right Neumann Problem (RNP for short). In details, according to the sign of δ_0, we could get the following table.

Table 1 Number of eigenvalues of RDP (or RNP).

	$\sigma_0 > 0$	$\sigma_0 < 0$
Number of Positive Eigenvalues (NPE)	m	$m+1$
Number of Negative Eigenvalues (NNE)	$T+1-m$	$T-m$

Then, the sign-changing time of the generalized Sturm's sequences will be obtained in Subsection 3.2 and then the sign-changing time of the eigenfunctions of RDP and RNP will be obtained subsequently. Furthermore, based on the discussion of Section 3, we define two new fundamental functions $f(\lambda)$ and $g(\lambda)$, and then get the interlacing properties of the eigenvalues and the oscillation properties of the corresponding eigenfunctions in Section 4. Different from the continuous version, it is noted that the sign of the weight function $r(t)$ and σ_i influences the number of positive (negative) eigenvalues. The following table shows that how $r(t)$ and σ_i influence the number of positive (negative) eigenvalues.

Table 2 Number of Eigenvalues of (1.1)–(1.3).

	$\sigma_0 > 0$		$\sigma_0 < 0$	
	$\sigma_1 > 0$	$\sigma_1 < 0$	$\sigma_1 > 0$	$\sigma_1 < 0$
NPE	$m+1$	m	$m+2$	$m+1$
NNE	$T+1-m$	$T+2-m$	$T-m$	$T+1-m$

2 Left-Definiteness of the Generalized Operator

Let

$$E = \{y \mid y : [1,T]_{\mathbb{Z}} \to \mathbb{C}\}.$$

Then E is a Hilbert space under the inner product $\langle y, z\rangle_E = \sum\limits_{t=1}^{T} y(t)\overline{z}(t)$.

Furthermore, Let $H := E \oplus \mathbb{C}^2$. Then H is a Hilbert space under the inner product

$$\langle Y, Z\rangle = ((y,\alpha,\mu),(z,\beta,\nu)) = \langle y, z\rangle_Y + p(0)\frac{1}{|\sigma_0|}\alpha\beta + p(T)\frac{1}{|\sigma_1|}\mu\nu.$$

Define an operator $L : D(L) \to Y \oplus \mathbb{C}^2$ as follows:

$$\begin{aligned} LY &= L(y,\alpha,\mu) \\ &= (-\nabla(p(t)\Delta y(t)) + q(t)y(t), -\varepsilon_0(a_0y(0) - c_0\Delta y(0)), \varepsilon_1(a_1y(T+1) - c_1\nabla y(T+1))), \end{aligned}$$

where $D(L) = \{(y,\alpha,\mu) \mid y \in Y, d_0\Delta y(0) - b_0y(0) = \alpha, d_1\nabla y(T+1) - b_1y(T+1) = \mu\}$, $\varepsilon_0 = \operatorname{sgn}\sigma_0$, $\varepsilon_1 = \operatorname{sgn}\sigma_1$. So, (1.1)–(1.3) is equivalent to $LY = \lambda SY$, where $S(y,\alpha,\mu) = (ry, -\varepsilon_0\alpha, \varepsilon_1\mu)$.

Lemma 2.1 *Suppose that* (A1), (A3) *and* (A4) *hold. Then the operator L is positive definite on* $D(L)$.

Proof Let $Y \in H$ be a nonzero element. Then

$$\begin{aligned} &\langle LY, Y\rangle \\ &= \sum_{t=1}^{T}(-\nabla(p(t)\Delta y(t)) + q(t)y(t))y(t) + p(0)\frac{1}{|\sigma_0|}(-b_0y(0) + d_0\Delta y(0))(-\varepsilon_0(a_0y(0) - c_0\Delta y(0))) \\ &\quad + p(T)\frac{1}{|\sigma_1|}(d_1\nabla y(T+1) - b_1y(T+1))(\varepsilon_1(a_1y(T+1) - c_1\nabla y(T+1))) \\ &= \sum_{t=1}^{T} q(t)(y(t))^2 - \sum_{t=0}^{T} p(t)y(t)\Delta y(t) + \sum_{t=0}^{T} p(t)y(t+1)\Delta y(t) + y(0)\Delta y(0)p(0) \end{aligned}$$

$$
\begin{aligned}
&- p(0)\frac{\varepsilon_0}{|\sigma_0|}[d_0a_0y(0)\Delta y(0) + c_0b_0y(0)\Delta y(0) - a_0b_0y(0)^2 - c_0d_0(\Delta y(0))^2] \\
&+ p(T)\frac{\varepsilon_1}{|\sigma_1|}[b_1c_1y(T+1)\nabla y(T+1) - a_1b_1y(T+1)^2 + a_1d_1y(T+1)\nabla y(T+1) \\
&- c_1d_1(\nabla y(T+1))^2] \\
= &\sum_{t=1}^{T} q(t)(y(t))^2 - \sum_{t=0}^{T} p(t)y(t)\Delta y(t) + \sum_{t=0}^{T} p(t)y(t+1)\Delta y(t) + y(0)\Delta y(0)p(0) \\
&- y(T+1)p(T)\nabla y(T+1) - p(0)\frac{1}{\sigma_0}[2d_0a_0y(0)\Delta y(0) - a_0b_0(y(0))^2 - c_0d_0(\Delta y(0))^2 \\
&+ \sigma_0 y(0)\Delta y(0)] + p(T)\frac{1}{\sigma_1}[2a_1d_1y(T+1)\nabla y(T+1) - a_1b_1y(T+1)^2 \\
&+ \sigma_1 y(T+1)\nabla y(T+1) - c_1d_1(\nabla y(T+1))^2] \\
= &\sum_{t=1}^{T} q(t)(y(t))^2 + \sum_{t=0}^{T} p(t)(\Delta y(t))^2 + p(0)\frac{1}{\sigma_0}\langle M_0 f_0, f_0\rangle_E - p(T)\frac{1}{\sigma_1}\langle M_1 f_1, f_1\rangle_E,
\end{aligned}
$$

where $f_0 = (y(0), \Delta y(0))^T$, $f_1 = (y(T+1), \nabla y(T+1))^T$. Since (A3), (A4) hold, we know $\sigma_0 M_0, -\sigma_1 M_1$ is positive definite. Therefore, $\langle LY, Y\rangle > 0$. The desired result is obtained.

3 Generalized Sturm's Squences and Its Properties

In this section, we try to consider the properties of the roots of the polynomial $y(t,\lambda) = 0$, which satisfy the initial condition

$$y(0,\lambda) = c_0 + d_0\lambda, \quad \Delta y(0,\lambda) = a_0 + b_0\lambda. \tag{3.1}$$

Then, by (1.1), the explicit expression of $y(t,\lambda)$ is obtained as follows:

$$
\begin{aligned}
&y(0,\lambda) = c_0 + d_0\lambda; \\
&y(1,\lambda) = (a_0 + c_0) + (b_0 + d_0)\lambda; \\
&y(2,\lambda) = \Big[1 + \frac{p(0)}{p(1)} + \frac{q(1)}{p(1)} - \lambda\frac{r(1)}{p(1)}\Big]y(1,\lambda) - \frac{p(0)}{p(1)}y(0,\lambda); \\
&\cdots \\
&y(t,\lambda) = \Big[1 + \frac{p(t-2)}{p(t-1)} + \frac{q(t-1)}{p(t-1)} - \lambda\frac{r(t-1)}{p(t-1)}\Big]y(t-1,\lambda) - \frac{p(t-2)}{p(t-1)}y(t-2,\lambda) \\
&\qquad\quad = (-1)^{t-1}\frac{r(t-1)r(t-2)\cdots r(1)}{p(t-1)p(t-2)\cdots p(1)}\lambda^t(b_0+d_0) + P_{t-1}(\lambda), \\
&\qquad t = 2, 3, 4, \cdots, T+1,
\end{aligned}
\tag{3.2}
$$

where $P_{t-1}(\lambda)$ is a polynomial of degree $t-1$ of λ. This kind of sequences is the generalized Sturm's sequence. Now, let us recall some basic properties of the generalized Sturm's sequence (3.2), which can be found in the references [23, 26, 28].

Lemma 3.1 (see [23, 28]) *Suppose that $y(t,\lambda)$ is a solution of the initial value problem $(1.1)_\lambda$, (2.1) and $y(t,\mu)$ is a solution of $(1.1)_\mu$, (2.1). Then for $t \in [1,T]_{\mathbb{Z}}$,*

$$(\mu - \lambda)\Big[\sum_{s=1}^{t} r(s)y(s,\lambda)y(s,\mu) - p(0)\sigma_0\Big] = p(t)\begin{vmatrix} y(t+1,\lambda) & y(t+1,\mu) \\ y(t,\lambda) & y(t,\mu)\end{vmatrix}. \tag{3.3}$$

Lemma 3.2 (see [23, 28]) *Suppose that $y(t,\lambda)$ is a solution of the initial value problem* $(1.1)_\lambda, (2.1)$*. Then, for $t \in [1,T]_{\mathbb{Z}}$, we have*

$$\sum_{s=1}^{t} r(s)(y(s,\lambda))^2 - p(0)\sigma_0 = p(t)\begin{vmatrix} y(t+1,\lambda) & \frac{\partial}{\partial\lambda}y(t+1,\lambda) \\ y(t,\lambda) & \frac{\partial}{\partial\lambda}y(t,\lambda) \end{vmatrix}. \tag{3.4}$$

Lemma 3.3 (see [26]) *Suppose that $y(t,\lambda)$ is a solution of the initial value problem* $(1.1)_\lambda$, (2.1)*. Then, for $t \in [1,T]_{\mathbb{Z}}$, we have*

$$\begin{aligned} \lambda\sum_{t=1}^{T} r(t)(y(t))^2 = {} & \sum_{t=0}^{T} p(t)(\Delta y(t))^2 + \sum_{t=1}^{T} q(t)(y(t))^2 \\ & + p(0)y(0)\Delta y(0) - p(T)y(T+1)\nabla y(T+1). \end{aligned} \tag{3.5}$$

Lemma 3.4 (see [23, 28]) *For $t \in \{1,\cdots,T+1\}$, $y(t,\lambda)=0$ and $y(t-1,\lambda)=0$ have no common zeros.*

Lemma 3.5 (see [23, 28]) *Suppose that $\lambda=\lambda_0$ is a root of $y(t,\lambda)=0$. Then, for $t\in\{1,\cdots,T\}$, $y(t-1,\lambda_0)y(t+1,\lambda_0) < 0$.*

Next, let us prove another useful properties for the Sturm's sequences. To get it, it is necessary to discuss the relations of the σ_0, a_0, b_0, c_0 and d_0.

Remark 3.1 Suppose that (A3) holds. Then $\sigma_0 M_0$ is positive definite. By the properties of positive definite matrix, it follows that $\sigma_0 a_0 b_0 > 0, \sigma_0 c_0 d_0 > 0$ and $\sigma_0 \det M_0 = \sigma_0^2(a_0b_0c_0d_0 - a_0^2d_0^2) = \sigma_0^3 a_0 d_0 > 0$. Therefore, $\det M_0 = \sigma_0 a_0 d_0 > 0$. Finally, from the inequalities $\sigma_0 a_0 d_0 > 0, \sigma_0 a_0 b_0 > 0, \sigma_0 c_0 d_0 > 0$, we get $a_0 c_0 > 0, b_0 d_0 > 0$.

Lemma 3.6 *Suppose that* (A1) *holds. If $\lambda = 0$, then $(a_0+c_0)\Delta y(t-1,0) > 0$ and $(a_0+c_0)y(t,0) > 0$, $t \in [0,T+1]_{\mathbb{Z}}$.*

Proof Without loss of generality, let $a_0 + c_0 > 0$. Then by Remark 3.1, it is easy to see that $y(0,0) = c_0 > 0$ and $y(1,0) = a_0 + c_0 > 0$. Then, $\Delta y(0,0) > 0$. Next, suppose that the result holds for $t = k$, that is, $\Delta y(k-1,0) > 0$ and $y(k,0) > 0$. Then for $t = k+1$, by (1.1), it follows that

$$\Delta y(k,0) = \frac{q(t)}{p(t)}y(k,0) + \frac{p(t-1)}{p(t)}\Delta y(k-1,0).$$

By the condition (A1), we get $\Delta y(k,0) > 0$, and $y(k+1,0) > 0$ subsequently. Therefore, the induction method implies that the results hold.

3.1 Interlacing properties of zeros of Sturm's sequence

In this subsection, let us consider the interlacing properties of the roots of $y(t,\lambda) = 0$.

For $t \in [2,T+1]_{\mathbb{Z}}$, let

$$S_t^+ = \{k \in [1,t-1]_{\mathbb{Z}} \mid r(k) > 0\}, \quad S_t^- = \{k \in [1,t-1]_{\mathbb{Z}} \mid r(k) < 0\}.$$

Let m_t be the number of S_t^+, $t-1-m_t$ be the number of S_t^-. Meanwhile, if $t=0$ and $t=1$, we define $m_t = 0$ and $t-1-m_t = 0$ for convenience.

Lemma 3.7 *For $t \in [0, T+1]_{\mathbb{Z}}$, $y(t, \lambda)$ has the following asymptotic behavior as $\lambda \to \pm\infty$. In details,*

(a) *if $b_0 + d_0 > 0$, then*

$$\lim_{\lambda\to\pm\infty} y(0,\lambda) = \pm\infty, \quad \lim_{\lambda\to\pm\infty} y(1,\lambda) = \pm\infty,$$
$$\lim_{\lambda\to-\infty} (-1)^{t-m_t} y(t,\lambda) = +\infty, \quad \lim_{\lambda\to+\infty} (-1)^{m_t} y(t,\lambda) = +\infty;$$

(b) *if $b_0 + d_0 < 0$, then*

$$\lim_{\lambda\to\pm\infty} y(0,\lambda) = \mp\infty, \quad \lim_{\lambda\to\mp\infty} y(1,\lambda) = \mp\infty,$$
$$\lim_{\lambda\to-\infty} (-1)^{t+m_t-1} y(t,\lambda) = +\infty, \quad \lim_{\lambda\to+\infty} (-1)^{m_t+1} y(t,\lambda) = +\infty.$$

Proof In fact, by Remark 3.1, if $b_0 + d_0 > 0$, then $d_0 > 0$, if $b_0 + d_0 < 0$, then $d_0 < 0$. Therefore, the asymptotic behavior of $y(0, \lambda)$ and $y(1, \lambda)$ holds for $\lambda \to \pm\infty$.

For $t \in [2, T+1]_{\mathbb{Z}}$, by (2.2),

$$\begin{aligned} y(t,\lambda) &= (-1)^{t-1} \frac{r(t-1)r(t-2)\cdots r(1)}{p(t-1)p(t-2)\cdots p(1)} \lambda^t (b_0 + d_0) + P_{t-1}(\lambda) \\ &= (-1)^{m_t} \frac{|r(t-1)||r(t-2)|\cdots|r(1)|}{p(t-1)p(t-2)\cdots p(1)} \lambda^t (b_0 + d_0) + P_{t-1}(\lambda). \end{aligned}$$

Then, it is not difficult to see that (a) and (b) hold.

Now, we focus on the interlacing properties of the generalized Sturm's sequence. For the sake of simplicity, let $\lambda_{t,+i}$ denote the positive root(s) of $y(t, \lambda) = 0$ and $\lambda_{t,-(t-1-i)}$ denote the negative root(s) of $y(t, \lambda) = 0$. By Remark 3.1, the roots of $y(0, \lambda) = 0$ and $y(1, \lambda) = 0$ satisfy the following relation:

If $\sigma_0 > 0$, then

$$\lambda_{0,-1} = -\frac{c_0}{d_0} < \lambda_{1,-1} = -\frac{a_0 + c_0}{b_0 + d_0} < 0. \tag{3.6}$$

If $\sigma_0 < 0$, then

$$\lambda_{0,+1} = -\frac{c_0}{d_0} > \lambda_{1,+1} = -\frac{a_0 + c_0}{b_0 + d_0} > 0. \tag{3.7}$$

Furthermore, for $t \in [1, T]_{\mathbb{Z}}$, the following interlacing result holds.

Lemma 3.8 *For $t \in [1, T]_{\mathbb{Z}}$, the roots of $y(t, \lambda) = 0$ and $y(t+1, \lambda) = 0$ are simple, real and interlacing each other. In details,*

(a) *if $\sigma_0 < 0$ and $r(t) > 0$, then*

$$\begin{aligned} &0 < \lambda_{t+1,+i} < \lambda_{t,+i} < \lambda_{t+1,+(i+1)}, \quad i = 1, 2, \cdots, m_t + 1; \\ &0 > \lambda_{t+1,-j} > \lambda_{t,-j} > \lambda_{t+1,-(j+1)}, \\ &\lambda_{t+1,-(t-m_t-1)} > \lambda_{t,-(t-m_t-1)}, \quad j = 1, \cdots, t - m_t - 2; \end{aligned}$$

(b) *if $\sigma_0 < 0$ and $r(t) < 0$, then*

$$0 < \lambda_{t+1,+i} < \lambda_{t,+i} < \lambda_{t+1,+(i+1)}, \quad \lambda_{t+1,+(m_t+1)} < \lambda_{t,+(m_t+1)}, \quad i = 1, 2, \cdots, m_t;$$

$$0 > \lambda_{t+1,-j} > \lambda_{t,-j} > \lambda_{t+1,-(j+1)}, \quad j = 1, 2, \cdots, t - m_t - 1;$$

(c) *if* $\sigma_0 > 0$ *and* $r(t) > 0$, *then*

$$0 < \lambda_{t+1,+i} < \lambda_{t,+i} < \lambda_{t+1,+(i+1)}, \quad i = 1, 2, \cdots, m_t;$$
$$0 > \lambda_{t+1,-j} > \lambda_{t,-j} > \lambda_{t+1,-(j+1)}, \quad \lambda_{t+1,-(t-m_t)} > \lambda_{t,-(t-m_t)}, \quad j = 1, 2, \cdots, t - m_t - 1;$$

(d) *if* $\sigma_0 > 0$ *and* $r(t) < 0$, *then*

$$0 < \lambda_{t+1,+i} < \lambda_{t,+i} < \lambda_{t+1,+(i+1)}, \quad \lambda_{t+1,+m_t} < \lambda_{t,+m_t}, \quad i = 1, 2, \cdots, m_t - 1;$$
$$0 > \lambda_{t+1,-j} > \lambda_{t,-j} > \lambda_{t+1,-(j+1)}, \quad j = 1, 2, \cdots, t - m_t.$$

Proof We only prove the case (a). Others could be proved similarly. First, we prove that (a) holds for $y(1, \lambda) = 0$ and $y(2, \lambda) = 0$. In this case, $r(1) > 0$, $m_1 = 0$ and $m_2 = 1$. Now, the proof will be divided into two cases.

Case I $b_0 + d_0 > 0$. Then, $y(0, \lambda)$ and $y(1, \lambda)$ are both increasing linear function with respect to λ. On the other hand, it follows from (3.7) that $\lambda_{0,+1} > \lambda_{1,+1} > 0$, and then, by the fact that $\delta_0 < 0$, we obtain

$$y(0, \lambda_{1,+1}) = c_0 + d_0\lambda_{1,+1} = \frac{\sigma_0}{b_0 + d_0} < 0.$$

Combining this with Lemma 3.5, it follows that $y(2, \lambda_{1,+1}) > 0$. On the other hand, Lemma 3.7 implies that

$$\lim_{t\to\pm\infty} y(1, \lambda) = \pm\infty \quad \text{and} \quad \lim_{t\to\pm\infty} y(2, \lambda) = -\infty.$$

Therefore, $y(2, \lambda) = 0$ has exact 2 real roots. Furthermore, by Remark 3.1, $\sigma_0 < 0$ implies that $a_0 < 0$ and $a_0 + c_0 < 0$. Combining this with Lemma 3.6, we have that $y(2, \lambda) = 0$ has two positive roots $\lambda_{2,+1}$ and $\lambda_{2,+2}$ satisfying $0 < \lambda_{2,+1} < \lambda_{1,+1} < \lambda_{2,+2}$. This means (a) holds for $y(1, \lambda) = 0$ and $y(2, \lambda) = 0$, if $b_0 + d_0 > 0$.

Case II $b_0 + d_0 < 0$. Then (3.7) still implies that $\lambda_{0,+1} > \lambda_{1,+1} > 0$ and $\sigma_0 < 0$ implies that

$$y(0, \lambda_{1,+1}) = c_0 + d_0\lambda_{1,+1} = \frac{\sigma_0}{b_0 + d_0} > 0.$$

From Lemmas 3.5 and 3.7, $y(2, \lambda_{1,+1}) < 0$ and

$$\lim_{t\to\pm\infty} y(1, \lambda) = \mp\infty \quad \text{and} \quad \lim_{t\to\pm\infty} y(2, \lambda) = +\infty.$$

Therefore, by Lemma 3.6, we obtain that $y(2, \lambda) = 0$ has exact two different real roots $\lambda_{2,+1}$ and $\lambda_{2,+2}$ such that $0 < \lambda_{2,+1} < \lambda_{1,+1} < \lambda_{2,+2}$. Therefore, (a) holds.

Next, suppose that (a) holds for $y(k, \lambda) = 0$ and $y(k + 1, \lambda) = 0$. That is to say, if $\sigma_0 < 0$ and $r(k) > 0$, then the roots of these two equations satisfy the following interlacing properties:

$$\begin{aligned} &0 < \lambda_{k+1,+i} < \lambda_{k,+i} < \lambda_{k+1,+(i+1)}, \quad i = 1, 2, \cdots, m_k + 1; \\ &0 > \lambda_{k+1,-j} > \lambda_{k,-j} > \lambda_{k+1,-(j+1)}, \\ &\lambda_{k+1,-(t-m_k-1)} > \lambda_{k,-(t-m_k-1)}, \quad j = 1, 2, \cdots, t - m_k - 2; \end{aligned} \tag{3.8}$$

Finally, let us consider the interlacing properties of the roots of $y(k+1,\lambda)=0$ and $y(k+2,\lambda)=0$. By (2.2),

$$y(k+2,\lambda)=(-1)^{m_{k+2}}\frac{|r(k+1)||r(k)|\cdots|r(1)|}{p(k+1)p(k)\cdots p(1)}\lambda^{k+2}(b_0+d_0)+P_{k+1}(\lambda).$$

Then the following two cases should be discussed.

Case I $r(k+1)>0$ and $b_0+d_0>0$. In this case, $m_{k+2}=m_{k+1}+1=m_k+2$ and

$$\lim_{\lambda\to-\infty}(-1)^{k+2-m_{k+2}}y(k+2,\lambda)=+\infty,\quad \lim_{\lambda\to+\infty}(-1)^{m_{k+2}}y(k+2,\lambda)=+\infty. \tag{3.9}$$

Now, let us focus on the sign of $y(k,\lambda)$ and $y(k+2,\lambda)$ at the roots of $y(k+1,\lambda)=0$, i.e., at the points $\lambda=\lambda_{k+1,\pm i}$. In fact, from Lemma 3.5 and (3.8), it follows that

$$(-1)^{m_k}y(k,\lambda_{k+1,+m_k+2})>0,(-1)^{m_k+1}y(k,\lambda_{k+1,+m_k+1})>0,\cdots,(-1)^{2m_k+1}y(k,\lambda_{k+1,+1})>0.$$

This together with Lemma 3.5 implies that

$$\begin{aligned}(-1)^{m_k}y(k+2,\lambda_{k+1,+m_k+2})&<0,\\(-1)^{m_k+1}y(k+2,\lambda_{k+1,+m_k+1})&<0,\\&\cdots,\\(-1)^{2m_k+1}y(k+2,\lambda_{k+1,+1})&<0.\end{aligned}\tag{3.10}$$

Meanwhile, since $\sigma_0<0$ and $b_0,d_0>0$, we know from Remark 3.1 that $a_0<0$ and $a_0+c_0<0$. Therefore, $y(k+2,0)<0$. Combining this with (3.9)–(3.10), we know that $y(k+2,\lambda)=0$ has m_k+2 positive roots with

$$0<\lambda_{k+2,+i}<\lambda_{k+1,+i}<\lambda_{k+2,+i+1},\quad i=1,2,\cdots,m_k+1.$$

Next, let us consider the distribution of the negative roots of $y(k+1,\lambda)=0$ and $y(k+2,\lambda)=0$. In fact, by (3.8), at the point $\lambda=\lambda_{k+1,-j}$, we get

$$\begin{gathered}(-1)^{k-m_k-1}y(k,\lambda_{k+1,-(k-m_k-1)})>0,(-1)^{k-m_k-2}y(k,\lambda_{k+1,-(k-m_k-2)})>0,\\\cdots,(-1)^2y(k,\lambda_{k+1,-2})>0,(-1)^1y(k,\lambda_{k+1,-1})>0.\end{gathered}$$

Therefore, Lemma 3.5 implies that

$$\begin{gathered}(-1)^{k-m_k-1}y(k+2,\lambda_{k+1,-(k-m_k-1)})<0,(-1)^{k-m_k-2}y(k+2,\lambda_{k+1,-(k-m_k-2)})<0,\\\cdots,(-1)^2y(k+2,\lambda_{k+1,-2})<0,(-1)^1y(k+2,\lambda_{k+1,-1})<0.\end{gathered}$$

Combining this with (3.9) and the fact that $y(k+2,0)<0$, we get that $y(k+2,\lambda)=0$ has $k-1-m_k$ negative roots which satisfy

$$\begin{aligned}&0>\lambda_{k+2,-j}>\lambda_{k+1,-j}>\lambda_{k+2,-(j+1)},\\&\lambda_{k+2,-(t-m_k-1)}>\lambda_{k+1,-(t-m_k-1)},\quad j=1,2,\cdots,k-m_k-1.\end{aligned}$$

By the mathematical induction, (a) holds for $r(k+1)>0$ and $b_0+d_0>0$.

Case II $r(k+1) > 0$ and $b_0 + d_0 < 0$. In this case, $a_0 > 0$, $c_0 > 0$, $m_k = m_k + 1 = m_k + 2$ and by Lemma 3.7,

$$\begin{gathered}\lim_{\lambda\to-\infty}(-1)^{k+m_k-1}y(k,\lambda) = +\infty, \quad \lim_{\lambda\to+\infty}(-1)^{m_k+1}y(k,\lambda) = +\infty. \\ \lim_{\lambda\to-\infty}(-1)^{k+m_k+1}y(k+2,\lambda) = +\infty, \quad \lim_{\lambda\to+\infty}(-1)^{m_k+1}y(k+2,\lambda) = +\infty. \end{gathered} \tag{3.11}$$

Furthermore, by Lemma 3.5 and (3.8), we get

$$\begin{gathered}(-1)^{m_k+1}y(k,\lambda_{k+1,+m_k+2}) > 0, (-1)^{m_k+2}y(k,\lambda_{k+1,+m_k+1}) > 0, \\ \cdots, (-1)^{2m_k+2}y(k,\lambda_{k+1,+1}) > 0\end{gathered}$$

and

$$\begin{gathered}(-1)^{m_k+1}y(k+2,\lambda_{k+1,+m_k+2}) < 0, (-1)^{m_k+2}y(k+2,\lambda_{k+1,+m_k+1}) < 0, \\ \cdots, (-1)^{2m_k+2}y(k+2,\lambda_{k+1,+1}) < 0.\end{gathered}$$

Therefore, if follows from (3.11) that $y(k+2,\lambda) = 0$ has at least $m_k + 3$ positive roots satisfying (a). On the other hand,

$$\begin{gathered}(-1)^{k-m_k}y(k,\lambda_{k+1,-(k-m_k-1)}) > 0, (-1)^{k-m_k-1}y(k,\lambda_{k+1,-(k-m_k-2)}) > 0, \\ \cdots, (-1)^{3}y(k,\lambda_{k+1,-2}) > 0, (-1)^{2}y(k,\lambda_{k+1,-1}) > 0\end{gathered}$$

and

$$\begin{gathered}(-1)^{k-m_k}y(k+2,\lambda_{k+1,-(k-m_k-1)}) < 0, (-1)^{k-m_k-1}y(k+2,\lambda_{k+1,-(k-m_k-2)}) < 0, \\ \cdots, (-1)^{3}y(k+2,\lambda_{k+1,-2}) < 0, (-1)^{2}y(k+2,\lambda_{k+1,-1}) < 0.\end{gathered}$$

Combining this with (3.11) and the fact $y(k+2,0) > 0$, we get that $y(k+2,\lambda) = 0$ has at least $k - m_k - 1$ negative roots satisfying (a). Finally, since $y(k+2,\lambda)$ is a $(k+2)$th degree polynomial about λ, we know that those "at least" should be "exact".

In what follows, by the mathematical induction, we get that (a) holds for $\sigma_0 < 0$ and $r(t) > 0$. This completes the proof.

From Lemma 3.8, it is easy to get the following eigenvalue results for the Right Dirichlet problem (RDP for short): (1.1), (2.1) with the boundary condition $y(T+1) = 0$.

Theorem 3.1 *Suppose that* (A1)–(A3) *hold. Then Right Dirichlet problem has exact s real and simple positive eigenvalues λ^D_{+i}, and $T+1-s$ real and simple negative eigenvalues λ^D_{-l}, satisfying*

$$\lambda^D_{T+1-s} < \cdots < \lambda^D_{-2} < \lambda^D_{-1} < 0 < \lambda^D_{+1} < \lambda^D_{+2} < \cdots < \lambda^D_{+s}, \tag{3.12}$$

where

$$s = \begin{cases} m, & \sigma_0 > 0, \\ m+1, & \sigma_0 < 0. \end{cases}$$

Then by Theorem 3.1, we could get the distribution of eigenvalues of the Right Neumann problem (RNP for short): (1.1), (2.1) and boundary condition $\nabla y(T+1)=0$.

Theorem 3.2 *Suppose that* (A1)–(A3) *hold. Then Right Neumann problem has exact s real and simple positive eigenvalues λ^N_{+i}, $i=1,\cdots,s$ and $T-s+1$ real and simple negative eigenvalues λ^N_{-j}, $j=1,\cdots,T-s+1$. These eigenvalues satisfy the following interlacing properties*

$$\lambda^N_{-(j+1)}<\lambda^D_{-j}<\lambda^N_{-j}<0<\lambda^N_{+i}<\lambda^D_{+i}<\lambda^N_{+(i+1)}. \tag{3.13}$$

Proof Let

$$N(\lambda)=\frac{y(T,\lambda)}{y(T+1,\lambda)},\quad \lambda\neq\lambda^D_{+i}\ \text{ and }\ \lambda\neq\lambda^D_{-j}.$$

It follows from (2.2) that $\nabla y(T+1)$ is a $(T+1)$th degree polynomial of λ. Then, there are at most $T+1$ real λ such that $\nabla y(T+1,\lambda)=0$. This implies that $N(\lambda)$ has at most $T+1$ real zeros. Now, we only need to discuss that the equation $N(\lambda)=1$ has exact $T+1$ real roots which satisfy the conclusion.

By (2.2), it is easy to see that $\lim\limits_{\lambda\to\pm\infty}N(\lambda)=0$. Furthermore, by Lemma 3.6, we know that $(a_0+c_0)\Delta y(T,0)>0$ and $N(0)<1$. Meanwhile, no matter $a_0+c_0>0$ or $a_0+c_0<0$, by Lemma 3.8, it is not difficult to see that, there exists a small left neighbourhood $\mathcal{U}^o_-(\lambda^D_{+1};\delta_0)$ of λ^D_{+1}, such that $N(\lambda)>0$ for $\mathcal{U}^o_-(\lambda^D_{+1};\delta_0)$. This implies that $\lim\limits_{\lambda\to\lambda^D_{+1}-0}N(\lambda)=+\infty$. Therefore, $N(\lambda)=1$ has a positive root λ^N_{+1} in $(0,\lambda^D_{+1})$. Furthermore, by Lemma 3.8, there exists a small right neighbourhood $\mathcal{U}^o_+(\lambda^D_{+1};\delta_0)$ of λ^D_{+1}, such that $N(\lambda)<0$ for $\mathcal{U}^o_+(\lambda^D_{+1};\delta_0)$. This implies that $\lim\limits_{\lambda\to\lambda^D_{+1}+0}N(\lambda)=-\infty$. Similarly, by Lemma 3.8, we know that for some $\delta>0$ small enough,

$$\begin{aligned}
&N(\lambda)>0,\quad \lambda\in\mathcal{U}^o_-(\lambda^D_{+i};\delta)\quad\text{and}\quad N(\lambda)<0,\quad \lambda\in\mathcal{U}^o_+(\lambda^D_{+i};\delta),\quad i=1,2,\cdots,s-1,\\
&N(\lambda)<0,\quad \lambda\in\mathcal{U}^o_-(\lambda^D_{-j};\delta)\quad\text{and}\quad N(\lambda)>0,\quad \lambda\in\mathcal{U}^o_+(\lambda^D_{-j};\delta),\quad j=1,2,\cdots,T-s.
\end{aligned}$$

Therefore, $N(\lambda)$ has at least one zero point in each of these open intervals: $(0,\lambda^D_{+1})$, $(-\lambda^D_{+1},0)$, $(\lambda_{+i},\lambda_{+(i+1)})$ and $(\lambda^D_{-(j+1)},\lambda^D_{-j})$, where $i\in[1,s-1]_{\mathbb{Z}}$ and $j\in[1,T-s]_{\mathbb{Z}}$.

Meanwhile, $N(\lambda)<0$ for $\lambda\in\mathcal{U}^o_+(\lambda^D_{+s};\delta)$ and $N(\lambda)>0$ for $\lambda\in\mathcal{U}^o_-(\lambda^D_{-(T+1-s)};\delta)$. Combining this with the fact that $\lim\limits_{\lambda\to\pm\infty}N(\lambda)=0$, it is not difficult to see that $N(\lambda)$ does not have any zero in $(-\infty,\lambda^D_{-(T+1-s)})$ and $(\lambda^D_{+s},+\infty)$.

Overall, $N(\lambda)$ has exact s positive zeros λ^N_{+i} ($i\in[1,s]_{\mathbb{Z}}$) and $T+1-s$ negative zeros λ^N_{-j} ($j\in[1,T+1-s]_{\mathbb{Z}}$) such that the interlacing inequalities (3.13) holds.

3.2 The sign-changing rule of the generalized Sturm's sequence

In this section, we try to discuss the oscillation properties of $y(k,\lambda)$ about k. Then the position of the point $\lambda_{0,+1}$(or $\lambda_{0,-1}$) will be important for our discussion.

For the sake of convenience, if $\sigma_0>0$, let

$$\lambda^N_{-(T-m+2)}=\lambda^D_{-(T-m+2)}=-\infty,\quad \lambda^N_{+(m+1)}=\lambda^D_{+(m+1)}=+\infty,\quad \lambda^N_{+0}=\lambda^D_{+0}=\lambda^N_{-0}=\lambda^D_{-0}=0.$$

If $\sigma_0<0$, let

$$\lambda^N_{-(T-m+1)}=\lambda^D_{-(T-m+1)}=-\infty,\quad \lambda^N_{+(m+2)}=\lambda^D_{+(m+2)}=+\infty,\quad \lambda^N_{+0}=\lambda^D_{+0}=\lambda^N_{-0}=\lambda^D_{-0}=0.$$

Furthermore, without loss of generality, if $\sigma_0 > 0$, we suppose that there exists a nonnegative constant $W_- : 1 \le W_- \le T - m + 2$ such that

$$\lambda^D_{-W_-} \le \lambda_{0,-1} < \lambda^D_{-(W_- - 1)}.$$

If $\sigma_0 < 0$, we suppose that there exists a non-negative constant $W_+ : 1 \le W_+ \le m + 2$, such that

$$\lambda^D_{+(W_+ - 1)} < \lambda_{0,+1} \le \lambda^D_{+W_+}.$$

Theorem 3.3 *Suppose that* (A1)–(A3) *hold and* $\sigma_0 > 0$. *Then*

(i) *if* $\lambda \in (\lambda^D_{+(k-1)}, \lambda^D_{+k}]$ *and* $k \in [1, m+1]_{\mathbb{Z}}$, *then all solutions of* (1.1), (2.1) *change their signs exactly* $k - 1$ *times on* $[0, T+1]_{\mathbb{Z}}$;

(ii) *if* $\lambda \in [\lambda^D_{-k}, \lambda^D_{-(k-1)})$ *or* $\lambda \in (\lambda_{0,-1}, \lambda^D_{-(W_- - 1)})$, $k \in [1, W_- - 1]_{\mathbb{Z}}$, *then all solutions of* (1.1), (2.1) *change their signs exactly* $k - 1$ *times in* $[0, T+1]_{\mathbb{Z}}$;

(iii) *if* $\lambda \in [\lambda^D_{-k}, \lambda^D_{-(k-1)})$ *or* $\lambda \in (\lambda^D_{-W_-}, \lambda_{0,-1}]$, $k \in [W_-, T - m + 2,]_{\mathbb{Z}}$, *then all solutions of* (1.1), (2.1) *change their signs exactly* $k - 2$ *times in* $[0, T+1]_{\mathbb{Z}}$.

Proof Now, we only need to discuss the sign-changing time of the sequence

$$\{y(0, \lambda),\ y(1, \lambda),\ \cdots, y(T+1, \lambda)\}. \tag{3.14}$$

First, let us prove (i) holds. Since $\sigma_0 > 0$, it follows that $y(0, \lambda)$ and $y(1, \lambda)$ are both negative or positive for $\lambda \ge 0$. Now, we prove the result by using the method of induction on k, $k = 1, 2, \cdots, T+1$.

Step 1 Let us consider the case that $k = 2$. If $r(1) < 0$, then, by Lemma 3.8, we know that the two roots of $y(2, \lambda) = 0$ are both negative. This together with Lemmas 3.6–3.7 implies that $y(2, \lambda) > 0$ for $\lambda > 0$.

On the other hand, if $r(1) > 0$, then by Lemma 3.7, $y(2, \lambda) = 0$ has exact one negative root $\lambda_{2,-1}$ and one positive root $\lambda_{2,+1}$. Then, the following two cases happen for $r(1) > 0$.

Case I $b_0 + d_0 > 0$. Since $\sigma_0 > 0$, it follows from Remark 3.1 that $a_0 > 0$ and $c_0 > 0$. Then, by Lemmas 3.6–3.7, we get $y(2, 0) < 0$ and $\lim\limits_{\lambda \to \pm\infty} y(2, \lambda) = -\infty$. Therefore,

$$\begin{aligned}
&\operatorname{sgn}\{y(0, \lambda),\ y(1, \lambda),\ y(2, \lambda)\} = \{(-1)^0, (-1)^0, (-1)^0\} \quad \text{for } \lambda \in (0, \lambda_{2,+1});\\
&\operatorname{sgn}\{y(0, \lambda),\ y(1, \lambda),\ y(2, \lambda)\} = \{(-1)^0, (-1)^0, 0\} \quad \text{for } \lambda = \lambda_{2,+1};\\
&\operatorname{sgn}\{y(0, \lambda),\ y(1, \lambda),\ y(2, \lambda)\} = \{(-1)^0, (-1)^0, (-1)^1\} \quad \text{for } \lambda \in (\lambda_{2,+1}, +\infty).
\end{aligned}$$

Therefore, $\{y(0, \lambda),\ y(1, \lambda),\ y(2, \lambda)\}$ does not change its sign for $\lambda \in (0, \lambda_{2,+1}]$ and changes its sign exactly one time for $\lambda \in (\lambda_{2,+1}, +\infty)$. We get the desired result.

Case II $b_0 + d_0 < 0$. In this case, $a_0 < 0$ and $c_0 < 0$. By Lemmas 3.6–3.7, $y(2, 0) < 0$ and $\lim\limits_{\lambda \to \pm\infty} y(2, \lambda) = -\infty$. Then, Lemma 3.8($c$) implies that

$$\begin{aligned}
&\operatorname{sgn}\{y(0, \lambda),\ y(1, \lambda), y(2, \lambda)\} = \{(-1)^1, (-1)^1, (-1)^1\} \quad \text{for } \lambda \in (0, \lambda_{2,+1});\\
&\operatorname{sgn}\{y(0, \lambda),\ y(1, \lambda), y(2, \lambda)\} = \{(-1)^1, (-1)^1, 0\} \quad \text{for } \lambda = \lambda_{2,+1};\\
&\operatorname{sgn}\{y(0, \lambda),\ y(1, \lambda), y(2, \lambda)\} = \{(-1)^1, (-1)^1, (-1)^0\} \quad \text{for } \lambda \in (\lambda_{2,+1}, +\infty).
\end{aligned}$$

The result still holds for $k = 2$.

Step 2 Suppose that the result holds for $k = t$. This is to say, the sequence

$$\{y(0,\lambda), y(1,\lambda), \cdots, y(t,\lambda)\} \tag{3.15}$$

changes its sign exactly $k-1$ times for $\lambda \in (\lambda^D_{+(k-1)}, \lambda^D_{+k}]$.

Step 3 Now, we consider the case $k = t+1$, i.e., the sign-changing time of the sequence

$$\{y(0,\lambda), y(1,\lambda), \cdots, y(t,\lambda), y(t+1,\lambda)\}. \tag{3.16}$$

Without loss of generality, let $\lambda \in (\lambda_{t+1,+(i-1)}, \lambda_{t+1,+i}]$, $i = 2, \cdots, s_{t+1}$. According to Lemma 3.8, $\lambda_{t,+(i-2)} < \lambda_{t+1,+(i-1)} < \lambda_{t,+(i-1)} < \lambda_{t+1,+i} < \lambda_{t,+i}$. Furthermore, since $\lambda_{t+1,+(i-1)}$ is the $(i-1)$th zero of $y(t+1,\lambda)$ and $(a_0+c_0)y(t+1,0) > 0$, we know that

$$\mathrm{sgn} y(t+1,\lambda) = (-1)^{i-1}\mathrm{sgn}(a_0+c_0) \quad \text{for } \lambda \in (\lambda_{t+1,+(i-1)}, \lambda_{t+1,+i}). \tag{3.17}$$

If $\lambda \in (\lambda_{t+1,+(i-1)}, \lambda_{t,+(i-1)}]$, then $\lambda \in (\lambda_{t,+(i-2)}, \lambda_{t,+(i-1)}]$. According to Step 2, we know that the sequence (3.15) changes its sign exactly $i-2$ times and the sign of $y(t,\lambda)$ in this interval is $(-1)^{i-2}\mathrm{sgn}(a_0+c_0)$. This together with (3.17) implies that (3.16) changes its sign exactly $i-1$ times. On the other hand, if $\lambda = \lambda_{t,+(i-1)}$, then by Lemma 3.5, $y(t+1, \lambda_{t,+(i-1)})y(t-1, \lambda_{t,+(i-1)}) < 0$. Therefore, (3.16) still changes its sign exactly $i-1$ times.

If $\lambda \in (\lambda_{t,+(i-1)}, \lambda_{t+1,+i}]$, then $\lambda \in (\lambda_{t,+(i-1)}, \lambda_{t,+i})$. According to Step 2, we know that the sequence (3.15) changes its sign exactly $i-1$ times and the sign of $y(t,\lambda)$ in this interval is $(-1)^{i-1}\mathrm{sgn}(a_0+c_0)$. Combining this with (3.17), we know that $y(t+1,\lambda)$ and $y(t,\lambda)$ have the same sign in $(\lambda_{t,+(i-1)}, \lambda_{t+1,+i})$, which guarantees the sequence (3.16) changes its sign exactly $i-1$ times. Meanwhile, for $\lambda = \lambda_{t+1,+i}$, the result holds since (3.15) changes its sign exactly $i-1$ times at this point and $y(t+1, \lambda_{t+1,+i}) = 0$.

Therefore, the conclusion (i) holds by the induction method.

Next, let us prove (ii) and (iii) hold. Because of $\sigma_0 > 0$, we know that $\lambda_{1,-1} < \lambda_{0,-1} < 0$. So, there exists a non-negative constant $W'_- : 1 \le W'_- \le T-m+2$ such that

$$\lambda^D_{-W'_-} \le \lambda_{1,-1} = -\frac{a_0+c_0}{b_0+d_0} < \lambda^D_{-(W'_- -1)}.$$

Similar to the discussion for (i), it is not difficult to see that

$$\{y(1,\lambda), y(2,\lambda), \cdots, y(T+1,\lambda)\} \tag{3.18}$$

has the following two oscillation properties:

(P1) The sequence (3.18) changes its sign exactly $k-1$ times for $\lambda \in [\lambda^D_{-k}, \lambda^D_{-k-1})$, $k = 1, 2, \cdots, W'_- - 1$ and $W'_- - 1$ times for $\lambda \in (\lambda_{1,-1}, \lambda^D_{-(W'_- -1)})$.

(P2) The sequence (3.18) changes its sign exactly $k-2$ times, for $\lambda \in [\lambda^D_{-k}, \lambda^D_{-k-1})$, $k = W'_-, \cdots, T-m+2$ and $W'_- - 2$ times for $\lambda \in (\lambda^D_{-W'_-}, \lambda_{1,-1}]$.

Now, let us discuss the sign-changing time of the sequence (3.14).

First, when $\lambda \in (\lambda_{1,-1}, 0)$, then $(a_0+c_0)y(0,\lambda) > 0, (a_0+c_0)y(1,\lambda) > 0$. This means $y(0,\lambda)$ and $y(1,\lambda)$ have the same sign for $\lambda \in (\lambda_{1,-1}, 0)$. This together with the property (P1) implies that the sequence (3.14) changes its sign exactly $k-1$ times, when $\lambda \in [\lambda^D_{-k}, \lambda^D_{-k+1})$ or $\lambda \in (\lambda_{1,-1}, \lambda^D_{-(W'_--1)})$, $k \in [1, W'_- - 1]_{\mathbb{Z}}$.

Second, when $\lambda \in (\lambda_{0,-1}, \lambda_{1,-1}]$, then $(a_0+c_0)y(0,\lambda) > 0$ and $(a_0+c_0)y(1,\lambda) < 0$. Therefore, $y(0,\lambda)$ and $y(1,\lambda)$ have the opposite sign in this interval. This together with (P1) implies that the sequence (3.14) changes its sign exactly $k-1$ times, when $\lambda \in [\lambda^D_{-k}, \lambda^D_{-k+1})$ or $\lambda \in (\lambda^D_{-W'_-}, \lambda_{1,-1}]$ or $\lambda \in (\lambda_{0,-1}, \lambda^D_{-(W_--1)})$, $k \in [W'_-, (W_- - 1)]_{\mathbb{Z}}$.

Third, when $\lambda \in (-\infty, \lambda_{0,-1}]$, then $(a_0+c_0)y(0,\lambda) < 0$, and $(a_0+c_0)y(1,\lambda) < 0$. Therefore, when $\lambda \in [\lambda^D_{-k}, \lambda^D_{-k+1})$ or $\lambda \in (\lambda^D_{-W_-}, \lambda_{0,-1}]$, $k \in [W_-, T-m+2]_{\mathbb{Z}}$, the sequence (3.14) changes its sign exactly $k-2$ times. If $\lambda = \lambda_{0,-1}$, then $y(0,\lambda) = 0$ and the sequence (3.14) still changes its sign exactly $k-2$ times.

Final, when $\lambda \in (0, +\infty)$, then $(a_0+c_0)y(0,\lambda) > 0$, $(a_0+c_0)y(1,\lambda) > 0$. Therefore, when $\lambda \in (\lambda^D_{k-1}, \lambda^D_k]$, $k \in [1, m+1]_{\mathbb{Z}}$, the sequence (3.18) changes its sign exactly $k-1$ times.

Similar to the proof of Theorem 3.3, we could get the following oscillation properties for the case $\sigma_0 < 0$.

Theorem 3.4 *Suppose that* (A1)–(A3) *hold and* $\sigma_0 < 0$.

(i) *If* $\lambda \in [\lambda^D_{-k}, \lambda^D_{-k+1})$, $k \in [1, T-m+1]_{\mathbb{Z}}$, *then all solutions of* (1.1), (2.1) *change their signs exactly* $k-1$ *times in* $[0, T+1]_{\mathbb{Z}}$.

(ii) *If* $\lambda \in (\lambda^D_{+k-1}, \lambda^D_{+k}]$ *or* $\lambda \in (\lambda^D_{+(W_+-1)}, \lambda_{0,+1})$, $k \in [1, W_+ - 1]_{\mathbb{Z}}$, *then all solutions of* (1.1), (2.1) *change their signs exactly* $k-1$ *times in* $[0, T+1]_{\mathbb{Z}}$.

(iii) *If* $\lambda \in (\lambda^D_{+k-1}, \lambda^D_{+k}]$ *or* $\lambda \in [\lambda_{0,+1}, \lambda^D_{+W_+})$, $k \in [W_+, m+2]_{\mathbb{Z}}$, *then all solutions of* (1.1), (2.1) *change their signs exactly* $k-2$ *times in* $[0, T+1]_{\mathbb{Z}}$.

4 Spectral Properties for (1.1)–(1.3)

First, let us introduce a function

$$f(\lambda) = \frac{\nabla y(T+1,\lambda)}{\lambda y(T+1,\lambda)}, \quad \lambda \in \bigcup_{k=-(T-s+2)}^{+(s+1)} (\lambda^D_k, \lambda^D_{k+1}), \tag{4.1}$$

where s is defined in Theorem 3.1. Then, we get the following result.

Lemma 4.1 *Suppose that* (A1)–(A3) *hold. Then* $f(\lambda)$ *has the following properties*:

(i) *The graph of* $f(\lambda)$ *consists of* $T+3$ *branches* $\mathscr{E}_k, k \in [-(T-s+2), -1] \cup [+1, +(s+1)]_{\mathbb{Z}}$. *The branch intersects* $\lambda-$*axis at* $\lambda = \lambda^N_k$.

(ii) $f(\lambda)$ *is* (*strictly*) *decreasing as* λ *varies from* λ^D_k *to* λ^D_{k+1} *for* $k \in [-(T-s+2), +(s+1)]_{\mathbb{Z}}$.

(iii) *For* $k \in [-(T-s+1), +s]_{\mathbb{Z}}$, $f(\lambda) \to -\infty$ *as* $\lambda \uparrow \lambda^D_k$ *and* $f(\lambda) \to +\infty$ *as* $\lambda \downarrow \lambda^D_k$. *Meanwhile,* $f(\lambda) \to 0$ *as* $\lambda \to \pm\infty$.

Proof (i) Since $f(\lambda)$ has only one kind of zero point: $\lambda = \lambda^N_k$, it is easy to see that (i) holds.

(ii) For $\lambda \in \bigcup_{k=-(T-s+2)}^{+(s+1)} (\lambda^D_k, \lambda^D_{k+1})$, we have

$$f'(\lambda) = \frac{\partial}{\partial \lambda}\Big(\frac{\nabla y(T+1,\lambda)}{\lambda y(T+1,\lambda)}\Big)$$

$$
\begin{aligned}
&= -\frac{y(T+1,\lambda)\nabla y(T+1,\lambda)+\lambda\big[y(T+1,\lambda)\frac{\partial}{\partial\lambda}y(T,\lambda)-y(T,\lambda)\frac{\partial}{\partial\lambda}y(T+1,\lambda)\big]}{(\lambda y(T+1,\lambda))^2}\\
&= -\frac{p(T)y(T+1,\lambda)\nabla y(T+1,\lambda)+\lambda\sum\limits_{s=1}^{T}r(s)(y(s,\lambda))^2-p(0)\delta_0\lambda}{p(T)(\lambda y(T+1,\lambda))^2}\\
&= -\frac{p(0)y(0,\lambda)\Delta y(0,\lambda)+\sum\limits_{t=0}^{T}p(t)(\Delta y(t,\lambda))^2+\sum\limits_{t=1}^{T}q(t)(y(t,\lambda))^2-p(0)\delta_0\lambda}{p(T)(\lambda y(T+1,\lambda))^2}.
\end{aligned}
$$

According to Remark 3.1, we have $b_0d_0>0, \sigma_0a_0d_0>0$. Therefore,

$$
\begin{aligned}
&p(0)y(0,\lambda)\Delta y(0,\lambda)-p(0)\sigma_0\lambda\\
&=p(0)[(c_0+d_0\lambda)(a_0+b_0\lambda)-(b_0c_0-a_0d_0)\lambda]=p(0)(b_0d_0\lambda^2+2a_0d_0\lambda+a_0c_0)\geq 0.
\end{aligned}
$$

Further, we have $f'(\lambda)<0$. Therefore, (ii) holds.

(iii) The results of (ii) and the definition of $f(\lambda)$ imply the results of (iii) holds.

Let

$$g(\lambda)=\frac{a_1+b_1\lambda}{\lambda(c_1+d_1\lambda)},\quad \lambda\neq 0,\ -\frac{c_1}{d_1}.$$

According to (A4), if $\sigma_1>0$, then $-\frac{c_1}{d_1}>0$; if $\sigma_1<0$, then $-\frac{c_1}{d_1}<0$.

Lemma 4.2 *$g(\lambda)$ has the following properties:*

(i) *$g(\lambda)$ has two vertical asymptotes $\mu_1=\min\{0,-\frac{c_1}{d_1}\}$ and $\mu_2=\max\{0,-\frac{c_1}{d_1}\}$. Meanwhile, the graph of $g(\lambda)$ has three branches $\mathscr{F}_k, k=1,2,3$ and one zero point $-\frac{a_1}{b_1}$.*

(ii) *$g(\lambda)$ is (strictly) increasing for $(-\infty,\mu_1)\cup(\mu_1,\mu_2)\cup(\mu_2,+\infty)$.*

(iii) $\lim\limits_{s\to\mu_i^-}g(s)=+\infty,\ \lim\limits_{s\to\mu_i^+}g(s)=-\infty,\ \lim\limits_{s\to\infty}g(s)=0,\ i=1,2.$

Proof Obviously, (i) holds. Now, we only need to prove that (ii) holds. For $\lambda\in(-\infty,\mu_1)\cup(\mu_1,\mu_2)\cup(\mu_2,+\infty)$, we have

$$g'(\lambda)=\Big(\frac{a_1+b_1\lambda}{\lambda(c_1+d_1\lambda)}\Big)'=-\frac{a_1c_1+2a_1d_1\lambda+b_1d_1\lambda^2}{(a_1+b_1\lambda)^2}\geq 0.$$

Since (A4) holds, the last inequality holds. This implies that $g(\lambda)$ is (strictly) increasing on each of its branches. Further, (ii), (iii) hold.

In order to obtain the oscillation theory of problems (1.1)–(1.3), we give some notations. Let

$$l=\begin{cases}s, & \sigma_1<0,\\ s+1, & \sigma_1>0.\end{cases}$$

Then

$$l=\begin{cases}m, & \sigma_1<0,\ \sigma_0>0,\\ m+1, & \sigma_1<0,\ \sigma_0<0\ \text{ or }\ \sigma_1>0,\ \sigma_0>0,\\ m+2, & \sigma_1>0,\ \sigma_0<0.\end{cases}$$

If $\sigma_1<0$, then there exists a non-negative integer $L_-: 1\leq L_-\leq T-l+2$ such that

$$\lambda^N_{-L_-}\leq -\frac{a_1}{b_1}<\lambda^N_{-(L_--1)}. \tag{4.2}$$

Further, there exists a non-negative integer $K_- : 1 \leq K_- \leq T - l + 2$ such that

$$\lambda^D_{-K_-} \leq -\frac{c_1}{d_1} < \lambda^D_{-(K_- -1)} \tag{4.3}$$

Obviously, $L_- \leq K_-$. If $\sigma_1 > 0$, then there exists a non-negative integer $L_+ : 1 \leq L_+ \leq l + 1$, such that

$$\lambda^N_{+(L_+ -1)} < -\frac{a_1}{b_1} \leq \lambda^N_{+L_+} \tag{4.4}$$

Further, there exists a non-negative integer $K_+ : 1 \leq K_+ \leq l + 1$ such that

$$\lambda^D_{+(K_+ -1)} < -\frac{c_1}{d_1} \leq \lambda^D_{+K_+}. \tag{4.5}$$

Obviously, $L_+ \leq K_+$.

Theorem 4.1 *Suppose that* (A1)–(A4) *hold. Then* (1.1)–(1.3) *has l positive eigenvalues $\{\lambda_{+k}\}_{k=1}^{k=l}$, $T + 2 - l$ negative eigenvalues $\{\lambda_{-k}\}_{k=1}^{k=T+2-l}$. These $T + 2$ eigenvalues satisfy the following properties*:

(a) *If $\sigma_1 > 0$, then*

$$\begin{aligned}
&\lambda^D_{-(T+2-l)} < \lambda_{-(T+2-l)} < \lambda^N_{-(T+2-l)} < \cdots < \lambda^D_{-1} < \lambda_{-1} < \lambda^N_{-1} < 0 \\
&< \lambda^N_{+1} < \lambda_{+1} < \lambda^D_{+1} < \cdots < \lambda^N_{+(L_+-1)} < \lambda_{+(L_+-1)} < \lambda^D_{+(L_+-1)} < \lambda_{+(L_+)} \\
&\leq \lambda^N_{+L_+} < \lambda^D_{+L_+} < \cdots < \lambda_{+(K_+-1)} < \lambda^N_{+(K_+-1)} < \lambda^D_{+(K_+-1)} < \lambda_{+K_+} \\
&< \lambda^N_{+K_+} < \lambda_{+(K_++1)} \leq \lambda^D_{+K_+} < \cdots < \lambda^N_{+l-1} < \lambda_{+l} < \lambda^D_{+l-1}.
\end{aligned} \tag{4.6}$$

(b) *If $\sigma_1 < 0$, then*

$$\begin{aligned}
&\lambda^D_{-(T-l+1)} < \lambda_{-(T-l+2)} < \lambda^N_{-(T-l+1)} < \cdots < \lambda^D_{-K_-} \leq \lambda_{-(K_-+1)} < \lambda^N_{-K_-} \\
&< \lambda_{-K_-} < \lambda^D_{-(K_--1)} < \lambda^N_{-(K_--1)} < \lambda_{-(K_--1)} < \cdots < \lambda^D_{-L_-} \\
&< \lambda^N_{-L_-} \leq \lambda_{-L_-} < \lambda^D_{-(L_--1)} < \lambda_{-(L_--1)} < \lambda^N_{-(L_--1)} < \cdots < \lambda^D_{-1} \\
&< \lambda_{-1} < \lambda^N_{-1} < 0 < \lambda^N_{+1} < \lambda_{+1} < \lambda^D_{+1} < \cdots < \lambda^N_{+l} < \lambda_{+l} < \lambda^D_{+l}.
\end{aligned} \tag{4.7}$$

Proof We only show that (a) holds. Without loss of generality, let $\sigma_0 > 0$. Then $l = m+1$ and $T + 2 - l = T + 1 - m$. Now, the proof will be divided into two cases.

Case I $1 \leq K_+ \leq m + 1$. First, we consider the first branch $\mathscr{F}_1$ of $g(\lambda)$. From the monotonicity of $g(\lambda)$ and $f(\lambda)$, the first branch $\mathscr{F}_1$ of $g(\lambda)$ intersects the upper half of branch $\mathscr{E}_{-(T-m+1)}$ to branch $\mathscr{E}_{-1}$ of $f(\lambda)$ and does not intersect the branch $\mathscr{E}_{-(T-m+2)}$. Therefore, we can get the following interlacing inequalities

$$\lambda^D_{-(T-m+1)} < \lambda_{-(T-m+1)} < \lambda^N_{-(T-m+1)} < \cdots < \lambda^D_{-1} < \lambda_{-1} < \lambda^N_{-1} < 0. \tag{4.8}$$

Next, we consider the second branch $\mathscr{F}_2$ of $g(\lambda)$. From the monotonicity of $g(\lambda)$ and $f(\lambda)$, the second branch $\mathscr{F}_2$ of $g(\lambda)$ intersects branch $\mathscr{E}_{+1}$ to branch $\mathscr{E}_{+K_+}$ of $f(\lambda)$. More precisely, since $\lambda^N_{+(L_+-1)} < -\frac{a_1}{b_1} < \lambda^N_{+L_+}$, the second branch $\mathscr{F}_2$ of $g(\lambda)$ intersects the lower half of branch $\mathscr{E}_{+1}$ to branch $\mathscr{E}_{+(L_+-1)}$ of $f(\lambda)$ and intersects the upper half of branch $\mathscr{E}_{+L_+}$ to branch $\mathscr{E}_{+K_+}$ of $f(\lambda)$. Therefore, we can get the following interlacing inequality

$$0 < \lambda^N_{+1} < \lambda_{+1} < \lambda^D_{+1} < \cdots < \lambda^N_{+(L_+-1)} < \lambda_{+(L_+-1)} < \lambda^D_{+(L_+-1)}$$

and

$$\lambda_{+L_+} < \lambda^N_{+L_+} < \lambda^D_{+L_+} < \cdots < \lambda_{+(K_+-1)} < \lambda^N_{+(K_+-1)} < \lambda^D_{+(K_+-1)} < \lambda_{+K_+} < \lambda^N_{+K_+}.$$

We consider the third branch $\mathscr{F}_3$ of $g(\lambda)$. From the monotonicity of $g(\lambda)$ and $f(\lambda)$, the third branch $\mathscr{F}_3$ of $g(\lambda)$ intersects the lower half of branch $\mathscr{E}_{+K_+}$ to branch $\mathscr{E}_{+m}$ of $f(\lambda)$ and does not intersect the last branch $\mathscr{E}_{+(m+1)}$. Therefore, we can get the following interlacing inequality

$$\lambda^N_{+K_+} < \lambda_{+(K_++1)} < \lambda^D_{+K_+} < \cdots < \lambda^N_{+m} < \lambda_{+(m+1)} < \lambda^D_{+m}.$$

Finally, if $-\frac{a_1}{b_1} = \lambda^N_{+L_+}$, then $\lambda^N_{+L_+} = \lambda_{+L_+}$. So (4.6) holds.

Case II $K_+ = +(m+1)$. Then $\lambda^D_{+m} < -\frac{c_1}{d_1} < \lambda^D_{+(m+1)}$. Similar to the discussion of Case I, we obtain (4.8) holds. Therefore, we only consider how the branches $\mathscr{F}_2$ and $\mathscr{F}_3$ of $g(\lambda)$ intersect the branches of $f(\lambda)$. From the monotonicity of $g(\lambda)$ and $f(\lambda)$, the second branch $\mathscr{F}_2$ of $g(\lambda)$ intersects the upper half of branch $\mathscr{E}_{+L_+}$ to branch $\mathscr{E}_{+(m+1)}$ of $f(\lambda)$, the third branch $\mathscr{F}_3$ of $g(\lambda)$ does not intersect $f(\lambda)$. So, we can get the following interleaving inequality

$$\lambda_{+L_+} \le \lambda^N_{+L_+} < \lambda^D_{+L_+} < \cdots < \lambda_{+m} < \lambda^N_{+m} < \lambda^D_{+m} < \lambda_{+(m+1)}.$$

Therefore, (4.6) holds.

For the sake of convenience, let

$$K_1 = \min\{\lambda_{0,-1}, \mu_1\}, \quad K_2 = \max\{\lambda_{0,-1}, \mu_1\} \quad \text{if} \quad \sigma_1 < 0,$$
$$K_3 = \min\{\lambda_{0,+1}, \mu_2\}, \quad K_4 = \max\{\lambda_{0,+1}, \mu_2\} \quad \text{if} \quad \sigma_1 > 0.$$

Theorem 4.2 *Suppose that* (A1)–(A4) *hold. If* $\sigma_0 > 0$ *and* $\sigma_1 < 0$*, then*

(i) *if* $K_2 < \lambda_{-k}$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(ii) *if* $K_1 < \lambda_{-k} \le K_2$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-2$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iii) *if* $\lambda_{-k} \le K_1$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-3$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iv) *the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$.

Proof Without loss of generality, suppose that $K_1 = \mu_1$. Then $W_- < K_-$. Other cases can be obtained similarly.

First, from Theorem 4.1, if $k \in [1, K_-]_{\mathbb{Z}}$, then $\lambda^D_{-k} < \lambda_{-k} < \lambda^D_{-(k-1)}$. Therefore, by Theorem 3.3, if $\lambda_k > \lambda_{0,-1}$, then the $-k$th eigenfunction $y_{-k}(t)$ changes its sign exactly $k-1$ times for $t \in [1, T+1]_{\mathbb{Z}}$. If $\mu_1 < \lambda_{-k} < \lambda_{0,-1}$, then the $-k$th eigenfunction $y_{-k}(t)$ changes its sign exactly $k-2$ times for $t \in [1, T+1]_{\mathbb{Z}}$.

Second, from Theorem 4.1, if $k \in [K_- + 1, T - m + 2]_{\mathbb{Z}}$, then $\lambda^D_{-(k-1)} < \lambda_{-k} < \lambda^D_{-(k-2)}$. Therefore, by Theorem 3.3, if $\lambda_k < \mu_1$, then the $-k$th eigenfunction $y_{-k}(t)$ changes its sign exactly $k-3$ times for $t \in [1, T+1]_{\mathbb{Z}}$.

Last, if $k \in [1, m+1]_{\mathbb{Z}}$, then $\lambda_{+k} \in (\lambda^D_{+(k-1)}, \lambda^D_k]$. According to Theorem 3.1, then the $+k$th eigenfunction $y_{+k}(t)$ changes its sign exactly $k-1$ times for $t \in [1, T+1]_{\mathbb{Z}}$.

Theorem 4.3 *Suppose that* (A1)–(A4) *hold. If* $\sigma_0 > 0$ *and* $\sigma_1 > 0$*, then*

(i) *if* $\lambda_{0,-1} < \lambda_{-k}$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(ii) *if* $\lambda_k \le \lambda_{0,-1}$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-2$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iii) *if* $\lambda_{+k} < \mu_2$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iv) *if* $\lambda_{+k} \ge \mu_2$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-2$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$.

Theorem 4.4 *Suppose that* (A1)–(A4) *hold. If* $\sigma_0 < 0$ *and* $\sigma_1 < 0$*, then*

(i) *if* $\mu_1 < \lambda_{-k}$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(ii) *if* $\lambda_{-k} \le \mu_1$*, then the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-2$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iii) *if* $\lambda_{+k} < \lambda_{0,+1}$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iv) *if* $\lambda_{+k} \ge \lambda_{0,+1}$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-2$ *times for* $t \in [1, T+1]_{\mathbb{Z}}$.

Theorem 4.5 *Suppose that* (A1)–(A4) *hold. If* $\sigma_0 < 0$ *and* $\sigma_1 > 0$*, then*

(i) *the* $-k$*th eigenfunction* $y_{-k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(ii) *if* $\lambda_{+k} < K_3$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-1$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iii) *if* $K_3 \le \lambda_{+k} < K_4$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-2$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$;

(iv) *if* $\lambda_{+k} \ge K_4$*, then the* $+k$*th eigenfunction* $y_{+k}(t)$ *changes its sign exactly* $k-3$ *times for* $t \in [0, T+1]_{\mathbb{Z}}$.

Remark 4.1 In order to better verify the results of this paper, we use another method to verify the spectra of the following left-definite difference operator

$$\nabla(\Delta y(t)) = -\lambda r(t) y(t), \quad t \in [1, 3]_{\mathbb{Z}}, \tag{4.9}$$

$$(1-3\lambda) y(0) = (2-\lambda)\Delta y(0), \quad (1-2\lambda) y(4) = \Big(-\frac{2}{3} + \lambda\Big)\nabla y(4), \tag{4.10}$$

where $r(1) = 1, r(2) = 1, r(3) = -1, a_0 = 1, b_0 = -3, c_0 = 2, d_0 = -1, a_1 = 1 = -d_1, b_1 = -2, c_1 = -\frac{2}{3}$. Suppose that initial conditions $y(0,\lambda) = 2-\lambda, y(1,\lambda) = 3-4\lambda$. Therefore, by (3.2), we obtain

$$\begin{aligned}
y(2,\lambda) &= (2-\lambda)(3-4\lambda) + \lambda - 2 = 4\lambda^2 - 10\lambda + 4;\\
y(3,\lambda) &= (2-\lambda)[(2-\lambda)(3-4\lambda) + \lambda - 2] + 4\lambda - 3 = -4\lambda^3 + 18\lambda^2 - 20\lambda + 5;\\
y(4,\lambda) &= (2+\lambda)[(2-\lambda)[(2-\lambda)(3-4\lambda) + \lambda - 2] + 4\lambda - 3] - (2-\lambda)(3-4\lambda) - \lambda + 2\\
&= -4\lambda^4 + 10\lambda^3 + 12\lambda^2 - 25\lambda + 6.
\end{aligned}$$

In order to obtain the eigenvalues of (4.9)–(4.10), we first consider the eigenvalues of Right Dirichlet problem (i.e., $y(4) = 0$) and Right Neumann problem (i.e., $\nabla y(4) = 0$). By using

Matlab 7.0, it is not difficult to see that $y(4,\lambda)=0$ has one negative eigenvalue and three positive eigenvalues, i.e., $\lambda_{-1}^D,\ \lambda_{+1}^D,\ \lambda_{+2}^D,\ \lambda_{+3}^D$. Similarly, $\nabla y(4,\lambda)=0$ has one negative eigenvalue and three positive eigenvalues, i.e., $\lambda_{-1}^N,\ \lambda_{+1}^N,\ \lambda_{+2}^N,\ \lambda_{+3}^N$. Then

$$\lambda_{-1}^D=-1.7064,\quad \lambda_{+1}^D=0.2884,\quad \lambda_{+2}^D=1.0703,\quad \lambda_{+3}^D=2.8477;$$
$$\lambda_{-1}^N=-0.5000,\quad \lambda_{+1}^N=0.1771,\quad \lambda_{+2}^N=1.0000,\quad \lambda_{+3}^N=2.8229.$$

Further, we can get the following interlacing inequality

$$\lambda_{-1}^D<\lambda_{-1}^N<0<\lambda_{+1}^N<\lambda_{+1}^D<\lambda_{+2}^N<\lambda_{+2}^D<\lambda_{+3}^N<\lambda_{+3}^D.$$

Next, suppose that

$$f_1(\lambda)=\frac{1-\frac{2}{3}}{\lambda}\nabla y(4)=\frac{\lambda-\frac{2}{3}}{\lambda}(-4\lambda^4+14\lambda^3-6\lambda^2-5\lambda+1),$$
$$g_1(\lambda)=\frac{1-2\lambda}{\lambda}y(4)=\frac{1-2\lambda}{\lambda}(-4\lambda^4+10\lambda^3+12\lambda^2-25\lambda+6).$$

So, in order to get the eigenvalues of (4.9)–(4.10), we need to find the intersection of $f_1(\lambda)$ and $g_1(\lambda)$. It can be seen that $f_1(\lambda)$ has five zeros $\lambda_{-1}^N,\lambda_{+1}^N,\lambda_{+2}^N,\lambda_{+3}^N$ and $\lambda=\frac{2}{3}$. Meanwhile,

$$\lim_{\lambda\to-\infty}f_1(\lambda)=-\infty,\quad \lim_{\lambda\to+\infty}f_1(\lambda)=-\infty.$$

$f_1(\lambda)$ has an asymptote $\lambda=0$, and

$$\lim_{\lambda\downarrow 0}f_1(\lambda)=+\infty,\quad \lim_{\lambda\uparrow 0}f_1(\lambda)=-\infty.$$

$g_1(\lambda)$ has five zeros, i.e., $\lambda_{-1}^D,\lambda_{+1}^D,\lambda_{+2}^D,\lambda_{+3}^D,\frac{1}{2}$, and

$$\lim_{\lambda\to-\infty}g_1(\lambda)=+\infty,\quad \lim_{\lambda\to+\infty}g_1(\lambda)=+\infty.$$

$g_1(\lambda)$ has an asymptote $\lambda=0$, and

$$\lim_{\lambda\downarrow 0}g_1(\lambda)=-\infty,\quad \lim_{\lambda\uparrow 0}g_1(\lambda)=+\infty.$$

Therefore, (4.9)–(4.10) have two negative eigenvalues λ_{-1} and three positive eigenvalues $\lambda_{+1},\lambda_{+2},\lambda_{+3},\lambda_{+4}$, which satisfy the following interlacing inequality

$$\lambda_{-1}^D<\lambda_{-1}<\lambda_{-1}^N<0<\lambda_{+1}^N<\lambda_{+1}<\lambda_{+1}^D<\lambda_{+2}<\lambda_{+2}^N<\lambda_{+3}<\lambda_{+2}^D<\lambda_{+3}^N<\lambda_{+4}<\lambda_{+3}^D.$$

In this example, we can see that $K_+=L_+=2$, $\sigma_0=-5$ and $\sigma_1=\frac{2}{3}$, therefore, the conclusion is the same to the conclusion of Theorem 4.1.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Ahn, H. J., On random transverse vibrations of rotating beam with tip mass, *Quart. J. Mech. Appl. Math.*, **39**, 1983, 97–109.

[2] Aliyev, Z. S., On the basis properties of the root functions of a boundary value problem with a spectral parameter in the boundary conditions, *Dokl. Math.*, **87**, 2013, 137–139.

[3] Aliyev, Z. S. and Dun'yamalieva, A. A., Defect basis property of a system of root functions of a Sturm-Liouville problem with spectral parameter in the boundary conditions, *Differ. Equ.*, **51**, 2015, 1249–1266.

[4] Aliyev, Z. S. and Guliyeva, S. B., Spectral properties of a fourth-order eigenvalue problem with spectral parameter in the boundary conditions, *Filomat*, **32**, 2018, 2421–2431.

[5] Aliyev, Z. S. and Kerimov, N. B., Spectral properties of the differential operators of the fourth-order with eigenvalue parameter dependent boundary condition, *Int. J. Math. Math. Sci.*, 2012, 28 pp.

[6] Aliyev, Z. S. and Namazov, F. M., Spectral properties of a fourth-order eigenvalue problem with spectral parameter in the boundary conditions, *Electron. J. Differential Equations*, **307**, 2017, 1–11.

[7] Aliyev, Z. S. and Namazov, F. M., Spectral properties of the equation of a vibrating rod at both ends of which the masses are concentrated, *Banach J. Math. Anal.*, **14**, 2020, 585–606.

[8] Aliyev, Z. S. and Namazov, F. M., Some properties of eigenfunctions for the equation of vibrating beam with a spectral parameter in the boundary conditions, *J. Differential Equations*, **269**, 2020, 1383–1400.

[9] Atkinson, F., Discrete and Continuous Boundary Problems, Academic Press, New York, 1964.

[10] Belinskiy, B., Dauer, J. P. and Xu, Y., Inverse scattering of acoustic waves in an ocean with ice cover, *Applicable Analysis*, **61**, 1996, 255–283.

[11] Binding, P. A. and Browne, P. J., Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Proc. Edinburgh Math. Soc.*, **37**, 1993, 57–72.

[12] Binding, P. A. and Browne, P. J., Left definite Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Differ. Int. Equ.*, **12**, 1999, 167–182.

[13] Binding, P. A. and Volkmer, H., A Prüfer angle approach to semidefinite Sturm-Liouville problems with coupling boundary conditions, *J. Differential Equations*, **255**, 2013, 761–778.

[14] Coddington, E. A. and Levison, N., Theory of Ordinary Differential Equations, McGraw-Hill Book Company Inc., New York-Toromto-London, 1955.

[15] Constantin, A., A general-weighted Sturm-Liouville problem, *Ann. Scuola Norm. Sup. Pisa*, **24**(4), 1997, 767–782.

[16] Ćurgus, B., Dijksma, A. and Read, T., The linearization of boundary eigenvalue problems and reproducing kernel Hilbert spaces, *Linear Algebra Appl.*, **329**, 2001, 97–136.

[17] Dijksma, A. and de Snoo, H. S. V., Symmetric and selfadjoint relations in Kreĭn spaces II, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **12**, 1987, 199–216.

[18] Dijksma, A., Langer, H. and de Snoo, H. S. V., Symmetric Sturm-Liouville operators with eigenvalue depending boundary conditions, *Amer. Math. Soc.*, **8**, 1986, 87–116.

[19] Došlý, O. and Kratz, W., Oscillation theorems for symplectic difference systems, *J. Difference Equ. Appl.*, **13**, 2007, 585–605.

[20] Fulton, C. and Pruess, S., Numerical methods for a singular eigenvalue problem with eigenparameter in the boundary conditions, *J. Math. Anal. Appl.*, **71**, 1979, 431–462.

[21] Gao, C., Li, X. and Ma, R., Eigenvalues of a linear fourth-order differential operator with squared spectral parameter in a boundary condition, *Mediterr. J. Math.*, **15**, 2018, 14 pp.

[22] Gao, C., Li, X. and Zhang, F., Eigenvalues of discrete Sturm-Liouville problems with nonlinear eigenparameter dependent boundary conditions, *Quaest. Math.*, **41**, 2018, 773–797.

[23] Gao, C., Lv, L. and Wang, Y., Spectra of a discrete Sturm-Liouville problem with eigenparameter-dependent boundary conditions in Pontryagin space, *Quaest. Math.*, **44**, 2021, 217–242.

[24] Gao, C. and Ma, R., Eigenvalues of discrete linear second-order periodic and antiperiodic eigenvalue problems with sign-changing weight, *Linear Algebra Appl.*, **467**, 2015, 40–56.

[25] Gao, C. and Ma, R., Eigenvalues of discrete Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Linear Algebra Appl.*, **503**, 2016, 100–119.

[26] Gao, C., Ma, R. and Zhang, F., Spectrum of discrete left definite Sturm-Liouville problems with eigenparameter-dependent boundary conditions, *Linear Multilinear Algebra*, **65**, 2017, 1905–1923.

[27] Gao, C. and Ran, M., Spectral properties of a fourth-order eigenvalue problem with quadratic spectral parameters in a boundary condition, *AIMS Math.*, **5**, 2020, 904–922.

[28] Gao, C., Wang, Y. and Lv, L., Spectral properties of discrete Sturm-Liouville problems with two squared eigenparameter-dependent boundary conditions, *Acta Math. Sci. Ser. B* (*Engl. Ed.*), **40**, 2020, 755–781.

[29] Gao, C., Zhang, F. and Ran, M., Eigenvalues of discrete Sturm-Liouville problems with sign-changing weight and coupled boundary conditions, *Oper. Matrices*, **14**, 2020, 491–513.

[30] Guo, H. and Qi, J., Sturm-Liouville problems involving distribution weights and an application to optimal problems, *J. Optim. Theory Appl.*, **184**, 2020, 842–857.

[31] Harmsen, B. J. and Li, A., Discrete Sturm-Liouville problems with parameter in the boundary conditions, *J. Difference Equ. Appl.*, **8**, 2002, 969–981.

[32] Harmsen, B. J. and Li, A., Discrete Sturm-Liouville problems with nonlinear parameter in the boundary conditions, *J. Difference Equ. Appl.*, **13**, 2007, 639–653.

[33] Hartman, P., Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, *Trans. Amer. Math. Soc.*, **246**, 1978, 1–30.

[34] Ince, E. L., Ordinary Differential Equations, Dover, New York, 1926.

[35] Jirari, A., Second-order Sturm-Liouville difference equations and orthogonal polynomials, *Mem. Amer. Math. Soc.*, **113**, 1995.

[36] Kerimov, N. B. and Aliyev, Z. S., On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in the boundary condition, *Differ. Equ.*, **43**, 2007, 905–915.

[37] Kerimov, N. B. and Maris, E. A., On the uniform convergence of Fourier series expansions for Sturm-Liouville problems with a spectral parameter in the boundary conditions, *Results Math.*, **73**, 2018, 16 pp.

[38] Kerimov, N. B. and Poladov, R. G., Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Dokl. Math.*, **85**, 2012, 8–13.

[39] Luo, H., Spectral theory of linear weighted Sturm-Liouville eigenvalue problems, *Acta Math. Sci.*, **37B**, 2017, 427–449.

[40] Ma, R. and Gao, C., Spectrum of discrete second-order difference operator with sign-changing weight and its applications, *Discrete Dyn. Nat. Soc.*, **2014**, 2014, 9 pp.

[41] Ma, R., Gao, C. and Lu, Y., Spectrum of discrete second-order Neumann boundary value problems with sign-changing weight, *Abstr. Appl. Anal.*, 2013, 10 pp.

[42] Ma, R., Gao, C. and Lu, Y., Spectrum theory of second-order difference equations with indefinite weight, *J. Spectr. Theory*, **8**, 2018, 971–985.

[43] Ma, R. and Ma, H., Existence of sign-changing periodic solutions of second order difference equations, *Appl. Math. Comput.*, **203**, 2008, 463–470.

[44] Poisson, M., Sur la manière d'éxperimer les fonctions par des series de quantités, et sur l'usage de cette transformation dans la résolution de différents probléḿs, **XI**, École Polytechnique de Paris, Paris, 1820.

[45] Sun, H. and Shi, Y., Eigenvalues of second-order difference equations with coupled boundary conditions, *Linear Algebra Appl.*, **414**, 2006, 361–372.

[46] Wang, Y. and Shi, Y. M., Eigenvalues of second-order difference equations with periodic and antiperiodic boundary conditions, *J. Math. Anal. Appl.*, **309**, 2005, 56–69.

[47] Zhang, M., The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, *J. London Math. Soc.*, **64**, 2001, 125–143.