Chin. Ann. Math. Ser. B
46(1), 2025, 1–24
DOI: 10.1007/s11401-025-0001-8

Chinese Annals of Mathematics, Series B

On Hempel Pairs and Turaev-Viro Invariants*

Yi LIU[1]

Abstract Surface bundles arising from periodic mapping classes may sometimes have non-isomorphic, but profinitely isomorphic fundamental groups. Pairs of this kind have been discovered by Hempel. This paper exhibits examples of nontrivial Hempel pairs where the mapping tori can be distinguished by some Turaev-Viro invariants, and also examples where they cannot be distinguished by any Turaev-Viro invariants.

Keywords Profinite completion, Mapping class group, Turaev-Viro invariant
2020 MR Subject Classification 57K31, 57K16

1 Introduction

Let S be a connected closed orientable surface. Denote by $\mathrm{Mod}(S)$ the mapping class group of S, whose elements are the isotopy classes of orientation-preserving self-homeomorphisms of S.

A Hempel pair as we call refers to a pair of periodic mapping classes $[f_A], [f_B] \in \mathrm{Mod}(S)$ of identical order $d \geq 1$, such that $[f_B] = [f_A^k]$ holds for some integer k coprime to d. Hence $[f_A] = [f_B^{k^*}]$ holds for any congruence inverse k^* of k modulo d, (that is, $k^*k \equiv 1 \bmod d$). Hempel studied such pairs in [11], and found out that the fundamental groups of their mapping tori M_A and M_B always have identical collections of (isomorphism types of) finite quotient groups. This is equivalent to saying that the profinite completions of $\pi_1(M_A)$ and $\pi_1(M_B)$ are isomorphic groups. Hempel discovered examples of such pairs with non-isomorphic fundamental groups. This is equivalent to saying that M_A and M_B are not homeomorphic 3-manifolds.

We call $[f_A]$ and $[f_B]$ a nontrivial Hempel pair, if M_A and M_B are not homeomorphic. There are no nontrivial Hempel pairs when S is a sphere or a torus, as the condition forces $d = 1$ or $d \in \{1, 2, 3, 4, 6\}$, and hence $k = \pm 1$. One may obtain a nontrivial Hempel pair of order 5 when S has genus 2. Nontrivial Hempel pairs are a source of distinct 3-manifold pairs that cannot be distinguished by their profinite fundamental groups. Among 3-manifolds, the question as to which topological invariants are determined by the profinite fundamental group has stimulated a lot of fruitful study in recent years. See [14, Section 9] and [16] for past surveys on that fast-growing topic.

Manuscript received February 2, 2024. Revised August 21, 2024.
[1]Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China.
E-mail: liuyi@bicmr.pku.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11925101) and the National Key R&D Program of China (No. 2020YFA0712800).

Turaev-Viro invariants are topological invariants of closed 3-manifolds, originally constructed using quantum $6j$-symbols in [24]. That they are generally not profinite invariants is evident from their explicit values on all lens spaces (see [21]). Meanwhile, there are many homeomorphically distinct torus bundles over a circle, whose monodromies are Anosov and mutually conjugate up to inverse in every congruence quotient of $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$. These torus bundles have isomorphic profinite fundamental groups. Funar shows that no Turaev-Viro invariants (associated to any spherical fusion categories) distinguish these torus bundles (see [8, Proposition 1.1]).

In this paper, we take up the question as to whether Turaev-Viro invariants distinguish nontrivial Hempel pairs. We shall content ourselves with the SU(2) and the SO(3) Turaev-Viro invariants, as they are mentioned the most often. The SU(2) series can be fully listed as $\mathrm{TV}_{r,s}$ for any integer $r \geq 3$ and any integer s coprime to r; the SO(3) series can be fully listed as $\mathrm{TV}'_{r,s}$ for any odd integer $r \geq 3$ and any even integer s coprime to r. More economically, one could focus on $\mathrm{TV}_{r,1}$ (r even) and $\mathrm{TV}'_{r,r-1}$ (r odd), together with $\mathrm{TV}_{3,1}$ and $\mathrm{TV}_{3,2}$, which depend only on the $\mathbb{Z}/2\mathbb{Z}$ cohomology ring. These determine all the other ones. See Section 3 for the notations and more review.

Our conclusion can be summarized as follows.

Theorem 1.1 (1) *For any integer $d \geq 5$ other than 6, there exists some nontrivial Hempel pair of order d, such that the mapping tori can be distinguished by some* SU(2) *Turaev-Viro invariant, and if d is odd, also by some* SO(3) *Turaev-Viro invariant.*

(2) *For any prime integer $p \geq 5$, there exists some nontrivial Hempel pair of order p, such that the mapping tori cannot be distinguished by any* SU(2) *or* SO(3) *Turaev-Viro invariants.*

Theorem 1.1 is proved in Section 5 by exhibiting concrete families of examples. Our simplest distinguishable Hempel pair exists with order d on genus $d-2 \geq 3$. Our simplest indistinguishable nontrivial Hempel pair exists with prime order p on genus $\frac{p-1}{2} \geq 2$.

The technical heart of this paper is the following calculation.

Theorem 1.2 *Let $a \geq 3$ be an integer and $b_1, \cdots, b_n$ be integers coprime to a. Let M be a Seifert fiber space with orientable orbifold base and orientable Seifert fibration, and with symbol $(g; (a, b_1), \cdots, (a, b_n))$. Suppose that $g \geq 0$, $a > n \geq 0$ and $b_1 + \cdots + b_n = 0$.*

(1) *If there exists some integer b^* coprime to a and $\nu_1, \cdots, \nu_n \in \{\pm 1\}$, such that $b^* b_j \equiv \nu_j \bmod a$ holds for all $j \in \{1, \cdots, n\}$, then for any s coprime to a,*

$$\mathrm{TV}_{a,s}(M) = \frac{a^{n+2g-2}}{2^{2n+2g-4}} \cdot \frac{1}{\sin^{2n+4g-4}\left(\dfrac{\pi b^* s}{a}\right)},$$

and moreover, if a is odd and s is even,

$$\mathrm{TV}'_{a,s}(M) = \frac{a^{n+2g-2}}{2^{2n+4g-4}} \cdot \frac{1}{\sin^{2n+4g-4}\left(\dfrac{\pi b^* s}{a}\right)}.$$

(2) *Otherwise, for any integer $r \geq 3$ divisible by a and any integer s coprime to r,*

$$\mathrm{TV}_{r,s}(M) = 0,$$

and moreover, if r is odd and s is even,

$$\mathrm{TV}'_{r,s}(M) = 0.$$

See Section 2 for review on Seifert fiber spaces and the standard notation for their symbols.

Theorem 1.2 is proved in Section 4, by applying a general formula for calculating the Witten-Reshetikhin-Turaev invariant τ_r of oriented closed Seifert fiber spaces. The exact formula we invoke is due to Hansen [9], while similar calculation in special cases or with likewise strategy also appear in many other places (for instance, see [1, 13, 19, 22]).

Theorem 1.2 is formulated by first testing samples of Seifert fiber spaces (on computer), and then observing interesting phenomena. Luckily, we find the assumptions as in Theorem 1.2, which greatly simplify the situation and the answer.

In Section A, we prove a splitting formula

$$\mathrm{TV}_{r,s}(M) = \mathrm{TV}_{3,1}(M) \cdot \mathrm{TV}'_{r,r-s}(M)$$

for r odd and s odd. This complements a former formula

$$\mathrm{TV}_{r,s}(M) = \mathrm{TV}_{3,2}(M) \cdot \mathrm{TV}'_{r,s}(M)$$

for r odd and s even, proved by Detcherry-Kalfagianni-Yang [7, Theorem 2.9]. Our proof in Section A automatically includes the non-orientable case, although the orientable case suffices for our application.

This paper is organized as follows. In Section 2, we recall the preliminary description of periodic mapping tori as Seifert fiber spaces with vanishing rational Euler number. In Section 3, we review Turaev-Viro invariants. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.1. In Section A, we give an elementary proof of the aforementioned splitting formula regarding SU(2) Turaev-Viro invariants at odd r and odd s.

2 Periodic Mapping Tori

Let S be a connected orientable closed surface. The mapping class group $\mathrm{Mod}(S)$ consists of all the isotopy classes of orientation-preserving self-homeomorphism of S. For any mapping class $[f] \in \mathrm{Mod}(S)$, we denote by M_f the mapping torus

$$M_f = \frac{S \times \mathbb{R}}{(f(x), r) \sim (x, r+1)},$$

which naturally fibers over the oriented circle $\mathbb{R}/\mathbb{Z}$ with fiber type S and (backward) monodromy type $[f]$. The mapping torus M_f is a connected orientable closed 3-manifold, whose homeomorphism type depends only on $[f]$.

A periodic mapping class refers to a mapping class of finite order. In this case, the mapping torus is a Seifert fiber space. The Seifert fibration is orientable over an orientable base orbifold, and has vanishing rational Euler number. Moreover, the Seifert fibration on any periodic mapping torus is unique up to isotopy. Conversely, any closed Seifert fiber space with orientable orbifold base and orientable Seifert fibration of vanishing rational Euler number arises as the mapping torus of some periodic mapping class. Moreover, the genus of surface and the conjugacy class of the periodic mapping class up to inverse are both unique. This way, periodic mapping classes can be described equivalently by indicating the symbol of its mapping torus, as a Seifert fiber space (see [2, Chapter 1]).

The most general symbol describes any (connected, compact) Seifert fiber space with all features, allowing possibly non-orientable orbifold base, non-orientable Seifert fibration, and non-empty boundary. For discussing periodic mapping classes, we only need to consider closed Seifert fiber spaces with orientable orbifold base and orientable Seifert fibration, whose (possibly non-normalized) symbol is denoted as

$$(g;(a_1,b_1),\cdots,(a_n,b_n)), \tag{2.1}$$

where $g \geq 0$ is an integer, $a_j \geq 1$ is an integer and b_j is an integer coprime to a_j for all $j \in \{1,\cdots,n\}$. This symbol presents a Seifert fiber space (with standard orientation) constructed as follows.

Take a product 3-manifold $\Sigma \times S^1$ of a connected closed oriented surface Σ of genus g and an oriented circle S^1; take n disjoint embedded disks $D_1,\cdots,D_n \subset \Sigma$; remove the solid tori $D_j \times S^1$ from $\Sigma \times S^1$, and refill with solid tori in other ways, such that the slopes $a_j[\partial D_j]+b_j[S^1]$ on $\partial D_j \times S^1$ bound disks in the new solid tori.

The base of this Seifert fiber space is a connected, closed, oriented 2-orbifold of genus g with n cone points of order $a_1,\cdots,a_n$ (ordinary and negligible if $a_j = 1$). Its orbifold Euler characteristic equals $2-2g-\sum\limits_j \left(1-\frac{1}{a_j}\right)$. The rational Euler number of the Seifert fibration equals $-\sum\limits_j \frac{b_j}{a_j}$, where the minus sign comes from our convention on the orientation of ∂D_j (as induced by the orientation of D_j, rather than $S \setminus (D_1 \cup \cdots \cup D_n)$.

The following operations of the symbol do not change the homeomorphism type of the resulting Seifert fiber space: Re-ordering all (a_j,b_j); inserting or deleting a term $(1,0)$; replacing one (a_j,b_j) with (a_j,b_j+a_j) and another $(a_{j'},b_{j'})$ with $(a_{j'},b_{j'}-a_{j'})$ simultaneously; or replacing all (a_j,b_j) with $(a_j,-b_j)$ simultaneously. Note that only the last operation changes the orientation-preserving homeomorphism type of the resulting Seifert fiber space.

For homeomorphic periodic mapping tori, their symbols are all related by finitely many steps of the above operations. This is a special case of the classification of Seifert fiber spaces (see [10, Chapter 12, 15, Theorem 1.10]).

The following Lemmas 2.1–2.2 actually appear in [11] in equivalent forms. We include quick (and slightly different) proofs here for the reader's reference.

Lemma 2.1 *Let M be a closed Seifert fiber space with orientable orbifold base and orientable Seifert fibration, and with symbol $(g;(a_1,b_1),\cdots,(a_n,b_n))$. If the Seifert fibration has vanishing rational Euler characteristic, then M is homeomorphic to the mapping torus M_f of a periodic mapping class $[f]\in\mathrm{Mod}(S)$. Moreover, $[f]$ has order $d=\mathrm{lcm}(a_1,\cdots,a_n)$, and S has genus $1+(g-1)d+\sum_j\left(1-\frac{1}{a_j}\right)\frac{d}{2}$.*

Proof The orbifold fundamental group of the base $\mathcal{O}$ has a presentation with generators $x_1,y_1,\cdots,x_g,y_g,s_1,\cdots,s_n$ and relations $[x_1,y_1]\cdots[x_g,y_g]=s_1\cdots s_n$, and $s_1^{a_1}=\cdots=s_n^{a_n}=1$. Under the assumption of vanishing rational Euler number, the assignments $x_i\mapsto 0 \bmod d$, $y_i\mapsto 0 \bmod d$, and $s_j\mapsto -\frac{b_jd}{a_j} \bmod d$ yield a well-defined, surjective homomorphism $\pi_1(\mathcal{O})\to\mathbb{Z}/d\mathbb{Z}$, where $d=\mathrm{lcm}(a_1,\cdots,a_n)$. The kernel of the homorphism corresponds to a cyclic orbifold covering $S\to\mathcal{O}$ of degree d, where S has no singular cone points. The generator $1\in\mathbb{Z}/d\mathbb{Z}$ corresponds to a deck transformation, representing a periodic mapping class $[f]\in\mathrm{Mod}(S)$ of order d. It is elementary to check that M_f is homeomorphic to M. The Euler characteristic of the surface S is equal to d times the rational Euler number of $\mathcal{O}$, which implies the asserted genus of S.

Lemma 2.2 *Let S be a connected, close, orientable surface, and $[f]\in\mathrm{Mod}(S)$ be a periodic mapping class of order d. If the mapping torus M_f has symbol $(g;(a_1,b_1),\cdots,(a_n,b_n))$, then for any integer k coprime to d, the mapping torus M_{f^k} of the iterate $[f^k]\in\mathrm{Mod}(S)$ has symbol $(g;(a_1,b_1k^*),\cdots,(a_n,b_nk^*))$, where k^* is any integer satisfying $kk^*\equiv 1 \bmod d$.*

Proof The mapping torus M_{f^k} naturally cyclically cover M_f of degree k. Since f has order d, we see that $M_f=M_{f^{kk^*}}$ cyclically covers M_{f^k} of degree k^*. Consider the covering $M_f\to M_{f^k}$. The pull-back of the Seifert fibration of M_{f^k} is a Seifert fibration on M_f. If M_f has symbol $(g;(a_1,b_1),\cdots,(a_n,b_n))$, by definition, the preimage of $(\Sigma\setminus(D_1\cup\cdots\cup D_n))\times S^1\subset M_{f^k}$ is also a product $(\Sigma\setminus(D_1\cup\cdots\cup D_n))\times S^1\subset M_f$, while ordinary fibers (that is, $*\times S^1\subset M_f$) cyclically cover the ordinary fibers in M_{f^k} with degree k^*. Since the slope $a_j[D_j]+b_j[S^1]$ on $\partial D_j\times S^1$ is homotopically trivial in M_f, the slope $a_j[D_j]+b_jk^*[S^1]$ on $\partial D_j\times S^1$ is homotopically trivial in M_k. Therefore, $(g;(a_1,b_1k^*),\cdots,(a_n,b_nk^*))$ is a symbol of M_{f^k}.

3 Turaev-Viro Invariants

Turaev-Viro invariants are topological invariants of closed 3-manifolds, arising from representation theory of quantum groups at roots of unity. Throughout this paper, we only discuss Turaev-Viro invariants pertaining to the most basic quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. In this setting, there are essentially two series, namely, the SU(2) Turaev-Viro invariants TV_r for all integers $r\geq 3$, and the SO(3) Turaev-Viro invariants TV'_r for all odd integers $r\geq 3$.

Throughout this paper, a closed manifold only means a compact manifold with empty boundary, possibly disconnected and possibly non-orientable.

3.1 Abstract versions

Let $q^{\frac{1}{2}}$ be a square root of a root of unity $q \neq \pm 1$. Denote by $r \geq 3$ the order of q. Note that $q^{\frac{1}{2}}$ must have order $2r$ when r is even, but may have order $2r$ or r when r is odd. We denote by $\mathbb{Q}(q^{\frac{1}{2}})$ the abstract cyclotomic field generated by $q^{\frac{1}{2}}$.

For any closed 3-manifold M, the Turaev-Viro invariant $\mathrm{TV}(M; q^{\frac{1}{2}})$ can be defined at $q^{\frac{1}{2}}$. If r is odd, the refined Turaev-Viro invariant $\mathrm{TV}'(M; q^{\frac{1}{2}})$ can be defined at $q^{\frac{1}{2}}$ of order r. Both $\mathrm{TV}(M; q^{\frac{1}{2}})$ and $\mathrm{TV}'(M; q^{\frac{1}{2}})$ are topological invariants of M, with totally real values in $\mathbb{Q}(q^{\frac{1}{2}})$. We call TV and TV$'$ the abstract SU(2) and the abstract SO(3) Turaev-Viro invariants at $q^{\frac{1}{2}}$ (or at level $r-2$), respectively.

More details about the actual construction of these invariants are not be necessary in the sequel, except Section A. We record them below for the sake of clarity.

Let $\mathscr{T} = (V, E, F, T)$ be any finite simplicial 3-complex, where the items denote the sets of vertices, edges, faces and tetrahedra, respectively. Denote $I_r = \{0, 1, \cdots, r-2\}$. A triple $(i, j, k) \in I_r \times I_r \times I_r$ is said to be admissible if the numbers $\frac{i+j+k}{2}$, $\frac{i+j-k}{2}$, $\frac{j+k-i}{2}$ and $\frac{k+i-j}{2}$ all stay in I_r. An admissible coloring of $(M, \mathscr{T})$ (of level $r-2$) refers to a map $c\colon E \to I_r$, such that on any face, the edge colors (that is, values of c) form an admissible triple. Denote by $\mathcal{A}_r = \mathcal{A}_r(M, \mathscr{T})$ the set of all admissible colorings of level $r-2$. If r is odd, denote by $I'_r = \{0, 2, \cdots, r-3\}$ the subset of even elements of I_r. Denote by $\mathcal{A}'_r = \mathcal{A}'_r(M, \mathscr{T})$ the subset of $\mathcal{A}_r$ consisting of admissible colorings with values in I'_r.

Let M be a (possibly disconnected, possibly non-orientable) closed 3-manifold. Take a triangulation $\mathscr{T} = (V, E, F, T)$ of M. Then invariants TV and TV$'$ of M at $q^{\frac{1}{2}}$ can be expressed in terms of $\mathscr{T}$ as follows:

$$\mathrm{TV}(M; q^{\frac{1}{2}}) = \left(\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}{-2r}\right)^{|V|} \cdot \sum_{c \in \mathcal{A}_r} |\mathscr{T}|_c\,; \tag{3.1}$$

if r is odd and $q^{\frac{1}{2}}$ has order r,

$$\mathrm{TV}'(M; q^{\frac{1}{2}}) = \left(\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}{-r}\right)^{|V|} \cdot \sum_{c \in \mathcal{A}'_r} |\mathscr{T}|_c\,, \tag{3.2}$$

where $|V|$ denotes the number of vertices in $\mathscr{T}$ and $|\mathscr{T}|_c$ is explained in Notation 3.1 below. Note that our notations follow the reformulation as in [7, Appendix A], where the factors in $|e|_c$, $|f|_c$ and $|t|_c$ are grouped slightly differently than those in [24], so as to avoid unnecessary square roots.

As it turns out, the values of the above expressions in $\mathbb{Q}(q^{\frac{1}{2}})$ are independent of the auxiliary choice of the triangulation $\mathscr{T}$ (see [24]). Therefore, $\mathrm{TV}(M; q^{\frac{1}{2}})$ and $\mathrm{TV}'(M; q^{\frac{1}{2}})$ are indeed topological invariants of M.

Notation 3.1 (1) Denote

$$[n]! = \begin{cases} [1] \cdot [2] \cdot \cdots \cdot [n], & n = 1, 2, \cdots, r-1, \\ 1, & n = 0, \end{cases}$$

where

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Note that the quantum integers $[1], [2], \cdots, [r-1]$ take totally real nonzero values in $\mathbb{Q}(q^{\frac{1}{2}})$, as q is a primitive r-th root of unity ($r \geq 3$).

(2) For any $e \in E$ and $c \in \mathcal{A}_r$,

$$|e|_c = (-1)^i \cdot [i+1],$$

where i is the color of e under c.

(3) For any $f \in F$ and $c \in \mathcal{A}_r$,

$$|f|_c = (-1)^{-S} \cdot \frac{[S-i]! \cdot [S-j]! \cdot [S-k]!}{[S+1]!},$$

where i, j, k are the edge colors of f under c and $S = \frac{i+j+k}{2}$.

(4) For any $t \in T$ and $c \in \mathcal{A}_r$, denote

$$|t|_c = \sum_z \frac{(-1)^z \, [z+1]!}{\prod_a [z - T_a]! \cdot \prod_b [Q_b - z]!},$$

where $(i,j,k), (i,m,n), (j,l,n), (k,l,m)$ are the edge colors of the faces of t under c and $T_1 = \frac{i+j+k}{2}$, $T_2 = \frac{i+m+n}{2}$, $T_3 = \frac{j+l+n}{2}$, $T_4 = \frac{k+l+m}{2}$, $Q_1 = \frac{i+j+l+m}{2}$, $Q_2 = \frac{i+k+l+n}{2}$, $Q_3 = \frac{j+k+m+n}{2}$. The index a ranges in $\{1,2,3,4\}$, b ranges in $\{1,2,3\}$ and z is from $\max_a T_a$ to $\min_b Q_b$.

(5) For any $c \in \mathcal{A}_r$, denote

$$|\mathscr{T}|_c = \prod_{e \in E} |e|_c \cdot \prod_{f \in F} |f|_c \cdot \prod_{t \in T} |t|_c.$$

3.2 Specialized versions

In the literature, the SU(2) and the SO(3) Turaev-Viro invariants often refer to specialization of the abstract versions at customary complex roots of unity. To be precise, these refer to the numerical quantities TV_r and TV'_r below.

Notation 3.2 Let $r \geq 3$ be an integer and s be an integer coprime to r. The following expressions are all evaluated by specializing $\mathbb{Q}(q^{\frac{1}{2}}) \to \mathbb{C}$.

(1) Denote

$$\mathrm{TV}_{r,s}(M) = \mathrm{TV}\Big(M; q^{\frac{1}{2}} = \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{r}}\Big).$$

(2) If r is odd and s is even, denote

$$\mathrm{TV}'_{r,s}(M) = \mathrm{TV}'\Big(M; q^{\frac{1}{2}} = \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{r}}\Big).$$

(3) In informal discussion, we often write TV_r and TV'_r, assuming s implicitly fixed.

Lemma 3.1 *Let M, N be any closed 3-manifolds. Then we have*

(1) $\mathrm{TV}_{r,s}(M) \in \mathbb{R}$,

(2) $\mathrm{TV}_{r,s}(M \sqcup N) = \mathrm{TV}_{r,s}(M) \cdot \mathrm{TV}_{r,s}(N)$,

(3) $\mathrm{TV}_{r,s}(S^2 \times S^1) = 1$.

If r is odd and s is even, the same statements hold with $\mathrm{TV}'_{r,s}$ in place of $\mathrm{TV}_{r,s}$.

The first statement follows immediately from the observation that TV and TV′ can be written as rational functions over $\mathbb{Q}$ in $[1], \cdots, [r-1]$ and $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2$ (see (3.1)–(3.2), Notation 3.1). Note $[n] = \sin\left(\frac{\pi s n}{r}\right) / \sin\left(\frac{\pi s}{r}\right)$ and $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 = -4\sin^2\left(\frac{\pi s}{r}\right)$ evaluated at $q^{\frac{1}{2}} = \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{r}}$. The second statement is also obvious by definition. The last statement appears in Turaev and Viro's original paper [24, Section 8.1.B].

Lemma 3.2 *Let M be any closed 3-manifold. Then we have*

(1) $\mathrm{TV}_{r,s}(M) = \mathrm{TV}_{r,-s}(M) = \mathrm{TV}_{r,s+2r}(M)$. *If r is odd and s is even, the same identities hold with TV' in place of TV.*

(2) *If s is odd, $\mathrm{TV}_{r,s}(M)$ is Galois conjugate to $\mathrm{TV}_{r,1}(M)$; if s is even (hence r odd), $\mathrm{TV}_{r,s}(M)$ is Galois conjugate to $\mathrm{TV}_{r,r-1}(M)$. If r is odd and s is even, the same statements hold with $\mathrm{TV}'_{r,s}$ in place of $\mathrm{TV}_{r,s}$.*

(3) *If r is odd,*

$$\mathrm{TV}_{r,s}(M) = \begin{cases} \mathrm{TV}_{3,2}(M) \cdot \mathrm{TV}'_{r,s}(M), & s \text{ even}, \\ \mathrm{TV}_{3,1}(M) \cdot \mathrm{TV}'_{r,r-s}(M), & s \text{ odd}. \end{cases}$$

(4) *Denote by β_i the dimension of $H_i(M; \mathbb{Z}/2\mathbb{Z})$ over $\mathbb{Z}/2\mathbb{Z}$ and $w_1 \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ the first Stifel-Whitney class of M. Then we have*

$$\mathrm{TV}_{3,2}(M) = 2^{-\beta_0(M)+\beta_2(M)}$$

and

$$\mathrm{TV}_{3,1}(M) = 2^{-\beta_0(M)} \cdot \sum_t (-1)^{\langle t^3 + w_1^2 t,\, [M] \rangle},$$

where the index t ranges over $H^1(M; \mathbb{Z}/2\mathbb{Z})$.

The first and the second statements are again obvious properties of the defining expressions. The third statement is proved when s is even by Detcherry-Kalfagianni-Yang [7, Theorem 2.9], and can be easily derived from well-known facts when s is odd, assuming M orientable (see Lemmas 3.4–3.5). We supply a proof without assuming orientability in Section A.

The formulas for $\mathrm{TV}_{3,s}$ in the fourth statement are due to Turaev-Viro [24, Section 9.3.A].

3.3 Relation to Witten-Reshetikhin-Turaev invariants

For connected oriented closed 3-manifolds, the Witten-Reshetikhin-Turaev invariants are invariant under orientation-preserving homeomorphisms. These invariants were suggested by

Witten [26], and the first mathematically rigorous construction was due to Reshetikhin-Turaev [17].

Following Kirby-Melvin [12], we denote by τ_r the SU(2) Witten-Reshetikhin-Turaev invariants, defined for any integer $r \geq 3$, and by τ'_r the SO(3) Witten-Reshetikhin-Turaev invariants, defined for any odd integer $r \geq 3$. Note that τ_r is slight modification of the original construction by Reshetikhin-Turaev [17], differing by a factor of absolute value 1 (depending on the first Betti number), and τ'_r is introduced by Kirby-Melvin [12, Section 8]. The invariant τ_r corresponds to the $4r$-th primitive complex root of unity $q^{\frac{1}{4}} = \mathrm{e}^{\sqrt{-1}\cdot\frac{2\pi}{4r}}$ by construction, whereas τ'_r is obtained by modifying the defining expression of τ_r when r is odd. Upon suitable interpretation, τ'_r appears to correspond to the $2r$-th primitive complex root of unity $q^{\frac{1}{4}} = \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi(r-1)}{4r}}$ (compare [4, Theorem B]). Both τ_r and τ'_r take values in $\mathbb{C}$.

The following properties characterize the normalization of τ_r and τ'_r.

Lemma 3.3 *Let M, N be any connected oriented closed 3-manifolds. Then we have*
(1) $\tau_r(M) = \overline{\tau_r(-M)}$,
(2) $\tau_r(M \# N) = \tau_r(M) \cdot \tau_r(N)$,
(3) $\tau_r(S^2 \times S^1) = \sqrt{\frac{r}{2}} / \sin\left(\frac{\pi}{r}\right)$. *Hence,* $\tau_r(S^3) = 1$.
If r is odd, the same statements hold with τ'_r in place of τ_r, and $\sqrt{\frac{r}{4}}$ in place with $\sqrt{\frac{r}{2}}$.

See [4, Proposition 6.10] and [12, Section 1 and (5.11)].

Lemma 3.4 *Let M be any connected oriented closed 3-manifold. If r is odd,*

$$\tau_r(M) = \begin{cases} \tau_3(M) \cdot \tau'_r(M), & r \equiv 3 \bmod 4, \\ \overline{\tau_3(M)} \cdot \tau'_r(M), & r \equiv 1 \bmod 4. \end{cases}$$

See [12, Corollary 8.9, Theorem 8.10, Theorem 6.11] for characterization of $\tau_3(M)$ in terms of classical topological invariants.

The Turaev-Viro invariants TV_r and TV'_r are essentially the absolute value squares of Witten-Reshetikhin-Turaev invariants, or precisely as follows.

Lemma 3.5 *Let M be a connected closed 3-manifold and $r \geq 3$ be any integer.*
(1) *If M is oriented, then*

$$\frac{\mathrm{TV}_{r,1}(M)}{\mathrm{TV}_{r,1}(S^3)} = |\tau_r(M)|^2,$$

and if r is odd, then

$$\frac{\mathrm{TV}'_{r,r-1}(M)}{\mathrm{TV}'_{r,r-1}(S^3)} = |\tau'_r(M)|^2.$$

(2) *If M is non-orientable, then*

$$\frac{\mathrm{TV}_{r,1}(M)}{\mathrm{TV}_{r,1}(S^3)} = \tau_r(W),$$

and if r is odd, then

$$\frac{\mathrm{TV}'_{r,r-1}(M)}{\mathrm{TV}'_{r,r-1}(S^3)} = \tau'_r(W),$$

where W denotes the orientable connected double cover of M. Note that W can be canonically constructed, with a canonical orientation and an orientation-reserving deck transformation.

(3)

$$\mathrm{TV}_{r,1}(S^3) = \frac{2}{r} \cdot \sin^2\left(\frac{\pi}{r}\right),$$

and if r is odd, then

$$\mathrm{TV}'_{r,r-1}(S^3) = \frac{4}{r} \cdot \sin^2\left(\frac{\pi}{r}\right).$$

See [23, Theorems 4.1.1 and 4.4.1] for systematic proofs of the formulas regarding TV_r and [18] for an elegant proof for the oriented case regarding TV_r. The formulas regarding TV'_r can be easily derived from the TV_r formulas using Lemmas 3.2 and 3.4. The values for $\mathrm{TV}_{r,1}(S^3)$ and $\mathrm{TV}'_{r,r-1}(S^3)$ can be obtained immediately by taking $M = S^2 \times S^1$ and applying Lemmas 3.1 and 3.4.

3.4 Perspective of TQFT

The Turaev-Viro invariants fit into the general framework of $(2+1)$-dimensional topological quantum field theories (TQFT for short) (see [3]). We describe below focusing on the SU(2) Turaev-Viro invariants. The discussion regarding SO(3) Turaev-Viro invariants is completely similar.

Turaev and Viro constructed a functor $\mathscr{Z}^{\mathrm{TV}}$ from the (2+1)-dimensional cobordism category to the category of Hermitian vector spaces (and linear homomorphisms) over the abstract cyclotomic field $\mathbb{K} = \mathbb{Q}(q^{\frac{1}{2}})$, with respect to the involution $*$ on $\mathbb{K}$ as provided by the Galois transformation $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$. To any oriented closed surface S, there is a finite-dimensional vector space $\mathscr{Z}^{\mathrm{TV}}(S)$, equipped with a nondegenerate Hermitian pairing

$$\langle \_, \_ \rangle \colon \mathscr{Z}^{\mathrm{TV}}(S) \times \mathscr{Z}^{\mathrm{TV}}(S) \to \mathbb{K}$$

(Being Hermitian means $\mathbb{K}$-sesquilinear and $*$-symmetric). To any cobordism M from S_0 to S_1 (that is, an oriented compact 3-manifold M with bipartite boundary $\partial M = \partial_- M \sqcup \partial_+ M = (-S_0) \sqcup S_1$, up to boundary-fixing homomeorphisms), there is a linear homomorphism

$$\mathscr{Z}^{\mathrm{TV}}(M) \colon \mathscr{Z}^{\mathrm{TV}}(S_0) \to \mathscr{Z}^{\mathrm{TV}}(S_1).$$

The assignment $\mathscr{Z}^{\mathrm{TV}}$ is functorial, and satisfies Atiyah's Hermitian TQFT axioms (see [3, Section 2]): $\mathscr{Z}^{\mathrm{TV}}(-S) = \mathscr{Z}^{\mathrm{TV}}(S)^*$, $\mathscr{Z}^{\mathrm{TV}}(-M) = \mathscr{Z}^{\mathrm{TV}}(M)^*$, $\mathscr{Z}^{\mathrm{TV}}(S' \sqcup S'') = \mathscr{Z}^{\mathrm{TV}}(S') \otimes_{\mathbb{K}} \mathscr{Z}^{\mathrm{TV}}(S'')$ and $\mathscr{Z}^{\mathrm{TV}}(\emptyset) = \mathbb{K}$ ($\emptyset$ denoting the empty surface, having a unique orientation by convention).

Turaev and Viro showed that TV comes from a functor $\mathcal{Z}^{\mathrm{TV}}$ as above, in the sense that the identity

$$\mathrm{TV}(M; q^{\frac{1}{2}}) = \mathcal{Z}^{\mathrm{TV}}(M)$$

holds for any closed 3-manifold M. Here, M is treated as a cobordism between empty surfaces and $\mathcal{Z}^{\mathrm{TV}}(\emptyset, \emptyset) = \mathrm{End}_{\mathbb{K}}(\mathbb{K})$ is identified as $\mathbb{K}$ (see [24, Section 2.3]).

Being a TQFT functor, $\mathcal{Z}^{\mathrm{TV}}$ naturally induces a $\mathbb{K}$-linear representation

$$\mathrm{Mod}(S) \to \mathrm{GL}(\mathcal{Z}^{\mathrm{TV}}(S))$$

of the mapping class group $\mathrm{Mod}(S)$ for any oriented closed surface S (see [24, Section 2.4]). Specializing $q^{\frac{1}{2}}$ to complex roots of unity as in Notation 3.2, we obtain a complex linear representation, denoted as

$$\rho^{\mathrm{TV}}_{r,s} : \mathrm{Mod}(S) \to \mathrm{GL}(\mathcal{Z}^{\mathrm{TV}}_{r,s}(S)) \tag{3.3}$$

for each integer $r \geq 3$ and any integer s coprime to r. These representations preserve the specialized Hermitian pairings $\langle \_, \_ \rangle_{r,s}$, whose signatures depend on both r and s, but are not necessarily (Hilbert) unitary. We refer to these as the SU(2) Turaev-Viro TQFT representations (at level $r-2$) of $\mathrm{Mod}(S)$.

The following formula is useful implication of TQFT axioms (see [3, Section 2]).

Lemma 3.6 *Let $r \geq 3$ be any integer and s be any integer coprime to r. For any oriented closed surface Σ and any mapping class $[f] \in \mathrm{Mod}(S)$,*

$$\mathrm{TV}_{r,s}(M_f) = \mathrm{tr}_{\mathbb{C}}(\rho^{\mathrm{TV}}_{r,s}([f])),$$

where M_f denotes the mapping torus of f.

If S is connected, the complex dimension of the representation $\rho^{\mathrm{TV}}_{r,s}$ depends only on r and genus g of S. This fact can be derived from Lemmas 3.2 and 3.6 by considering the mapping torus of $f = \mathrm{id}_S$. Verlinde type formulas for these dimensions can be derived from Lemma 3.5 and known formulas about Witten-Reshetikhin-Turaev invariants (see [5, Corollary 1.16]). However, Witten-Reshetikhin-Turaev invariants only come from generalized TQFTs, which require extra structures for resolving framing anomaly (see [5]). That is why they only naturally lead to projective linear representations of surface mapping class groups.

4 Calculations

This section is devoted to the proof of Theorem 1.2.

To restate our task, we consider a closed Seifert fiber space M with orientable orbifold base and orientable fibration, and with symbol $(g; (a_1, b_1), \cdots, (a_n, b_n))$, such that

$$a_1 = a_2 = \cdots = a_n = a$$

and

$$b_1 + b_2 + \cdots + b_n = 0.$$

Moreover, we suppose $a \geq 3$ and $a > n \geq 0$. We compute TV_r and TV'_r of M for $r = a$, and once they vanish, we show that they must also vanish for any r divisible by a.

We invoke the following explicit formula for computing the Witten-Reshetikhin-Turaev invariant τ_r of Seifert fiber spaces. Recall $\tau_r(S^1 \times S^2) = \sqrt{\frac{r}{2}}/\sin\left(\frac{\pi}{r}\right)$ (see Lemma 3.3).

Lemma 4.1 *Let M be a closed Seifert fiber space with orientable orbifold base and orientable fibration, and with symbol $(g; (a_1, b_1), \cdots, (a_n, b_n))$, where $g, n \geq 0$ are integers, $a_j \geq 0$ and b_j are coprime pairs of integers for $j = 1, \cdots, n$. Orient M by orienting the base and the fibers, such that the rational Euler number of the Seifert fibration is*

$$E = -\sum_j \frac{b_j}{a_j}.$$

Then

$$\frac{\tau_r(M)}{\tau_r(S^2 \times S^1)} = \frac{r^{g-1} \cdot U_r \cdot Z_r}{2^{n+g-1}\sqrt{\prod_j a_j}},$$

where

$$Z_r = \sum_{(\gamma, \boldsymbol{\mu}, \boldsymbol{m})} \left\{ \frac{\mathrm{e}^{\sqrt{-1}\cdot\frac{\pi\gamma^2 E}{2r}}}{\sin^{n+2g-2}\left(\frac{\pi\gamma}{r}\right)} \cdot \prod_j \mu_j \mathrm{e}^{\sqrt{-1}\cdot\left(\frac{-\pi(2rm_j+\mu_j)\gamma}{a_j r} + \frac{-2\pi(rm_j^2+\mu_j m_j)b_j^*}{a_j}\right)} \right\}$$

and

$$U_r = (-1)^n \cdot \mathrm{e}^{\sqrt{-1}\cdot\left(\frac{3\pi}{2r}-\frac{3\pi}{4}\right)\cdot\mathrm{sgn}(E)} \cdot \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi(E+12\cdot\sum_j \mathrm{s}(b_j,a_j))}{2r}}.$$

Here, any j ranges over $\{1, \cdots, n\}$ and $(\gamma, \boldsymbol{\mu}, \boldsymbol{m}) = (\gamma, (m_1, \cdots, m_n), (\mu_1, \cdots, \mu_n))$ ranges over $\{1, 2, \cdots, r-1\} \times \{\pm 1\}^n \times \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$. The notation b_j^* denotes any congruence inverse of b_j modulo a_j, namely, $b_j b_j^* \equiv 1 \bmod a_j$; $\mathrm{sgn}(E)$ denotes the sign of E, with value ± 1 or 0; $\mathrm{s}(b_j, a_j)$ denotes the Dedekind sum $(4a_j)^{-1} \cdot \sum_{l \in \{1,2,\cdots,a_j-1\}} \cot\left(\frac{\pi l}{a_j}\right)\cot\left(\frac{\pi l b_j}{a_j}\right)$.

See [9, Theorem 8.4] for this formula and Remark 4.1 below for clarification about different notations and normalizations. Hansen actually obtained the most general formula that applies to any orientable closed Seifert fiber space, including those with non-orientable orbifold base. We have only stated here the case with orientable orbifold base. An equivalent formula for this case is formerly obtained by Rozansky [19].

Remark 4.1 Our notation τ_r agrees with Kirby-Melvin [12], differing from Hansen's notation by our factor $\frac{1}{\tau_r(S^2 \times S^1)}$. In [9, Section 8], Hansen's $\tau_r(M)$ is defined as $\tau_{(\mathcal{V}_t,\mathcal{D})}(M)$ therein; as pointed out in [9, Appendix A], Kirby-Melvin's $\tau_r(M)$ is equal to $\tau_A(M) = \mathcal{D} \cdot \tau_{(\mathcal{V}_t,\mathcal{D})}(M)$, where $\mathcal{D}$ is specified as $\sqrt{\frac{r}{2}}/\sin\left(\frac{\pi}{r}\right)$ in the equation (38) therein. More directly, one may check by evaluating the formula in [9, Theorem 8.4] for S^3 (setting $g = 0$, $b = 1$ and $n = 0$) and $S^2 \times S^1$ (setting $g = 0$, $b = 0$ and $n = 0$) in the simplest case $r = 3$.

Lemma 4.2 *Under the assumptions of Theorem* 1.2 *and assuming* r *divisible by* a*, the term* $Z_r = Z_r(M)$ *as in Lemma* 4.1 *becomes*

$$Z_r = \sum_{(\gamma,\boldsymbol{\mu})} \Big\{ \frac{\mathrm{e}^{\sqrt{-1}\cdot\frac{-\pi\gamma\sum_j \mu_j}{ar}} \cdot \prod_j \mu_j}{\sin^{n+2g-2}\left(\frac{\pi\gamma}{r}\right)} \cdot \prod_j \sum_{m_j} \mathrm{e}^{\sqrt{-1}\cdot\frac{-2\pi(\gamma+b_j^*\mu_j)m_j}{a}} \Big\},$$

where $(\gamma, \boldsymbol{\mu}, \boldsymbol{m})$ *ranges over* $\{1, \cdots, r-1\} \times \{\pm 1\}^n \times (\mathbb{Z}/a\mathbb{Z})^n$.

Proof In the expression of Z_r in Lemma 4.1, if any a_j divides r, we can ignore the term rm_j^2 in the exponent of the j-th factor in the product; if $E = 0$, we can ignore the factor that involves γ^2 on the exponent. Therefore, under these conditions, the expression of Z_r can be rearranged into

$$\begin{aligned} Z_r(M) &= \sum_{(\gamma,\boldsymbol{\mu})} \Big\{ \frac{\prod_j \mu_j \mathrm{e}^{\sqrt{-1}\cdot\frac{-\pi\gamma\mu_j}{a_j r}}}{\sin^{n+2g-2}\left(\frac{\pi\gamma}{r}\right)} \cdot \sum_{\boldsymbol{m}} \mathrm{e}^{\sqrt{-1}\cdot\sum_j\left(\frac{-2\pi\gamma m_j}{a_j} + \frac{-2\pi b_j^*\mu_j m_j}{a_j}\right)} \Big\} \\ &= \sum_{(\gamma,\boldsymbol{\mu})} \Big\{ \frac{\mathrm{e}^{\sqrt{-1}\cdot\sum_j\frac{-\pi\gamma\mu_j}{a_j r}} \cdot \prod_j \mu_j}{\sin^{n+2g-2}\left(\frac{\pi\gamma}{r}\right)} \cdot \sum_{\boldsymbol{m}} \mathrm{e}^{\sqrt{-1}\cdot\sum_j\frac{-2\pi(\gamma+b_j^*\mu_j)m_j}{a_j}} \Big\} \\ &= \sum_{(\gamma,\boldsymbol{\mu})} \Big\{ \frac{\mathrm{e}^{\sqrt{-1}\cdot\sum_j\frac{-\pi\gamma\mu_j}{a_j r}} \cdot \prod_j \mu_j}{\sin^{n+2g-2}\left(\frac{\pi\gamma}{r}\right)} \cdot \prod_j \sum_{m_j} \mathrm{e}^{\sqrt{-1}\cdot\frac{-2\pi(\gamma+b_j^*\mu_j)m_j}{a_j}} \Big\}, \end{aligned}$$

where $(\gamma, \boldsymbol{\mu})$ ranges in $\{1, 2, \cdots, r-1\} \times \{\pm 1\}^n$, j in $\{1, \cdots, n\}$ and m_j in $\mathbb{Z}/a_j\mathbb{Z}$. In particular, the simplification applies as we assume $a_1 = a_2 = \cdots = a_n = a$, $b_1 + b_2 + \cdots + b_n = 0$ and r divisible by a.

Lemma 4.3 *Under the assumptions of Theorem* 1.2 *and assuming* r *divisible by* a*, if there does not exist any* $\boldsymbol{\mu} \in \{\pm 1\}^n$ *that satisfies the congruence equations*

$$b_1^*\mu_1 \equiv b_2^*\mu_2 \equiv \cdots \equiv b_n^*\mu_n \mod a,$$

then

$$Z_r = 0,$$

where Z_r *is the term as in Lemma* 4.1.

Proof For any fixed $(\gamma, \boldsymbol{\mu})$, we observe

$$\sum_{m_j \in \mathbb{Z}/a\mathbb{Z}} \mathrm{e}^{\sqrt{-1}\cdot\frac{-2\pi(\gamma+b_j^*\mu_j)m_j}{a}} = \begin{cases} a, & \text{if } \gamma + b_j^*\mu_j \equiv 0 \bmod a, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, in the simplified expression of Z_r as in Lemma 4.2, the summand corresponding to $(\gamma, \boldsymbol{\mu})$ is nonzero if and only if $\gamma + b_j^*\mu_j \equiv 0 \bmod a$ holds for all $j \in \{1, \cdots, n\}$. For the sum Z_r to be nonzero, there has to be some $\gamma \in \{1, 2, \cdots, r-1\}$ that satisfies the above condition for some $\boldsymbol{\mu} \in \{\pm 1\}^n$, then there has to be some $\boldsymbol{\mu}$ that satisfies $b_1^*\mu_1 \equiv b_2^*\mu_2 \equiv \cdots \equiv b_n^*\mu_n \mod a$.

Lemma 4.4 *Under the assumptions of Theorem* 1.2*, if there exists some integer b^* coprime to a and some $\boldsymbol{\nu} \in \{\pm 1\}^n$, such that $b^* \equiv b_j^*\nu_j \bmod a$ holds for all $j \in \{1, \cdots, n\}$, then*

$$Z_a = \frac{2 \cdot a^n \cdot \prod_j \nu_j}{\sin^{n+2g-2}\left(\frac{\pi b^*}{a}\right)},$$

where Z_a is the term as in Lemma 4.1 *with $r = a$.*

Proof Possibly after permuting $\{1, \cdots, n\}$, we may assume $\nu_j = 1$ for $j = 1, \cdots, m$ and $\nu_j = -1$ for $j = m+1, \cdots, n$. We may also assume $b^* \in \{1, 2, \cdots, a-1\}$ without changing its residue class modulo a.

For $r = a \geq 3$, there are only two nonzero summands in Z_r and their corresponding to $(\gamma, \boldsymbol{\mu})$ are

$$(\gamma, \boldsymbol{\mu}) = (a - b^*, (\underbrace{1, \cdots, 1}_{m}, \underbrace{-1, \cdots, -1}_{n-m}))$$

and

$$(\gamma, \boldsymbol{\mu}) = (b^*, (\underbrace{-1, \cdots, -1}_{m}, \underbrace{1, \cdots, 1}_{n-m})).$$

By our assumption $\sum_j b_j = 0$ in Theorem 1.2, $(2m - n)b \equiv mb + (n-m)(-b) \equiv \sum_j b_j = 0 \bmod a$, so $n - 2m$ is divisible by a. By our assumption $a > n \geq 0$ in Theorem 1.2, we must have $|n - 2m| < a$, hence $n - 2m = 0$. So, we observe

$$(-1)^{n-m} = (-1)^m,$$

which is useful below.

We compute

$$\begin{aligned} Z_a(M) &= \frac{(-1)^{n-m} \cdot \mathrm{e}^{\sqrt{-1} \cdot \frac{-\pi(2m-n)(a-b^*)}{a^2}}}{\sin^{n+2g-2}\left(\frac{\pi(a-b^*)}{a}\right)} \cdot a^n + \frac{(-1)^m \cdot \mathrm{e}^{\sqrt{-1} \cdot \frac{-\pi(n-2m)b^*}{a^2}}}{\sin^{n+2g-2}\left(\frac{\pi b^*}{a}\right)} \cdot a^n \\ &= \frac{2 \cdot a^n \cdot (-1)^{n-m}}{\sin^{n+2g-2}\left(\frac{\pi b^*}{a}\right)} \\ &= \frac{2 \cdot a^n \cdot \prod_j \nu_j}{\sin^{n+2g-2}\left(\frac{\pi b^*}{a}\right)} \end{aligned}$$

as desired.

Lemma 4.5 *Let M be a closed Seifert fiber space with orientable orbifold base and orientable fibration, and with symbol $(g;(a_1,b_1),\cdots,(a_n,b_n))$, where $g,n\geq 0$ are integers, $a_j\geq 1$ and b_j are coprime pairs of integers for $j=1,\cdots,n$, and satisfy $\frac{b_1}{a_1}+\cdots+\frac{b_n}{a_n}=0$. If $a_1,\cdots,a_n$ are all odd, then*

$$\mathrm{TV}_{3,1}(M)=\mathrm{TV}_{3,2}(M)=2^{2g}.$$

Proof Denote

$$\mathrm{lcm}(a_1,\cdots,a_n)=d.$$

If $a_1,\cdots,a_n$ are all odd, d is also odd.

The fundamental group of M has a presentation with generators

$$x_1,y_1,\cdots,x_g,y_g,s_1,\cdots,s_n,f$$

and relations

$$\begin{cases} s_1\cdots s_n=[x_1,y_1]\cdots[x_g,y_g],\\ x_if=fx_i,\quad i=1,\cdots,g,\\ y_if=fy_i,\quad i=1,\cdots,g,\\ s_jf=fs_j,\quad j=1,\cdots,n,\\ s_j^{a_j}f^{b_j}=1,\quad j=1,\cdots,n.\end{cases}$$

With $\mathbb{Z}/2\mathbb{Z}$ coefficients, we can eliminate any $[s_j]$ using the relation $a_j[s_j]+b_j[f]=0$, since a_j is odd. Then the first relation is equivalent to $-\big(\frac{b_1d}{a_1}+\cdots+\frac{b_nd}{a_n}\big)[f]=0$ over $\mathbb{Z}/2\mathbb{Z}$, having no effect by our assumption $\frac{b_1}{a_1}+\cdots+\frac{b_n}{a_n}=0$. It follows that $H_1\big(M;\mathbb{Z}/2\mathbb{Z}\big)$ is freely generated by $[x_1],[y_1],\cdots,[x_g],[y_g],[f]$ over $\mathbb{Z}/2\mathbb{Z}$. Then the $\mathbb{Z}/2\mathbb{Z}$ Betti numbers of M are $\beta_0=\beta_3=1$ and $\beta_2=\beta_1=2g+1$, by the Poincaré duality with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Therefore, we obtain

$$\mathrm{TV}_{3,2}(M)=2^{2g}$$

by Lemma 3.2.

Since M is homeomorphic to the mapping torus of a periodic surface automorphism of order d, there is a cyclic cover $\widetilde{M}\to M$ of degree d, and $\widetilde{M}$ is a product of a closed orientable surface with a circle. Since d is odd, the induced homomorphism $H^*(M;\mathbb{Z}/2\mathbb{Z})\to H^*(\widetilde{M};\mathbb{Z}/2\mathbb{Z})$ is injective (because of the Poincaré duality pairing with $\mathbb{Z}/2\mathbb{Z}$ coefficients and the isomorphism on the top dimension). However, $H^1(\widetilde{M};\mathbb{Z}/2\mathbb{Z})$ contains no element whose cube is nontrivial, (by the Künneth theorem which determines the cohomology ring of $\widetilde{M}$ over $\mathbb{Z}/2\mathbb{Z}$). It follows that $t^3=0\in H^3\big(M;\mathbb{Z}/2\mathbb{Z}\big)$ holds for any $t\in H^1\big(M;\mathbb{Z}/2\mathbb{Z}\big)$. Moreover, the first Stiefel-Whitney class w_1 of M vanishes, as M is orientable. Therefore, we obtain

$$\mathrm{TV}_{3,1}(M)=2^{2g}$$

by Lemma 3.2.

Under the assumptions of Theorem 1.2, we compute the Turaev-Viro invariants in Theorem 1.2 as follows.

Suppose that $b^*b_j \equiv \nu_j \bmod a$ holds for some integer b^* coprime to a and some $\nu_j \in \{\pm 1\}$, and for all $j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}\mathrm{TV}_{a,1}(M) &= \mathrm{TV}_{a,1}(S^3) \cdot |\tau_a(M)|^2 \\ &= \Big| \frac{\tau_a(M)}{\tau_a(S^2 \times S^1)} \Big|^2 \\ &= \Big| \frac{a^{g-1}}{2^{n+g-1} a^{\frac{n}{2}}} \cdot \frac{2 \cdot a^n}{\sin^{n+2g-2}\left(\frac{\pi b^*}{a}\right)} \Big|^2 \\ &= \frac{a^{n+2g-2}}{2^{2n+2g-4}} \cdot \frac{1}{\sin^{2n+4g-4}\left(\frac{\pi b^*}{a}\right)}\end{aligned}$$

by Lemmas 3.5, 4.1 and 4.4.

When a is even, we apply Galois conjugacy (transforming $\mathrm{e}^{\sqrt{-1}\cdot\frac{\pi}{a}} \mapsto \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{a}}$) for any s coprime to a, obtaining

$$\mathrm{TV}_{a,s}(M) = \frac{a^{n+2g-2}}{2^{2n+2g-4}} \cdot \frac{1}{\sin^{2n+4g-4}\left(\frac{\pi b^* s}{a}\right)}$$

by Lemma 3.2.

When a is odd, we apply Lemmas 3.2 and 4.5, obtaining

$$\mathrm{TV}_{a,a-1}(M) = \mathrm{TV}_{3,2}(M) \cdot \mathrm{TV}'_{a,a-1}(M) = \frac{\mathrm{TV}_{3,2}(M)}{\mathrm{TV}_{3,1}(M)} \cdot \mathrm{TV}_{a,1}(M) = \mathrm{TV}_{a,1}(M)$$

and

$$\mathrm{TV}'_{a,a-1}(M) = \frac{\mathrm{TV}_{a,1}(M)}{\mathrm{TV}_{3,1}(M)} = \frac{1}{2^{2g}} \cdot \mathrm{TV}_{a,1}(M).$$

Again, we apply Galois conjugacy (transforming $\mathrm{e}^{\sqrt{-1}\cdot\frac{\pi}{a}} \mapsto \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{a}}$ or $\mathrm{e}^{\sqrt{-1}\cdot\frac{\pi(a-1)}{a}} \mapsto \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{a}}$ depending on s odd or even), obtaining for any s coprime to a

$$\mathrm{TV}_{a,s}(M) = \frac{a^{n+2g-2}}{2^{2n+2g-4}} \cdot \frac{1}{\sin^{2n+4g-4}\left(\frac{\pi b^* s}{a}\right)},$$

and if s is even,

$$\mathrm{TV}'_{a,s}(M) = \frac{a^{n+2g-2}}{2^{2n+4g-4}} \cdot \frac{1}{\sin^{2n+4g-4}\left(\frac{\pi b^* s}{a}\right)}$$

by Lemma 3.2. This completes the computation of the nonvanishing values in Theorem 1.2.

Suppose the otherwise case. Then, for all $r \geq 3$ divisible by a, we obtain

$$\mathrm{TV}_{r,1}(M) = 0$$

by Lemmas 3.5, 4.1 and 4.3. Similarly, we derive $\mathrm{TV}_{r,r-1}(M) = \mathrm{TV}'_{r,r-1}(M) = 0$ in this case, using Lemmas 3.2 and 4.5. Finally, by Galois conjugacy (see Lemma 3.2), we see that $\mathrm{TV}_{r,s}(M) = 0$ and $\mathrm{TV}'_{r,s}(M) = 0$ for any applicable s.

This completes the proof of Theorem 1.2.

5 Examples

In this section, we prove Theorem 1.1 by exhibiting nontrivial Hempel pairs that can or cannot be distinguished by Turaev-Viro invariants. See Example 5.1 for our distinguishable ones and Example 5.2 for our indistinguishable ones.

We need the following lemma for verifying our examples. A rationality statement would be good enough for our application, but the integrality here is also well-known to experts.

Lemma 5.1 *Let S be a connected orientable closed surface of genus $g \geq 0$ and $[f] \in \mathrm{Mod}(S)$ be a periodic mapping class of order $d \geq 1$. Then, for any integer $r \geq 3$ coprime to d and integer s coprime to r,*

$$\mathrm{TV}_{r,s}(M_f) \in \mathbb{Z},$$

and if r is odd and s is even,

$$\mathrm{TV}'_{r,s}(M_f) \in \mathbb{Z},$$

where M_f denotes the mapping torus of f.

Proof By Lemma 3.6, $\mathrm{TV}_{r,s}(M_f) \in \mathbb{C}$ is equal to the trace of the Turaev-Viro TQFT representation $\rho^{\mathrm{TV}}_{r,s}([f]) \in \mathrm{GL}(\mathcal{Z}^{\mathrm{TV}}_{r,s}(S))$. If $[f] \in \mathrm{Mod}(S)$ is periodic of order d, the eigenvalues of $\rho^{\mathrm{TV}}_{r,s}([f])$ are all complex roots of unity of order divisible by d. In particular, $\mathrm{TV}_{r,s}(M_f)$ is an algebraic integer.

On the other hand, entries of $\rho^{\mathrm{TV}}_{r,s}([f])$ lie in the cyclotomic subfield $\mathbb{Q}\big(\mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{r}}\big)$ of $\mathbb{C}$. For any roots of unity $\zeta_m, \zeta_n \in \mathbb{C}$ of coprime orders $m, n \geq 1$, respectively, it is an elementary fact that $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n)$ equals $\mathbb{Q}$ (see [25, Chapter 2, Proposition 2.4]). Applying with $m|d$ and $n = r$ $\big($taking $\zeta_n = \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{r}}$, or if r is odd and s is even, $\zeta_n = \mathrm{e}^{\sqrt{-1}\cdot\frac{\pi(r-s)}{r}}\big)$, we obtain $\mathrm{TV}_{r,s}(M_f) \in \mathbb{Q}$. Together with the algebraic integrality, we obtain $\mathrm{TV}_{r,s}(M_f) \in \mathbb{Z}$.

With Lemmas 3.2 and 4.5, one may derive $\mathrm{TV}'_{r,s}(M_f) \in \mathbb{Q}$ from $\mathrm{TV}_{r,s}(M_f) \in \mathbb{Z}$. To obtain $\mathrm{TV}'_{r,s}(M_f) \in \mathbb{Z}$, it is possible to appeal to a similar lemma as Lemma 3.6 with a TQFT functor associated to TV'. In fact, this case has been established by Detcherry and Kalfagianni. We refer to [6, Corollary 6.1] for their proof of this case.

Example 5.1 (Distinguishable pairs) Let $g \geq 0$, $d = 5$ or $d \geq 7$ be integers and k be an integer coprime to d. Let M_A and M_B be closed Seifert fiber spaces with orientable orbifold base and orientable Seifert fibration. We assign their symbols as

$$M_A\colon (g; (d,1), (d,1), (d,-1), (d,-1)),$$
$$M_B\colon (g; (d,k^*), (d,k^*), (d,-k^*), (d,-k^*)),$$

where k^* is an integer satisfing $k^*k \equiv 1 \bmod p$. The 3-manifold M_A is homeomorphic to the mapping torus of some periodic mapping class $[f_A] \in \mathrm{Mod}(S)$ of order d, where S is a connected orientable closed surface of genus $dg + d - 2$ (see Lemma 2.1). The 3-manifold M_B is homeomorphic to the mapping torus of the iterate mapping class $[f_B] = [f_A^k]$ (see Lemma 2.2).

By Theorem 1.2, we compute

$$\mathrm{TV}_{d,1}(M_A) = \frac{d^{2g+2}}{2^{2g+4}} \cdot \frac{1}{\sin^{4g+4}\left(\dfrac{\pi}{d}\right)},$$
$$\mathrm{TV}_{d,1}(M_B) = \frac{d^{2g+2}}{2^{2g+4}} \cdot \frac{1}{\sin^{4g+4}\left(\dfrac{\pi k}{d}\right)}.$$

The values are equal if and only if $k \equiv \pm 1 \bmod d$, namely, $[f_B] = [f_A]$ or $[f_B] = [f_A^{-1}]$. For k other than these values, one may also check that $\mathrm{TV}_{d,s}(M_A) \neq \mathrm{TV}_{d,s}(M_B)$ for any integer s coprime to d, and if d is odd, $\mathrm{TV}'_{d,s}(M_A) \neq \mathrm{TV}'_{d,s}(M_B)$ for any even integer s coprime to d. Under our assumption on d, such k does exist, so M_A and M_B form a nontrivial Hempel pair.

Example 5.2 (Indistinguishable pairs) Let $g \geq 0$ be any integers, $p \geq 5$ be a prime integer and k be an integer coprime to p. Let M_A and M_B be closed Seifert fiber spaces with orientable orbifold base and orientable Seifert fibration. We assign their symbols as

$$M_A\colon (g; (p,1), (p,1), (p,-2)),$$
$$M_B\colon (g; (p,k^*), (p,k^*), (p,-2k^*)),$$

where k^* is an integer satisfying $k^*k \equiv 1 \bmod p$. Obtain the connected orientable closed surface S of genus $pg + \frac{p-1}{2}$ and the periodic mapping classes $[f_B] = [f_A^k]$ of order p, similarly as in the previous example (see Lemmas 2.1 and 2.2).

Again, $[f_A]$ and $[f_B]$ form a nontrivial Hempel pair exactly when $k \not\equiv \pm 1 \bmod p$, existing under the assumption on p.

If $r \geq 3$ is divisible by p, applying Theorem 1.2 and Lemma 5.1, we see that $\mathrm{TV}_{r,s}(M_A) = \mathrm{TV}_{r,s}(M_B) = 0$ holds for any s coprime to r, and moreover, if r is odd and s is even, $\mathrm{TV}'_{r,s}(M_A) = \mathrm{TV}'_{r,s}(M_B) = 0$ also holds.

If $r \geq 3$ is not divisible by p, then it is coprime to p. By Lemma 5.1, $\mathrm{TV}_{r,s}(M_A)$ and $\mathrm{TV}_{r,s}(M_B)$ are rational. Since $\mathrm{TV}_{r,s}(M_A)$ and $\mathrm{TV}_{r,s}(M_B)$ are the traces of the Turaev-Viro TQFT representations $\rho^{\mathrm{TV}}_{r,s}$ of the periodic mapping class $[f_A]$ and $[f_B] = [f_A^k]$, respectively, the eigenvalues of $[f_A]$ are roots of unity of order dividing d and the eigenvalues of $[f_B]$ are their Galois conjugates under the transformation $\mathrm{e}^{\sqrt{-1}\cdot\frac{2\pi}{d}} \mapsto \mathrm{e}^{\sqrt{-1}\cdot\frac{2\pi k}{d}}$. Then by the rationality, we obtain $\mathrm{TV}_{r,s}(M_A) = \mathrm{TV}_{r,s}(M_B)$ for any s coprime to r. Moreover, if r is odd and s is even, we apply Lemmas 3.2 and 4.5 to deduce $\mathrm{TV}'_{r,s}(M_A) = \mathrm{TV}'_{r,s}(M_B)$.

A Splitting of Turaev-Viro Invariants at Odd Levels

In this appendix section, we prove the formula in Lemma 3.2 about $\mathrm{TV}_{r,s}$ when r and s are both odd. We restate this part as a separate theorem and make a couple of remarks regarding former results.

Theorem A.1 *Let M be any closed 3-manifold. Let $r \geq 3$ be an odd integer and s be an integer coprime to r. Adopt Notation 3.2.*

If s is odd, then the following formula holds: $\mathrm{TV}_{r,s}(M) = \mathrm{TV}_{3,1}(M) \cdot \mathrm{TV}'_{r,r-s}(M)$.

Remark A.1 (1) Sokolov obtains a canonical splitting of $\mathrm{TV}_{r,s}$ into the sum of three refined invariants [20]. When r is odd and s is odd, one may identify the three refined invariants (the zeroth, the first and the second in Sokolov's definition) as $\mathrm{TV}'_{r,r-s}$, $\frac{\mathrm{TV}_{r,s}-\mathrm{TV}_{r,r-s}}{2}$ and $\frac{\mathrm{TV}_{r,s}+\mathrm{TV}_{r,r-s}}{2} - \mathrm{TV}'_{r,r-s}$. In this case, the splitting of $\mathrm{TV}_{r,s}$ is proportional to the splitting of $\mathrm{TV}_{3,1}$. Similarly, when r is odd and s is even, the splitting of $\mathrm{TV}_{r,s}$ is proportional to the splitting of $\mathrm{TV}_{3,2}$. Compare [5, Theorem 1.5].

(2) In the same paper, Sokolov quickly points out Lemma A.4 below, with assumption of orientability. See the formula (1) in [20, Proof of Lemma 2.2].

The rest of this section is devoted to the proof of Theorem A.1.

Our strategy is to derive needed ingredients from the proof of Detcherry, Kalfagianni and Yang for the case with r odd and s even (see [7, Appendix A]). We count sign change from their case for individual terms in the defining state-sum expression, and verify that overall, the factors $\mathrm{TV}_{3,1}$ and $\mathrm{TV}_{r,s}$ are result of proportional change from factors in their case.

We denote by $\mathrm{ev}_{r,s}\colon \mathbb{Q}(q^{\frac{1}{2}}) \to \mathbb{C}$ the evaluation which assigns the abstract root of unity $q^{\frac{1}{2}}$ to be $\mathrm{e}^{\sqrt{-1}\cdot\frac{\pi s}{r}}$.

For any odd integer $r \geq 3$, recall that $I_r = \{0, 1, \cdots, r-2\}$ denotes the set of colors on this level. It contains the subset of even colors $I'_r = \{0, 2, \cdots, r-3\}$ and also $I_3 = \{0, 1\}$. For any finite simplicial 3-complex $\mathscr{T} = (V, E, F, T)$. From any coloring $c\colon E \to I_r$, we obtain a pair of colorings $c_3\colon E \to I_3$ and $c'\colon E \to I'_r$ as follows: For any $e \in E$, we assign $c_3(e) = 0$ and $c'(e) = c(e)$ if $c(e)$ is even, or $c_3(e) = 1$ and $c'(e) = r - 2 - c(e)$ if $c(e)$ is odd. By observation, this operation preserves admissible colorings and yields a bijective correspondence between $\mathcal{A}_r$ and $\mathcal{A}_3 \times \mathcal{A}'_r$.

Lemma A.1 *Let $\mathscr{T} = (V, E, F, T)$ be any finite simplicial 3-complex. Let $r \geq 3$ be an odd integer and s be an integer coprime to r. Adopt Notation 3.1. Identify $\mathcal{A}_r = \mathcal{A}_3 \times \mathcal{A}'_r$. If s is even, then, for any $x \in E \sqcup F \sqcup T$ and any $c = (c_3, c') \in \mathcal{A}_r$,*

$$\mathrm{ev}_{r,s}(|x|_c) = \mathrm{ev}_{3,2}(|x|_{c_3}) \cdot \mathrm{ev}_{r,s}(|x|_{c'})$$

and hence

$$\mathrm{ev}_{r,s}(|\mathscr{T}|_c) = \mathrm{ev}_{3,2}(|\mathscr{T}|_{c_3}) \cdot \mathrm{ev}_{r,s}(|\mathscr{T}|_{c'}).$$

The identities in Lemma A.1 are key to the proof of Detcherry-Kalfagianni-Yang for the s even case (see [7, Theorem 2.9]). We refer to [7, Lemma A.4] for the proof. Note that the identity (A.1) therein essentially relies on the parity assumption of s.

For proving the s odd case, our next few lemmas examine the sign difference between $\mathrm{ev}_{r,s}(|x|_c)$ and $\mathrm{ev}_{r,r-s}(|x|_c)$, and between $\mathrm{ev}_{r,s}(|\mathscr{T}|_c)$ and $\mathrm{ev}_{r,r-s}(|\mathscr{T}|_c)$.

Lemma A.2 *Let $\mathscr{T} = (V, E, F, T)$ be any finite simplicial 3-complex. Let $r \geq 3$ be an integer and s be an integer coprime to r. Adopt Notation 3.1. Then, for any $x \in E \sqcup F \sqcup T$ and any $c \in \mathcal{A}_r$,*

$$\mathrm{ev}_{r,s}(|x|_c) = (-1)^{\delta(x,c)} \cdot \mathrm{ev}_{r,r-s}(|x|_c),$$

where $\delta(x, c) \in \mathbb{Z}$ is assigned as follows.

(1) *For $x \in E$, having color i under c,*

$$\delta(x, c) = i.$$

(2) *For $x \in F$, having edge colors (i, j, k) under c,*

$$\delta(x, c) = \frac{i^2 + j^2 + k^2}{2}.$$

(3) *For $x \in T$, having edge colors $(i, j, k), (i, m, n), (j, m, n), (k, l, n)$ on each face under c,*

$$\delta(x, c) = i + j + k + l + m + n + \frac{il + jm + kn}{2}.$$

Proof We prove for $x \in T$. The formulas with $x \in E$ and $x \in F$ can be proved by similar means, and are simpler. Suppose the tetrahedron x has edge colors i, j, k, l, m, n given by c. Note that the value of the quantum integer only change by a sign determined by its parity

$$\mathrm{ev}_{r,s}([w]) = (-1)^{w-1} \cdot \mathrm{ev}_{r,r-s}([w]).$$

For any fixed z, the total sign change of the summand is -1 to the power

$$\begin{aligned}&\frac{(z+1)z}{2} + \sum_a \frac{(z - T_a)(z - T_a - 1)}{2} + \sum_b \frac{(Q_b - z)(Q_b - z - 1)}{2}\\ &= 4z^2 - 2zW + \frac{1}{2} \cdot \Big(\sum_a T_a^2 + \sum_b Q_b^2\Big),\end{aligned}$$

where $W = i + j + k + l + m + n = \sum_a T_a = \sum_b Q_b$ is an integer. The first two terms are even integers, having no effect to the total change of sign and the last term is independent of z. Therefore, we obtain

$$\mathrm{ev}_{r,s}(|x|_c) = (-1)^{\delta(x,c)} \cdot \mathrm{ev}_{r,r-s}(|x|_c),$$

where

$$\delta(x, c) \equiv \frac{1}{2} \cdot \Big(\sum_a T_a^2 + \sum_b Q_b^2\Big) \mod 2.$$

The expression $\sum_a T_a^2 + \sum_b Q_b^2$ is equal to the sum of all the quadratic monomials in i, j, k, l, m, n, namely, $i^2 + ij + \cdots + in + j^2 + jk + \cdots + mn$. We rearrange

$$\sum_a T_a^2 + \sum_b Q_b^2 = X^2 + Y^2 + Z^2 + XY + XZ + YZ - il - jm - kn,$$

where $X = i+l$, $Y = j+m$ and $Z = k+n$. Note that the parity pattern of (X, Y, Z) may only be $(0, 0, 0)$ or $(1, 1, 1)$, up to permutation of the components. Indeed, the only possible patterns of $(i_3+l_3, j_3+m_3, k_3+n_3)$ are $(0, 0, 0)$, $(1, 1, 1)$ and $(0, 2, 2)$, up to permutation of the components, because of admissible coloring. So, the part $X^2 + Y^2 + Z^2 + XY + XZ + YZ = (X + Y)(X + Z)+Y^2+Z^2$ is congruent to 0 modulo 4 if $(X, Y, Z) \equiv (0, 0, 0) \bmod 2$, or congruent to 2 modulo 4 if $(X, Y, Z) \equiv (1, 1, 1) \bmod 2$. In both cases, we see that $\frac{X^2+Y^2+Z^2+XY+XZ+YZ}{2} \equiv X + Y + Z \bmod 2$. This yields $\delta(x, c) \equiv X+Y+Z-\frac{il+jm+kn}{2} \equiv i+j+k+l+m+n+\frac{il+jm+kn}{2} \bmod 2$, as desired.

Lemma A.3 *Let $\mathscr{T} = (V, E, F, T)$ be any finite simplicial 3-complex. Let $r \geq 3$ be an odd integer and s be an integer coprime to r. Adopt Notation 3.1. Identify $\mathcal{A}_r = \mathcal{A}_3 \times \mathcal{A}'_r$. Let $\delta\colon (E \sqcup F \sqcup T) \times \mathcal{A}_r \to \mathbb{Z}$ be expressed as in Lemma A.2.*

(1) *For any $x \in E \sqcup F$ and any $c = (c_3, c') \in \mathcal{A}_r$,*

$$\delta(x, c) \equiv \delta(x, c_3) \mod 2,$$

treating $\mathcal{A}_3$ as a subset of $\mathcal{A}_r$.

(2) *For $x \in T$, having edge colors $(i, j, k), (i, m, n), (j, m, n), (k, l, n)$ on each face under c,*

$$\delta(x, c) \equiv \delta(x, c_3) + \lambda(x, c) \mod 2,$$

where

$$\lambda(x, c) = \frac{i_3l' + l_3i' + j_3m' + m_3j' + k_3n' + n_3k'}{2}.$$

Proof We make use of the relation

$$c(e) = c_3(e) \cdot (r - 2 - c'(e)) + (1 - c_3(e)) \cdot c'(e) = (r - 2) \cdot c_3(e) - c'(e) - 2 \cdot c_3(e) \cdot c'(e)$$

for any $e \in E$. Since r is odd and i', l' are even, we obtain

$$\begin{aligned} il &= ((r-2)i_3 - i' - 2i_3i') \cdot ((r-2)l_3 - l' - 2l_3l') \\ &\equiv (r-2)^2 i_3l_3 - (r-2) \cdot (i_3l' + l_3i') \\ &\equiv i_3l_3 + (i_3l' + l_3i') \mod 4 \end{aligned}$$

and similarly we manipulate jk and kn. Taking the sum, we obtain

$$il + jm + kn \equiv i_3l_3 + j_3m_3 + k_3n_3 + 2 \cdot \lambda(x, c) \mod 4.$$

Moreover, we observe

$$i + j + k + l + m + n \equiv i_3 + j_3 + k_3 + l_3 + m_3 + n_3 \mod 2.$$

By Lemma A.2, the above congruence equalities imply $\delta(x, c) \equiv \delta(x, c_3) + \lambda(x, c) \bmod 2$, as desired.

For any $c_3 \in \mathcal{A}_3$, there is a canonical subsurface $\mathcal{S}(c_3) \subset M$, in normal position with respect to $\mathscr{T}$, such that $c_3(e)$ indicates the number of intersection points of any edge $e \in E$ with $\mathcal{S}(c_3)$. The subsurface $\mathcal{S}(c_3)$ is formed by taking one normal disk in each tetrahedron that has nonzero color, and then taking their union matching the sides. The types of the normal disks (among four triangular types and three quadrilateral types for each tetrahedron) are forced by the admissible coloring c_3. The subsurface $\mathcal{S}(c_3)$ is closed, as M does not have boundary.

Lemma A.4 *Let $(M, \mathscr{T})$ be any triangulated closed 3-manifold. Let $r \geq 3$ be an odd integer and s be an integer coprime to r. Adopt Notation 3.1. Identify $\mathcal{A}_r = \mathcal{A}_3 \times \mathcal{A}'_r$. Then, for any $c = (c_3, c') \in \mathcal{A}_r$,*

$$\mathrm{ev}_{r,s}(|\mathscr{T}|_c) = (-1)^{\chi(\mathcal{S}(c_3))} \cdot \mathrm{ev}_{r,r-s}(|\mathscr{T}|_c),$$

where $\chi(\mathcal{S}(c_3))$ denotes the Euler characteristic of the normal subsurface $\mathcal{S}(c_3) \subset M$ determined by c_3.

Proof For all $x \in E \sqcup F \sqcup T$, the nonempty intersections $x \cap \mathcal{S}(c_3)$ give rise to a polygonal cell decomposition of $\mathcal{S}(c_3)$. Denote by $\nu_0, \nu_1, \nu_{2,\triangle}, \nu_{2,\square}$ the numbers of vertices, edges, triangular normal disks and quadrilateral normal disks in $\mathcal{S}(c_3)$, respectively.

Let $\delta \colon (E \sqcup F \sqcup T) \times \mathcal{A}_r \to \mathbb{Z}$ be expressed as in Lemma A.2. For any $x \in E \sqcup F \sqcup T$, it is direct to check that $\delta(x, c_3) = 1$ if and only if $x \cap \mathcal{S}(c_3)$ is a vertex or an edge or a quadrilateral normal disk of $\mathcal{S}(c_3)$, otherwise $\delta(x, c_3) = 0$, treating $\mathcal{A}_3$ as a subset of $\mathcal{A}_r$. This means $\nu_0 = \sum_{x \in E} \delta(x, c_3)$, $\nu_1 = \sum_{x \in F} \delta(x, c_3)$ and $\nu_{2,\square} = \sum_{x \in T} \delta(x, c_3)$. Moreover, the relation $3 \cdot \nu_{2,\triangle} + 4 \cdot \nu_{2,\square} = 2 \cdot \nu_1$ implies that $\nu_{2,\triangle}$ is even, as $\mathcal{S}(c_3)$ is closed. By Lemma A.3, we obtain

$$\begin{aligned}\chi(\mathcal{S}(c_3)) &= \nu_0 - \nu_1 + \nu_{2,\triangle} + \nu_{2,\square} \\ &\equiv \sum_{x \in E \sqcup F \sqcup T} \delta(x, c_3) \\ &\equiv \sum_{x \in E \sqcup F \sqcup T} \delta(x, c) + \sum_{x \in T} \lambda(x, c) \mod 2.\end{aligned}$$

Therefore, to derive the asserted formula $\mathrm{ev}_{r,s}(|\mathscr{T}|_c) = (-1)^{\chi(\mathcal{S}(c_3))} \cdot \mathrm{ev}_{r,r-s}(|\mathscr{T}|_c)$ from Lemma A.2, it remains to prove

$$\sum_{x \in T} \lambda(x, c) \equiv 0 \mod 2.$$

To this end, we observe that $\sum_{x \in T} \lambda(x, c)$ is the sum of $\frac{c'(e)}{2}$, where e ranges over the edges of x whose opposite edge in x meets $\mathcal{S}(c_3)$. For any edge $e^* \in E$, the link of e^* refers to the union of all the opposite edges $e_1, \cdots, e_h$ in all the tetrahedra $t_1, \cdots, t_h \in T$ that contain e^*, denoted as $\mathrm{lk}(e^*) \subset M$. The link $\mathrm{lk}(e^*)$ is a contractible loop in M (bounding a disk transverse to e^*). On the other hand, any edge either misses $\mathcal{S}(c_3)$, or meets $\mathcal{S}(c_3)$ at exactly one point. Then the number of edges in $\mathrm{lk}(e^*)$ that meet $\mathcal{S}(c_3)$ must be even and the integer $\frac{c'(e^*)}{2}$ contributes exactly this even number of times to $\sum_{x \in T} \lambda(x, c)$. Because $\sum_{x \in T} \lambda(x, c)$ is the total of the contribution from each edge $e^* \in E$, we conclude $\sum_{x \in T} \lambda(x, c) \equiv 0 \bmod 2$ as desired, completing the proof.

Lemma A.5 *Let $(M, \mathscr{T})$ be any triangulated closed 3-manifold. Let $r \geq 3$ be an odd integer and s be an integer coprime to r. Adopt Notation 3.1. Identify $\mathcal{A}_r = \mathcal{A}_3 \times \mathcal{A}'_r$. If s is odd, then*

$$\mathrm{ev}_{r,s}(|\mathscr{T}|_c) = \mathrm{ev}_{3,1}(|\mathscr{T}|_{c_3}) \cdot \mathrm{ev}_{r,r-s}(|\mathscr{T}|_{c'}).$$

Proof By Lemma A.4, we obtain $\mathrm{ev}_{r,s}(|\mathscr{T}|_c) = (-1)^{\chi(\mathcal{S}(c_3))}\mathrm{ev}_{r,r-s}(|\mathscr{T}|_c)$ and $\mathrm{ev}_{3,1}(|\mathscr{T}|_{c_3}) = (-1)^{\chi(\mathcal{S}(c_3))} \cdot \mathrm{ev}_{3,2}(|\mathscr{T}|_c)$. Then the asserted identity follows from the s even case (see Lemma A.1).

To complete the proof of Theorem A.1, we observe

$$\mathrm{ev}_{r,s}(Y_r) = \begin{cases} \mathrm{ev}_{3,2}(Y_3) \cdot \mathrm{ev}_{r,s}(Y'_r), & s \text{ even}, \\ \mathrm{ev}_{3,1}(Y_3) \cdot \mathrm{ev}_{r,r-s}(Y'_r), & s \text{ odd}, \end{cases} \tag{A.1}$$

where $Y_r = -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}{2r}$ and $Y'_r = -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}{r}$ (indeed, $\mathrm{ev}_{r,s}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = -\sqrt{-1} \cdot 2\sin\left(\frac{\pi s}{r}\right)$).

Let M be any closed 3-manifold. If r is odd and s is even, we obtain

$$\begin{aligned} \mathrm{TV}_{r,s}(M) &= \mathrm{ev}_{r,s}\Big(Y_r \cdot \sum_{c \in \mathcal{A}_r} |\mathscr{T}|_c\Big) \\ &= \mathrm{ev}_{r,s}(Y_r) \cdot \sum_{c \in \mathcal{A}_r} \mathrm{ev}_{r,s}(|\mathscr{T}|_c) \\ &= \mathrm{ev}_{3,1}(Y_3) \cdot \mathrm{ev}_{r,r-s}(Y'_r) \cdot \sum_{c_3 \in \mathcal{A}_3} \sum_{c' \in \mathcal{A}'_r} \mathrm{ev}_{3,1}(|\mathscr{T}|_c) \cdot \mathrm{ev}_{r,r-s}(|\mathscr{T}|_{c'}) \\ &= \mathrm{ev}_{3,1}\Big(Y_3 \cdot \sum_{c_3 \in \mathcal{A}_3} |\mathscr{T}|_c\Big) \cdot \mathrm{ev}_{r,r-s}\Big(Y'_r \cdot \sum_{c' \in \mathcal{A}'_r} |\mathscr{T}|_{c'}\Big) \\ &= \mathrm{TV}_{3,1}(M) \cdot \mathrm{TV}'_{r,r-s}(M) \end{aligned}$$

by (3.1)–(3.2), (A.1) and Lemma A.5. This completes the proof of Theorem A.1.

Acknowledgement The author would like to thank Yuxuan Yang for valuable comments on a preliminary version of this paper.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Andersen, J. E., The Witten-Reshetikhin-Turaev invariants of finite order mapping tori, I., *J. Reine Angew. Math.*, **681**, 2013, 1–38.

[2] Aschenbrenner, M., Friedl, S. and Wilton, H., 3-Manifold Groups, EMS Series of Lectures in Mathematics, Europen Mathematical Society (EMS), Zürich, 2015.

[3] Atiyah, M., Topological quantum field theories, *Publ. Math. IHES,* **68**, 1989, 175–186.

[4] Blanchet, C., Habegger, N., Masbaum, G. and Vogel, P., Three-manifold invariants derived from the Kauffman bracket, *Topol.*, **31**, 1992, 685–699.

[5] Blanchet, C., Habegger, N., Masbaum, G. and Vogel, P., Topological quantum field theories derived from the Kauffman bracket, *Topol.*, **34**, 1995, 883–927.

[6] Detcherry, R. and Kalfagianni, E., Quantum representations and monodromies of fibered links., *Adv. Math.*, **351**, 2019, 676–701.

[7] Detcherry, R., Kalfagianni, E. and Yang, T., Turaev-Viro invariants, colored Jones polynomials, and volume, *Quantum Topol.*, **9**, 2018, 775–813.

[8] Funar, L., Torus bundles not distinguished by TQFT invariants, *Geom. Topol.*, **17**, 2013, 2289–2344.

[9] Hansen, S. K., Reshetikhin-Turaev invariants of Seifert 3-manifolds and a rational surgery formula, *Algebr. Geom. Topol.*, **1**, 2001, 627–686.

[10] Hempel, J., 3-Manifolds, AMS Chelsea Publishing, Providence, RI, 2004.

[11] Hempel, J., Some 3-manifold groups with the same finite quotients, 2014, arxiv:1409.3509v2.

[12] Kirby, R. and Melvin, P., The 3-manifold invariants of Witten and Reshetikhin-Turaev for $\mathrm{sl}(2;\mathbb{C})$, *Invent. Math.*, **105**, 1991, 473–545.

[13] Lawrence, R. and Rozansky, L., Witten-Reshetikhin-Turaev invariants of Seifert manifolds, *Comm. Math. Phys.*, **205**, 1999, 287–314.

[14] Liu, Y. and Sun, H., Toward and after virtual specialization in 3-manifold topology, *Surveys in Differential Geometry*, 2020 Surveys in 3-manifold Topology and Geometry, **25**, Int. Press. Boston, MA, 2020, 215–252.

[15] Orlik, P., Seifert Manifolds, Springer-Verlag, Berlin-New York, 1972.

[16] Reid, A., Profinite rigidity, Proceedings of the International Congress of Mathematicians — Rio de Janeiro 2018. Vol. II. Invited Lectures, World Sci. Publ., Hackensack, NJ, 2018, 1193–1216.

[17] Reshetikhin, N. and Turaev, V. G., Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.*, **103**, 1991, 547–597.

[18] Roberts, J. Skein theory and Turaev-Viro invariants, *Topol.*, **34**, 1995, 771–787.

[19] Rozansky, L., Residue formulas for the large k asymptotics of Witten's invariants of Seifert manifolds. The case of SU(2), *Comm. Math. Phys.*, **178**, 1996, 27–60.

[20] Sokolov, M. V., The Turaev-Viro invariant for 3-manifolds is a sum of three invariants, *Canad. Math. Bull.*, **39**, 1996, 468–475.

[21] Sokolov, M. V., Which lens spaces are distinguished by Turaev-Viro invariants? *Mat. Zametki*, **61**, 1997, 468–470 (in Russian).

[22] Taniguchi, T., The Turaev-Viro invariants of all orientable closed Seifert fibered manifolds, *Tokyo J. Math.*, **30**, 2007, 497–522.

[23] Turaev, V. G., Quantum Invariants of Knots and 3-Manifolds, De Gruyter Studies in Mathematics, **18**, Walter de Gruyter & Co., Berlin, 1994.

[24] Turaev, V. G. and Viro, O. Ya., State sum invariants of 3-manifolds and quantum $6j$-symbols, *Topol.*, **31**, 1992, 865–902.

[25] Washington, L. C., Introduction to Cyclotomic Fields, Second edition. Graduate Texts in Mathematics, **83**. Springer-Verlag, New York, 1997.

[26] Witten, E., Quantum field theory and the Jones polynomial, *Comm. Math. Phys.*, **121**, 1989, 351–399.