Chin. Ann. Math. Ser. B
45(6), 2024, 943–956
DOI: 10.1007/s11401-024-0046-0

Chinese Annals of Mathematics, Series B

Open Cones and K-Theory for ℓ^p Roe Algebras*

Jianguo ZHANG[1]

Abstract In this paper, the author verifies the ℓ^p coarse Baum-Connes conjecture for open cones and shows that the K-theory for ℓ^p Roe algebras of open cones is independent of $p \in [1, \infty)$. Combined with the result of Fukaya and Oguni, he gives an application to the class of coarsely convex spaces that includes geodesic Gromov hyperbolic spaces, CAT(0)-spaces, certain Artin groups and Helly groups equipped with the word length metric.

Keywords K-Theory, ℓ^p Roe algebras, Open cones
2000 MR Subject Classification 19K99, 46H99

1 Introduction

The coarse Baum-Connes conjecture provides an algorithm to compute the higher indices of generalized elliptic operators on open Riemannian manifolds which lie in the K-theory of Roe algebras associated to underlying manifolds (cf. [15, 28, 32]). The conjecture has some significant applications to topology and geometry, in particular, to the Novikov conjecture (cf. [16, 31]). The coarse Baum-Connes conjecture has been verified for a large class of spaces such as open cones (cf. [15]), metric spaces with finite asymptotic dimension (cf. [33]) and metric spaces which can be coarsely embedded into a Hilbert space (cf. [34]), as well as disproved for large spheres (cf. [33]) and expanders (cf. [13]).

In recent years, the ℓ^p coarse Baum-Connes conjecture for $p \in [1, \infty)$ (cf. Conjecture 2.1) gained attention, one of the motivations is to compute the K-theory for ℓ^p Roe algebras (cf. Definition 2.2). In [35], based on the work of Yu [33], the author and Zhou proved that the ℓ^p coarse Baum-Connes conjecture holds for spaces with finite asymptotic dimension for any $p \in [1, \infty)$ and the K-theory for ℓ^p Roe algebras of such spaces is independent of $p \in (1, \infty)$. In [29], Shan and Wang proved that the injective part of the ℓ^p coarse Baum-Connes conjecture is true for metric spaces with bounded geometry which can be coarsely embedded into a simply connected complete Riemannian manifold of non-positive sectional curvature for any $p \in (1, \infty)$. On the other hand, in [6], Chung and Nowak proved that certain expanders are still counterexamples to the ℓ^p coarse Baum-Connes conjecture for any $p \in (1, \infty)$.

Manuscript received March 20, 2022. Revised September 14, 2023.
[1]School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119, China; Research Center for Operator Algebras, East China Normal University, Shanghai 200062, China.
E-mail: jgzhang15@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12171156, 12271165, 12301154).

There are also some results in the literature concerning the structure, rigidity, quasi-locality and K-theory of ℓ^p uniform Roe algebras, L^p group algebras and L^p operator crossed products. Please refer to [3–5, 10–11, 19–21, 25–26, 30].

In this paper, we consider the ℓ^p coarse Baum-Connes conjecture for open cones (cf. Definition 3.2) and hence obtain a formula to compute the K-theory for ℓ^p Roe algebras of open cones. The main result of the paper is the following theorem that generalizes Higson and Roe's result on the coarse Baum-Connes conjecture for open cones in [15] to all $p \in [1, \infty)$.

Theorem 1.1 (cf. Theorem 3.1) *Let $\mathcal{O}M$ be the open cone over a compact metric space M, then for any $p \in [1, \infty)$, the ℓ^p coarse Baum-Connes conjecture holds for $\mathcal{O}M$.*

In [35], the author and Zhou showed that the left-hand side of the ℓ^p coarse Baum-Connes conjecture does not depend on $p \in (1, \infty)$ for metric spaces with bounded geometry. In Section 5, we expand this result to all metric spaces and all $p \in [1, \infty)$. Thus we have the following corollary.

Corollary 1.1 (cf. Corollary 3.1) *Let $\mathcal{O}M$ be the open cone over a compact metric space M, then for any $p \in [1, \infty)$, the group $K_*(B^p(\mathcal{O}M))$ is isomorphic to the group $K_*(B^2(\mathcal{O}M))$, i.e., the K-theory for the ℓ^p Roe algebra of $\mathcal{O}M$ does not depend on $p \in [1, \infty)$.*

As an application of the above theorem and corollary, combined with the result of Fukaya and Oguni [9], we show that the ℓ^p coarse Baum-Connes conjecture holds for any proper coarsely convex space (cf. Definition 4.1) and the K-theory for ℓ^p Roe algebras of such spaces is independent of $p \in [1, \infty)$. The class of proper coarsely convex space includes geodesic Gromov hyperbolic spaces, CAT(0)-spaces, certain Artin groups and Helly groups equipped with the word length metric. Please see Section 4 for more examples.

The paper is organized as follows. In Section 2, we recall some facts of the ℓ^p coarse Baum-Connes conjecture for $p \in [1, \infty)$. In Section 3, we give the proof of Theorem 1.1 and obtain Corollary 1.1. In Section 4, we discuss the applications of main results and show some examples. In the end, we discuss the p-independency of the left-hand side of the conjecture in Section 5.

2 Preliminaries

In this section, we briefly recall some facts about ℓ^p coarse Baum-Connes conjecture for $1 \leq p < \infty$. We refer the reader to [6, 35] for more details.

Let X be a proper metric space, i.e., every closed ball in X is compact. The proper metric space is a separable space since a compact metric space is separable. Choosing a countable dense subset Z_X in X, and then there is a natural action ρ of $\mathrm{Bol}(X)$ on ℓ^p-space $\ell^p(Z_X) \otimes \ell^p = \ell^p(Z_X, \ell^p)$ by point-wise multiplication, where $\mathrm{Bol}(X)$ is the Banach algebra of all bounded Borel functions on X and ℓ^p is the ℓ^p-space of all p-summable sequences on the non-negative integers $\mathbb{N}$. In what follows, we will omit ρ if there is no ambiguity and let χ_U be the characteristic function on subset U of X and let $p \in [1, \infty)$.

Definition 2.1 *Let X and Y be two proper metric spaces, Z_X and Z_Y be two countable* *dense subsets of X and Y, respectively. Consider a bounded linear operator $T : \ell^p(Z_X) \otimes \ell^p \to \ell^p(Z_Y) \otimes \ell^p$.*

(1) *The support of T, denoted by* $\mathrm{supp}(T)$, *is defined to be the set consisting of all points (x, y) in $X \times Y$ such that $\chi_V T \chi_U \neq 0$ for all open neighborhoods U of x and V of y.*

(2) *The propagation of T, denoted by* $\mathrm{prop}(T)$, *is defined to be* $\sup\{d(x_1, x_2) : (x_1, x_2) \in \mathrm{supp}(T)\}$. *Here we assume X and Y are the same.*

(3) *T is called to be locally compact, if $T\chi_K$ and $\chi_{K'}T$ are compact operators for all compact subsets K in X and K' in Y.*

Definition 2.2 *Let X be a proper metric space. The ℓ^p Roe algebra of X, denoted by $B^p(X)$, is defined to be the norm closure of the algebra of all locally compact operators acting on $\ell^p(Z_X) \otimes \ell^p$ with finite propagation.*

Remark 2.1 The ℓ^p Roe algebra $B^p(X)$ is a Banach algebra. When $p = 2$, the algebra $B^2(X)$ is a C^*-algebra, called Roe algebra (please refer to [16, 27, 31]). The ℓ^p Roe algebra $B^p(X)$ is non-canonically independent of the countable dense subset Z_X of X and its K-theory is canonically independent of Z_X (please refer to [35, Corollary 2.9]).

Definition 2.3 *A Borel map f from a proper metric space X to another proper metric space Y is called coarse if*

(1) *f is proper, i.e., the inverse image of any bounded set is bounded,*

(2) *for any $R \geq 0$, there exists $S \geq 0$ such that $d(f(x), f(x')) \leq S$ for all elements $x, x' \in X$ satisfying $d(x, x') \leq R$.*

The following lemma tells us that every coarse map induces a homomorphism between two ℓ^p Roe algebras. Please refer to [35, Lemma 2.8] for the proof.

Lemma 2.1 *Let f be a coarse map from a proper metric space X to another proper metric space Y, then for any $\varepsilon > 0$, there exists an isometric operator $V_f : \ell^p(Z_X) \otimes \ell^p \to \ell^p(Z_Y) \otimes \ell^p$ and a contractive operator $V_f^+ : \ell^p(Z_Y) \otimes \ell^p \to \ell^p(Z_X) \otimes \ell^p$ such that*

$$\mathrm{supp}(V_f) \subseteq \{(x, y) \in X \times Y : d(f(x), y) \leq \varepsilon\},$$

$$\mathrm{supp}(V_f^+) \subseteq \{(y, x) \in Y \times X : d(f(x), y) \leq \varepsilon\}.$$

Moreover, the pair (V_f, V_f^+) gives rise to a homomorphism $ad_f : B^p(X) \to B^p(Y)$ defined by

$$ad_f(T) = V_f T V_f^+$$

for any element $T \in B^p(X)$.

And the map $(ad_f)_*$ induced by ad_f on K-theory depends only on f, not on the choice of the pair (V_f, V_f^+).

Definition 2.4 *Let X be a proper metric space, the ℓ^p localization algebra of X, denoted by $B_L^p(X)$, is defined to be the norm closure of the algebra of all bounded and uniformly norm-continuous functions u from $[0, \infty)$ to $B^p(X)$ such that*

$$\sup_{t \in [0,\infty)} \mathrm{prop}(u(t)) < \infty \quad \textit{and} \quad \mathrm{prop}(u(t)) \to 0,\ t \to \infty.$$

Let f be a uniformly continuous coarse map from a proper metric space X to another proper metric space Y and $\{\varepsilon_k\}$ be a decreasing sequence of positive numbers satisfying $\varepsilon_k \to 0$ as $k \to \infty$. By Lemma 2.1, for each ε_k, there exists an isometric operator V_k and a contractive operator V_k^+. Then for $t \in [0, \infty)$, define

$$V_f(t) = R(t-k)(V_k \oplus V_{k+1})R^*(t-k),$$

$$V_f^+(t) = R(t-k)(V_k^+ \oplus V_{k+1}^+)R^*(t-k)$$

for all $k \leq t \leq k+1$, where

$$R(t) = \begin{pmatrix} \cos(\frac{\pi t}{2}) & \sin(\frac{\pi t}{2}) \\ -\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix}.$$

Similar to Lemma 2.1, we have the following lemma. Please refer to [35, Lemma 2.21] for the proof.

Lemma 2.2 *Let f and $\{\varepsilon_k\}$ be as above, then the pair $(V_f(t), V_f^+(t))$ induces a homomorphism Ad_f from $B_L^p(X)$ to $B_L^p(Y) \otimes M_2(\mathbb{C})$ defined by*

$$Ad_f(u)(t) = V_f(t)(u(t) \oplus 0)V_f^+(t)$$

for any element $u \in B_L^p(X)$ and $t \in [0, \infty)$, such that

$$\operatorname{prop}(Ad_f(u)(t)) \leq \sup_{(x,x') \in \operatorname{supp}(u(t))} d(f(x), f(x')) + 4\varepsilon_k$$

for all $t \in [k, k+1]$. Moreover, the induced map $(Ad_f)_*$ on K-theory depends only on f and not on the choice of the pair $(V_f(t), V_f^+(t))$.

Now we are ready to formulate the ℓ^p coarse Baum-Connes conjecture for $p \in [1, \infty)$. Let X be a proper metric space and then consider the evaluation-at-zero homomorphism

$$e_0 : B_L^p(X) \to B^p(X),$$

which induces a homomorphism on K-theory

$$e_0 : K_*(B_L^p(X)) \to K_*(B^p(X)).$$

Let C be a locally finite and uniformly bounded cover for X. The nerve space N_C associated with C is defined to be the simplicial complex whose set of vertices equals C and where a finite subset $\{U_0, \cdots, U_n\} \subseteq C$ spans an n-simplex in N_C if and only if $\bigcap_{i=0}^n U_i \neq \emptyset$. Endow N_C with the spherical metric, i.e., the path metric whose restriction to each simplex $\{U_0, \cdots, U_n\}$ is given by

$$d\Big(\sum_{i=0}^n t_i U_i, \sum_{i=0}^n s_i U_i\Big) = d_{S^n}\Big(\Big\{\frac{t_j}{(\sum_i t_i^2)^{\frac{1}{2}}}\Big\}_{j=0}^n, \Big\{\frac{s_j}{(\sum_i s_i^2)^{\frac{1}{2}}}\Big\}_{j=0}^n\Big),$$

where d_{S^n} is the standard Riemannian metric on the unit n-sphere. The distance of two points in different connected components is defined to be infinity by convention.

Definition 2.5 (cf. [28]) *A sequence of locally finite and uniformly bounded covers $\{C_k\}$ of metric space X is called an anti-Čech system of X, if there exists a sequence of positive numbers $R_k \to \infty$ such that for each k,*

(1) *every set U in C_k has diameter less than R_k,*

(2) *any subset of diameter less than R_k in X is contained in some member of C_{k+1}.*

An anti-Čech system always exists (please refer to [28, Lemma 3.15]). By the property of the anti-Čech system, for every pair $k_2 > k_1$, there exists a simplicial map $i_{k_1k_2}$ from $N_{C_{k_1}}$ to $N_{C_{k_2}}$ such that $i_{k_1k_2}$ maps a simplex $\{U_0, \cdots, U_n\}$ in $N_{C_{k_1}}$ to a simplex $\{U'_0, \cdots, U'_n\}$ in $N_{C_{k_2}}$ satisfying $U_i \subseteq U'_i$ for all $0 \le i \le n$. Thus by Lemmas 2.1–2.2, $i_{k_1k_2}$ gives rise to the following directed systems of groups

$$(\mathrm{ad}_{i_{k_1k_2}})_* : K_*(B^p(N_{C_{k_1}})) \to K_*(B^p(N_{C_{k_2}})),$$
$$(\mathrm{Ad}_{i_{k_1k_2}})_* : K_*(B^p_L(N_{C_{k_1}})) \to K_*(B^p_L(N_{C_{k_2}})).$$

The following conjecture is called the ℓ^p coarse Baum-Connes conjecture.

Conjecture 2.1 Let X be a proper metric space, $\{C_k\}_{k=0}^{\infty}$ be an anti-Čech system of X, then the evaluation-at-zero homomorphism

$$e_0 : \lim_{k\to\infty} K_*(B^p_L(N_{C_k})) \to \lim_{k\to\infty} K_*(B^p(N_{C_k})) \cong K_*(B^p(X))$$

is an isomorphism.

Remark 2.2 The ℓ^p coarse Baum-Connes conjecture for X does not depend on the choice of the anti-Čech system. When $p = 2$, the above conjecture is the known coarse Baum-Connes conjecture and has important applications to topology and geometry (cf. [16, 31]).

Let $B^p_{L,0}(X) = \{u \in B^p_L(X) : u(0) = 0\}$. Then there exists an exact sequence

$$0 \to B^p_{L,0}(X) \to B^p_L(X) \to B^p(X) \to 0.$$

Thus by six-term exact sequence in K-theory, we have the following reduction.

Lemma 2.3 *Let X be a proper metric space, $\{C_k\}_{k=0}^{\infty}$ be an anti-Čech system of X, then the ℓ^p coarse Baum-Connes conjecture is true if and only if*

$$\lim_{k\to\infty} K_*(B^p_{L,0}(N_{C_k})) = 0.$$

The following lemma generalizes [35, Proposition 5.20] from finite-dimensional simplicial complexes to general proper metric spaces and tells us that the left-hand side of the ℓ^p coarse Baum-Connes conjecture is independent of p. We will prove it in Section 5.

Lemma 2.4 *Let X be a proper metric space, then for any $p \in [1, \infty)$, the group $K_*(B^p_L(X))$ is isomorphic to the group $K_*(B^2_L(X))$, i.e., the group $K_*(B^p_L(X))$ does not depend on $p \in [1, \infty)$.*

Combining the above lemma with the ℓ^p coarse Baum-Connes conjecture, we have the following result concerning the p-independency of K-theory for ℓ^p Roe algebras.

Corollary 2.1 *Let X be a proper metric space. If for all $p \in [1,\infty)$, the ℓ^p coarse Baum-Connes conjecture is true for X, then the K-theory of ℓ^p Roe algebra $K_*(B^p(X))$ does not depend on $p \in [1,\infty)$.*

3 Main Results

In this section, we will prove that the ℓ^p coarse Baum-Connes conjecture holds for open cones and thus the K-theory of ℓ^p Roe algebras of open cones is independent of $p \in [1,\infty)$. The idea of the proof comes from [15]. Firstly, we recall a concept in coarse geometry.

Definition 3.1 *Let $f, g : X \to Y$ be two coarse maps between proper metric spaces. f and g are called to be coarsely homotopic, if there exists a metric subspace $Z = \{(x,t) : 0 \le t \le t_x\}$ of $X \times \mathbb{R}$ and a coarse map $h : Z \to Y$, such that*

(1) *the map from X to $\mathbb{R}$ given by $x \mapsto t_x$ satisfies that for any $R \ge 0$, there exists $S \ge 0$ such that $|t_x - t_{x'}| \le S$ for all elements $x, x' \in X$ with $d(x,x') \le R$,*

(2) $h(x,0) = f(x)$,

(3) $h(x,t_x) = g(x)$.

The map f is called a coarse homotopy equivalence map if there exists a coarse map $f' : Y \to X$ such that $f'f$ and ff' are coarsely homotopic to the identities id_X and id_Y, respectively. Call X and Y to be coarsely homotopy equivalent if there exists a coarse homotopy equivalence map from X to Y.

Remark 3.1 By [14, remark on page 349], any coarse homotopy is coarsely equivalent to a continuous coarse homotopy (i.e., h is a continuous map in the above definition).

The following lemma implies that the ℓ^p coarse Baum-Connes conjecture is permanent under the coarse homotopy equivalence. The proof relies on Mayer-Vietoris principle and please refer to [16, Proposition 12.4.12] for more details.

Lemma 3.1 *For any $p \in [1,\infty)$, let X, Y be two proper metric spaces and $f, g : X \to Y$ be two coarse maps. Let $KX_*^p(X)$ and $KX_*^p(Y)$ represent the left side of the ℓ^p coarse Baum-Connes conjecture (i.e., Conjecture 2.1) for X and Y, respectively. If f is coarsely homotopic to g, then they induce same homomorphisms: $f_* = g_* : KX_*^p(X) \to KX_*^p(Y)$ and $(ad_f)_* = (ad_g)_* : K_*(B^p(X)) \to K_*(B^p(Y))$. Moreover, if f is a coarse homotopy equivalence map, then we have the following commutative diagram and two vertical homomorphisms are isomorphisms*

$$\begin{array}{ccc} KX_*^p(X) & \xrightarrow{e_0} & K_*(B^p(X)) \\ {\scriptstyle f_*}\downarrow\cong & & {\scriptstyle (ad_f)_*}\downarrow\cong \\ KX_*^p(Y) & \xrightarrow{e_0} & K_*(B^p(Y)). \end{array}$$

Secondly, let us recall the definition of open cones.

Definition 3.2 *Let (M, d_M) be a compact metric space with diameter at most 2. The open cone over M, denoted by $\mathcal{O}M$, is defined to be the quotient space $\mathbb{R}_{\ge 0} \times M/(\{0\} \times M)$ with the* *following metric*

$$d((t,x),(s,y)) = |t-s| + \min\{t,s\} d_M(x,y)$$

for any $(t,x),(s,y) \in \mathcal{O}M$.

Obviously, the open cone is a proper metric space. By [8, Proposition B.1] and [15, Proposition 4.3], we can simplify the left side of the ℓ^p coarse Baum-Connes conjecture for open cones by the following lemma.

Lemma 3.2 *Let $\mathcal{O}M$ be the open cone over a compact metric space M, then there exists an anti-Čech system $\{C_k\}$ of $\mathcal{O}M$ such that*

$$i_* : K_*(B_L^p(\mathcal{O}M)) \to \lim_{k\to\infty} K_*(B_L^p(N_{C_k}))$$

is an isomorphism, where i_* is induced by a family of maps $i_k : \mathcal{O}M \to N_{C_k}$ that maps an element x in $\mathcal{O}M$ to an element $U_k^{(i)}$ in N_{C_k} such that $x \in U_k^{(i)}$.

Moreover, the following diagram is commutative

$$\begin{array}{ccc} K_*(B_L^p(\mathcal{O}M)) & & \\ \downarrow{\scriptstyle i_*} & \searrow^{e_0} & \\ \lim\limits_{k\to\infty} K_*(B_L^p(N_{C_k})) & \xrightarrow[e_0]{} & K_*(B^p(\mathcal{O}M)). \end{array}$$

Thus the map i_* induces an isomorphism from $K_*(B_{L,0}^p(\mathcal{O}M))$ to $\lim\limits_{k\to\infty} K_*(B_{L,0}^p(N_{C_k}))$.

Combining the above lemma with Lemma 3.1, we have the following lemma.

Lemma 3.3 *Let $\mathcal{O}M$ be the open cone over a compact metric space M and $f : \mathcal{O}M \to \mathcal{O}M$ be a coarse map defined by $f((t,x)) = (\frac{t}{2}, x)$ for any $(t,x) \in \mathcal{O}M$, then f is coarsely homotopic to the identity map i and they induce the same homomorphism on $K_*(B_{L,0}^p(\mathcal{O}M))$, where $B_{L,0}^p(\mathcal{O}M) = \{u \in B_L^p(\mathcal{O}M) : u(0) = 0\}$.*

Proof Let $Z = \{((t,x),s) : 0 \le s \le \frac{1}{2}\}$ and $h : Z \to \mathcal{O}M$ defined by $h((t,x),s) = ((1-s)t, x)$. Then h is a coarse homotopy connecting i and f. Let $\{C_k\}$ be an anti-Čech system of $\mathcal{O}M$. Then by Lemma 3.1, f and i induce same homomorphisms on $\lim\limits_{k\to\infty} K_*(B_L^p(N_{C_k}))$ and $K_*(B^p(\mathcal{O}M))$. Thus they induce the same homomorphism on $\lim\limits_{k\to\infty} K_*(B_{L,0}^p(N_{C_k}))$ by the five lemma. Therefore, by Lemma 3.2 and the five lemma, they induce the same homomorphism on $K_*(B_{L,0}^p(\mathcal{O}M))$.

Now we begin to show and prove the main theorem of this article.

Theorem 3.1 *Let $\mathcal{O}M$ be the open cone over a compact metric space M, then for any $p \in [1,\infty)$, the ℓ^p coarse Baum-Connes conjecture holds for $\mathcal{O}M$.*

Proof By Lemmas 2.3 and 3.2, it is sufficient to prove $K_*(B_{L,0}^p(\mathcal{O}M)) = 0$.

Choose a countable dense subset Z_M in M. Let $E^p = \ell^p(\mathbb{Q}_{\ge 0} \times Z_M) \otimes \ell^p$ and $E^{p,\infty} = \bigoplus\limits_{n=0}^{\infty} (\ell^p(\mathbb{Q}_{\ge 0} \times Z_M) \otimes \ell^p)$, where $\bigoplus$ is the ℓ^p-direct sum. Then there exists a natural action of

$\mathrm{Bol}(\mathcal{O}M)$ on $E^{p,\infty}$. Thus similar to the definition of the ℓ^p Roe algebra on E^p (denoted by $B^p(\mathcal{O}M; E^p)$), we can define the ℓ^p Roe algebra on $E^{p,\infty}$, denoted by $B^p(\mathcal{O}M; E^{p,\infty})$. So do $B^p_L(\mathcal{O}M; E^{p,\infty})$ and $B^p_{L,0}(\mathcal{O}M; E^{p,\infty})$. Denote $B^p_{L,0}(\mathcal{O}M)$ by $B^p_{L,0}(\mathcal{O}M; E^p)$.

Let f be a map from $\mathcal{O}M$ to $\mathcal{O}M$ given by $f((t,x)) = (\frac{t}{2}, x)$ and define two linear operators V, V^+ on E^p by $V(\delta_{(t,x)}) = \delta_{(\frac{t}{2},x)}$, $V^+(\delta_{(t,x)}) = \delta_{(2t,x)}$, where $\delta_{(t,x)} \in E^p$ maps (t', x') to 1 for $t' = t$ as well as $x' = x$ and to 0 for others. Then we have $V^+V = VV^+ = I$ and $\mathrm{supp}(V) = \{((t,x), f((t,x))) : (t,x) \in \mathcal{O}M\}$ as well as $\mathrm{supp}(V^+) = \{(f((t,x)), (t,x)) : (t,x) \in \mathcal{O}M\}$.

Define a map $\phi : B^p_{L,0}(\mathcal{O}M; E^p) \to B^p_{L,0}(\mathcal{O}M; E^{p,\infty})$ by

$$\phi(u)(s) = 0 \oplus Vu(s)V^+ \oplus \cdots \oplus V^{n+1}u(s-n)(V^+)^{n+1} \oplus \cdots$$

for any $u \in B^p_{L,0}(\mathcal{O}M; E^p)$ and let $u(s) = 0$ for $s < 0$. Now we show that ϕ is well-defined. Firstly, for any $s \geq 0$, the operator $\phi(u)(s)$ is locally compact since $u(s-n) = 0$ for all $n \geq s$. Secondly, by direct computation, we have that $\mathrm{supp}(V^{n+1}u(s-n)(V^+)^{n+1})$ is equal to the set

$$\left\{\left(\left(\frac{t}{2^{n+1}}, x\right), \left(\frac{t'}{2^{n+1}}, x'\right)\right) : ((t,x),(t',x')) \in \mathrm{supp}(u(s-n))\right\},$$

which implies that

$$\mathrm{prop}(V^{n+1}u(s-n)(V^+)^{n+1}) = \frac{\mathrm{prop}(u(s-n))}{2^{n+1}}.$$

Thus $\mathrm{prop}(\phi(u)(s)) \to 0$ as $s \to \infty$. Finally, $\phi(u)(0) = 0$ is obvious and thus ϕ is well-defined.

Define $\psi : B^p_{L,0}(\mathcal{O}M; E^p) \to B^p_{L,0}(\mathcal{O}M; E^{p,\infty})$ by

$$\psi(u)(s) = u(s) \oplus 0 \oplus \cdots \oplus 0 \oplus \cdots$$

for any $u \in B^p_{L,0}(\mathcal{O}M; E^p)$. Now we prove ψ induces an isomorphism ψ_* on the K-theory level. Let $T : E^p \to E^{p,\infty}$ given by $T(\xi) = (\xi, 0, \cdots, 0, \cdots)$ for any $\xi \in E^p$ and $T^+ : E^{p,\infty} \to E^p$ given by $T^+(\xi_0, \xi_1, \cdots, \xi_n, \cdots) = \xi_0$ for any $(\xi_0, \xi_1, \cdots, \xi_n, \cdots) \in E^{p,\infty}$. Then $T^+T = I$ and $\psi(u)(s) = Tu(s)T^+$. Identify $E^{p,\infty}$ with $\ell^p(\mathbb{Q}_{\geq 0} \times Z_M) \otimes (\bigoplus \ell^p)$, then the natural isometric isomorphism between ℓ^p and $\bigoplus \ell^p$ gives an isometric isomorphism U from E^p to $E^{p,\infty}$ which is the identity operator on $\ell^p(\mathbb{Q}_{\geq 0} \times Z_M)$. Define

$$W = \begin{pmatrix} TU^{-1} & I - TT^+ \\ 0 & UT^+ \end{pmatrix}.$$

Then W is an invertible operator on $E^{p,\infty} \oplus E^{p,\infty}$ with zero propagation and we have

$$\psi(u)(s) \oplus 0 = W(Uu(s)U^{-1} \oplus 0)W^{-1}$$

for any $u \in B^p_{L,0}(\mathcal{O}M; E^p)$. Thus ψ_* is an isomorphism from the group $K_*(B^p_{L,0}(\mathcal{O}M; E^p))$ to the group $K_*(B^p_{L,0}(\mathcal{O}M; E^{p,\infty}))$.

The homomorphism $\phi + \psi : B^p_{L,0}(\mathcal{O}M; E^p) \to B^p_{L,0}(\mathcal{O}M; E^{p,\infty})$ is given by

$$(\phi + \psi)(u)(s) = u(s) \oplus Vu(s)V^+ \oplus \cdots \oplus V^{n+1}u(s-n)(V^+)^{n+1} \oplus \cdots.$$

Define a homotopy $\Psi_\lambda : B^p_{L,0}(\mathcal{O}M; E^p) \to B^p_{L,0}(\mathcal{O}M; E^{p,\infty})$ by

$$(\Psi)_\lambda(u)(s) = u(s) \oplus Vu(s-\lambda)V^+ \oplus \cdots \oplus V^{n+1}u(s-n-\lambda)(V^+)^{n+1} \oplus \cdots,$$

where $\lambda \in [0,1]$. Then $\Psi_0 = \phi + \psi$ and $S\big(\bigoplus_{n=0}^{\infty} V\big)(\Psi_1(u)(s))\big(\bigoplus_{n=0}^{\infty} V^+\big)S^+ = \phi(u)(s)$ for any $u \in B^p_{L,0}(\mathcal{O}M; E^p)$, where $S, S^+ : E^{p,\infty} \to E^{p,\infty}$ are defined by $S(\xi_0, \xi_1, \cdots) = (0, \xi_0, \xi_1, \cdots)$ and $S^+(\xi_0, \xi_1, \xi_2, \cdots) = (\xi_1, \xi_2, \cdots)$, respectively. Thus we have the following relations at K-theory level

$$\phi_* + \psi_* = (\Psi_0)_* = (\Psi_1)_* = \phi_*,$$

(the last equation as above is due to Lemmas 2.2 and 3.3) which implies that $\psi_* = 0$. But we have shown that ψ_* is an isomorphism from $K_*(B^p_{L,0}(\mathcal{O}M; E^p))$ to $K_*(B^p_{L,0}(\mathcal{O}M; E^{p,\infty}))$. Therefore, $K_*(B^p_{L,0}(\mathcal{O}M)) = 0$.

Remark 3.2 A proper metric space X is scaleable if there is a continuous and proper map $f : X \to X$ which is coarsely homotopic to the identity map and satisfies that $d(f(x), f(x')) \leq \frac{1}{2}d(x, x')$ for all $x, x' \in X$. By Lemma 3.3, every open cone is scaleable space. Actually, the above theorem widely holds for all scaleable spaces by a similar proof. In [15], Higson and Roe have proved this theorem for $p = 2$ by using different languages.

Combining the above theorem and Corollary 2.1, we have the following corollary.

Corollary 3.1 *Let $\mathcal{O}M$ be the open cone over a compact metric space M, then for any $p \in [1, \infty)$, the group $K_*(B^p(\mathcal{O}M))$ is isomorphic to the group $K_*(B^2(\mathcal{O}M))$, i.e., the K-theory for ℓ^p Roe algebra of $\mathcal{O}M$ does not depend on $p \in [1, \infty)$.*

4 Applications

In this section, we will show some applications of the main theorem (i.e., Theorem 3.1) based on the result of Fukeya and Oguni in [9].

The following concept comes from [9, Definition 3.1].

Definition 4.1 *Let X be a metric space. Let $\lambda \geq 1$, $k \geq 0$, $E \geq 1$ and $C \geq 0$ be constants. Let $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-decreasing function. Let $\mathcal{L}$ be a family of (λ, k)-quasi-geodesic segments. The metric space X is $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex, if $\mathcal{L}$ satisfies the following*:

(1) *For $x_1, x_2 \in X$, there exists a quasi-geodesic segment $\gamma \in \mathcal{L}$ with $\gamma : [0, a] \to X$ such that $\gamma(0) = x_1$ and $\gamma(a) = x_2$,*

(2) *let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$, then for $t \in [0, a]$, $s \in [0, b]$ and $0 \leq c \leq 1$, we have that*

$$d(\gamma(ct), \eta(cs)) \leq cEd(\gamma(t), \eta(s)) + (1-c)Ed(\gamma(0), \eta(0)) + C$$

and

$$|t - s| \leq \theta(d(\gamma(0), \eta(0)) + d(\gamma(t), \eta(s))).$$

We call a metric space X to be a coarsely convex space, if there exist data $\lambda, k, E, C, \theta, \mathcal{L}$ such that X is $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex.

Remark 4.1 For a $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex space X, in [9, Section 4], Fukeya and Oguni constructed the ideal boundary ∂X of X as a set of equivalence classes of quasi-geodesic rays which can be approximated by quasi-geodesic segments in $\mathcal{L}$.

The following result is the main theorem in [9].

Lemma 4.1 *Let X be a proper coarsely convex space, then X is coarsely homotopy equivalent to $\mathcal{O}\partial X$, the open cone over the ideal boundary of X.*

Combining this lemma with Theorem 3.1 and Lemma 3.1, we have the following result.

Theorem 4.1 *Let X be a proper coarsely convex space, then X satisfies the ℓ^p coarse Baum-Connes conjecture for any $p \in [1, \infty)$.*

Remark 4.2 In [9], Fukaya and Oguni have proved this theorem for p=2 by using Higson and Roe's result in [15].

By Corollary 2.1, we have the following corollary.

Corollary 4.1 *Let X be a proper coarsely convex space, then the K-theory of ℓ^p Roe algebra $K_*(B^p(X))$ does not depend on $p \in [1, \infty)$.*

Example 4.1 The following examples are proper coarsely convex spaces:

(1) Geodesic Gromov hyperbolic spaces (cf. [12]),

(2) CAT(0)-spaces, more generally, Busemann non-positively curved spaces (cf. [1, 24]),

(3) systolic groups with the word length metric (cf. [23]), especially, Artin groups of almost large type (cf. [17]) and graphical small cancellation groups (cf. [22]),

(4) Helly groups with the word length metric (cf. [2]), especially, weak Garside groups of finite type and FC-type Artin groups (cf. [18]),

(5) products of proper coarsely convex spaces.

By the above theorem and corollary, these spaces satisfy the ℓ^p coarse Baum-Connes conjecture for any $p \in [1, \infty)$ and the K-theory of their ℓ^p Roe algebras does not depend on $p \in [1, \infty)$.

5 Proof of Lemma 2.4

In this section, we will give a proof of Lemma 2.4 based on the results of [35, Section 5] where the authors have proved this lemma for all finite-dimensional simplicial complexes. This answers a question raised by Chung and Nowak in [6].

In what follows, we assume X is a locally compact, second countable, Hausdorff space, X^+ is the one-point compactification of X and Z_X is a countable dense subset in X. Firstly, similar to [31, Definition 6.2.3], we extend the definition of the ℓ^p localization algebra to all locally compact, second countable, Hausdorff spaces.

Definition 5.1 *The algebra of all bounded linear operators on $\ell^p(Z_X) \otimes \ell^p$ is denoted by $\mathfrak{B}(\ell^p(Z_X) \otimes \ell^p)$. Define $B^p_{TL}[X]$ to be the collection of all bounded functions u from $[0, \infty)$ to $\mathfrak{B}(\ell^p(Z_X) \otimes \ell^p)$ such that*

(1) *for any compact subset K of X, there exists a positive number t_K such that $\chi_K u(t)$ and $u(t)\chi_K$ are compact operators for all $t \geq t_K$, moreover, the functions $t \mapsto \chi_K u(t)$ and $t \mapsto u(t)\chi_K$ are uniformly norm continuous when restricted to $[t_K, \infty)$,*

(2) *for any open neighborhood U of the diagonal in $X^+ \times X^+$, there exists a positive number t_U such that* $\mathrm{supp}(u(t)) \subseteq U$ *for all $t \geq t_U$.*

The ℓ^p localization algebra of X, denoted by $B^p_{TL}(X)$, is defined to be the norm closure of $B^p_{TL}[X]$ with the norm $||u|| = \sup ||u(t)||$.

Remark 5.1 The above definition is non-canonically independent of the countable dense subset Z_X of X, and its K-theory is canonically independent of Z_X. When X is a proper metric space, these two ℓ^p localization algebras of X defined by Definitions 2.4 and 5.1 are isomorphic at the K-theory level. Please refer to [31, Chapter 6] for the details.

For $p \in (1, \infty)$, let q be the dual number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Define $\mathfrak{B}^*(\ell^p(Z_X) \otimes \ell^p)$ to be the collection of all linear operators T acting on $C_c(Z_X \times \mathbb{N})$ and satisfying that there exists a constant C such that $||T\xi||_{\ell^p(Z_X)\otimes\ell^p} \leq C||\xi||_{\ell^p(Z_X)\otimes\ell^p}$ and $||T\xi||_{\ell^q(Z_X)\otimes\ell^q} \leq C||\xi||_{\ell^q(Z_X)\otimes\ell^q}$ for any $\xi \in C_c(Z_X \times \mathbb{N})$, where $C_c(Z_X \times \mathbb{N})$ is the linear space of all continuous functions on $Z_X \times \mathbb{N}$ with compact support. Then $\mathfrak{B}^*(\ell^p(Z_X) \otimes \ell^p)$ is a Banach algebra equipped with the norm $||T||_{\max} = \max\{||T||_{\mathfrak{B}(\ell^p(Z_X)\otimes\ell^p)}, ||T||_{\mathfrak{B}(\ell^q(Z_X)\otimes\ell^q)}\}$. In order to use the Riesz-Thorin interpolation theorem to build the relationship between operators acting on ℓ^p and on ℓ^2, we choose $q = 3$ for $p = 1$, then we can similarly define $\mathfrak{B}^*(\ell^1(Z_X) \otimes \ell^1)$ to be the collection of all linear operators S acting on $C_c(Z_X \times \mathbb{N})$ which can be boundedly extended to $\ell^1(Z_X) \otimes \ell^1$ and $\ell^3(Z_X) \otimes \ell^3$.

Definition 5.2 *Let p, q be as above. Define $B^{p,*}_{TL}[X]$ to be the collection of all bounded functions u from $[0, \infty)$ to $\mathfrak{B}^*(\ell^p(Z_X) \otimes \ell^p)$ such that*

(1) *for any compact subset K of X, there exists a positive number t_K such that $\chi_K u(t)$ and $u(t)\chi_K$ are compact operators on $\ell^p(Z_X) \otimes \ell^p$ and $\ell^q(Z_X) \otimes \ell^q$ for all $t \geq t_K$, moreover, the functions $t \mapsto \chi_K u(t)$ and $t \mapsto u(t)\chi_K$ are uniformly norm continuous when restricted to $[t_K, \infty)$,*

(2) *for any open neighborhood U of the diagonal in $X^+ \times X^+$, there exists a positive number t_U such that* $\mathrm{supp}(u(t)) \subseteq U$ *for all $t \geq t_U$.*

The dual ℓ^p localization algebra of X, denoted by $B^{p,*}_{TL}(X)$, is defined to be the norm closure of $B^{p,*}_{TL}[X]$ with the norm $||u|| = \sup ||u(t)||_{\max}$.

There is an inclusion homomorphism $i : B^{p,*}_{TL}(X) \to B^p_{TL}(X)$. On the other hand, by the Riesz-Thorin interpolation theorem, we also have a contractive homomorphism $\varphi : B^{p,*}_{TL}(X) \to B^2_{TL}(X)$. In [35], the authors have proved that i and φ induce two group isomorphisms at the K-theory level for any finite-dimensional simplicial complex X. More generally, we have the following lemma.

Lemma 5.1 *Let i and φ be two homomorphisms as above, then they induce two group isomorphisms at the K-theory level for all locally compact, second countable, Hausdorff space X.*

Proof The proof for i and the proof for φ are very similar. So we just prove the lemma for φ. By [35, Proposition 5.18], we have the following claim.

Claim: The homomorphism φ induces a group isomorphism at the K-theory level for any finite dimensional simplicial complex.

Let X^+ be the one-point compactification of X, then we have $K_*(B^{p,*}_{TL}(X))$ is isomorphic to $K_*(B^{p,*}_{LT}(X^+))/K_*(B^{p,*}_{TL}(\{\infty\}))$ and $K_*(B^2_{TL}(X))$ is isomorphic to $K_*(B^2_{TL}(X^+))/K_*(B^2_{TL}(\{\infty\}))$ by the six-term exact sequence. Thus, we just need to prove the lemma for any compact space X.

We can endow X with a metric that induces the topology since X is a metrizable space. Then X can be represented as the inverse limit of a sequence of finite simplicial complexes by the following construction (cf. [7, Theorem 2.8]). Fix a sequence of finite covers $\{C_k\}$ which satisfies that $\sup_{U\in C_k} \mathrm{diam(U)} \to 0$ as $k \to \infty$ and each element of C_{k+1} is contained in an element of C_k. Then a sequence of nerve spaces of C_k produces an inverse system $\cdots \to N_{C_3} \to N_{C_2} \to N_{C_1}$ and X is its inverse limit.

By [31, Chapter 6] and [35, Section 5], two families of functors from the category of locally compact, second countable, Hausdorff spaces to the category of abelian groups that are given by $F^{p,*}_* : X \mapsto K_*(B^{p,*}_{TL}(X))$ and $F^2_* : X \mapsto K_*(B^2_{TL}(X))$, respectively, satisfy the axioms of generalized Steenrod homology theory. That means (1) $F^{p,*}_*$ and F^2_* are homotopy functors; (2) $F^{p,*}_*(X)$ and $F^2_*(X)$ have Mayer-Vietoris sequence for a pair of closed subsets of X; (3) cluster axiom: If X is a countable disjoint union of spaces X_j, then the inclusions $X_j \to X$ induce two families of isomorphisms $\prod_j F^{p,*}_*(X_j) \cong F^{p,*}_*(X)$ and $\prod_j F^2_*(X_j) \cong F^2_*(X)$. Then by [16, Proposition 7.3.4], we have the following commutative diagram about the Milnor exact sequences:

$$\begin{array}{ccccccccc}
0 \to & \varprojlim^1 K_{*+1}(B^{p,*}_{TL}(N_{C_k})) & \to & K_*(B^{p,*}_{TL}(X)) & \to & \varprojlim K_*(B^{p,*}_{TL}(N_{C_k})) & \to 0 \\
& \downarrow {\scriptstyle \varprojlim^1 \varphi_*} & & \downarrow {\scriptstyle \varphi_*} & & \downarrow {\scriptstyle \varprojlim \varphi_*} & \\
0 \to & \varprojlim^1 K_{*+1}(B^2_{TL}(N_{C_k})) & \to & K_*(B^2_{TL}(X)) & \to & \varprojlim K_*(B^2_{TL}(N_{C_k})) & \to 0.
\end{array}$$

Thus by the five lemma and claim, we complete the proof.

Finally, Lemma 2.4 is an obvious corollary of the above lemma.

Acknowledgements The author would like to thank Jiawen Zhang, Dapeng Zhou and referees for providing many valuable suggestions and comments.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Bridson, M. R. and Haefliger, A., Metric Spaces of Non-positive Curvature, Grundlehren der Mathematischen Wissenschaften, Fundamental Principles of Mathematical Sciences, **319**, Springer-Verlag, Berlin, 1999.

[2] Chalopin, J., Chepoi, V., Genevois, A., et al., Helly groups, 2020, arXiv: 2002.06895.

[3] Chung, Y. C., Dynamical complexity and K-theory of L^p operator crossed products, *J. Topol. Anal.*, **13**(3), 2021, 809–841.

[4] Chung, Y. C. and Li, K., Rigidity of ℓ^p Roe-type algebras, *Bull. Lond. Math. Soc.*, **50**(6), 2018, 1056-1070.

[5] Chung, Y. C. and Li, K., Structure and K-theory of ℓ^p uniform Roe algebras, *J. Noncommut. Geom.*, **15**(2), 2021, 581–614.

[6] Chung, Y. C. and Nowak, P. W., Expanders are counterexamples to the ℓ^p coarse Baum-Connes conjecture, *J. Noncommut. Geom.*, **17**(1), 2023, 305–331.

[7] Dydak, J., Cohomological dimension theory, Handbook of Geometric Topology, North-Holland, Amsterdam, 2002, 423–470.

[8] Fukaya, T. and Oguni, S.-I., The coarse Baum-Connes conjecture for relatively hyperbolic groups, *J. Topol. Anal.*, **4**(1), 2012, 99–113.

[9] Fukaya, T. and Oguni, S.-I., A coarse Cartan-Hadamard theorem with application to the coarse Baum-Connes conjecture, *J. Topol. Anal.*, **12**(3), 2020, 857–895.

[10] Gardella, E. and Thiel, H., Group algebras acting on L^p-spaces, *J. Fourier Anal. Appl.*, **21**(6), 2015, 1310–1343.

[11] Gardella, E. and Thiel, H., Representations of p-convolution algebras on L^q-spaces, *Trans. Amer. Math. Soc.*, **371**(3), 2019, 2207–2236.

[12] Ghys, E. and de la Harpe, P., Espaces Métriques Hyperboliques, Surles groupes hyperboliques d'après Mikhael Gromov (Bern, 1988), Progr. Math., **83**, Birkäuser Boston, Boston, MA, 1990, 27–45.

[13] Higson, N., Lafforgue, V. and Skandalis, G., Counterexamples to the Baum-Connes conjecture, *Geom. Funct. Anal.*, **12**(2), 2002, 330–354.

[14] Higson, N. and Roe, J., A homotopy invariance theorem in coarse cohomology and K-theory, *Trans. Amer. Math. Soc.*, **345**(1), 1994, 347–365.

[15] Higson, N. and Roe, J., On the Coarse Baum-Connes Conjecture, Novikov Conjectures, Index Theorems and Rigidity, London Math. Soc. Lecture Note Ser., **227**, Cambridge Univ. Press, Cambridge, 1995, 227–254.

[16] Higson, N. and Roe, J., Analytic K-homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[17] Huang, J. and Osajda, D., Large-type Artin groups are systolic, *Proc. Lond. Math. Soc.* (3), **120**(1), 2020, 95–123.

[18] Huang, J. and Osajda, D., Helly meets Garside and Artin, *Invent. Math.*, **225**(2), 2021, 395–426.

[19] Kasparov, G. G. and Yu, G., The Baum-Connes conjecture and actions on ℓ^p spaces, in preparation.

[20] Li, K., Wang, Z. and Zhang, J., A quasi-local characterisation of L^p-Roe algebras, *J. Math. Anal. Appl.*, **474**(2), 2019, 1213–1237.

[21] Liao, B. and Yu, G., K-theory of group Banach algebras and Banach property RD, 2017, arXiv:1708.01982.

[22] Osajda, D. and Prytuła, T., Classifying spaces for families of subgroups for systolic groups, *Groups Geom. Dyn.*, **12**(3), 2018, 1005–1060.

[23] Osajda, D. and Przytycki, P., Boundaries of systolic groups, *Geom. Topol.*, **13**(5), 2009, 2807–2880.

[24] Papadopoulos, A., Metric spaces, convexity and nonpositive curvature, IRMA Lectures in Mathematics and Theoretical Physics, **6**, European Mathematical Society (EMS), Zürich, 2005.

[25] Phillips, N. C., Crossed products of L^p operator algebras and the K-theory of Cuntz algebras on L^p spaces, 2013, arXiv:1309.6406.

[26] Phillips, N. C., Simplicity of reduced group Banach algebras, 2019, arXiv:1909.11278.

[27] Roe, J., An index theorem on open manifolds, I, II, *J. Differential Geom.*, **27**(1), 1988, 87–113, 115–136.

[28] Roe, J., Coarse cohomology and index theory on complete Riemannian manifolds, *Mem. Amer. Math. Soc.*, **104**(497), 1993, p.90.

[29] Shan, L. and Wang, Q., The coarse geometric ℓ^p-Novikov conjecture for subspaces of nonpositively curved manifolds, *J. Noncommut. Geom.*, **15**(4), 2021, 1323–1354.

[30] Špakula, J. and Zhang, J., Quasi-locality and property A, *J. Funct. Anal.*, **278**(1), 2020, 108299.

[31] Willett, R. and Yu, G., Higher index theory, Cambridge studies in advanced mathematics, **189**, Cambridge University Press, Cambridge, 2020.

[32] Yu, G., Coarse Baum-Connes conjecture, *K-Theory*, **9**(3), 1995, 199–221.

[33] Yu, G., The Novikov conjecture for groups with finite asymptotic dimension, *Ann. of Math.* (2), **147**(2), 1998, 325–355.

[34] Yu, G., The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.*, **139**(1), 2000, 201–240.

[35] Zhang, J. and Zhou, D., L^p coarse Baum-Connes conjecture and K-theory for L^p Roe algebras, *J. Noncommut. Geom.*, **15**(4), 2021, 1285–1322.