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Critical Perturbation Results for a Mixed Boundary Value Problem

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Abstract Let K be a given positive function on a bounded domain Ω of $\mathbb{R}^n$, $n \geq 3$. The authors consider a nonlinear variational problem of the form: $-\Delta u = K|u|^{\frac{4}{n-2}}u$ in Ω with mixed Dirichlet-Neumann boundary conditions. It is a non-compact variational problem, in the sense that the associated energy functional J fails to satisfy the Palais-Smale condition. This generates concentration and blow-up phenomena. By studying the behaviors of non-precompact flow lines of a decreasing pseudogradient of J, they characterize the points where blow-up phenomena occur, the so-called critical points at infinity. Such a characterization combined with tools of Morse theory, algebraic topology and dynamical system, allow them to prove critical perturbation results under geometrical hypothesis on the boundary part in which the Neumann condition is prescribed.

Keywords Critical elliptic equations, Variational methods, Asymptotic analyzes, Critical points at infinity
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1 Introduction

Let Ω be a connected bounded domain of $\mathbb{R}^n$, $n \geq 3$, whose boundary $\partial\Omega$ is Lipschitz-continuous and splitted into two parts Γ_0 and Γ_1 having positive $(n-1)$-dimensional Hausdorff measure. Given a positive function $K : \overline{\Omega} \to \mathbb{R}$. We deal with the analysis of critical nonlinear problems of the form:

$$\begin{cases} -\Delta u = K|u|^{\frac{4}{n-2}}u & \text{in } (\Omega), \\ u = 0 & \text{on } (\Gamma_0), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (\Gamma_1), \end{cases} \tag{1.1}$$

where ν is the out-ward unit normal to the boundary $\partial\Omega$.

Such problems may appear in different branches of the applied sciences. For example in the theory of viscoelastic fluids, when modeling the slip of a fluid along solid walls. Namely, Dirichlet condition on Γ_0 means that no-slip condition of the viscoelastic medium is imposed on this part, while on Γ_1 the impermeability condition is used. See for example the Kelvin Voigt fluid model in [9] and [26]. It appears also when modeling problems of the boundary control

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of flows in a domain Ω with boundary $\partial\Omega$ divided on several parts which differ in physical properties (see [24–25]).

The natural space where we look for solutions of problem (1.1) is

$$V(\Omega) = \{u \in H^1(\Omega),\ \text{s.t. } u = 0 \text{ on } \Gamma_0\}.$$

It is straightforward to see that the positive solutions of problem (1.1) correspond to the positive critical points of the functional

$$J^K(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x}{\Big(\displaystyle\int_\Omega K|u|^{\frac{2n}{n-2}} \mathrm{d}x\Big)^{\frac{n-2}{n}}},$$

subjected to the constraint $u \in \Sigma$, where

$$\Sigma = \Big\{u \in V(\Omega), \|u\|^2 := \int_\Omega |\nabla u|^2 \mathrm{d}x = 1\Big\}.$$

The main difficulty arising in studying problem (1.1) and its related functional J^K comes from the presence of the critical exponent of the Sobolev embedding $V(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$. In virtue of the compactness defect of this embedding, J^K does not satisfy the Palais-Smale condition and therefore the classical variational methods do not work in the present setting. It is also interesting to note that the analysis of blow-up phenomena of problem (1.1) presents an additional difficulty compablack with the one of [7] concerning the case of homogenous Dirichlet boundary conditions. Indeed, in [7] (see also [21, 30]) blow-up phenomena occur only in the interior of the domain. However for the mixed problem (1.1), the picture is more complicated. Namely, blow-up phenomena occur in the interior of the domain as well as on the part of the boundary where the Neumann condition is prescribed (see [18, 22]). When $K = 1$ on $\overline{\Omega}$, an analysis of the minimising sequences of the functional J^1 has been established by Lions, Pacella and Tricarico [22]. As a consequence of it, it is that $J^1(u)$ may have a minimum on Σ (even if Ω is bounded) and therefore some existence results have been derived for a certain class of bounded domains (see [22, Corollaries 2.1–2.2]). However, it is proved that there are other conditions, obtained by a Pohozaev-type identity (see [22]), that guarantee the infimum of J^1 is not achieved. For more results on problem (1.1) for $K = 1$, we refer to [1, 10, 14–15, 17, 27]. For previous perturbation results on homogeneous boundary value problems, we refer the reader to [2–4, 11–12, 19–20, 23].

In this paper, we consider the case of non-constant function K. As far as we know, there is still no research on problem (1.1) dealing with this case. Our aim is to study the lack of compactness of the problem and derive some existence results. Our main tool of the proof is the critical points at infinity theory of Bahri [5]. For Bahri's theory and its applications we refer to [6–7, 13] and the references therein.

From now on we suppose that boundary of the domain Ω is smooth with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and the infimum of $J^1(u)$ is not achieved on Σ. As examples of such domains we may consider the domains bounded by two concentric spheres. If Γ_1 describes the interior sphere, it is proved in [28] by using certain "isoperimetric arguments" that the infimum of J^1 is not achieved whatever the radius of the two spheres. For other examples of domains we refer to [22].

Denoting H the mean curvature of Γ_1, $K_{/\Gamma_1}$ the restriction of K on Γ_1 and $\mathrm{Crit}(K_{/\Gamma_1})$ the set of critical points of $K_{/\Gamma_1}$. We shall prove the following existence results.

Theorem 1.1 *Let $n \geq 5$. Assume that $K_{/\Gamma_1}$ is a Morse function such that*

$$L(y) := \widehat{C}\frac{1}{K(y)}\frac{\partial K}{\partial \nu}(y) - \overline{C}H(y) \neq 0, \quad \forall y \in \mathrm{Crit}(K_{/\Gamma_1}),$$

where $\overline{C} = \pi^{\frac{n-1}{2}}(n-2)^2\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})\Gamma(n)}$, $\Gamma(s) = \int_0^\infty \frac{\mathrm{e}^{-t}}{t^{1-s}}\mathrm{d}t$, $s > 0$ *is the Gamma function and* $\widehat{C} = n(n-2)^2\int_{\mathbb{R}^n_+} z_n\frac{|z|^2-1}{(1+|z|^2)^{n+1}}\mathrm{d}z$. *If*

$$\sum_{y \in \mathrm{Crit}(K_{/\Gamma_1}), L(y)>0} (-1)^{n-1-\mathrm{ind}(K_{/\Gamma_1}, y)} \neq \chi(\Gamma_1),$$

then problem (1.1) *has a positive solution provided K is close to* (1). *Here* $\mathrm{ind}(K_{/\Gamma_1}, y)$ *denotes the Morse index of $K_{/\Gamma_1}$ at y and $\chi(\Gamma_1)$ denotes the Euler-Poincaré characteristic of* Γ_1.

Theorem 1.2 *Let $n \geq 5$. Assume that $K_{/\Gamma_1}$ admits an absolute maximum y_0 such that $L(y_0) < 0$. Then* (1.1) *has a positive solution provided K is close to* 1.

We organize the rest of this paper as follows. In Section 2 we prove some preliminary results related to the variational structure associated to problem (1.1). In Section 3 we provide asymptotic expansions of the associated energy functional leading to describe the behavior of the non-compact gradient flowlines at infinity and identify their possible ends, the so-called critical points at infinity. Many useful estimates in this Section have been extracted from the work of Rey [29]. In Section 4 we compute the topological contributions of the critical points at infinity near the infimum of the associated variational functional and use it to prove our existence results.

2 Variational Tools

Problem (1.1) has a variational structure. There is a one to one correspondence between the positive solutions of (1.1) and the positive critical points of the Euler-Lagrange functional

$$J(u) = J^K(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x}{\Big(\displaystyle\int_\Omega K u^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{n-2}{n}}},$$

subjected to the constraint $u \in \Sigma$, where

$$\Sigma = \Big\{u \in V(\Omega),\ \|u\|^2 := \int_\Omega |\nabla u|^2 \mathrm{d}x = 1\Big\}.$$

In virtue of the compactness defect of the embedding $V(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, the functional J fails to satisfy the Palais-Smale condition. Following the concentration and compactness principle of Grossi and Pacella [18] and Lions, Pacella and Tricarico [22], we describe in the next all the possible neighborhoods in Σ where the positive sequences failing Palai-Smale condition can stay there.

Let $a \in \Omega \cup \Gamma_1$ and $\lambda > 0$. We consider

$$\delta_{(a,\lambda)}(x) = \frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{n-2}{2}}}$$

and

$$\varphi_{(a,\lambda)}(x) = \psi_a(x)\delta_{(a,\lambda)}(x), \quad x \in \mathbb{R}^n,$$

where ψ_a is a C^∞ cut-off function defined by

$$\psi_a(x) = 1, \quad \text{if } x \in B\Big(a, \frac{\rho}{2}\Big)$$

and

$$\psi_a(x) = 0, \quad \text{if } x \in B(a,\rho)^c.$$

Here ρ is a positive constant sufficiently small in such a way that $\varphi_{(a,\lambda)} = 0$ on Γ_0.

Let h be a positive integer and $0 \le \ell \le h$. For $\varepsilon > 0$ small enough we define

$$\begin{aligned} W(h,\ell,\varepsilon) = \Big\{ u = \sum_{i=1}^{\ell} \alpha_i \varphi_{(a_i,\lambda_i)} + \sum_{i=\ell+1}^{h} \alpha_i \varphi_{(a_i,\lambda_i)} + v \in \Sigma, \ \text{s.t. } \alpha_1, \cdots, \alpha_h > 0, \\ \lambda_1, \cdots, \lambda_h > \varepsilon^{-1}, (a_1, \cdots, a_\ell) \in \Gamma_1^\ell, (a_{l+1}, \cdots, a_h) \in \Omega^{h-\ell} \text{ and } \|v\| < \varepsilon \\ \text{satisfying } (V_0) \text{ with } |\alpha_i^{\frac{4}{n-2}} K(a_i) J(u)^{\frac{n}{n-2}} - n(n-2)| < \varepsilon, \ \forall i = 1, \cdots, h, \\ \lambda_i \mathrm{d}(a_i, \partial\Omega) > \varepsilon^{-1}, \ \forall i = \ell+1, \cdots, h \text{ and } \varepsilon_{ij} < \varepsilon, \ \forall 1 \le i \ne j \le h \Big\}. \end{aligned}$$

Here $\varepsilon_{ij} = \big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\big)^{-\frac{n-2}{2}}$ and

$$(V_0) : \langle v, \phi \rangle = \int_\Omega \nabla\phi\nabla v \mathrm{d}x = 0, \quad \forall \phi \in \Big\{ \varphi_{(a_i,\lambda_i)}, \frac{\partial \varphi_{(a_i,\lambda_i)}}{\partial a_i}, \frac{\partial \varphi_{(a_i,\lambda_i)}}{\partial \lambda_i}, i = 1, \cdots, h \Big\}.$$

Let

$$\Sigma^+ = \{u \in \Sigma, u \ge 0\}.$$

Proposition 2.1 *Let $(u_k)_k$ be a sequence of Σ^+ such that $J(u_k) \to c$, $\partial J(u_k) \to 0$ and $(u_k)_k$ converges weakly to zero. Then there exist $h \ge 1$, $0 \le \ell \le h$ and subsequence of $(u_k)_k$ denoted again $(u_k)_k$ such that $u_k \in W(h, \ell, \varepsilon_k)$, $\forall k \gg 1$. Here $\varepsilon_k > 0, \varepsilon_k \to 0$ as $k \to \infty$. Moreover if $K = 1$ on $\overline{\Omega}$, $J^1(u) \to c_0\big((2h-\ell)\frac{S_n}{2}\big)^{\frac{2}{n}}$, where $c_0 = n^{\frac{n-2}{n}}(n-2)^{\frac{n+2}{n}}$ and $S_n = \int_{\mathbb{R}^n} \frac{|z|^2}{(1+|z|^2)^n} \mathrm{d}z$.*

Proof The proof follows from [7, Propositions 1–2, 18, Section 2].

To demonstrate our existence results we will focus our study on a specific open part of Σ that we will describe in the sequel. Our used topological argument will avoid all the rest of this part. Let b be a positive constant such that

$$c_0\Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + 3b < c_0 S_n^{\frac{2}{n}}. \tag{2.1}$$

We have the following result.

Proposition 2.2 *Let $n \geq 3$. There exists a fixed positive constant δ_b (which depends only on b) such that if $\|K-1\|_{L^\infty(\overline{\Omega})} < \delta_b$, then the following hold:*

(i) *For any $u \in W(h,\ell,\varepsilon)$, $0 \leq \ell \leq h$ with $(h,\ell) \neq (1,1)$, we have*

$$J(u) > c_0\Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + 3b.$$

(ii) *For any $u \in W(1,1,\varepsilon)$, we have*

$$J(u) < c_0\Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + b.$$

Proof We will denote $O(\varepsilon)$ any function on a and λ such that $|O(\varepsilon)| \leq c\,\varepsilon$. Let

$$u = \sum_{i=1}^{h} \alpha_i \varphi_{(a_i,\lambda_i)} + v \in W(h,\ell,\varepsilon).$$

Recall that

$$J(u) = \frac{\|u\|^2}{\Big(\int_\Omega K u^{\frac{2n}{n-2}} \mathrm{d}x\Big)^{\frac{n-2}{n}}} = \frac{N}{D}.$$

We have

$$N = \sum_{i=1}^{h} \alpha_i^2 \|\varphi_{(a_i,\lambda_i)}\|^2 + O(\varepsilon) = (n-2)^2\Big(\sum_{i=1}^{\ell} \alpha_i^2 \frac{S_n}{2} + \sum_{i=\ell+1}^{h} \alpha_i^2 S_n\Big) + O(\varepsilon).$$

Concerning the denominator,

$$\begin{aligned} D^{\frac{n}{n-2}} &= \int_\Omega K\Big(\sum_{i=1}^{h} \alpha_i \varphi_{(a_i,\lambda_i)}\Big)^{\frac{2n}{n-2}} \mathrm{d}x + O(\varepsilon) \\ &= \sum_{i=1}^{h} \alpha_i^{\frac{2n}{n-2}} \int_\Omega K \varphi_{(a_i,\lambda_i)}^{\frac{2n}{n-2}} \mathrm{d}x + O(\varepsilon). \end{aligned}$$

By an expansion of K around a_i, we have

$$D^{\frac{n}{n-2}} = \frac{n-2}{n}\Big(\sum_{i=1}^{\ell} \alpha_i^{\frac{2n}{n-2}} K(a_i)\frac{S_n}{2} + \sum_{i=\ell+1}^{h} \alpha_i^{\frac{2n}{n-2}} K(a_i) S_n\Big) + O(\varepsilon).$$

It follows that

$$D = \Big(\frac{n-2}{n}\Big)^{\frac{n-2}{n}} \Big(\sum_{i=1}^{\ell} \alpha_i^{\frac{2n}{n-2}} K(a_i)\frac{S_n}{2} + \sum_{i=\ell+1}^{h} \alpha_i^{\frac{2n}{n-2}} K(a_i) S_n\Big)^{\frac{n-2}{n}} + O(\varepsilon).$$

Therefore

$$J(u) = c_0 \frac{\displaystyle\sum_{i=1}^{\ell} \alpha_i^2 \frac{S_n}{2} + \sum_{i=\ell+1}^{h} \alpha_i^2 S_n}{\Big(\displaystyle\sum_{i=1}^{\ell} \alpha_i^{\frac{2n}{n-2}} K(a_i)\frac{S_n}{2} + \sum_{i=\ell+1}^{h} \alpha_i^{\frac{2n}{n-2}} K(a_i) S_n\Big)^{\frac{n-2}{n}}} + O(\varepsilon).$$

Using the fact that $\alpha_i^{\frac{4}{n-2}} K(a_i) = n(n-2)J(u)^{-\frac{n-2}{n}} + O(\varepsilon)$, $\forall i = 1, \cdots, h$, we get

$$J(u) = c_0 \Big(\sum_{i=1}^{\ell} \frac{S_n}{2K(a_i)^{\frac{n-2}{2}}} + \sum_{i=\ell+1}^{h} \frac{S_n}{K(a_i)^{\frac{n-2}{2}}} \Big)^{\frac{2}{n}} + O(\varepsilon).$$

Denote $M = \sup_{\overline{\Omega}} K$ and $m = \inf_{\overline{\Omega}} K$. For any $0 \le \ell \le h$ such that $(h, \ell) \ne (1, 1)$, we have

$$J(u) \ge c_0 \frac{1}{M^{\frac{n-2}{n}}} S_n^{\frac{2}{n}} + O(\varepsilon). \tag{2.2}$$

For $(h, \ell) = (1, 1)$, we have

$$J(u) \le c_0 \frac{1}{m^{\frac{n-2}{n}}} \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + O(\varepsilon). \tag{2.3}$$

Let $\theta > 0$ such that

$$c_0 S_n^{\frac{2}{n}} - \theta > c_0 \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + 3b \quad \text{and} \quad c_0 \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + \theta < c_0 \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + b.$$

From (2.2)–(2.3), there exists $\varepsilon_b > 0$ such that $\forall 0 < \varepsilon < \varepsilon_b$, we have

$$J(u) \ge c_0 \frac{1}{M^{\frac{n-2}{n}}} S_n^{\frac{2}{n}} - \theta, \quad \text{if } (h, \ell) \ne (1, 1)$$

and

$$J(u) \le c_0 \frac{1}{m^{\frac{n-2}{n}}} \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + \theta, \quad \text{if } (h, \ell) = (1, 1).$$

It follows that there exists $\delta_b > 0$ which depends only on b such that if $\|K - 1\|_{L^\infty(\overline{\Omega})} < \delta_b$, we have

$$J(u) > c_0 \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + 3b, \quad \text{if } (h, \ell) \ne (1, 1)$$

and

$$J(u) < c_0 \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} + b, \quad \text{if } (h, \ell) = (1, 1).$$

This completes the proof.

We now introduce the following notations. For $c > 0$ and $\gamma > 0$, we define

$$J_c = \{u \in \Sigma, \ J(u) \le c\}$$

and

$$V_\gamma(\Sigma^+) = \Big\{ u \in \Sigma, \ \|u^-\|_{L^{\frac{2n}{n-2}}}^{\frac{4}{n-2}} < \gamma \Big\},$$

where $u^- = \max(-u, 0)$. The following proposition extends the result of Proposition 2.1 to Palais-Smale sequences in

$$J_{c_0(\frac{S_n}{2})^{\frac{2}{n}} + 3b} \cap V_\gamma(\Sigma^+).$$

Proposition 2.3 *Let $b > 0$ satisfying (2.1) and $\widetilde{b} = c_0\big(\frac{S_n}{2}\big)^{\frac{2}{n}} + 3b$. There exists $\gamma = \gamma(b) > 0$ such that for any sequence $(u_k)_k \subset J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, $J(u_k) \to c$, $\partial J(u_k) \to 0$ and $(u_k)_k$ converges weakly to zero, there exists a subsequence of $(u_k)_k$ denoted again $(u_k)_k$ such that $(u_k) \in W(1,1,\varepsilon_k)$, $\forall k \gg 1$. Here $\varepsilon_k > 0, \varepsilon_k \to 0$ as $k \to \infty$.*

Proof For any $u \in \Sigma$ and $\psi \in V(\Omega)$, we have

$$\langle \partial J(u), \psi\rangle = 2J(u)\Big(\langle u, \psi\rangle - J(u)^{\frac{n}{n-2}} \int_\Omega K|u|^{\frac{4}{n-2}} u\psi \mathrm{d}x\Big).$$

Let $(u_k)_k$ be a sequence satisfying the assumptions of Proposition 2.3. For any $k \geq 1$, we have after denoting $u^- = \max(-u, 0)$, $u_k^+ = \max(u_k, 0)$ and $u_k^- = \max(-u_k, 0)$,

$$\begin{aligned}
\langle \partial J(u_k), -u_k^-\rangle &= 2J(u_k)\Big[-\langle u_k^+ - u_k^-, u_k^-\rangle + J(u_k)^{\frac{n}{n-2}} \int_\Omega K\|u_k\|^{\frac{4}{n-2}}(u_k^+ - u_k^-)(u_k^-)\mathrm{d}x\Big]\\
&= 2J(u_k)\Big[\|u_k^-\|^2 - J(u_k)^{\frac{n}{n-2}}\Big(\int_{x, u(x)\geq 0} K|u_k^+ - u_k^-|^{\frac{4}{n-2}}(u_k^-)^2\mathrm{d}x\\
&\quad + \int_{x, u(x)\leq 0} K|u_k^+ - u_k^-|^{\frac{4}{n-2}}(u_k^-)^2\mathrm{d}x\Big)\Big]\\
&= 2J(u_k)\Big[\|u_k^-\|^2 - J(u_k)^{\frac{n}{n-2}} \int_\Omega K(u_k^-)^{\frac{2n}{n-2}}\mathrm{d}x\Big].
\end{aligned}$$

Using the fact that $\alpha_0 \leq J(u_k) \leq \widetilde{b}$, $\forall k$, where $\alpha_0 = \inf_\Sigma J$, we get

$$\langle \partial J(u_k), -u_k^-\rangle \geq 2\alpha_0\|u_k^-\|^2 - 2\widetilde{b}^{\frac{2(n-1)}{n-2}} M\|u_k^-\|^{\frac{2n}{n-2}}_{L^{\frac{2n}{n-2}}},$$

and by Sobolev inequality, we obtain that

$$\langle \partial J(u_k), -u_k^-\rangle \geq 2\alpha_0\|u_k^-\|^2\Big(1 - \widetilde{b}^{\frac{2(n-1)}{n-2}} \frac{M\Sigma_n^2}{\alpha_0}\|u_k^-\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}}\Big),$$

where $\Sigma_n = \Sigma_n(\Omega)$ denotes the best constant of the embedding $V(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}$. Let

$$\gamma = \frac{1}{2}\frac{\alpha_0}{M\Sigma_n^2}\widetilde{b}^{-\frac{2(n-1)}{n-2}}.$$

Observe that if $\|u_k^-\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}} < \gamma$, for any k, then

$$\langle \partial J(u_k), -u_k^-\rangle \geq \alpha_0\|u_k^-\|^2 \text{ and thus } \|u_k^-\| \to 0,$$

since $\partial J(u_k) \to 0$ as $k \to \infty$. Using the fact that

$$|\|u_k\| - \|u_k^+\|| \leq \|u_k^-\| \quad \text{and} \quad \|u_k\| = 1, \quad \forall k \geq 1,$$

we get

$$\Big\|\frac{u_k^+}{\|u_k^+\|} - u_k\Big\| \to 0 \quad \text{as } k \to \infty,$$

and therefore

$$\Big|J\Big(\frac{u_k^+}{\|u_k^+\|}\Big) - J(u_k)\Big| \to 0 \quad \text{and} \quad \Big\|\partial J\Big(\frac{u_k^+}{\|u_k^+\|}\Big) - \partial J(u_k)\Big\| \to 0 \quad \text{as } k \to \infty,$$

since ∂J is a bounded and Lipschitz vector field in $J_{\widetilde{b}}$. It follows that $\big(\frac{u_k^+}{\|u_k^+\|}\big)_k$ satisfies the conditions of Proposition 2.1. Using the results of Propositions 2.1–2.2, there exits $\varepsilon'_k > 0, \varepsilon'_k \to 0$ such that $\frac{u_k^+}{\|u_k^+\|} \in W(1,1,\varepsilon'_k)$, $\forall k$. Consequently, we write

$$\frac{u_k^+}{\|u_k^+\|} = \alpha^k \varphi_{(a^k,\lambda^k)} + v^k \quad \text{with } \|v^k\| < \varepsilon'_k,\ \forall k.$$

Let $\varepsilon_k = \big\|u_k - \frac{u_k^+}{\|u_k^+\|}\big\| + \varepsilon'_k$. We then have

$$\|u_k - \alpha^k \varphi_{(a^k,\lambda^k)}\| < \varepsilon_k,\ \forall k.$$

Equivalently, see ([7, Appendix A]), $u_k \in W(1,1,\varepsilon_k), \forall k$.

We now prove the following result.

Proposition 2.4 *Let b, $\widetilde{b}$ and $\gamma = \gamma(b)$ be the constants subjected to Proposition 2.3. There exists a bounded pseudogradient Y_b in $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, such that for any flowline $s \mapsto \eta(s,u)$ of Y_b converging weakly to zero and with initial data $(0,u)$, $u \in J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, the following holds: For any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that for any $s > s_\varepsilon$, $\eta(s,u) \in W(1,1,\varepsilon)$.*

To prove Proposition 2.4, we first define on Σ a continuous and Lipschitz vector field X by

$$X(u) = \|u^-\|^2 u + u^-.$$

Lemma 2.1 *Let $u \in J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$. We then have*
(i) $\langle \partial J(u), X(u)\rangle \le 0$,
(ii) $\langle \partial J(u), X(u)\rangle \le -c_b$ *and* $\|\partial J(u)\| \ge \dfrac{c_b}{2}$, *if* $u \in (V_{\frac{\gamma}{2}}(\Sigma^+))^c$.
Here c_b is a positive constant independent of u.

Proof Let $u \in J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$. Using the fact that $\|X(u)\| \le 2$ and $\langle X(u), u\rangle = 0$, the computations of Proposition 2.3 yield

$$\begin{aligned}
\langle \partial J(u), X(u)\rangle &= 2J(u)\Big(J(u)^{\frac{n}{n-2}} \int_\Omega K(u^-)^{\frac{2n}{n-2}} \mathrm{d}x - \|u^-\|^2\Big)\\
&\le 2\Big(\widetilde{b}^{\frac{2(n-1)}{n-2}} M \|u^-\|^{\frac{2n}{n-2}}_{L^{\frac{2n}{n-2}}} - \alpha_0 \|u^-\|^2\Big)\\
&\le 2\alpha_0 \|u^-\|^2 \Big(\frac{\widetilde{b}^{\frac{2(n-1)}{n-2}} M \Sigma_n^2}{\alpha_0} \|u^-\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}} - 1\Big)\\
&\le -\alpha_0 \|u^-\|^2,
\end{aligned}$$

since $\gamma = \frac{1}{2}\frac{\alpha_0}{M\Sigma_n^2} \widetilde{b}^{-\frac{2(n-1)}{n-2}}$. Inequality (i) follows.

Now if $\|u^-\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}} \ge \frac{\gamma}{2}$, then by Sobolev inequality we have

$$\|u^-\|^2 \ge \Big(\frac{\gamma}{2}\Big)^{\frac{n-2}{2}} \Sigma_n^{-2},$$

and therefore

$$\langle \partial J(u), X(u)\rangle \le -\alpha_0 \Big(\frac{\gamma}{2}\Big)^{\frac{n-2}{2}} \Sigma_n^{-2} := -c_b.$$

In addition, we have

$$|\langle \partial J(u), X(u)\rangle| \leq 2\|\partial J(u)\|,$$

it follows that $\|\partial J(u)\| \geq \frac{c_b}{2}$. Inequality (ii) is valid.

Lemma 2.2 *Let $u \in J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$ and let $s \mapsto u(s)$ be a movement of X with $u(0) = u$. Then $u(s) \in J_b \cap V_\gamma(\Sigma^+)$, $\forall s \geq 0$.*

Proof For $s \geq 0$, we have $\dot{u}(s) = \|u^-(s)\|^2 u(s) + u^-(s)$. We claim that

$$u^-(s) = u^-(0)\mathrm{e}^{\int_0^s \|u^-(t)\|^2 \mathrm{d}t - s}. \tag{2.4}$$

Indeed, let $g(s) = u(s)\mathrm{e}^{-\int_0^s \|u^-(t)\|^2 \mathrm{d}t}$. We have

$$\dot{g}(s) = u^-(s)\mathrm{e}^{-\int_0^s \|u^-(t)\|^2 \mathrm{d}t} = g^-(s).$$

Therefore, $g(s) = g^+(0) - g^-(0)\mathrm{e}^{-s}$. Consequently,

$$g^-(s) = g^-(0)\mathrm{e}^{-s} = u^-(0)\mathrm{e}^{-s} = u^-(s)\mathrm{e}^{-\int_0^s \|u^-(t)\|^2 \mathrm{d}t}.$$

Hence claim (2.4) follows. Let $s < s'$. We have

$$\|u^-(s')\|_{L^{\frac{2n}{n-2}}} = \|u^-(0)\|_{L^{\frac{2n}{n-2}}} \mathrm{e}^{\int_0^{s'} \|u^-(t)\|^2 \mathrm{d}t - s'} \leq \|u^-(s)\|_{L^{\frac{2n}{n-2}}},$$

since $\displaystyle\int_s^{s'} \|u^-(t)\|^2 \mathrm{d}t \leq s' - s$. It follows that $s \mapsto \|u^-(s)\|_{L^{\frac{2n}{n-2}}}$ is a decreasing function and therefore the flowline $u(s)$ remains in $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, $\forall s \geq 0$.

Proof of Proposition 2.4 We first note that by the expansion of Proposition 2.2 and by Sobolev embedding there exists a positive constant $\varepsilon_b > 0$ such that for any $0 < \varepsilon < \varepsilon_b$, we have $W(1, 1, \varepsilon) \subset J_{\widetilde{b}} \cap V_{\frac{\gamma}{2}}(\Sigma^+)$. Define

$$Y_b(u) = -\chi(u)\partial J(u) + (1 - \chi(u))X(u), \quad u \in J_{\widetilde{b}} \cap V_\gamma(\Sigma^+),$$

where $\chi(u) = 1$ in $J_{\widetilde{b}} \cap V_{\frac{\gamma}{2}}(\Sigma^+)$ and $\chi(u) = 0$ in $J_{\widetilde{b}} \cap (V_{\frac{3}{4}\gamma}(\Sigma^+))^c$.

Let $u \in J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$ and let $s \mapsto \eta(s, u)$ be a flow line of Y_b converging weakly to zero. By Lemmas 2.1–2.2, $\eta(s, u)$ stays in $J_{\widetilde{b}} \cap V_{\frac{\gamma}{2}}(\Sigma^+)$ for any $s \geq 0$. Let $\varepsilon > 0$ ($\varepsilon < \varepsilon_b$). We claim that there exists $s_\varepsilon > 0$ such that $\eta(s, u) \in W(1, 1, \varepsilon)$, $\forall s \geq s_\varepsilon$.

Indeed, outside $W(1, 1, \varepsilon)$, $\eta(s, u)$ has to satisfy

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}J(\eta(s)) &= \langle \partial J(\eta(s)), Y_b(\eta(s))\rangle \\ &= -\chi(\eta(s))\|\partial J(\eta(s))\|^2 + (1 - \chi(\eta(s)))\langle \partial J(\eta(s)), X(\eta(s))\rangle.\end{aligned}$$

Observe that from Proposition 2.3, there exists $c_\varepsilon > 0$ such that $\|\partial J(u)\| \geq c_\varepsilon$, $\forall u \in W(1, 1, \varepsilon)^c$. Moreover by Lemma 2.1, we have $\langle \partial J(u), X(u)\rangle \leq -c_b$ and $\|\partial J(u)\| \geq c_b$, $\forall u \in (V_{\frac{\gamma}{2}}(\Sigma^+))^c$. Thus there exists a positive constant $c_{b\varepsilon}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s}J(\eta(s)) \leq -c_{b\varepsilon}.$$

Hence our claim follows from the fact J is lower bounded.

According to the result of Proposition 2.4 and in order to characterize the ends of non-precompact flow lines in $J_b \cap V_\gamma(\Sigma^+)$, the so-called critical points at infinity (see [5]), we focus in what follows on detailing the analysis of the functional J in $W(1,1,\varepsilon)$. Note that in $W(1,1,\varepsilon)$ the pseudogradient Y_b coincides with the genuine gradient vector field $(-\partial J)$.

3 Analysis at Infinity

In this section we provide asymptotic expansions of the functional J and its gradient in $W(1,1,\varepsilon)$ on function of α_1, λ_1, a_1 and v. This will clarify the behavior of the functional at infinity and hint to describe the concentration phenomenon of the problem in $W(1,1,\varepsilon)$.

Proposition 3.1 *Let $n \geq 5$. There exists $\varepsilon_1 > 0$ such that for any $u = \alpha_1 \varphi_{(a_1,\lambda_1)} + v \in W(1,1,\varepsilon)$, $0 < \varepsilon < \varepsilon_1$, we have*

$$J(u) = \frac{\widetilde{S}}{2^{\frac{2}{n}}} \frac{1}{K(a)^{\frac{n-2}{n}}} \Big[1 - \frac{2(n(n-2))^{\frac{n-2}{2}}}{\widetilde{S}^{\frac{n}{2}}} \Big(\overline{C} H(a) - \widehat{C} \frac{1}{K(a)} \frac{\partial K}{\partial \nu}(a)\Big) \frac{1}{\lambda}\Big]$$
$$+ f(v) + Q(v,v) + o\Big(\frac{1}{\lambda}\Big) + o(\|v\|^2),$$

where

$$\widetilde{S} = n(n-2)\Big(\int_{\mathbb{R}^n} \frac{\mathrm{d}z}{(1+|z|^2)^n}\Big)^{\frac{2}{n}}, \quad f(v) = -\frac{4}{\alpha} \frac{(n(n-2))^{\frac{n}{2}}}{K(a)\widetilde{S}^{\frac{n}{2}}} \int_\Omega K \varphi_{(a_1,\lambda_1)}^{\frac{n+2}{n-2}} v \mathrm{d}x,$$

$$Q(v,v) = \frac{2(n(n-2))^{\frac{n-2}{2}}}{\alpha^2 S^{\frac{n}{2}}} \Big(\|v\|^2 - \frac{n(n+2)}{K(a)} \int_\Omega K \varphi_{(a_1,\lambda_1)}^{\frac{4}{n-2}} v^2 \mathrm{d}x\Big),$$

$\overline{C}$ and $\widehat{C}$ are defined in Theorem 1.1. Here and in the sequel, $O(f(a,\lambda))$ denotes any function on a and λ, such that $|O(f(a,\lambda))|$ is bounded by $c|f(a,\lambda)|$, where c is a positive constant independent of a and λ, and $o(f(a,\lambda))$ denotes any function on a and λ such that $|o(f(a,\lambda))|$ is dominated by $|f(a,\lambda)|g(\varepsilon_1)$, where $g(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$.

Proof Let $\varepsilon > 0$ and $u = \alpha \varphi_{(a,\lambda)} + v \in W(1,1,\varepsilon)$,

$$J(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x}{\Big(\displaystyle\int_\Omega K u^{\frac{2n}{n-2}} \mathrm{d}x\Big)^{\frac{n-2}{n}}} = \frac{N}{D}.$$

Since v satisfies the orthogonality condition (V_0), we have

$$N = \alpha^2 \|\varphi_{(a,\lambda)}\|^2 + \|v\|^2.$$

Observe that

$$\|\varphi_{(a,\lambda)}\|^2 = \int_\Omega |\nabla \varphi_{(a,\lambda)}|^2 \mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega |\nabla\delta_{(a,\lambda)}|^2\psi_a^2 \mathrm{d}x + 2\int_\Omega \nabla\delta_{(a,\lambda)}\nabla\psi_a\delta_{(a,\lambda)}\psi_a \mathrm{d}x + \int_\Omega |\nabla\psi_a|^2\delta_{(a,\lambda)}^2 \mathrm{d}x \\
&= \mathrm{I}_1 + 2\mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}
$$

The first integral can be estimated as follows

$$
\mathrm{I}_1 = \int_\Omega |\nabla\delta_{(a,\lambda)}|^2 \mathrm{d}x + \int_\Omega |\nabla\delta_{(a,\lambda)}|^2(\psi_a^2 - 1)\mathrm{d}x.
$$

By elementary computation,

$$
\int_\Omega |\nabla\delta_{(a,\lambda)}|^2(1-\psi_a^2)\mathrm{d}x \le \int_{|x-a|>\frac{\varrho}{2}} |\nabla\delta_{(a,\lambda)}|^2 \mathrm{d}x \le O\Big(\frac{1}{\lambda^{n-2}}\Big),
$$

and by estimate [29, (D.6)],

$$
\int_\Omega |\nabla\delta_{(a,\lambda)}|^2 \mathrm{d}x = \frac{\widetilde{S}^{\frac{n}{2}}}{2(n(n-2))^{\frac{n-2}{2}}} - C'\frac{H(a)}{\lambda} + o\Big(\frac{1}{\lambda}\Big),
$$

since $a \in \partial\Omega$. Here, H is the mean curvature of $\partial\Omega$ and

$$
C' = \frac{(n-2)^2}{2}\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{n+3}{2})\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})\Gamma(n)}.
$$

Thus, we write

$$
\mathrm{I}_1 = \frac{\widetilde{S}^{\frac{n}{2}}}{2(n(n-2))^{\frac{n-2}{2}}} - C'\frac{H(a)}{\lambda} + o\Big(\frac{1}{\lambda}\Big).
$$

For I_2, we have

$$
\mathrm{I}_2 = -(n-2)\int_\Omega \nabla\psi_a(x)\frac{\lambda^n(x-a)}{(1+\lambda^2|x-a|^2)^{n-1}}\psi_{a_1}(x)\mathrm{d}x.
$$

Using the fact that $\nabla\psi_a = 0$ in $B(a,\frac{\varrho}{2})\cap\Omega$, we get

$$
|\mathrm{I}_2| \le c\int_{|x-a|\ge\frac{\varrho}{2}} \frac{\lambda^n}{(1+\lambda^2|x-a|^2)^{n-1}}\mathrm{d}x \le c\int_{|z|>\lambda\frac{\varrho}{2}} \frac{\mathrm{d}z}{(1+|z|^2)^{n-1}} = O\Big(\frac{1}{\lambda^{n-2}}\Big).
$$

In the same way,

$$
|\mathrm{I}_3| \le \frac{c}{\lambda^2}\int_{|z|>\lambda\frac{\varrho}{2}} \frac{\mathrm{d}z}{(1+|z|^2)^{n-2}} = O\Big(\frac{1}{\lambda^{n-2}}\Big).
$$

Therefore,

$$
N = \frac{\alpha^2\widetilde{S}^{\frac{n}{2}}}{2(n(n-2))^{\frac{n-2}{2}}}\Big[1 - C'\frac{2(n(n-2))^{\frac{n-2}{2}}}{\widetilde{S}^{\frac{n}{2}}}\frac{H(a)}{\lambda} + \frac{2(n(n-2))^{\frac{n-2}{2}}}{\alpha^2\widetilde{S}^{\frac{n}{2}}}\|v\|^2 + o\Big(\frac{1}{\lambda}\Big)\Big]. \tag{3.1}
$$

We now expand the denominator D,

$$
\begin{aligned}
D^{\frac{n}{n-2}} &= \int_\Omega K(\alpha\varphi_{(a,\lambda)} + v)^{\frac{2n}{n-2}}\mathrm{d}x \\
&= \int_\Omega K(\alpha\varphi_{(a,\lambda)})^{\frac{2n}{n-2}}\mathrm{d}x + \frac{2n}{n-2}\int_\Omega K(\alpha\varphi_{(a,\lambda)})^{\frac{n+2}{n-2}}v\mathrm{d}x
\end{aligned}
$$

$$+\frac{n(n+2)}{(n-2)^2}\int_\Omega K(\alpha\varphi_{(a,\lambda)})^{\frac{4}{n-2}}v^2\mathrm{d}x$$
$$+O\Big(\int_\Omega(\alpha\varphi_{(a,\lambda)})^{\frac{4}{n-2}-1}\inf(\alpha\varphi_{(a,\lambda)},|v|)^3\mathrm{d}x\Big)+O\Big(\int_\Omega|v|^{\frac{2n}{n-2}}\mathrm{d}x\Big). \tag{3.2}$$

Observe that

$$\int_\Omega K(\alpha\varphi_{(a,\lambda)})^{\frac{2n}{n-2}}\mathrm{d}x=\alpha^{\frac{2n}{n-2}}\int_{B(a,\rho)\cap\Omega}K(\delta_{(a,\lambda)}\psi_a)^{\frac{2n}{n-2}}\mathrm{d}x. \tag{3.3}$$

By expanding K around a, we get

$$\begin{aligned}\int_{B(a,\rho)\cap\Omega}K(\delta_{(a,\lambda)}\psi_a)^{\frac{2n}{n-2}}\mathrm{d}x&=K(a)\int_{B(a,\rho)\cap\Omega}(\delta_{(a,\lambda)}\psi_a)^{\frac{2n}{n-2}}\mathrm{d}x\\&\quad+\int_{B(a,\rho)\cap\Omega}\nabla K(a)(x-a)(\delta_{(a,\lambda)}\psi_a)^{\frac{2n}{n-2}}\mathrm{d}x\\&\quad+O\Big(\int_{\mathbb{R}^n}\frac{\lambda^n|x-a|^2}{(1+\lambda^2|x-a|^2)^n}\mathrm{d}x\Big)\\&=\mathrm{I}_1'+\mathrm{I}_2'+O\Big(\frac{1}{\lambda^2}\Big).\end{aligned} \tag{3.4}$$

We have

$$\mathrm{I}_1'=K(a)\Big(\int_\Omega\delta_{(a,\lambda)}^{\frac{2n}{n-2}}\mathrm{d}x-\int_\Omega\delta_{(a,\lambda)}^{\frac{2n}{n-2}}(1-\psi_a^{\frac{2n}{n-2}})\mathrm{d}x\Big).$$

Using the fact that

$$\int_\Omega\delta_{(a,\lambda)}^{\frac{2n}{n-2}}(1-\psi_a^{\frac{2n}{n-2}})\mathrm{d}x\le\int_{|x-a|\ge\frac{\rho}{2}}\delta_{(a,\lambda)}^{\frac{2n}{n-2}}\mathrm{d}x\le O\Big(\frac{1}{\lambda^n}\Big),$$

and by estimate [29, (D.17)],

$$\int_\Omega\delta_{(a,\lambda)}^{\frac{2n}{n-2}}\mathrm{d}x=\frac{\widetilde{S}^{\frac{n}{2}}}{2(n(n-2))^{\frac{n}{2}}}-C''\frac{H(a)}{\lambda}+o\Big(\frac{1}{\lambda}\Big),$$

where $C''=\frac{1}{2}\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-1}{2})\Gamma(n)}$, we get

$$\mathrm{I}_1'=K(a)\Big(\frac{\widetilde{S}^{\frac{n}{2}}}{2(n(n-2))^{\frac{n}{2}}}-C''\frac{H(a)}{\lambda}\Big)+o\Big(\frac{1}{\lambda}\Big). \tag{3.5}$$

To estimate the second integral in (3.4), we use orthonormal basis $(e_1,\cdots,e_n)$ of $\mathbb{R}^n$, such that $(e_1,\cdots,e_{n-1})$ is a basis of the tangent space of Γ_1 at a and $e_n=-\nu_a=-\nu$. We have

$$\begin{aligned}\mathrm{I}_2'&=\int_\Omega\nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{2n}{n-2}}\mathrm{d}x-\int_\Omega\nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{2n}{n-2}}(1-\psi_a^{\frac{2n}{n-2}})\mathrm{d}x\\&=\sum_{i=1}^n\frac{\partial K}{\partial e_i}(a)\int_\Omega(x-a)_i\frac{\lambda^n}{(1+\lambda^2|x-a|^2)^n}\mathrm{d}x+O\Big(\frac{1}{\lambda^n}\Big)\\&=\frac{1}{\lambda}\sum_{i=1}^n\frac{\partial K}{\partial e_i}(a)\int_{\lambda(\Omega-a)}\frac{z_i}{(1+|z|^2)^n}\mathrm{d}z+O\Big(\frac{1}{\lambda^n}\Big).\end{aligned}$$

Observe that when λ tends to ∞, $\lambda(\Omega - a)$ tends to the half space π^+ containing $0_{\mathbb{R}^n}$ in its boundary. Using a suitable coordinates system, we may assume that

$$\pi^+ = \Big\{ \sum_{i=1}^{n} z_i e_i, z_i \in \mathbb{R},\ \forall i = 1, \cdots, n-1 \text{ and } z_n > 0 \Big\}.$$

Thus, for any $i = 1, \cdots, n$, we write

$$\int_{\lambda(\Omega-a)} \frac{z_i}{(1+|z|^2)^n} \mathrm{d}z = \int_{\pi^+} \frac{z_i}{(1+|z|^2)^n} \mathrm{d}z + o(1),$$

where $o(1) \to \infty$ as $\lambda \to \infty$.

Let $i = 1, \cdots, n-1$. By oddness properties, we have

$$\int_{\pi^+} \frac{z_i}{(1+|z|^2)^n} \mathrm{d}z = 0$$

and for $i = n$,

$$\int_{\pi^+} \frac{z_n}{(1+|z|^2)^n} \mathrm{d}z =: C_1.$$

Thus,

$$\mathrm{I}_2' = -C_1 \frac{1}{\lambda} \frac{\partial K}{\partial \nu}(a) + o\Big(\frac{1}{\lambda}\Big). \tag{3.6}$$

It follows from (3.2)–(3.6) that

$$\begin{aligned} D^{\frac{n}{n-2}} = \alpha^{\frac{2n}{n-2}} \frac{K(a)\widetilde{S}^{\frac{n}{2}}}{2(n(n-2))^{\frac{n}{2}}} \Big[1 - \frac{2C''}{\widetilde{S}^{\frac{n}{2}}} (n(n-2))^{\frac{n}{2}} \frac{H(a)}{\lambda} - 2C_1 \frac{(n(n-2))^{\frac{n}{2}}}{K(a)\widetilde{S}^{\frac{n}{2}}} \frac{1}{\lambda} \frac{\partial K}{\partial \nu}(a) \\ + \frac{1}{\alpha} \frac{4n}{n-2} \frac{(n(n-2))^{\frac{n}{2}}}{K(a)\widetilde{S}^{\frac{n}{2}}} \int_\Omega K \varphi_{(a,\lambda)}^{\frac{n+2}{n-2}} v \mathrm{d}x + \frac{2}{\alpha^2} \frac{n(n+2)}{(n-2)^2} \frac{(n(n-2))^{\frac{n}{2}}}{K(a)\widetilde{S}^{\frac{n}{2}}} \int_\Omega K \varphi_{(a,\lambda)}^{\frac{4}{n-2}} v^2 \mathrm{d}x \\ + o\Big(\frac{1}{\lambda}\Big) + o(\|v\|^2) \Big]. \end{aligned} \tag{3.7}$$

Thus

$$\begin{aligned} D = \alpha^2 \frac{\widetilde{S}^{\frac{n-2}{2}}}{2^{\frac{n-2}{n}}} \frac{K(a)^{\frac{n-2}{n}}}{(n(n-2))^{\frac{n-2}{2}}} \Big[1 - \frac{n-2}{n} (n(n-2))^{\frac{n}{2}} \frac{2}{\widetilde{S}^{\frac{n}{2}}} \Big(C'' \frac{H(a)}{\lambda} + C_1 \frac{1}{K(a)\lambda} \frac{\partial K}{\partial \nu}(a) \Big) \\ + \frac{4}{\alpha} \frac{(n(n-2))^{\frac{n}{2}}}{K(a)\widetilde{S}^{\frac{n}{2}}} \int_\Omega K \varphi_{(a,\lambda)}^{\frac{n+2}{n-2}} v \mathrm{d}x + \frac{2}{\alpha^2} \frac{n+2}{n-2} \frac{(n(n-2))^{\frac{n}{2}}}{K(a)\widetilde{S}^{\frac{n}{2}}} \int_\Omega K \varphi_{(a,\lambda)}^{\frac{4}{n-2}} v^2 \mathrm{d}x \\ + o\Big(\frac{1}{\lambda}\Big) + o(\|v\|^2) \Big]. \end{aligned} \tag{3.8}$$

After recalling that $C' - (n-2)^2 C'' = \overline{C}$ (see [29, (4.14)]), the expansion of Proposition 3.1 follows from (3.1) and (3.8).

Next we focus on improving the above expansion by analysing the behavior of J with respect to the v-variable.

Let $u = \alpha \varphi_{(a,\lambda)} \in W(1, 1, \varepsilon)$ and

$$\mathcal{A}(\alpha, a, \lambda) = \{ v \in V(\Omega),\ v \ \text{ satisfies } (V_0) \}.$$

We consider the minimization problem

$$\min\{J(\alpha\varphi_{(a,\lambda)}+v),\ v\in\mathcal{A}(\alpha,a,\lambda) \text{ and } \|v\|<\varepsilon\}.$$

Proposition 3.2 *Let $n\geq 5$. There exists $\varepsilon_1>0$ such that for any $u=\alpha\varphi_{(a,\lambda)}\in W(1,1,\varepsilon)$, $0<\varepsilon<\varepsilon_1$, the above minimization problem has a unique solution $\overline{v}=\overline{v}(\alpha,a,\lambda)$ satisfying*

$$\|\overline{v}\|=O\Big(\frac{1}{\lambda}\Big).$$

In addition, there exists a change of variables $V=v-\overline{v}$ such that

$$J(u+v)=J(u+\overline{v})+\|V\|^2.$$

Proof Arguing as in [5, Proposition 5.4]. There exists $\varepsilon_1>0$ such that for any $u=\alpha\varphi_{(a,\lambda)}\in W(1,1,\varepsilon)$, $0<\varepsilon<\varepsilon_1$, there exists $\alpha_0>0$ such that for any $v\in\mathcal{A}(\alpha,a,\lambda)$, $\|v\|<\varepsilon$, the quadratic form $Q(v,v)$ defined in Proposition 3.1 satisfies

$$Q(v,v)\geq\alpha_0\|v\|^2.$$

Let

$$g(v)=J(u+v)\equiv f(v)+Q(v,v).$$

The coercivity of $Q(v,v)$ implies the existence of a unique critical point $\overline{v}$ in $A(\alpha,a,\lambda)$, $\|\overline{v}\|<\varepsilon$ minimizing $g(v)$. It follows that for any $h\in\mathcal{A}(\alpha,a,\lambda)$,

$$Q(\overline{v},h)=-\frac{1}{2}f(h)$$

and therefore

$$\|\overline{v}\|<\frac{1}{2\alpha_0}\|f\|_{\mathcal{L}(\mathcal{A}(\alpha,a,\lambda))}.$$

The estimate of $\|\overline{v}\|$ follows from the estimate of $\|f\|$ in the space of linear forms on $\mathcal{A}(\alpha,a,\lambda)$. For any $v\in\mathcal{A}(\alpha,a,\lambda)$ we have

$$\begin{aligned} f(v)&=\int_\Omega K(\alpha\varphi_{(a,\lambda)})^{\frac{n+2}{n-2}}v\mathrm{d}x=\int_{B(a,\frac{\varrho}{2})\cap\Omega}K(\alpha\delta_{(a,\lambda)})^{\frac{n+2}{n-2}}v\mathrm{d}x+O\Big(\frac{\|v\|}{\lambda^{\frac{n+2}{2}}}\Big)\\ &=\alpha^{\frac{n+2}{n-2}}K(a)\int_{B(a,\frac{\varrho}{2})\cap\Omega}\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}v\mathrm{d}x+O\Big(\frac{\|v\|}{\lambda}\Big). \end{aligned}\tag{3.9}$$

Observe that

$$\begin{aligned} \mathrm{I}:=\int_{B(a,\frac{\varrho}{2})\cap\Omega}\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}v\mathrm{d}x&=\int_\Omega\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}v\mathrm{d}x+O\Big(\frac{\|v\|}{\lambda^{\frac{n+2}{2}}}\Big)\\ &=\frac{1}{n(n-2)}\int_\Omega-\Delta\delta_{(a,\lambda)}v\mathrm{d}x+O\Big(\frac{\|v\|}{\lambda^{\frac{n+2}{2}}}\Big)\\ &=\frac{1}{n(n-2)}\Big(\int_\Omega\nabla\delta_{(a,\lambda)}\nabla v\mathrm{d}x-\int_{\Gamma_1}\frac{\partial\delta_{(a,\lambda)}}{\partial\nu}v\mathrm{d}\sigma\Big)+O\Big(\frac{\|v\|}{\lambda^{\frac{n+2}{2}}}\Big)\\ &=\frac{1}{n(n-2)}\Big(\int_{B(a,\frac{\varrho}{2})\cap\Omega}\nabla\varphi_{(a,\lambda)}\nabla v\mathrm{d}x+\int_{B(a,\frac{\varrho}{2})^c\cap\Omega}\nabla\delta_{(a,\lambda)}\nabla v\mathrm{d}x \end{aligned}$$

$$- \int_{\Gamma_1} \frac{\partial \delta_{(a,\lambda)}}{\partial \nu} v \mathrm{d}\sigma\Big) + O\Big(\frac{\|v\|}{\lambda^{\frac{n+2}{2}}}\Big).$$

Using the fact that v satisfies (V_0), we have

$$\int_{B(a,\frac{\rho}{2})\cap\Omega} \nabla\varphi_{(a,\lambda)}\nabla v \mathrm{d}x = -\int_{B(a,\frac{\rho}{2})^c\cap\Omega} \nabla\varphi_{(a,\lambda)}\nabla v \mathrm{d}x = O\Big(\frac{\|v\|}{\lambda^{\frac{n-2}{2}}}\Big).$$

Moreover, by Hölder's inequalities,

$$\Big|\int_{\Gamma_1} \frac{\partial \delta_{(a,\lambda)}}{\partial \nu} v \mathrm{d}\sigma\Big| \leq \Big(\int_{\Gamma_1} |v|^{\frac{2(n-1)}{n-2}} \mathrm{d}\sigma\Big)^{\frac{n-2}{2(n-1)}} \Big(\int_{\Gamma_1} \Big|\frac{\partial \delta_{(a,\lambda)}}{\partial \nu}\Big|^{\frac{2(n-1)}{n}} \mathrm{d}\sigma\Big)^{\frac{n}{2(n-1)}}.$$

Using the fact that the embedding $V(\Omega) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\Gamma_1)$ is continuous and

$$\Big(\int_{\Gamma_1} \Big|\frac{\partial \delta_{(a,\lambda)}}{\partial \nu}\Big|^{\frac{2(n-1)}{n}} \mathrm{d}\sigma\Big)^{\frac{n}{2(n-1)}} = O\Big(\frac{1}{\lambda}\Big)$$

(see [29, (D.49)]), we get

$$\mathrm{I} = O\Big(\frac{\|v\|}{\lambda}\Big). \tag{3.10}$$

The estimate of $\|\overline{v}\|$ follows from (3.9)–(3.10).

To prove the results of this paper, we need to establish deformation lemmas near the infimum of the functional J. These deformation lemmas will be realized using decreasing flow lines of a suitable pseudogradient W in $W(1,1,\varepsilon)$. The result of Proposition 3.2 shows that $W(1,1,\varepsilon)$ can be parameterized by the variables $(\alpha, a, \lambda, \overline{v}, V)$. On the V-space, we define a pseudogradient as Bahri did in [6] by setting $\dot{V} = -\mu V,\ \mu > 1$. This shows that $V(s) = \mathrm{e}^{-\mu s}V(0)$ will be very small at $s = 1$, taking μ large enough. It follows that in order to perform our deformations, we can work as if $V = 0$. The construction will extend with the same properties to a neighborhood of zero.

In order to define W on the $(\alpha, a, \lambda, \overline{v})$-variables, we introduce the following two propositions providing the asymptotic expansions of $\frac{\partial J}{\partial \lambda}$ and $\frac{\partial J}{\partial a}$.

Proposition 3.3 *Let $n \geq 5$. There exists $\varepsilon_1 > 0$ such that for any $u = \alpha\varphi_{(a,\lambda)} \in W(1,1,\varepsilon)$, $0 < \varepsilon < \varepsilon_1$, we have*

$$\Big\langle \partial J(u), \alpha\lambda \frac{\partial \varphi_{(a,\lambda)}}{\partial \lambda}\Big\rangle = J(u)\alpha^2\Big(\overline{C}H(a) - \widehat{C}\frac{1}{K(a)}\frac{\partial K}{\partial \nu}(a)\Big)\frac{1}{\lambda} + o\Big(\frac{1}{\lambda}\Big),$$

where $\overline{C}$ and $\widehat{C}$ are the constants subjected to Theorem 1.1.

Proof Let $\varepsilon > 0$ and $u = \alpha\varphi_{(a,\lambda)} \in W(1,1,\varepsilon)$. For any $h \in V(\Omega)$, we have

$$\langle \partial J(u), h\rangle = 2J(u)\Big[\langle u, h\rangle - J(u)^{\frac{n}{n-2}} \int_\Omega K u^{\frac{n+2}{n-2}} h \mathrm{d}x\Big].$$

Therefore,

$$\Big\langle \partial J(u), \alpha\lambda \frac{\partial \varphi_{(a,\lambda)}}{\partial \lambda}\Big\rangle = 2J(u)\Big[\alpha^2\Big\langle \varphi_{(a,\lambda)}, \lambda\frac{\partial \varphi_{(a,\lambda)}}{\partial \lambda}\Big\rangle$$

$$- J(u)^{\frac{n}{n-2}}\alpha^{\frac{2n}{n-2}}\int_\Omega K\varphi_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\varphi_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x\Big]. \tag{3.11}$$

Arguing as in the proof of Proposition 3.1,

$$\Big\langle\varphi_{(a,\lambda)},\lambda\frac{\partial\varphi_{(a,\lambda)}}{\partial\lambda}\Big\rangle = \int_\Omega \nabla\delta_{(a,\lambda)}\nabla\Big(\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big)\mathrm{d}x + O\Big(\frac{1}{\lambda^{n-2}}\Big). \tag{3.12}$$

Using estimate [29, (D.7)], we get

$$\Big\langle\varphi_{(a,\lambda)},\lambda\frac{\partial\varphi_{(a,\lambda)}}{\partial\lambda}\Big\rangle = \frac{C'}{2}\frac{H(a)}{\lambda} + o\Big(\frac{1}{\lambda}\Big), \tag{3.13}$$

where C' is defined in the proof of Proposition 3.1. Concerning the remainder integral of (3.11), we have

$$R_1 := \int_\Omega K\varphi_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\varphi_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x = \int_{B(a,\frac{\varrho}{2})\cap\Omega} K\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x + O\Big(\frac{1}{\lambda^n}\Big). \tag{3.14}$$

Expanding K around a, we get

$$\begin{aligned} R_1 = K(a)\int_\Omega \delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x + \int_\Omega \nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x \\ + O\Big(\int_{\mathbb{R}^n}|x-a|^2\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x\Big) + O\Big(\frac{1}{\lambda^n}\Big). \end{aligned}$$

Using estimate [29, (D.18)],

$$\int_\Omega \delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x = \frac{n-2}{2n}C''\frac{H(a)}{\lambda} + o\Big(\frac{1}{\lambda}\Big),$$

where C'' is defined in the proof of Proposition 3.1. Moreover,

$$\int_\Omega \nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x = \frac{n-2}{2}\sum_{i=1}^n\frac{\partial K}{\partial e_i}(a)\int_\Omega (x-a)_i\frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}}\mathrm{d}x,$$

where $(e_1,\cdots,e_{n-1})$ is an orthonormal basis of the tangent space of Γ_1 at a and $e_n = -\nu_a = -\nu$. By a change of variables, we find

$$\int_\Omega \nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x = \frac{n-2}{2}\frac{1}{\lambda}\sum_{i=1}^n\frac{\partial K}{\partial e_i}(a)\int_{\pi^+} z_i\frac{1-|z|^2}{(1+|z|^2)^{n+1}}\mathrm{d}z + o\Big(\frac{1}{\lambda}\Big),$$

since, as $\lambda\to\infty$, $\lambda(\Omega-a)\to\pi^+$, the half space defined in the proof of Proposition 3.1. Using oddness arguments, we get

$$\int_\Omega \nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\mathrm{d}x = \frac{n-2}{2}C_2\frac{1}{\lambda}\frac{\partial K}{\partial\nu}(a) + o\Big(\frac{1}{\lambda}\Big),$$

where $C_2 = \displaystyle\int_{\pi^+} z_n\frac{|z|^2-1}{(1+|z|^2)^{n+1}}\mathrm{d}z$. Therefore,

$$R_1 = \frac{n-2}{2n}C''K(a)\frac{H(a)}{\lambda} + \frac{n-2}{2}C_2\frac{1}{\lambda}\frac{\partial K}{\partial\nu}(a) + o\Big(\frac{1}{\lambda}\Big). \tag{3.15}$$

Using relation $|\alpha^{\frac{4}{n-2}}K(a)J(u)^{\frac{n}{n-2}}-n(n-2)|<\varepsilon$, we write for $0<\varepsilon<\varepsilon_1$, ε_1 is small enough,

$$\alpha^{\frac{4}{n-2}}K(a)J(u)^{\frac{n}{n-2}}=n(n-2)+o(1). \tag{3.16}$$

This with estimates (3.11), (3.13) and (3.15), yields after recalling that $C'-(n-2)^2C''=\overline{C}$ and $n(n-2)^2C_2=\widehat{C}$ the requiblack expansion.

In the next proposition, we denote by $(a)_k$, $k=1,\cdots,n-1$, the system of coordinates of $a\in\Gamma_1$ in the orthonormal basis $(e_1,\cdots,e_{n-1})$ of the tangent space of Γ_1 at a. We have

Proposition 3.4 *Let $n\geq 5$ and $u=\alpha\varphi_{(a,\lambda)}\in W(1,1,\varepsilon)$. For any $k=1,\cdots,n-1$, it holds*

$$\Big\langle \partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial(a)_k}\Big\rangle=-2(n-2)C_3\alpha^2J(u)\frac{1}{K(a)\lambda}\frac{\partial K}{\partial e_k}(a)+o\Big(\frac{1}{\lambda}\Big),$$

where $C_3=\frac{1}{n}\displaystyle\int_{\mathbb{R}^n_+}\frac{|z|^2}{(1+|z|^2)^{n+1}}\mathrm{d}z$.

Proof Let $u=\alpha\varphi_{(a,\lambda)}\in W(1,1,\varepsilon)$. We have

$$\begin{aligned}&\Big\langle \partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial(a)_k}\Big\rangle\\&=2J(u)\Big[\alpha^2\Big\langle\varphi_{(a,\lambda)},\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial(a)_k}\Big\rangle-\alpha^{\frac{2n}{n-2}}J(u)^{\frac{n}{n-2}}\int_\Omega K\varphi_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x\Big].\end{aligned} \tag{3.17}$$

Using estimate [29, (D.8)], We have

$$\Big\langle\varphi_{(a,\lambda)},\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial(a)_k}\Big\rangle=O\Big(\frac{1}{\lambda^2}\Big).$$

Moreover,

$$\begin{aligned}R_2&:=\int_\Omega K\varphi_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x=\int_{B(a,\frac{\rho}{2})\cap\Omega}K\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x+O\Big(\frac{1}{\lambda^n}\Big)\\&=K(a)\int_\Omega\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x+\int_\Omega\nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x+O\Big(\frac{1}{\lambda^2}\Big).\end{aligned} \tag{3.18}$$

From [29, (D.19)], we have

$$\int_\Omega\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x=O\Big(\frac{1}{\lambda^2}\Big).$$

In addition, by a preceding argument,

$$\int_\Omega\nabla K(a)(x-a)\delta_{(a,\lambda)}^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial(a)_k}\mathrm{d}x=(n-2)\frac{1}{\lambda}\sum_{i=1}^n\frac{\partial K}{\partial e_i}(a)\int_{\pi^+}\frac{z_iz_k}{(1+|z|^2)^{n+1}}\mathrm{d}z+o\Big(\frac{1}{\lambda}\Big).$$

By oddness, we have

$$\int_{\pi^+}\frac{z_iz_k}{(1+|z|^2)^{n+1}}\mathrm{d}z=0,\quad\forall i\neq k,$$

and hence

$$R_2 = (n-2)\frac{1}{\lambda}\frac{\partial K}{\partial e_k}(a)\int_{\pi^+}\frac{|z_k^2}{(1+|z|^2)^{n+1}}\mathrm{d}z + o\Big(\frac{1}{\lambda}\Big). \tag{3.19}$$

Combining estimates (3.17)–(3.19) with relation $\alpha^{\frac{4}{n-2}}J(u)^{\frac{n}{n-2}}K(a) = n(n-2)+o(1)$, we get the desiblack expansion.

We now describe the concentration phenomenon of problem (1.1) in $W(1,1,\varepsilon)$.

Let $\mathrm{Crit}(K_{/\Gamma_1})$ be the set of critical points of $K_{/\Gamma_1}$. For any $y \in \mathrm{Crit}(K_{/\Gamma_1})$ we denote $L(y) = \widehat{C}\frac{1}{K(y)}\frac{\partial K}{\partial \nu}(y) - \overline{C}H(y)$. We set

$$C^\infty = \{y \in \mathrm{Crit}(K_{/\Gamma_1}),\ L(y) > 0\}. \tag{3.20}$$

Proposition 3.5 *Let $n \geq 5$. Assume that $L(y) \neq 0$ for every $y \in \mathrm{Crit}(K_{/\Gamma_1})$. There exists a bounded pseudogradient W in $W(1,1,\varepsilon)$ satisfying*

(i) $\langle \partial J(u), W(u)\rangle \leq -\frac{c}{\lambda}$,

(ii) $\langle \partial J(u+\overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha,a,\lambda)}(W(u))\rangle \leq -\frac{c}{\lambda}$ *for any $u = \alpha\varphi_{(a,\lambda)} \in W(1,1,\varepsilon)$. Here the positive constant c is independent of u.*

(iii) *The only case when the parameter $\lambda(s)$ of a flow line $u(s)$ of W increases and tends to ∞ is when the concentration point $a(s)$ is close to $y \in C^\infty$.*

The following result is an immediate corollary of the above proposition.

Corollary 3.1 *Given a function $K\colon \overline{\Omega} \to \mathbb{R}$ satisfying the conditions of Proposition 3.5. Then the critical points at infinity of J in $W(1,1,\varepsilon)$ are*

$$\frac{\sqrt{2}(n(n-2))^{\frac{n-2}{4}}}{\widetilde{S}^{\frac{n}{4}}}\varphi_{(y,\infty)}, \quad y \in C^\infty,$$

where C^∞ is defined in (3.20).

We decompose the proof of Proposition 3.5 into two-lemmas. Each lemma describes the concentration phenomenon in a specific region of $W(1,1,\varepsilon)$. Let r be a positive constant such that

$$r < \frac{1}{4}\inf\{\mathrm{d}(y_i,y_j),\ y_i \neq y_j \in \mathrm{Crit}(K_{/\Gamma_1})\}$$

and satisfying

$$L(y)L(x) > 0, \quad \forall x \in B_{\Gamma_1}(y,r) \text{ and } \forall y \in \mathrm{Crit}(K_{/\Gamma_1}).$$

Here, $B_{\Gamma_1}(y,r)$ denotes the ball in Γ_1 of center y and radius r. Define

$$V^1_{\Gamma_1}(1,\varepsilon) = \Big\{u = \alpha\varphi_{(a,\lambda)} + \overline{v} \in W(1,1,\varepsilon),\ \text{s.t. } a \notin \bigcup_{y\in\mathrm{Crit}(K_{/\Gamma_1})} B_{\Gamma_1}\Big(y,\frac{r}{2}\Big)\Big\},$$

$$V^2_{\Gamma_1}(1,\varepsilon) = \{u = \alpha\varphi_{(a,\lambda)} + \overline{v} \in W(1,1,\varepsilon),\ \text{s.t. } a \in B_{\Gamma_1}(y,r), y \in \mathrm{Crit}(K_{/\Gamma_1})\}.$$

Lemma 3.1 *There exists a bounded pseudogradient W_1 in $V_{\Gamma_1}(1,\varepsilon)$ satisfying inequalities* (i) *and* (ii) *of Proposition* 3.5 *such that for any flowline $u(s) = \alpha(s)\varphi_{(a(s),\lambda(s))}$ of W_1, $\lambda(s) = \lambda(0)$ for any $s \geq 0$ as long as $u(s)$ stays in $V^1_{\Gamma_1}(1,\varepsilon)$.*

Proof We will denote by c any positive constant independent of u. Let $u = \alpha\varphi_{(a,\lambda)} \in V^1_{\Gamma_1}(1,\varepsilon)$. We move the concentration point a according to the differential equation

$$\dot{a} = \frac{1}{\lambda}\frac{\nabla_T K(a)}{|\nabla_T K(a)|},$$

where $\nabla_T K(a) = \sum_{i=1}^{n-1} \frac{\partial K}{\partial e_i}(a) e_i$ and $(e_1, \cdots, e_{n-1})$ is the orthonormal basis of the tangent space of Γ_1 at a. The corresponding vector field is

$$V_1(u) = \alpha\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial a}\frac{\nabla_T K(a)}{|\nabla_T K(a)|}.$$

Using the expansion of Proposition 3.4, we have

$$\langle \partial J(u), V_1(u)\rangle = -2J(u)(n-2)\frac{\alpha^2 C_3}{K(a)\lambda}|\nabla_T K(a)| + o\Big(\frac{1}{\lambda}\Big).$$

Using the fact that $|\nabla_T K|$ is lower bounded out side $\bigcup_{y\in\mathrm{Crit}(K_{/\Gamma_1})} B_{\Gamma_1}\big(1, \frac{r}{2}\big)$, we get

$$\langle \partial J(u), V_1(u)\rangle \le -\frac{c}{\lambda}. \tag{3.21}$$

Thus assertion (i) of Proposition 3.5 follows. Observe that in Proposition 3.2, $\|\overline{v}\|^2$ is small with respect to $\frac{1}{\lambda}$. Therefore from inequality (3.21) we obtain that

$$\Big\langle \partial J(u) + \overline{v}, V_1(u) + \frac{\partial\overline{v}}{\partial(\alpha, a, \lambda)}(V_1(u))\Big\rangle \le -\frac{c}{\lambda}. \tag{3.22}$$

Let

$$W_1(u) = V_1(u) - \langle u, V_1(u)\rangle u.$$

From (3.21)–(3.22), W_1 satisfies the requirements of Lemma 3.1.

Lemma 3.2 *There exists a bounded pseudogradient W_2 in $V^2_{\Gamma_1}(1,\varepsilon)$ satisfying inequalities* (i) *and* (ii) *of Proposition* 3.5*. Moreover the only case where $\lambda(s)$ increases and tends to ∞ under the action of W_2, when $a(s)$ is close to $y \in \mathrm{Crit}(K_{/\Gamma_1})$ such that $L(y) > 0$.*

Proof Let $u = \alpha\varphi_{(a,\lambda)} \in V^2_{\Gamma_1}(1,\varepsilon)$. If $L(y) < 0$, We move the parameters λ and a according to

$$\dot{\lambda} = -\lambda \quad \text{and} \quad \dot{a} = \frac{1}{\lambda}\nabla_T K(a).$$

The associated vector field is

$$V_2(u) = -\alpha\lambda\frac{\partial\varphi_{(a,\lambda)}}{\partial\lambda} + \alpha\frac{1}{\lambda}\frac{\partial\varphi_{(a,\lambda)}}{\partial a}\nabla_T K(a).$$

Using the expansions of Propositions 3.3–3.4, we have

$$\langle \partial J(u), V_2(u)\rangle = \alpha^2 J(u)\Big[\frac{L(y)}{\lambda} - 2C_3\frac{n-2}{K(a)} + \frac{|\nabla_T K(a)|^2}{\lambda}\Big] + o\Big(\frac{1}{\lambda}\Big).$$

Using the fact that $L(y) < 0$, we get

$$\langle \partial J(u), V_2(u)\rangle \leq -\frac{c}{\lambda}.$$

Now if $L(y) > 0$, we define

$$\dot{\lambda} = \lambda \quad \text{and} \quad \dot{a} = \frac{1}{\lambda}\nabla_T K(a).$$

The corresponding vector field is

$$V_2'(u) = \alpha\lambda\frac{\partial \varphi_{(a,\lambda)}}{\partial \lambda} + \alpha\frac{1}{\lambda}\frac{\partial \varphi_{(a,\lambda)}}{\partial a}\nabla_T K(a).$$

It satisfies by the expansions of Propositions 3.3–3.4,

$$\langle \partial J(u), V_2'(u)\rangle = -\alpha^2 J(u)\Big[\frac{L(y)}{\lambda} + 2C_3\frac{n-2}{K(a)} + \frac{|\nabla_T K(a)|^2}{\lambda}\Big] + o\Big(\frac{1}{\lambda}\Big).$$

Let $\widetilde{V}_2$ be a convex combination of V_2 and V_2'. We have

$$\langle \partial J(u), \widetilde{V}_2(u)\rangle \leq -\frac{c}{\lambda} \tag{3.23}$$

and

$$\Big\langle \partial J(u+\overline{v}), \widetilde{V}_2(u) + \frac{\partial \overline{v}}{\partial(\alpha, a, \lambda)}(\widetilde{V}_2(u))\Big\rangle \leq -\frac{c}{\lambda}. \tag{3.24}$$

The requiblack pseudogradient of Lemma 3.2 is

$$W_2(u) = \widetilde{V}_2(u) - \langle u, \widetilde{V}_2(u)\rangle u.$$

Proof of Proposition 3.5 Let W be a convex combination of W_1 and W_2 defined in Lemmas 3.1 and 3.2, respectively. Using (3.21)–(3.24), W satisfies the desiblack assertions.

4 Proof of Existence Results

We start this section with a few lemmas. Let

$$J^1(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x}{\Big(\displaystyle\int_\Omega u^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{n-2}{n}}}, \quad u \in \Sigma$$

be the variational functional associated to problem (1.1) when the function $K = 1$ on $\overline{\Omega}$. Under the assumption that the infimum of J^1 is not achieved, we prove the following results:

Lemma 4.1 *Let c_0 and S_n be the constants subjected to Proposition* 2.1. *There exists $\eta_1 > 0$ such that J^1 has no critical point in $J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_1}$.*

Proof We first note that by [18, Lemma 3.5] (see [22]), we have

$$\inf_{u\in\Sigma} J^1(u) = c_0\Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} = \frac{\widetilde{S}}{2^{\frac{2}{n}}},$$

where $\widetilde{S}$ is defined in Proposition 3.1. Arguing by a contradiction and suppose that for any $k \geq 1$ there exists $u_k \in J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\frac{1}{k}}$ such that $\partial J^1(u_k) = 0$. Denote w as a weak limit of the minimizing sequence $(u_k)_k$. Since the infimum of J^1 is not achieved, it follows by a blow-up argument of Lions, Pacella and Tricarico [22, Theorem 2.2] that $w = 0$ and $u_k \in W(1,1,\varepsilon)$, $\forall k \gg 1$. Performing an expansion like the one of Proposition 3.3, taking $K = 1$ on $\overline{\Omega}$ we find that,

$$\left|\left\langle \partial J(u), \alpha\lambda \frac{\partial \varphi_{(a,\lambda)}}{\partial \lambda}\right\rangle\right| = 2J(u)\frac{c_4}{\lambda}(1 + o(1)) \neq 0, \quad \forall\, u = \alpha\varphi_{(a,\lambda)} + \overline{v} \in W(1,1,\varepsilon).$$

It follows that $W(1,1,\varepsilon)$ does not contain any critical point of J^1. The obtained contradiction yields the proof.

Lemma 4.2 *Let η_1 be the constant subjected to Lemma* 4.1*. We then have*

$$J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_1} \text{ is homotopical equivalent to } \Gamma_1.$$

Proof Let $\varepsilon > 0$ small enough and fix $\lambda_0 > \frac{1}{\varepsilon}$. Define

$$\begin{aligned} \phi_{\lambda_0} : \Gamma_1 &\to W(1,1,\varepsilon) \\ a &\mapsto \frac{\varphi_{(a,\lambda)}}{\|\varphi_{(a,\lambda)}\|}. \end{aligned}$$

By the expansion of Proposition 3.1 (when $K = 1$), the mapping ϕ_{λ_0} is valued under the level $c_0\left(\frac{S_n}{2}\right)^{\frac{2}{n}} + \eta_1$, provided ε is small. Denote again

$$\Phi_{\lambda_0} : \Gamma_1 \to J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_1}.$$

We shall prove that Φ_{λ_0} is a homotopy equivalence.

Claim 1 There exists $\eta_2 > 0$ ($\eta_2 < \eta_1$) such that

$$J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_2} \subset W(1,1,\varepsilon).$$

Indeed, if not, there exists a sequence $(u_k)_k$ in $J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\frac{1}{k}}$ such that $u_k \notin W(1,1,\varepsilon)$. Using [22, Theorem 2.2], the minimizing sequence $(u_k)_k$ is relatively compact. This contradicts the fact that the infimum of J^1 is not achieved and confirms Claim 1. Let

$$i\colon J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_2} \hookrightarrow W(1,1,\varepsilon)$$

be the natural injection. Observe that by Lemma 4.1, J^1 has no critical points under the level $c_0\left(\frac{S_n}{2}\right)^{\frac{2}{n}} + \eta_1$. Therefore by the expansion of Proposition 3.1 (taking $K = 1$), the only critical value at infinity of J^1 under the level $c_0\left(\frac{S_n}{2}\right)^{\frac{2}{n}} + \eta_1$ is $c_0\left(\frac{S_n}{2}\right)^{\frac{2}{n}}$. It follows that J^1 has no critical points nor critical points at infinity between the levels $c_0\left(\frac{S_n}{2}\right)^{\frac{2}{n}} + \eta_2$ and $c_0\left(\frac{S_n}{2}\right)^{\frac{2}{n}} + \eta_1$. Thus the existence of a strong retract by deformation

$$r\colon J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_1} \to J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_2}.$$

Let

$$\begin{aligned}&P\colon W(1,1,\varepsilon)\to\Gamma_1,\\&u=\alpha\varphi_{(a,\lambda)}+\overline{v}\mapsto P(u)=a\end{aligned}$$

be the natural projection. Denote

$$g=P\circ i\circ r.$$

It satisfies

$$\phi_{\lambda_0}\circ g\sim \mathrm{id}_{J^1_{c_0(\frac{S_n}{2})^{\frac{2}{n}}+\eta_1}}\qquad\text{and}\qquad g\circ\phi_{\lambda_0}\sim \mathrm{id}_{\Gamma_1}.$$

The mapping Φ_{λ_0} is then an equivalence of homotopy and the result of Lemma 4.2 follows.

Let

$$J(u)=J^K(u)=\frac{\displaystyle\int_\Omega|\nabla u|^2\mathrm{d}x}{\Big(\displaystyle\int_\Omega Ku^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{n-2}{n}}},\quad u\in\Sigma.$$

We then have the following lemma.

Lemma 4.3 *Let η be a positive constant and $0<\delta<1$. If $\|K-1\|_{L^\infty(\Omega)}\le\delta$, then*

$$J(u)=J^1(u)+O(\delta),\quad \forall u\in J_\eta.$$

Here $O(\delta)$ does not depend on u, it depends only on δ and the given constant η.

Proof For any $u\in\Sigma$, we have

$$\begin{aligned}J^1(u)&=\frac{1}{\Big(\displaystyle\int_\Omega u^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{n-2}{n}}}\\&=\frac{1}{\Big(\displaystyle\int_\Omega Ku^{\frac{2n}{n-2}}\mathrm{d}x+\int_\Omega(1-K)u^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{n-2}{n}}}\\&=J(u)\frac{1}{\Big(1+\Big(\displaystyle\int_\Omega Ku^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{-1}\int_\Omega(1-K)u^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{\frac{n-2}{n}}}.\end{aligned}$$

Observe that by Sobolev inequality, we have

$$\Big|\int_\Omega(1-K)u^{\frac{2n}{n-2}}\mathrm{d}x\Big|\le\sum_n^{\frac{2n}{n-2}}\|K-1\|_{L^\infty(\overline{\Omega})},\quad\forall u\in\Sigma.$$

Moreover

$$\Big(\int_\Omega Ku^{\frac{2n}{n-2}}\mathrm{d}x\Big)^{-1}=J(u)^{\frac{n}{n-2}}\le\eta^{\frac{n}{n-2}},\quad\forall u\in J_\eta.$$

Consequently

$$J^1(u) = J(u)(1 + O(\|K-1\|_{L^\infty(\overline{\Omega})})).$$

This completes the proof.

We now state the proof our existence results.

Proof of Theorem 1.1 Let η_1 be the constant provided by Lemma 4.1 and $b(0 < b \leq \eta_1)$, $\widetilde{b}$ and $\gamma = \gamma(b)$ be the constants subjected to Proposition 2.3. Following Lemma 4.3, there exists a positive constant δ_b such that if $\|K-1\|_{L^\infty_{(\overline{\Omega})}} \leq \delta_b$, then

$$J_{\widetilde{b}-2b} \subset J^1_{\widetilde{b}-b} \subset J_{\widetilde{b}}. \tag{4.1}$$

Moreover by Proposition 2.2, we have

$$J(u) > \widetilde{b}, \quad \forall u \in W(h, \ell, \varepsilon) \text{ with } (h, \ell) \neq (1, 1) \tag{4.2}$$

and

$$J(u) < \widetilde{b} - 2b, \quad \forall u \in W(1, 1, \varepsilon). \tag{4.3}$$

Let $\widetilde{W}$ be a global vector filed on $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$ constructed by a convex combination of Y_b in $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+) \backslash W\big(1, 1, \frac{\varepsilon}{2}\big)$ and W in $W(1, 1, \varepsilon)$, where Y_b and W are the pseudogradients defined in Proposition 2.4 and Proposition 3.5, respectively. We use the flow of $\widetilde{W}$ to deform $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$. If we assume that J has no critical points in $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, it follows from (4.2)–(4.3) and the result of Corollary 3.1 that

$$J_{\widetilde{b}} \cap V_\gamma(\Sigma^+) \text{ retracts, by deformation on } \bigcup_{y \in C^\infty} W_u(y)_\infty, \tag{4.4}$$

where $W_u(y)_\infty$ denotes the unstable manifold of the critical point at infinity $(y)_\infty$. Around each critical point at infinity $(y)_\infty$, J can be expanded as follows

$$J(\alpha\varphi_{a,\lambda} + \overline{v}) = c_0 \Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} \frac{1}{K(a)^{\frac{n-2}{n}}} \Big(1 + \frac{L(y)}{\lambda}\Big) \tag{4.5}$$

after a change of variables. It follows that the Morse index $i(y)_\infty$ of J at $(y)_\infty$ equals to the Morse index of $(K_{/\Gamma_1})^{-1}$ around its nondegenerate critical point y. See for example [8, Lemma 10] (see also estimates (258) and (259) of the same paper). Namely,

$$i(y)_\infty = n - 1 - \text{ind}(K_{/\Gamma_1}, y).$$

Let $\chi(M)$ be the Euler-Poincaré characteristic of a topological space M. Using the fact that dimension of $W_u(y)_\infty$ equals to $\mathrm{i}(y)_\infty$, we get from (4.4) that

$$\chi(J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)) = \sum_{y \in C^\infty} (-1)^{i(y)_\infty}. \tag{4.6}$$

We now claim the following result.

Claim 2

$$\chi(J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)) = \chi(\Gamma_1).$$

Indeed, since we have assumed that J has nocritical point in $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, it follows from (4.2)–(4.3) that

$$J_{\widetilde{b}} \cap V_\gamma(\Sigma^+) \text{ retracts, by deformation on } J_{\widetilde{b}-2b} \cap V_\gamma(\Sigma^+). \tag{4.7}$$

Assertions (4.1) and (4.7) yield

$$J^1_{\widetilde{b}-b} \cap V_\gamma(\Sigma^+) \text{ retracts, by deformation on } J_{\widetilde{b}-2b} \cap V_\gamma(\Sigma^+).$$

Thus by Lemma 4.2 we get

$$\chi(J_{\widetilde{b}-2b} \cap V_\gamma(\Sigma^+)) = \chi(J^1_{\widetilde{b}-b} \cap V_\gamma(\Sigma^+)) = \chi(\Gamma_1). \tag{4.8}$$

Claim 2 follows from (4.7)–(4.8) and yields with (4.6) to a contradiction with the assumption of Theorem 1.1. It results from such a contradiction that J has at least a critical point in $J_{\widetilde{b}} \cap V_\gamma(\Sigma^+)$, and hence the existence of solution ω of the problem

$$\begin{cases} -\Delta\omega = K|\omega|^{\frac{4}{n-2}}\omega & \text{in } (\omega), \\ \omega = 0 & \text{on } (\Gamma_0), \\ \dfrac{\partial\omega}{\partial\nu} = 0 & \text{on } (\Gamma_1) \end{cases}$$

in $V_\gamma(\Sigma^+)$. Multiplying the above system by ω^-, we get

$$\begin{aligned} \int_\Omega |\nabla\omega^-|^2 \mathrm{d}x &= \int_\Omega K(\omega^-)^{\frac{2n}{n-2}} \mathrm{d}x \\ &\le M\Sigma_n^2 \|\omega^-\|^2 \|\omega^-\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}}. \end{aligned}$$

It follows that if $\|\omega^-\| \neq 0$, then

$$\|\omega^-\|^{\frac{4}{n-2}}_{L^{\frac{2n}{n-2}}} \ge (M\Sigma_n^2)^{-1}.$$

Thus for $\gamma < (M\Sigma_n^2)^{-1}$, ω is a positive solution of (1.1).

Proof of Theorem 1.2 Let y_0 be an absolute maximum of $K_{/\Gamma_1}$. It follows from expansions of Propositions 2.2 and 3.1 that if $\|K-1\|_{L^\infty(\overline{\Omega})} \le \delta_b$, all the critical points at infinity of J are above $c_0\big(\frac{S_n}{2}\big)^{\frac{2}{n}} \frac{1}{K(y_0)^{\frac{n-2}{n}}}$.

Let $\lambda \gg 1$ and $u_0 = \frac{\varphi_{(y_0,\lambda)}}{\|\varphi_{(y_0,\lambda)}\|}$. By the expansion of Proposition 3.1, we have

$$J(u) = c_0\Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} \frac{1}{K(y_0)^{\frac{n-2}{n}}} \Big(1 - \frac{2(n(n-2))^{\frac{n-2}{n}}}{S^{\frac{n}{2}}} \frac{L(y_0)}{\lambda}(1+o(1))\Big).$$

Under the hypothesis of Theorem 1.2, we have

$$J(u_0) < c_0\Big(\frac{S_n}{2}\Big)^{\frac{2}{n}} \frac{1}{K(y_0)^{\frac{n-2}{n}}}.$$

Let $\eta(s,u_0)$ be the motion of $(-\partial J)$ such that $\eta(0,u_0) = u_0$. Using the fact that J decreases along $\eta(s,u_0)$, if we suppose that J has no critical point in Σ^+, then by Proposition 2.1 there exists $m = m(u_0) > 0$ such that $\|\partial J(\eta(s,u_0))\| \geq m, \forall s \geq 1$ and hence

$$J(\eta(s)) = J(u_0) - \int_0^s \|\partial J(\eta(s))\|^2 \leq J(u_0) - ms.$$

It follows that, $J(ms) \to -\infty$ as $s \to +\infty$. This is impossible since by the Sobolev embedding J is lower bounded. Hence the result holds.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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