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Essential Numerical Ranges of Linear Relations and Singular Discrete Linear Hamiltonian Systems*

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Abstract In the paper, a concept of the essential numerical range $W_e(T)$ of a linear relation T in a Hilbert space is given, other various essential numerical ranges $W_{ei}(T)$, $i = 1, 2, 3, 4$, are introduced, and relationships among $W_e(T)$ and $W_{ei}(T)$ are established. These results generalize relevant results obtained by Bögli et al. in [Bögli, S., Marletta, M. and Tretter, C., The essential numerical range for unbounded linear operators, *J. Funct. Anal.*, **279**(1), 2020, 108509]. Moreover, several fundamental properties of closed relations related to its operator parts are presented. In addition, singular discrete linear Hamiltonian systems including non-symmetric cases are considered, several properties for the associated minimal relations H_0 are derived, and the above results for abstract linear relations are applied to H_0.

Keywords Linear relation, Numerical range, Essential numerical range, Essential spectrum, Singular discrete linear Hamiltonian system
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1 Introduction

The study of fundamental theory of single-valued operators has a long history and their spectral theory has been investigated widely and deeply (cf. e.g. [12, 15, 27]). However, many multi-valued operators have been found in the study of some problems. For example, in the case that an operator is not densely defined, its adjoint is multi-valued; for symmetric linear differential expressions, the associated minimal operators are non-densely defined, and the associated maximal operators are multi-valued when the differential expressions do not satisfy the definiteness condition (cf. [13]); the minimal and maximal operators generated by discrete Hamiltonian systems are multi-valued or non-densely defined in general even though the corresponding definiteness conditions are satisfied (cf. [16–17]). Therefore, it is required to establish the theory of multi-valued linear operators.

Multi-valued linear operators are often called linear relations (briefly, relations) or subspaces of the related product spaces (cf. [1, 5, 26]). Linear relations include both single-valued and

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multi-valued operators. In 1950, von Neumann introduced linear relations in order to study adjoints of non-densely defined linear differential operators (cf. [26]). In 1961, Arens [1] initiated the study of linear relations. He decomposed a closed relation in the product space X^2 as an operator part and a purely multi-valued part. This decomposition provides a bridge between closed relations in X^2 and linear operators in X so that we can apply the theory of linear operators to study some properties of closed relations. His work was followed by many scholars (cf. [5–6, 11, 21–23] and references cited therein). The study of relations has been growing interest in recent years because of its wide applications. In 2011, Ren and Shi studied the defect indices and definiteness conditions, and later, gave out complete characterizations of self-adjoint extensions for discrete Hamiltonian systems (cf. [16–17]). Recently, Shi and Sun studied some spectral properties of discrete Hamiltonian systems (cf. [25]). In addition, some fundamental results of Hermitian relations were established (cf. [20–22]). In particular, applying the theory of Fredholm relations, Wilcox [28] introduced the concepts of different types of essential spectra of relations and obtained basic properties of them. However, compared with the operator theory, many important problems about relations have not been studied. We shall investigate essential numerical ranges of relations in a Hilbert space.

The essential numerical range of a linear operator is an important concept in the spectral analysis, and the original idea of it was to give a convex enclosure of the essential spectrum. It was introduced for bounded operators in a Hilbert space by Stampfli and Williams in 1968 (cf. [24]), and later, several other characterizations were given by Fillmore et al in [10]. For the further relevant research of it for bounded operators, the reader is referred to [2, 8, 14, 19]. More recently, the concept of essential numerical ranges has been generalized to unbounded operators in a Hilbert space by Bögli et al. [4] and fundamental properties of essential numerical ranges including possible equivalent characterizations were studied. Some examples were given in [4] to illustrate that many of properties for bounded operators do not carry over to the unbounded cases. Moreover, one of advantages of the essential numerical range is that it captures all possible spectral pollution in a unified and minimal way when approximating an operator T by projection methods or domain truncation methods for PDEs (cf. [4, Theorems 6.3 and 7.1]). Now, the essential numerical range has been introduced for linear operator pencils and discussed in details by Bögli and Marletta in [3].

In this paper, the concept of the essential numerical range $W_e(T)$ of a linear relation T in a Hilbert space is given, other various essential numerical ranges $W_{ei}(T)$, $i = 1, 2, 3, 4$, are introduced, and relationships among $W_e(T)$ and $W_{ei}(T)$ are established. Furthermore, singular discrete linear Hamiltonian systems which may be non-symmetric are considered, and the associated maximal, pre-minimal, and minimal relations H, H_{00} and H_0 are introduced in a product Hilbert space. It is noted that the sufficient and necessary conditions for the minimal relation H_0 to be an operator are given for singular symmetric discrete linear Hamiltonian systems (cf. [17]). We first extend them to non-symmetric case, and then derive a sufficient condition for the minimal relation H_0 to be not densely defined. Finally, we apply the above results for abstract linear relations to H_0.

The paper is organized as follows. In Section 2, some notations and basic concepts are introduced, and several fundamental properties of closed relations related to its operator parts are presented. In Section 3, the concept of the essential numerical range $W_e(T)$ of a relation T is given, other various essential numerical ranges $W_{ei}(T)$, $i = 1, 2, 3, 4$, are introduced, and relationships among $W_e(T)$ and $W_{ei}(T)$ are established. In Section 4, singular discrete linear Hamiltonian systems and their essential numerical ranges are discussed.

2 Basic Concepts and Fundamental Results About Linear Relations

In this section, we introduce some notations, recall some basic concepts, and give several fundamental properties of closed relations related to its operator parts.

We denote by $\mathbb{C}$ and $\mathbb{R}$ the sets of complex and real numbers, respectively, and by $\mathbb{N}$ the set of positive integer numbers, throughout this paper. Let X be a Hilbert space over $\mathbb{C}$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and $X^2 := X \times X$ be the product space with the following induced inner product, still denoted by $\langle \cdot, \cdot \rangle$ without any confusion:

$$\langle (x, f), (y, g) \rangle = \langle x, y \rangle + \langle f, g \rangle, \quad (x, f),\ (y, g) \in X^2.$$

A linear subspace $T \subset X^2$ is called a linear relation (briefly, relation) in X. A (linear) operator T in X is always identified with a relation in X via its graph $G(T)$. The domain $D(T)$, range $R(T)$ and null space $N(T)$ of T are respectively defined by

$$\begin{aligned} D(T) &:= \{x \in X : \ (x, f) \in T \text{ for some } f \in X\}, \\ R(T) &:= \{f \in X : \ (x, f) \in T \text{ for some } x \in X\}, \\ N(T) &:= \{x \in X : \ (x, 0) \in T\}. \end{aligned}$$

Further, we denote

$$T(x) := \{f \in X : (x, f) \in T\}, \quad T^{-1} := \{(f, x) : (x, f) \in T\}. \tag{2.1}$$

It is evident that $T(0) = \{0\}$ if and only if T can uniquely determine an operator from $D(T)$ into X whose graph is T. For convenience, if $T(0) = \{0\}$, then the relation T in X is called an operator in X. The restriction of T to M, denoted by $T|_M$, is defined by $T|_M := \{(x, f) \in X^2, x \in D(T) \cap M\}$. The adjoint relation T^* of T is defined by

$$T^* = \{(y, g) \in X^2 : \langle g, x \rangle = \langle y, f \rangle \text{ for all } (x, f) \in T\}.$$

Then, T is said to be Hermitian in X if $T \subset T^*$ and to be self-adjoint in X if $T = T^*$.

Let S and T be relations in X, $\alpha \in \mathbb{C}$. We denote

$$\begin{aligned} \alpha T &:= \{(x, \alpha f) : \ (x, f) \in T\}, \\ T + S &:= \{(x, f + g) : (x, f) \in T,\ (x, g) \in S\}, \\ ST &:= \{(x, f) \in X^2 : (x, g) \in T,\ (g, f) \in S \text{ for some } g \in X\}. \end{aligned}$$

Let S and T be orthogonal, i.e., $\langle (x, f), (y, g) \rangle = 0$ for all $(x, f) \in T$ and $(y, g) \in S$. Then we set

$$T \oplus S = T \dot{+} S,$$

where $T\dot{+}S := \{(x+y, f+g) : (x,f) \in T, (y,g) \in S\}$ and $T \cap S = \{(0,0)\}$.

Let T be a closed relation in X, i.e., $\overline{T} = T$, where $\overline{T}$ is the closure of T. Then $T(0)$ is a closed subspace of X. Arens [1] introduced the following decomposition for a closed relation T:

$$T = T_s \oplus T_\infty, \tag{2.2}$$

where

$$T_\infty := \{(0,g) \in X^2 : (0,g) \in T\}, \quad T_s := T \ominus T_\infty.$$

Then T_s is an operator with $D(T_s) = D(T)$. T_s and T_∞ are called the operator and pure multi-valued parts of T, respectively. Further, for a closed relation T in X, Arens [1] showed the following result.

Lemma 2.1 (cf. [1, Lemma 5.2]) *Let T be a closed relation in X. Then $T(0) = D(T^*)^\perp$, $\overline{D(T_s)} = T^*(0)^\perp$ and $R(T_s) \subset T(0)^\perp$.*

The following corollary can be proved by Lemma 2.1.

Corollary 2.1 *Let T be a closed relation in X. Then $\overline{D(T)} = T(0)^\perp$ if and only if $T^*(0) = T(0)$.*

Proof The result can be obtained by $\overline{D(T)} = \overline{D(T_s)} = T^*(0)^\perp$ and the fact that $T(0)$ and $T^*(0)$ are closed. This completes the proof.

Now, let T be a relation in X. By Q_T or simply Q, when there is no ambiguity about the relation T, we denote the natural quotient map $X \to X/\overline{T(0)}$. Clearly QT is a single-valued operator from X to $X/\overline{T(0)}$. For simplicity, we write $T(x)$ defined in (2.1) as Tx. The norm $\|T\|$ of T is given by

$$\|T\| := \|QT\| = \sup\{\|QTx\| : x \in D(T) \text{ with } \|x\| = 1\}.$$

If $\|T\|$ is finite, then T is said to be bounded. It is noted that $\|T\| = 0$ implies $R(QT) \subset \overline{T(0)}$. By $d(U,V) := \inf\{\|u-v\| : u \in U, v \in V\}$ we denote the distance between U and V, where U and V are non-empty subsets of X. From [7, Chapter II, Proposition 1.4], we have

$$\|QTx\| = d(Tx, T(0)) = d(Tx, 0), \quad x \in D(T). \tag{2.3}$$

Let $I := \{(x,x) : x \in X\}$ be the identity relation on X. We usually write $\lambda I - T$ as $\lambda - T$. For the following concepts, the reader is referred to [5, 21–22, 25, 28].

Definition 2.1 *Let T be a relation in X.*

(1) *The set $\rho(T) := \{\lambda \in \mathbb{C} : (\lambda - T)^{-1}$ is a bounded operator defined on $X\}$ is called the resolvent set of T.*

(2) *The set $\sigma(T) := \mathbb{C}\backslash\rho(T)$ is called the spectrum of T.*

(3) *For $\lambda \in \mathbb{C}$, if there exists $x \neq 0$ such that $\lambda x \in Tx$, then λ is called an eigenvalue of T, while x is called an eigenvector of T with respect to the eigenvalue λ. Further, the set of all the eigenvalues of T is called the point spectrum of T, denoted by $\sigma_p(T)$.*

(4) *The set $\sigma_e(T) := \{\lambda \in \mathbb{C} : \exists\ \{x_n\}_{n\in\mathbb{N}} \subset D(T)$ with $||x_n|| = 1,\ x_n \xrightarrow{w} 0,$ and $\|Q(\lambda - T)x_n\| \to 0,\ n \to \infty\}$ is called the essential spectrum of T.*

Remark 2.1 Edmunds and Evans [9] gave five distinct essential spectra $\sigma_{e_i}(T)$, $1 \le i \le 5$, of a closed operator T in terms of its semi-Fredholm properties in the case that X is a Banach space. If T is a closed and densely defined operator in X, then $\sigma_e(T)$ given in Definition 2.1 is the second type of essential spectra in [9] by $Q(\lambda - T) = \lambda - T$ and [9, Chapter IX, Theorem 1.3]. Recently, Wilcox generalized these concepts of essential spectra to relations. The essential spectrum $\sigma_e(T)$ given in Definition 2.1 is the second one of [28, Definition 2.2] by [28, Proposition 3.3].

Proposition 2.1 *Let T be a closed relation in X. Then $\sigma_e(T) = \sigma_e(T_s)$, and if, in addition, $D(T) \subset T(0)^{\perp}$, then $\sigma_p(T) = \sigma_p(T_s)$.*

Proof Let T be a closed relation in X. Then (2.2) holds. It follows from (2.2) that

$$\|Q(\lambda - T)x\| = \|Q(\lambda - T_s)x + QT(0)\| = \|Q(\lambda - T_s)x\|, \quad x \in D(T),$$

which implies that $\sigma_e(T) = \sigma_e(T_s)$ since $D(T) = D(T_s)$.

Next, we show $\sigma_p(T) = \sigma_p(T_s)$. It suffices to show

$$N(\lambda - T) = N(\lambda - T_s), \quad \lambda \in \mathbb{C}. \tag{2.4}$$

Clearly, $N(\lambda - T_s) \subset N(\lambda - T)$. Consequently, for $x \in N(\lambda - T)$, we have $(x, \lambda x) \in T$. Then, $\lambda x = T_s x + g$ for some $g \in T(0)$ by (2.2), which implies $\lambda x = T_s x$ and $g = 0$ since $D(T) = D(T_s) \subset T(0)^{\perp}$ and $R(T_s) \subset T(0)^{\perp}$ by Lemma 2.1. Therefore, $(x, \lambda x) \in T_s$ and then $N(\lambda - T) \subset N(\lambda - T_s)$. Hence, (2.4) holds. This completes the proof.

The following result can be easily verified by the closed graph theorem.

Lemma 2.2 *Let T be a closed relation in X. Then $\lambda \in \rho(T)$ if and only if $R(\lambda - T) = X$ and $N(\lambda - T) = \{0\}$.*

Now, let T be a closed relation in X. Then $R(T_s) \subset T(0)^{\perp}$ by Lemma 2.1. Further, if $D(T) \subset T(0)^{\perp}$, then T_s is an operator from $T(0)^{\perp}$ to $T(0)^{\perp}$. In the following, we consider T_s in $(T(0)^{\perp})^2$. For clarity, let $\widetilde{\rho}(T_s)$ be the resolvent set of T_s in the space $T(0)^{\perp}$, i.e.,

$$\widetilde{\rho}(T_s) := \{\lambda \in \mathbb{C} : (\lambda - T_s)^{-1} \text{ is a bounded operator defined on } T(0)^{\perp}\},$$

and let $\widetilde{\sigma}(T_s) := \mathbb{C} \backslash \widetilde{\rho}(T_s)$. Then we have the result below.

Proposition 2.2 *Let T be a closed relation in X with $D(T) \subset T(0)^{\perp}$. Then*

$$\rho(T) = \widetilde{\rho}(T_s) \quad \textit{and} \quad \sigma(T) = \widetilde{\sigma}(T_s).$$

Proof Let T be a closed relation in X with $D(T) \subset T(0)^{\perp}$. We shall show $\rho(T) = \widetilde{\rho}(T_s)$. By Lemma 2.2 and (2.4), it suffices to show that $R(\lambda - T) = X$ if and only if $R(\lambda - T_s) = T(0)^{\perp}$ for any $\lambda \in \mathbb{C}$.

First, suppose that $R(\lambda - T) = X$. Then there exists $(x, f) \in T$ such that $\lambda x - f = h$ for every $h \in T(0)^{\perp} \subset X$. By (2.2), there exists $g \in T(0)$ such that $f = T_s x + g$. From the above two equations, we have

$$h - (\lambda x - T_s x) = -g. \tag{2.5}$$

Note that $D(T) \subset T(0)^\perp$ and $R(T_s) \subset T(0)^\perp$ by Lemma 2.1. Then we get from (2.5) that $h - (\lambda x - T_s x) = -g = 0$. Hence, $h = \lambda x - T_s x$. Then $T(0)^\perp \subset R(\lambda - T_s)$, and consequently, $R(\lambda - T_s) = T(0)^\perp$ since $R(T_s) \subset T(0)^\perp$ and $D(T) \subset T(0)^\perp$.

Next, suppose that $R(\lambda - T_s) = T(0)^\perp$. Let $h \in X$. Then h can be written as $h = h_1 + h_2$ with $h_1 \in T(0)$ and $h_2 \in T(0)^\perp$. Note that $h_2 \in T(0)^\perp = R(\lambda - T_s)$. Then there exists $x \in D(T_s)$ such that $h_2 = \lambda x - T_s x$. Obviously, $(x, T_s x - h_1) \in T$ and it is evident that $h = \lambda x - (T_s x - h_1)$, which implies that $h \in R(\lambda - T)$. Then $X \subset R(\lambda - T)$, and hence $X = R(\lambda - T)$.

Based on the above discussions, $\rho(T) = \widetilde{\rho}(T_s)$, and hence $\sigma(T) = \widetilde{\sigma}(T_s)$. This completes the proof.

Remark 2.2 If T is Hermitian, then $T(0) \subset T^*(0)$. Hence, $D(T) \subset T(0)^\perp$ by Lemma 2.1, and then Proposition 2.2 holds if T is closed. Therefore, Proposition 2.2 extends [22, Theorem 2.1] for closed Hermitian relations to general closed relations in X.

3 Essential Numerical Ranges of Linear Relations and Equivalent Characterizations

In this section, the concept of the essential numerical range $W_e(T)$ of a linear relation T in a Hilbert space is given, other various essential numerical ranges $W_{ei}(T)$, $i = 1, 2, 3, 4$, are introduced, and relationships among $W_e(T)$ and $W_{ei}(T)$ are established. This section is divided into two parts.

3.1 Essential numerical ranges of linear relations

First, for a relation T in X, the numerical range is the set

$$W(T) := \{\langle f, x\rangle : \ (x, f) \in T, \ \|x\| = 1\},$$

which was given by Rofe-Beketov in [18]. From [18, Theorem 1], $W(T)$ is a convex subset of $\mathbb{C}$, and if $\dim D(T) < \infty$, then $W(T)$ is closed and bounded or $W(T) = \mathbb{C}$. Further, the following result holds.

Lemma 3.1 (cf. [18, Lemma 1]) *Let T be a relation in X. If there exists $h \in T(0)$ such that $h \notin D(T)^\perp$, then $W(T) = \mathbb{C}$. Consequently, if $W(T) \neq \mathbb{C}$, then $T(0) \perp D(T)$.*

Next, we define the essential numerical range $W_e(T)$ of a relation T in X as follows.

Definition 3.1 *Let T be a relation in X. Then the set*

$$W_e(T) := \{\lambda \in \mathbb{C} : \exists\ \{(x_n, f_n)\}_{n\in\mathbb{N}} \subset T \ \textit{with}\ \|x_n\| = 1, x_n \xrightarrow{w} 0, \textit{and}\ \langle f_n, x_n\rangle \to \lambda\}$$

is called the essential numerical range of T.

It is evident that $W_e(T) \subset \overline{W(T)}$ and $W_e(zT) = zW_e(T)$ and $W_e(T + zI) = W_e(T) + z$ for $z \in \mathbb{C}$ by the definitions. For the concept of essential numerical ranges of bounded operators in a Hilbert space, the reader is referred to [10, 24]. Recently, Bögli et al [4] introduced the concept of essential numerical ranges of general operators including unbounded operators as follows:

$$W_e(T) := \{\lambda \in \mathbb{C} : \exists\ \{x_n\}_{n\in\mathbb{N}} \subset D(T)\ with\ ||x_n|| = 1,\ x_n \xrightarrow{w} 0, and\ \langle Tx_n, x_n\rangle \to \lambda\},$$

which is a generalization of the corresponding concept for bounded operators. Clearly, Definition 3.1 extends the above concept to relations in X.

Now, let T be a closed relation in X. Then (2.2) holds. The relationships between the numerical, essential numerical ranges of the relation T in X and those of its operator part T_s are given as follows.

Proposition 3.1 *Let T be a closed relation in X. If $D(T) \subset T(0)^\perp$, then*

$$W(T) = W(T_s) \quad and \quad W_e(T) = W_e(T_s). \tag{3.1}$$

In particular, if $W(T) \neq \mathbb{C}$, then (3.1) *holds.*

Proof Let T be a closed relation in X. Then, for $(x, f) \in T$, there exists $g \in T(0)$ such that $f = T_s x + g$ by (2.2). If $D(T) \subset T(0)^\perp$, then $\langle f, x\rangle = \langle T_s x, x\rangle$, which yields (3.1). In addition, if $W(T) \neq \mathbb{C}$, then $D(T) \subset T(0)^\perp$ by Lemma 3.1. Then (3.1) holds by the above discussions. This completes the proof.

Remark 3.1 Here we point out that if $W(T) = \mathbb{C}$, then (3.1) may be false. In fact, let E be a non-zero closed subspace of X and P be an orthogonal projection in X onto $E^\perp$. Then we define $Tx = Px + E$ for every $x \in D(T) = X$. Clearly, T is a closed relation in X, $T(0) = E$, $T_s = \{(x, Px) \in X^2 : x \in X\}$ and $T_\infty = \{(0, g) \in X^2 : g \in E\}$. It is easy to see that $W(T) = \mathbb{C}$ by Lemma 3.1 and $W(T_s) \subset [0, 1]$. Hence $W(T) \neq W(T_s)$. In particular, in the case of $W(T) = \mathbb{C}$, we shall show that $W_e(T) = W(T) = \mathbb{C}$ by the conclusion (iv) of Proposition 3.3 below. However, $W_e(T_s) \subset \overline{W(T_s)} \subset [0, 1]$. Hence, we also have $W_e(T) \neq W_e(T_s)$.

It is noted that the inclusion $\sigma_e(T) \subset W_e(T)$ for T being an operator is immediate from the definitions. However, it is not obvious for relations. Now, we prove it as follows.

Proposition 3.2 *Let T be a relation in X. Then $\sigma_e(T) \subset W_e(T)$.*

Proof Let $\lambda \in \sigma_e(T)$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(T)$ satisfying $\|x_n\| = 1, x_n \xrightarrow{w} 0$ and $\|Q(\lambda - T)x_n\| \to 0$ as $n \to \infty$. For every fixed $n \in \mathbb{N}$, we get from (2.3) that

$$\|Q(\lambda - T)x_n\| = d((\lambda - T)x_n, 0) = \inf_{f_n\in Tx_n} \|\lambda x_n - f_n\|. \tag{3.2}$$

For convenience, we denote $\varepsilon_n = \inf_{f_n\in Tx_n} \|\lambda x_n - f_n\|$. Then there exists $f_n^{(n)} \in Tx_n$ such that $\|\lambda x_n - f_n^{(n)}\| \leq 2\varepsilon_n$ for $n \in \mathbb{N}$. It follows from (3.2) and $\|Q(\lambda - T)x_n\| \to 0$ that $\varepsilon_n \to 0$ as $n \to \infty$. Therefore,

$$\|\lambda x_n - f_n^{(n)}\| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

From (3.3), we have

$$|\langle f_n^{(n)}, x_n\rangle - \lambda| = |\langle f_n^{(n)} - \lambda x_n, x_n\rangle| \le \|f_n^{(n)} - \lambda x_n\|\|x_n\| \to 0 \quad \text{as } n \to \infty.$$

Then $\lambda \in W_e(T)$, and hence $\sigma_e(T) \subset W_e(T)$. This completes the proof.

For a relation T in X, the following proposition is a generalization of the corresponding [4, Propositions 2.2–2.4, Corollary 2.5] for operators.

Proposition 3.3 *Let T be a relation in X.*

(a) *$W_e(T)$ is a closed and convex subset of $\mathbb{C}$ and $\operatorname{conv}\sigma_e(T) \subset W_e(T)$, where $\operatorname{conv}\sigma_e(T)$ is convex hull of $\sigma_e(T)$.*

(b) *If $\overline{W(T)}$ is a line or a strip or if $W(T) = \mathbb{C}$, then $W_e(T) \neq \emptyset$.*

(c) *If there exist $z \in W_e(T)$ and $w \in \mathbb{C}\backslash\{0\}$ with $z + w(0,\infty) \subset W(T)$, then $z + w[0,\infty) \subset W_e(T)$.*

(d) *The following five results hold:*

(i) *If $W(T)$ is a line, then so is $W_e(T)$, and thus $W_e(T) = W(T)$.*

(ii) *If $\overline{W(T)}$ is a strip, then $W_e(T)$ is a strip or a line.*

(iii) *If $\overline{W(T)}$ is a half-plane and $W_e(T) \neq \emptyset$, then $W_e(T)$ is a half-plane.*

(iv) *If $W(T) = \mathbb{C}$, then $W_e(T) = \mathbb{C}$, and vice versa.*

(v) *If $\overline{W(T)}$ is or contains a sector and $W_e(T) \neq \emptyset$, then $W_e(T)$ contains each subsector with vertex in $W_e(T)$.*

Proof By Proposition 3.2 and the proof of [4, Proposition 2.2] with some corresponding changes, e.g., (x_n, Tx_n) and (y_n, Ty_n) in [4, Proposition 2.2] replaced by (x_n, f_n) and (y_n, g_n), respectively, the conclusion (a) can be proved. Further, conclusions (b) and (c) can be proved with a similar argument to those of [4, Proposition 2.3] and [4, Proposition 2.3], respectively, with some corresponding changes mentioned as the above. The conclusion (d) follows from (a–c). Therefore, we omit the details here.

Finally, let T be a self-adjoint relation in X. The extended essential spectrum $\widehat{\sigma}_e(T)$ of T is defined by

$$\widehat{\sigma}_e(T) := \begin{cases} \sigma_e(T), & T \text{ is bounded}, \\ \sigma_e(T) \cup \{+\infty\}, & T \text{ is unbounded from above}, \\ \sigma_e(T) \cup \{-\infty\}, & T \text{ is unbounded from below}, \\ \sigma_e(T) \cup \{\pm\infty\}, & T \text{ is unbounded from above and below}. \end{cases}$$

Then, it is a generalization of the concept of the extended essential spectrum of an operator in a Hilbert space defined in [4]. It is always assumed that X is a separable infinite dimensional Hilbert space in the remainder of this paper. The following result extends [19, Corollary 5.1] for bounded operators and [4, Theorem 3.8] for unbounded operators to relations.

Theorem 3.1 *Let T be a self-adjoint relation in X. Then*

$$W_e(T) = \begin{cases} \operatorname{conv}(\widehat{\sigma}_e(T)), & T \text{ is bounded}, \\ \operatorname{conv}(\widehat{\sigma}_e(T))\backslash\{\pm\infty\}, & T \text{ is unbounded}. \end{cases} \tag{3.4}$$

Proof Let T be a self-adjoint relation in X. Then T is closed and (2.2) holds. Since $W(T) \subset \mathbb{R}$, we have $D(T) \subset T(0)^{\perp}$ by Lemma 3.1. Together with $R(T_s) \subset T(0)^{\perp}$ by Lemma 2.1, we get that T_s is a self-adjoint operator from $T(0)^{\perp}$ to $T(0)^{\perp}$. The essential spectrum and essential numerical range of T_s in $(T(0)^{\perp})^2$ are denoted by $\sigma_e(T_s)$ and $W_e(T_s)$ without any confusion. Then $\sigma_e(T) = \sigma_e(T_s)$ by Proposition 2.1 and $W_e(T) = W_e(T_s)$ by Proposition 3.1. In addition, the boundedness of T and T_s are equivalent by $\|Q(\lambda - T)x\| = \|Q(\lambda - T_s)x\|$, $x \in D(T) = D(T_s)$. Then, from [19, Corollary 5.1] and [4, Theorem 3.8], it follows that

$$W_e(T_s) = \begin{cases} \operatorname{conv}(\widehat{\sigma}_e(T_s)), & T_s \text{ is bounded,} \\ \operatorname{conv}(\widehat{\sigma}_e(T_s)) \backslash \{\pm\infty\}, & T_s \text{ is unbounded,} \end{cases}$$

which implies that (3.4) holds by the above discussions. This completes the proof.

3.2 Equivalent characterizations of $W_e(T)$

In this subsection, we first introduce four varieties of essential numerical ranges $W_{ei}(T)$, $i = 1, 2, 3, 4$, of a relation T in X and study the relationships of them.

Now, let T be a relation in X, $\mathcal{V}$ be the set of all finite-dimensional subspaces of X, and $L(X)$ be the space of all bounded linear operators acting on X. For a relation T, we define

$$\begin{aligned} W_{e1}(T) &:= \bigcap_{V \in \mathcal{V}} \overline{W(T|_{V^{\perp} \cap D(T)})}\ ; \\ W_{e2}(T) &:= \bigcap_{\substack{K \in L(X) \\ \operatorname{rank} K < \infty}} \overline{W(T + G(K))}\ ; \\ W_{e3}(T) &:= \bigcap_{\substack{K \in L(X) \\ K \text{ compact}}} \overline{W(T + G(K))}\ ; \\ W_{e4}(T) &:= \{\lambda \in \mathbb{C} : \exists \{(e_n, h_n)\}_{n \in \mathbb{N}} \subset T \text{ with } \langle h_n, e_n \rangle \xrightarrow{n \to \infty} \lambda \\ &\qquad \text{and } \{e_n\} \text{ is an orthonormal sequence}\}, \end{aligned}$$

where $G(K)$ is the graph of the operator K. These various essential numerical ranges $W_{ei}(T), i = 1, 2, 3, 4$, were firstly introduced for bounded operators (cf. [10, 24]). They were extended to unbounded operators by Bögli et al. [4]. The essential numerical ranges $W_{ei}(T), i = 1, 2, 3, 4$, defined here are generalizations of the corresponding concepts for bounded and unbounded operators to relations.

It is known that $T = T_s \oplus T_\infty$ if T is a closed relation in X. Note that $R(T_s) \subset T(0)^{\perp}$ by Lemma 2.1. If $D(T) \subset T(0)^{\perp}$, then T_s is an operator from $T(0)^{\perp}$ to $T(0)^{\perp}$. For clarity, let $\widetilde{W}_e(T_s)$ and $\widetilde{W}_{ei}(T_s)$, $i = 1, 2, 3, 4$, be defined similarly to $W_e(T)$ and $W_{ei}(T)$ with T and X replaced by T_s and $T(0)^{\perp}$, respectively. Then, we have the following result.

Proposition 3.4 *Let T be a closed relation in X. If $D(T) \subset T(0)^{\perp}$, then*

$$W_e(T) = \widetilde{W}_e(T_s) \quad \text{and} \quad W_{ei}(T) = \widetilde{W}_{ei}(T_s), \quad i = 1, 2, 3, 4. \tag{3.5}$$

Proof Let T be a closed relation in X with $D(T) \subset T(0)^{\perp}$. It is evident that $W_e(T) = \widetilde{W}_e(T_s)$ and $W_{e4}(T) = \widetilde{W}_{e4}(T_s)$ by their definitions and $D(T_s) = D(T) \subset T(0)^{\perp}$.

Next, we shall show that $W_{ei}(T)=\widetilde{W}_{ei}(T_s),\ i=1,2,3$. First, we prove $W_{e1}(T)=\widetilde{W}_{e1}(T_s)$. It can be easily verified that $W_{e1}(T)\subset\widetilde{W}_{e1}(T_s)$. We shall show $\widetilde{W}_{e1}(T_s)\subset W_{e1}(T)$. Let V be an arbitrary finite-dimensional subspace of X and P be an orthogonal projection in X onto $T(0)^\perp$. Define a finite-dimensional subspace $\widetilde{V}$ of $T(0)^\perp$ by $\widetilde{V}:=PV$. Then, we claim that $V^\perp\cap D(T)=(T(0)^\perp\ominus\widetilde{V})\cap D(T)$, i.e.,

$$(X\ominus V)\cap D(T)=(T(0)^\perp\ominus\widetilde{V})\cap D(T). \tag{3.6}$$

In fact, for every $x\in(X\ominus V)\cap D(T)$ and $v\in V$, there exists a unique $x_1\in X$ such that

$$x=x_1-v \quad \text{with } \langle x,v\rangle=0. \tag{3.7}$$

Further, $x_1=\widetilde{x}_1+\widetilde{x}_2$ and $v=v_1+v_2$, where $\widetilde{x}_1=Px_1$, $\widetilde{x}_2=x_1-Px_1$, $v_1=Pv$ and $v_2=v-Pv$. Then from (3.7), one has

$$x=(\widetilde{x}_1-v_1)+(\widetilde{x}_2-v_2). \tag{3.8}$$

Note that $x\in T(0)^\perp$ by $D(T)\subset T(0)^\perp$, $\widetilde{x}_1-v_1\in T(0)^\perp$ and $\widetilde{x}_2-v_2\in T(0)$. Then (3.8) yields $x=\widetilde{x}_1-v_1$. In addition, it follows from $\langle x,v\rangle=0$ and $D(T)\subset T(0)^\perp$ that $\langle x,v_1\rangle=0$. Then $x\in(T(0)^\perp\ominus\widetilde{V})$, and consequently, $x\in(T(0)^\perp\ominus\widetilde{V})\cap D(T)$. Then " $\subset$ " in (3.6) holds, and the reverse inclusion is obvious. Therefore, (3.6) holds. In addition, for $(x,f)\in T$, there exists $h\in T(0)$ such that $f=T_sx+h$, and hence $\langle f,x\rangle=\langle T_sx,x\rangle$ by $D(T)\subset T(0)^\perp$. Then, together with (3.6) and $D(T)=D(T_s)$, we have

$$W(T|_{(X\ominus V)\cap D(T)})=W(T_s|_{(X\ominus V)\cap D(T)})=W(T_s|_{(T(0)^\perp\ominus\widetilde{V})\cap D(T_s)}).$$

Hence, $\widetilde{W}_{e1}(T_s)\subset W_{e1}(T)$.

Now, we prove $W_{e3}(T)=\widetilde{W}_{e3}(T_s)$. Let $\widetilde{K}\in L(T(0)^\perp)$ be compact. Then we define $K:=\widetilde{K}P$, where P is given on the above. Then $K\in L(X)$ and it is compact. On the other hand, for $(x,f)\in T$, one has $\langle f,x\rangle=\langle T_sx,x\rangle$ by $D(T)\subset T(0)^\perp$. It follows that

$$W(T+G(K))=W(T_s+G(\widetilde{K})). \tag{3.9}$$

Conversely, let $K\in L(X)$ be compact. Then we define $\widetilde{K}:=PK|_{T(0)^\perp}$. Clearly, $\widetilde{K}\in L(T(0)^\perp)$ and it is compact. For this pair of K and $\widetilde{K}$, (3.9) also holds. Then $W_{e3}(T)=\widetilde{W}_{e3}(T_s)$ by their definitions.

Finally, note that if K is a finite rank operator, then it is compact. Then by the above discussions, one sees that $W_{e2}(T)=\widetilde{W}_{e2}(T_s)$. This completes the proof.

In what follows, we shall study the relationships among $W_e(T)$ and $W_{ei}(T), i=1,2,3,4$, for a relation T in X. For bounded and unbounded operators, the relationships were established in [10, Theorem 5.1] and [4, Theorem 3.1], respectively. For a closed relation T in X, the following result can be derived by Proposition 3.4 and [4, Theorem 3.1].

Theorem 3.2 *Let T be a closed relation in X.*
(1) *If $D(T)\subset T(0)^\perp$, then*

$$W_{e1}(T)\subset W_{e4}(T)\subset W_{e2}(T)=W_{e3}(T)=W_e(T). \tag{3.10}$$

(2) *If* $T^*(0) = T(0)$*, then*

$$W_{e1}(T) \subset W_{e4}(T) = W_{e2}(T) = W_{e3}(T) = W_e(T), \tag{3.11}$$

in addition, if $W(T) \neq \mathbb{C}$*, then*

$$W_{ei}(T) = W_e(T), \quad i = 1, 2, 3, 4. \tag{3.12}$$

(3) *If* $\overline{D(T) \cap D(T^*)} = T(0)^\perp$*, then* (3.12) *holds.*

Proof Let T be a closed relation in X. Then $T = T_s \oplus T_\infty$ and T_s is an operator in $T(0)^\perp$ since $D(T_s) = D(T) \subset T(0)^\perp$ and $R(T_s) \subset T(0)^\perp$ by Lemma 2.1. Then, from [4, Theorem 3.1], we have

$$\widetilde{W}_{e1}(T_s) \subset \widetilde{W}_{e4}(T_s) \subset \widetilde{W}_{e2}(T_s) = \widetilde{W}_{e3}(T_s) = \widetilde{W}_e(T_s),$$

which, together with (3.5), implies that (3.10) holds.

If $T^*(0) = T(0)$, then $\overline{D(T)} = T(0)^\perp$ by Corollary 2.1. Hence, T_s is a densely defined operator in $T(0)^\perp$. Then, from [4, Theorem 3.1], we have

$$\widetilde{W}_{e1}(T_s) \subset \widetilde{W}_{e4}(T_s) = \widetilde{W}_{e2}(T_s) = \widetilde{W}_{e3}(T_s) = \widetilde{W}_e(T_s),$$

which, together with (3.5), implies that (3.11) holds. Further, if $W(T) \neq \mathbb{C}$, by [4, p. 12, Claim 1)], we get

$$\widetilde{W}_{ei}(T_s) = \widetilde{W}_e(T_s), \quad i = 1, 2, 3, 4. \tag{3.13}$$

Then (3.12) holds by (3.5) and (3.13).

If $\overline{D(T) \cap D(T^*)} = T(0)^\perp$, then $\overline{D(T)} = T(0)^\perp$, which implies that T_s is a densely defined operator from $T(0)^\perp$ to $T(0)^\perp$. Let $(T_s)^*$ be the adjoint operator of T_s in $T(0)^\perp$ and $(T^*)_s$ be the operator part of T^*. We claim that $(T_s)^* = (T^*)_s$. In fact, for $y \in D((T^*)_s) = D(T^*)$ and $(x, f) \in T_s \subset T$, we have $\langle (T^*)_s y, x\rangle = \langle g, x\rangle = \langle y, f\rangle$, where $(y, g) \in T^*$ and $g = (T^*)_s y + h$ for some $h \in T(0)$. Then $y \in D((T_s)^*)$ and $(T^*)_s y = (T_s)^* y$, and consequently, $(T^*)_s \subset (T_s)^*$. Conversely, for $y \in D((T_s)^*)$ and $x \in D(T_s)$, we have $\langle (T_s)^* y, x\rangle = \langle y, T_s x\rangle$. In addition, for $(x, f) \in T$, we have $\langle y, T_s x\rangle = \langle y, f\rangle$ by $y \in D((T_s)^*) \subset T(0)^\perp$. Then, it follows that

$$\langle (T_s)^* y, x\rangle = \langle y, f\rangle, \quad y \in D((T_s)^*),\ (x, f) \in T.$$

Then $y \in D(T^*) = D((T^*)_s)$, which, together with $(T^*)_s \subset (T_s)^*$, implies that $(T^*)_s = (T_s)^*$. Hence

$$\overline{D(T_s) \cap D((T_s)^*)} = \overline{D(T_s) \cap D((T^*)_s)} = \overline{D(T) \cap D(T^*)} = T(0)^\perp.$$

Then, by [4, Theorem 3.1], (3.13) holds. Together with (3.5), we get that (3.12) holds. This completes the proof.

Remark 3.2 (1) If T is self-adjoint, then $\overline{D(T) \cap D(T^*)} = \overline{D(T)} = T(0)^\perp$, and hence (3.12) holds. For a general relation T in X, if $W(T) \neq \mathbb{C}$ and some open connected component of $\mathbb{C}\setminus\overline{W(T)}$ is contained in $\rho(T)$, then $T(0)=T^*(0)$ by [11, Lemma 2.32], and hence (3.11) holds.

(2) It was shown that (3.12) holds for a bounded operator T on X by [10, Theorem 5.1], where $W_e(T)=W_{e1}(T)$ is not explicitly stated in [10, Theorem 5.1], but it can be read off from the proof. This work was extended to unbounded operators and it has been proved in [4, Theorem 3.1] that (3.10) holds for a general operator T, (3.11) holds for T to be densely defined, and (3.12) holds for T satisfying $\overline{D(T)}=X$ and $W(T)\neq\mathbb{C}$, or $\overline{D(T)\cap D(T^*)}=X$. Hence, Theorem 3.2 extends [10, Theorem 5.1] and [4, Theorem 3.1] for operators to closed relations.

Theorem 3.2 presents relationships among $W_e(T)$ and $W_{ei}(T), i=1,2,3,4$, for a closed relation T under certain conditions. Further, we give the relationships for general relations.

Theorem 3.3 *Let T be a relation in X. Then* (3.10) *holds. Further, if $D(T)$ is infinite-dimensional and $W(T)=\mathbb{C}$, then*

$$W_{e1}(T)\subset W_{e4}(T)=W_{e2}(T)=W_{e3}(T)=W_e(T)=\mathbb{C}. \tag{3.14}$$

In particular, if $\overline{D(T)}=X$ and $W(T)\neq\mathbb{C}$, then (3.12) *holds.*

Before proving Theorem 3.3, we first prepare the following lemma, which can be derived with a similar argument to that of [4, Lemma 2.7]. Therefore, we omit the details here.

Lemma 3.2 *Let T be a relation in X, $W(T)=\mathbb{C}$ and $(y,g)\in T$ with $\|y\|=1$. If $\{y\}^\perp\cap D(T)\neq\{0\}$, then, for every $\varepsilon>0$, there exists $(w_\varepsilon,h_\varepsilon)\in T$ with $\|w_\varepsilon\|=1$ such that*

$$W(T|_{\{w_\varepsilon\}^\perp\cap D(T)})=\mathbb{C},\quad |\langle h_\varepsilon,w_\varepsilon\rangle-\langle g,y\rangle|<\varepsilon. \tag{3.15}$$

Proof of Theorem 3.3 It is easy to see that $W_{e4}(T)\subset W_e(T)\subset W_{e3}(T)\subset W_{e2}(T)$. Since every compact operator is the norm limit of finite rank operators, we have $W_{e3}(T)\supset W_{e2}(T)$. Similar to the proof of [4, Theorem 3.1], the conclusion $W_{e1}(T)\subset W_{e4}(T)$ can be proved by inductively constructing a sequence. By the above discussions, one has

$$W_{e1}(T)\subset W_{e4}(T)\subset W_e(T)\subset W_{e3}(T)=W_{e2}(T). \tag{3.16}$$

In order to prove (3.10), it suffices to show $W_{e3}(T)=W_e(T)$. First, we consider the case of $W(T)\neq\mathbb{C}$. Let P be an orthogonal projection in X onto $T(0)^\perp$. Then $PT(0)=0$, which yields that PT is single-valued. Then, for every $(x,f)\in T$, there exists $g\in\overline{T(0)}$ such that $f=PTx+g$. If $W(T)\neq\mathbb{C}$, then $T(0)\perp D(T)$ by Lemma 3.1, which implies

$$\langle f,x\rangle=\langle PTx,x\rangle,\quad (x,f)\in T. \tag{3.17}$$

Then

$$W_e(T)=W_e(PT)\quad\text{and}\quad W_{e3}(T)=W_{e3}(PT). \tag{3.18}$$

Note that PT is an operator. Then, by [4, Theorem 3.1, formula (3.1)], we have $W_e(PT)=W_{e3}(PT)$. Hence, $W_{e3}(T)=W_e(T)$ by (3.18). If $W(T)=\mathbb{C}$, then $W_e(T)=\mathbb{C}$ by (iv) of Proposition 3.3, and hence $W_{e3}(T)=W_e(T)=\mathbb{C}$ by (3.16). This completes the proof of (3.10).

Next, we prove (3.14). Suppose that $W(T) = \mathbb{C}$. Then $W_{e2}(T) = W_{e3}(T) = W_e(T) = W(T) = \mathbb{C}$ by (iv) of Proposition 3.3 and (3.16). It remains to show that $W_{e4}(T) = \mathbb{C}$. It is noted that if $N \subset D(T)$ is a finite-dimensional subspace, then it can be easily verified that $N^\perp \cap D(T) \neq \{0\}$. Next, we shall show the result by mathematical induction method. Take $\lambda \in \mathbb{C}$. Since $W(T) = \mathbb{C}$, there exists $(y_1, g_1) \in T$ with $\|y_1\| = 1$ satisfying $\langle g_1, y_1\rangle = \lambda$. Noting that $\{y_1\}^\perp \cap D(T) \neq \{0\}$ by $\{y_1\} \subset D(T)$, and applying Lemma 3.2 with $\varepsilon = 1$ and (y, g) replaced by (y_1, g_1), one sees that there exists $(e_1, h_1) \in T$ with $\|e_1\| = 1$ such that

$$W(T|_{\{e_1\}^\perp \cap D(T)}) = \mathbb{C}, \quad |\langle h_1, e_1\rangle - \lambda| < 1.$$

Since $W(T|_{\{e_1\}^\perp \cap D(T)}) = \mathbb{C}$, we can choose $(y_2, g_2) \in T$ such that $\|y_2\| = 1$, $y_2 \in \{e_1\}^\perp$ and $\langle g_2, y_2\rangle = \lambda$. Next, it is assumed that we have constructed $(e_1, h_1), \cdots, (e_{n-1}, h_{n-1}) \in T$ and $(y_1, g_1), \cdots, (y_n, g_n) \in T$ such that $\|e_{n-1}\| = \|y_{n-1}\| = 1$, $y_n \in X_{n-1}^\perp \cap D(T)$, $W(T|_{X_{n-1}^\perp \cap D(T)}) = \mathbb{C}$ and

$$\langle g_n, y_n\rangle = \lambda, \quad |\langle h_{n-1}, e_{n-1}\rangle - \lambda| < \frac{1}{n-1}, \quad n > 1,$$

where $X_{n-1} := \{e_1, \cdots, e_{n-1}\}$. Note that $\{y_n\} \subset X_{n-1}^\perp \cap D(T)$ implies that $\{y_n\}^\perp \cap X_{n-1}^\perp \cap D(T) \neq \{0\}$. Then, applying Lemma 3.2 with $\varepsilon = \frac{1}{n}$, T replaced by $T|_{X_{n-1}^\perp \cap D(T)}$, and (y, g) replaced by (y_n, g_n), one sees that there exists $(e_n, h_n) \in T$ with $\|e_n\| = 1$ such that

$$W(T|_{\{e_1,\cdots,e_n\}^\perp \cap D(T)}) = W(T|_{\{e_n\}^\perp \cap X_{n-1}^\perp \cap D(T)}) = \mathbb{C} \tag{3.19}$$

and

$$|\langle h_n, e_n\rangle - \lambda| < \frac{1}{n}.$$

Now, (3.19) allows us to choose $(y_{n+1}, g_{n+1}) \in T$ with $\|y_{n+1}\| = 1$ satisfying

$$y_{n+1} \in \{e_1, \cdots, e_n\}^\perp \cap D(T) \quad \text{and} \quad \langle g_{n+1}, y_{n+1}\rangle = \lambda.$$

This completes the induction. It implies that there exists an orthonormal sequence $\{(e_n, h_n)\}_{n\in\mathbb{N}}$ satisfying $\langle h_n, e_n\rangle \xrightarrow{n\to\infty} \lambda$. Since λ is arbitrary, we get $W_{e4}(T) = \mathbb{C}$.

Finally, suppose that $\overline{D(T)} = X$ and $W(T) \neq \mathbb{C}$. Then, one has $T(0) = \{0\}$ by Lemma 3.1, which implies that T can determine an operator. Hence, (3.12) holds by [4, p. 12, Claim 1]. This completes the proof.

Remark 3.3 (1) From Theorem 3.3, for a general relation T, if $\overline{D(T)} = X$ and $W(T) \neq \mathbb{C}$, then (3.12) holds. Here, we point out that the restriction $W(T) \neq \mathbb{C}$ is necessary. In fact, let T be the relation defined in Remark 3.1 with E being finite-dimensional. Then (3.14) holds for this T by Theorem 3.3. However, $W_{e1}(T) \subset W(T|_{E^\perp \cap D(T)}) = \{1\}$ by the definition. Then $W_{e1}(T) \subset W_e(T)$ in (3.14) is strict, which implies that (3.12) is false for this T.

(2) From Theorem 3.3, $W_{e1}(T) \subset W_e(T)$ holds for a general relation T with $D(T)$ being infinite-dimensional and $W(T) = \mathbb{C}$. Here, we point out that $W_{e1}(T) = W_e(T)$ maybe hold. In fact, let T be the relation defined in Remark 3.1 with $E = X$. Then it follows from Lemma 3.1 that $W(T|_{V^\perp \cap D(T)}) = \mathbb{C}$ for every finite-dimensional subspace V. It implies that $W_{e1}(T) = \mathbb{C}$, and hence $W_{e1}(T) = W_e(T) = \mathbb{C}$.

(3) If $D(T)$ is finite-dimensional, then $W_{e1}(T) = W_{e4}(T) = \emptyset$, which implies that (3.14) is false.

4 Singular Discrete Linear Hamiltonian Systems and Their Essential Numerical Ranges

In this section, we consider singular discrete linear Hamiltonian systems which may be non-symmetric. First, we introduce the associated maximal, pre-minimal and minimal relations in a product Hilbert space. Then, some sufficient and necessary conditions for the minimal relation to be an operator are given under certain conditions, and a sufficient condition for the minimal relation to be not densely defined is derived. Finally, we consider various essential numerical ranges of the minimal relation H_0. This section is divided into two parts.

4.1 Singular discrete linear Hamiltonian systems and their minimal relations

Consider the following singular discrete linear Hamiltonian system

$$\mathcal{L}(y)(t) := J\Delta y(t) - P(t)R(y)(t) = \lambda W(t)R(y)(t), \quad t \in \mathcal{I}, \tag{4.1}$$

where

(a) $\mathcal{I} := \{t\}_{t=a}^{+\infty}$ is an integer set with a being a finite integer;

(b) Δ is the forward difference operator, i.e., $\Delta y(t) = y(t+1) - y(t)$;

(c) J is the canonical symplectic matrix, i.e., $J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$, where I_d is the $d \times d$ identity matrix;

(d) for $t \in \mathcal{I}$, the weight function $W(t) = \operatorname{diag}\{W_1(t), W_2(t)\}$, where $W_1(t)$ and $W_2(t)$ are $d \times d$ matrices and $W_1(t), W_2(t) \geq 0$;

(e) for $t \in \mathcal{I}$, $P(t) = \begin{pmatrix} -C(t) & D(t) \\ A(t) & B(t) \end{pmatrix}$, where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are $d \times d$ matrices;

(f) $R(y)(t) = (x^{\mathrm{T}}(t+1), u^{\mathrm{T}}(t))^{\mathrm{T}}$ is the partial right shift operator with $y(t) = (x^{\mathrm{T}}(t), u^{\mathrm{T}}(t))^{\mathrm{T}}$ for $x(t)$, $u(t) \in \mathbb{C}^d$;

(g) λ is a complex spectral parameter.

By the condition (e), system (4.1) can be written as

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t) + \lambda W_2(t)u(t), \\ \Delta u(t) = C(t)x(t+1) - D(t)u(t) - \lambda W_1(t)x(t+1), \quad t \in \mathcal{I}. \end{cases} \tag{4.2}$$

If $P = P^*$, i.e., $D = A^*$, B and C are Hermitian matrices, then system (4.1) is (formally) symmetric. To ensure the existence and uniqueness of the solution of any initial value problem associated with system (4.1), it is always assumed that

$(\mathbf{A}_1)$ $\qquad$ $I_d - A(t)$ and $I_d - D(t)$ are invertible in $\mathcal{I}$.

Next, we introduce the following space

$$l_W^2(\mathcal{I}) := \left\{ y = \{y(t)\}_{t\in\mathcal{I}} \subset \mathbb{C}^{2d} : \sum_{t\in\mathcal{I}} R(y)^*(t)W(t)R(y)(t) < +\infty \right\}$$

with the semi-inner product

$$\langle y, z\rangle_W := \sum_{t\in\mathcal{I}} R^*(z)(t)W(t)R(y)(t).$$

We denote $\|y\|_W := \langle y, y\rangle_W^{\frac{1}{2}}$ for $y \in l^2_W(\mathcal{I})$. Since the weight function $W(t)$ may be singular in $\mathcal{I}$, $\|\cdot\|_W$ is a semi-norm. Set

$$L^2_W(\mathcal{I}) := l^2_W(\mathcal{I})/\{y \in l^2_W(\mathcal{I}) : \|y\|_W = 0\}.$$

Then $L^2_W(\mathcal{I})$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_W$. For a function $y \in l^2_W(\mathcal{I})$, we denote by $\widetilde{y}$ the corresponding equivalent class in $L^2_W(\mathcal{I})$. Then, $\langle\widetilde{y}, \widetilde{z}\rangle_W = \langle y, z\rangle_W$ for any $\widetilde{y}, \widetilde{z} \in L^2_W(\mathcal{I})$. In order to avoid confusion, we write $\widetilde{0}$ to denote the zero element in $L^2_W(\mathcal{I})$.

Set

$$\begin{aligned} l^2_{W,0}(\mathcal{I}) : = \{y \in l^2_W(\mathcal{I}) :\ & \text{There exist } s, k \in \mathcal{I} \text{ with } s \le k \text{ such that} \\ & y(t) = 0 \text{ for } t \le s \text{ and } t \ge k+1\} \end{aligned}$$

and

$$\begin{aligned} H : = \{(\widetilde{y}, \widetilde{g}) \in (L^2_W(\mathcal{I}))^2 :\ & \text{There exists } y \in \widetilde{y} \text{ such that } \mathcal{L}(y)(t) = W(t)R(g)(t) \\ & \text{for any } g \in \widetilde{g},\ t \in \mathcal{I}\}, \\ H_{00} : = \{(\widetilde{y}, \widetilde{g}) \in H :\ & \text{There exists } y \in \widetilde{y} \text{ such that } y \in l^2_{W,0}(\mathcal{I}) \text{ and} \\ & \mathcal{L}(y)(t) = W(t)R(g)(t) \text{ for any } g \in \widetilde{g},\ t \in \mathcal{I}\}. \end{aligned}$$

Clearly, H and H_{00} are subspaces of $(L^2_W(\mathcal{I}))^2$. Then H, H_{00} and $H_0 := \overline{H}_{00}$ are called the maximal, pre-minimal and minimal relations associated with system (4.1), respectively. The definiteness condition for system (4.1) is given by

($\mathbf{A}_2$) there exists a finite subinterval $\mathcal{I}_0 := [s_0, t_0] = \{t\}_{t=s_0}^{t_0} \subset \mathcal{I}$ such that for all $\lambda \in \mathbb{C}$ any non-trivial solution y of system (4.1) satisfies that

$$\sum_{t\in\mathcal{I}_0} R(y)^*(t)W(t)R(y)(t) > 0.$$

Remark 4.1 (1) From ($\mathbf{A}_2$), for every $(\widetilde{y}, \widetilde{g}) \in H$, there exists a unique $y \in \widetilde{y}$ such that $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for any $g \in \widetilde{g}$ and $t \in \mathcal{I}$. In this case, we write $(y, \widetilde{g}) \in H$ for brevity.

(2) It was pointed out that H_0 may be non-densely defined or multivalued, even if ($\mathbf{A}_2$) holds, see [17, Section 6] or Theorem 4.1 and Proposition 4.2 below. This is an important difference between the discrete and continuous Hamiltonian systems.

Furthermore, we have the following result for the minimal relation H_0.

Proposition 4.1 *Assume that* ($\mathbf{A}_1$) *and* ($\mathbf{A}_2$) *hold. If $y \in D(H_0)$, then $y(a) = 0$.*

Proof Let $(y, \widetilde{g}) \in H_0$. Then there exists a sequence $\{(y_n, \widetilde{g}_n)\} \subset H_{00}$ such that

$$\|y_n - y\|_W \to 0 \quad \text{and} \quad \|g_n - g\|_W \to 0 \text{ for any } g \in \widetilde{g} \text{ and } g_n \in \widetilde{g}_n \quad \text{as } n \to \infty. \tag{4.3}$$

Let $y_{nk} = y_n - y_k$ and $g_{nk} = g_n - g_k$. Then y_{nk} is the solution of the initial value problem

$$\begin{cases} \mathcal{L}(z)(t) = W(t)R(g_{nk})(t), & t \in \mathcal{I}, \\ z(a) = 0. \end{cases}$$

Now, let $\Phi(t)$ be the fundamental solution matrix of the system $\mathcal{L}(z)(t) = 0$ on $\mathcal{I}$ with $\Phi(a) = I_d$. For every $t \in \mathcal{I}$, we define

$$H(t) := P(t)\begin{pmatrix} 0 & 0 \\ 0 & I_d \end{pmatrix} + J = \begin{pmatrix} 0 & -I_d + D(t) \\ I_d & B(t) \end{pmatrix}.$$

Since $I_d - D(t)$ is invertible, we get that $H(t)$ is invertible on $t \in \mathcal{I}$. Let $Y(t) := \Phi(t)C(t)$ be a solution of $\mathcal{L}(z)(t) = W(t)R(g_{nk})(t)$. Then we have

$$P(t)[R(\Phi)(t)C(t+1) - R(\Phi C)(t)] + J\Phi(t)\Delta C(t) = W(t)R(g_{nk})(t),$$

which yields that $H(t)\Phi(t)\Delta C(t) = W(t)R(g_{nk})(t)$ by the direct calculation. Hence, y_{nk} can be expressed by

$$y_{nk}(t) = \Phi(t)\sum_{s=a}^{t-1}\Phi^{-1}(s)H^{-1}(s)W(s)R(g_{nk})(s), \quad t \in \mathcal{I}. \tag{4.4}$$

Here, we use the convention that $\sum\limits_{s=a}^{a-1}\Phi^{-1}(s)H^{-1}(s)W(s)R(g_{nk})(s) = 0$.

Take $t_0 \in \mathcal{I} \setminus \{a\}$. Clearly, $\Phi(t)$ and $\Phi^{-1}(t)H^{-1}(t)$ are bounded on $[a, t_0] := \{t\}_{t=a}^{t_0} \subset \mathcal{I}$. Then, together with $\|g_{nk}\|_W \to 0$ and (4.4), we have $y_{nk}(t) \to 0$ for $t \in [a, t_0]$ as $n, k \to \infty$. Set

$$\widehat{y}(t) = \lim_{n\to\infty} y_n(t), \quad t \in \mathcal{I}.$$

Then $\widehat{y}(a) = 0$ and it is clear that $\lim\limits_{n\to\infty}\mathcal{L}(y_n)(t) = \mathcal{L}(\widehat{y})(t)$ for $t \in \mathcal{I}$, which, together with the second relation of (4.3) and $\mathcal{L}(y_n)(t) = W(t)R(g_n)(t),\ t \in \mathcal{I}$, yields that

$$\mathcal{L}(\widehat{y})(t) = W(t)R(g)(t) \quad \text{for any } g \in \widetilde{g},\ t \in \mathcal{I}.$$

Then $(\widetilde{\widehat{y}}, \widetilde{g}) \in H$. Clearly, $\|y - \widehat{y}\|_W = 0$. By (1) of Remark 4.1, we get $y(t) = \widehat{y}(t)$ for $t \in \mathcal{I}$. Then $y(a) = \widehat{y}(a) = 0$. This completes the proof.

In [17, Section 6], they give some sufficient and necessary conditions for the minimal relation H_0 to be an operator in the symmetric case; that is, $H_0(\widetilde{0}) = \{\widetilde{0}\}$. By Proposition 4.1, we extended these results to non-symmetric systems with a similar method, i.e., Theorem 4.1 below. First, we introduce some notations. Since $W_1(t)$ and $W_2(t)$ are Hermitian matrices for every $t \in \mathcal{I}$, one has

$$\mathbb{C}^d = \operatorname{Ker} W_1(t) \oplus \operatorname{Ran} W_1(t) = \operatorname{Ker} W_2(t) \oplus \operatorname{Ran} W_2(t), \quad t \in \mathcal{I}.$$

For every $t \in \mathcal{I}$, by $P_1(t)$ and $P_2(t)$ we denote the projection maps from $\mathbb{C}^d$ to $\operatorname{Ker} W_1(t)$ and $\operatorname{Ker} W_2(t)$, respectively. Then, the following result is obtained.

Theorem 4.1 *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold.*

(1) *In the case of* $W_1(t) > 0$ *for* $t \in \mathcal{I}$, H_0 *is an operator if and only if for all* $t \in \mathcal{I} \setminus \{a\}$,

$$\operatorname{Ker} W_2(t) \cap \operatorname{Ker} B(t) = \{0\} \quad \text{and} \quad \operatorname{Ran} W_2(t) \cap \operatorname{Ran}(B(t)P_2(t)) = \{0\}. \tag{4.5}$$

(2) *In the case of* $W_2(t) > 0$ *for* $\in \mathcal{I}$, H_0 *is an operator if and only if for all* $t \in \mathcal{I}$,

$$\operatorname{Ker} W_1(t) \cap \operatorname{Ker} C(t) = \{0\} \quad \text{and} \quad \operatorname{Ran} W_1(t) \cap \operatorname{Ran}(C(t)P_1(t)) = \{0\}. \tag{4.6}$$

Proof We first consider the sufficiency of assertion (1). Let $(\widetilde{0}, \widetilde{g}) \in H_0$. Then, by (1) of Remark 4.1, there exists a unique $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}} \in \widetilde{0}$ such that $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for any $g = (g_1^{\mathrm{T}}, g_2^{\mathrm{T}})^{\mathrm{T}} \in \widetilde{g}$, i.e.,

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t) + W_2(t)g_2(t), \\ \Delta u(t) = C(t)x(t+1) - D(t)u(t) - W_1(t)g_1(t+1), \quad t \in \mathcal{I}. \end{cases}$$

Then we have

$$\begin{cases} (I_d - A(t))x(t+1) - x(t) - B(t)u(t) = W_2(t)g_2(t), \\ (I_d - D(t))u(t) - u(t+1) + C(t)x(t+1) = W_1(t)g_1(t+1), \quad t \in \mathcal{I}. \end{cases} \tag{4.7}$$

It follows from $W_1(t) > 0$ for $t \in \mathcal{I}$ and $y \in \widetilde{0}$ that

$$x(t+1) = 0 \quad \text{and} \quad W_2(t)u(t) = 0, \quad t \in \mathcal{I}, \tag{4.8}$$

which implies $u(t) \in \operatorname{Ker} W_2(t)$. In addition, $x(a) = 0$ by Proposition 4.1. Then, by (4.7) and the first relation of (4.8), we have

$$\begin{cases} -B(t)u(t) = W_2(t)g_2(t), \\ (I_d - D(t))u(t) - u(t+1) = W_1(t)g_1(t+1), \quad t \in \mathcal{I}. \end{cases} \tag{4.9}$$

From $u(t) \in \operatorname{Ker} W_2(t)$, the first relation of (4.9), and the second relation of (4.5), one has $-B(t)u(t) = W_2(t)g_2(t) = 0$ for $t \in \mathcal{I} \setminus \{a\}$. Therefore, $u(t) \in \operatorname{Ker} W_2(t) \cap \operatorname{Ker} B(t)$. Then, by the first relation of (4.5), one has $u(t) = 0$ for $t \in \mathcal{I} \setminus \{a\}$, which, together with $u(a) = 0$ by Proposition 4.1, yields that $u(t) = 0$ for $t \in \mathcal{I}$. Hence, $g_1(t+1) = 0$ for $t \in \mathcal{I}$ by the second relation of (4.9), $W_1(t) > 0$ and $u(t) = 0$ for $t \in \mathcal{I}$. In addition, one has $W_2(t)g_2(t) = 0$ for $t \in \mathcal{I}$ by the first relation of (4.9) and $u(t) = 0$ for $t \in \mathcal{I}$. Then $g \in \widetilde{0} \in L^2_W(\mathcal{I})$, which implies $H_0(\widetilde{0}) = \{\widetilde{0}\}$. Hence, H_0 is an operator.

Next, we consider the necessity of assertion (1). Let H_0 be an operator. Suppose on the contrary that the first relation of (4.5) does not hold. Then there exist $t_0 \in \mathcal{I} \setminus \{a\}$ and $0 \neq \xi \in \mathbb{C}^d$ such that $\xi \in \operatorname{Ker} W_2(t_0) \cap \operatorname{Ker} B(t_0)$. Since $W_1(t) > 0$ and $I_d - D(t)$ is invertible for $t \in \mathcal{I}$, there exist $0 \neq \eta_1, \eta_2 \in \mathbb{C}^d$ such that $W_1(t_0 - 1)\eta_1 = -\xi$ and $W_1(t_0)\eta_2 = (I_d - D(t_0))\xi$. Set

$$y(t) = \begin{cases} (0^{\mathrm{T}}, \xi^{\mathrm{T}})^{\mathrm{T}}, & t = t_0, \\ 0, & t \in \mathcal{I} \setminus \{t_0\}, \end{cases} \qquad R(g)(t) = \begin{cases} (\eta_1^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}}, & t = t_0 - 1, \\ (\eta_2^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}}, & t = t_0, \\ 0, & t \in \mathcal{I} \setminus \{t_0 - 1, t_0\}. \end{cases}$$

Then $y \in \widetilde{0}$, $g \notin \widetilde{0}$, and it is easy to verify that $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in \mathcal{I}$. It follows that $\widetilde{g} \in H_0(\widetilde{0})$, and hence H_0 is not an operator. This is a contradiction. Hence, the first relation of (4.5) holds. Now, suppose on the contrary that the second relation of (4.5) does not hold. Then there exist $t_0 \in \mathcal{I} \setminus \{a\}$ and $0 \neq \nu \in \mathbb{C}^d$ such that $\nu \in \operatorname{Ran} W_2(t_0) \cap \operatorname{Ran}(B(t_0)P_2(t_0))$. It follows from $\nu \in \operatorname{Ran}(B(t_0)P_2(t_0))$ that there exists $0 \neq \xi \in \operatorname{Ker} W_2(t_0)$ such that $-B(t_0)\xi = \nu$. In addition, $\nu \in \operatorname{Ran} W_2(t_0)$ implies that there exists $0 \neq \zeta$ such that $W_2(t_0)\zeta = \nu$. Since $W_1(t) > 0$ and $I_d - D(t)$ is invertible for $t \in \mathcal{I}$, there exist $0 \neq \eta_1, \eta_2 \in \mathbb{C}^d$ such that $W_1(t_0 - 1)\eta_1 = -\xi$ and $W_1(t_0)\eta_2 = (I_d - D(t_0))\xi$. Set

$$y(t) = \begin{cases} (0^{\mathrm{T}}, \xi^{\mathrm{T}})^{\mathrm{T}}, & t = t_0, \\ 0, & t \in \mathcal{I} \setminus \{t_0\}, \end{cases} \qquad R(g)(t) = \begin{cases} (\eta_1^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}}, & t = t_0 - 1, \\ ({\eta_2}^{\mathrm{T}}, \zeta^{\mathrm{T}})^{\mathrm{T}}, & t = t_0, \\ 0, & t \in \mathcal{I} \setminus \{t_0 - 1, t_0\}. \end{cases}$$

Then $y \in \widetilde{0}$, $g \notin \widetilde{0}$, and it is easy to verify that $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in \mathcal{I}$. It follows that $\widetilde{g} \in H_0(\widetilde{0})$, and hence H_0 is not an operator. This is a contradiction. Hence, (4.5) holds. Assertion (2) can be proved similarly. This completes the proof.

The following specific conditions for H_0 to be an operator can be obtained by Theorem 4.1 directly.

Corollary 4.1 *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold.*

(1) H_0 *is an operator if one of the following conditions holds:*

(i) $W_1(t) > 0$ *for* $t \in \mathcal{I}$ *and* $W_2(t) > 0$ *for* $t \in \mathcal{I} \setminus \{a\}$;

(ii) $W_1(t) > 0$ *for* $t \in \mathcal{I}$ *and* $B(t) \equiv I_d$ *for* $t \in \mathcal{I} \setminus \{a\}$;

(iii) $W_2(t) > 0$ *for* $t \in \mathcal{I}$ *and* $C(t) \equiv I_d$ *for* $t \in \mathcal{I}$.

(2) *Let* $W_1(t) > 0$ *for* $t \in \mathcal{I}$ *and* $W_2(t) \equiv 0$ *for* $t \in \mathcal{I} \setminus \{a\}$. *Then* H_0 *is an operator if and only if* $B(t)$ *is invertible for each* $t \in \mathcal{I} \setminus \{a\}$.

(3) *Let* $W_1(t) \equiv 0$ *for* $t \in \mathcal{I}$ *and* $W_2(t) > 0$ *for* $t \in \mathcal{I}$. *Then* H_0 *is an operator if and only if* $C(t)$ *is invertible for each* $t \in \mathcal{I}$.

Next, we present a sufficient condition for H_0 to be non-densely defined in any significant case

Proposition 4.2 *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold. If there exists* $t_0 \in \mathcal{I}$ *such that* $W(t_0) \neq 0$, *then* H_0 *is non-densely defined.*

Proof Let $t_0 \in \mathcal{I}$ be the minimal point such that $W(t_0) \neq 0$, i.e., $W(t) = 0$ for $a \leq t \leq t_0 - 1$. The proof is divided into two cases:

Case 1 $t_0 = a$. If $W_2(a) \neq 0$, then there exists $0 \neq \xi \in \mathbb{C}^d$ such that $W_2(a)\xi \neq 0$. Take $z(a) = (0, \xi^{\mathrm{T}})^{\mathrm{T}}$ and $z(t) = 0$, $t \geq a + 1$. It is clear that $\widetilde{z} \in L^2_W(\mathcal{I})$. For any $(y, \widetilde{g}) \in H_0$, it follows from Proposition 4.1 that $y(a) = 0$, and consequently,

$$\|z - y\|^2{}_W \geq \xi^* W_2(a)\xi > 0,$$

which implies that there exists no sequence $\{(y_n, \widetilde{g}_n)\} \subset H_0$ such that $\widetilde{y}_n \to \widetilde{z}$ in $L^2_W(\mathcal{I})$. Hence, H_0 is non-densely defined in $L^2_W(\mathcal{I})$. On the other hand, if $W_2(a) = 0$, then $W_1(a) \neq 0$ by the assumption. Take $\widetilde{z} \in L^2_W(\mathcal{I})$ and $z = (\eta^{\mathrm{T}}, \nu^{\mathrm{T}})^{\mathrm{T}} \in \widetilde{z}$ with $\eta(a+1) \neq 0$. For any $(y, \widetilde{g}) \in H_0$ with $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$, it follows from Proposition 4.1 that $y(a) = 0$. Consequently, by the first relation of (4.7), $(\mathbf{A}_1)$ and $W_2(a) = 0$, we get $x(a+1) = 0$. Then

$$\|z - y\|^2{}_W \geq \eta(a+1)^* W_1(a)\eta(a+1) > 0.$$

Therefore, H_0 is non-densely defined in $L^2_W(\mathcal{I})$.

Case 2 $t_0 > a$. If $W_2(t_0) \neq 0$, then we take $\widetilde{z} \in L^2_W(\mathcal{I})$ and $z = (\eta^{\mathrm{T}}, \nu^{\mathrm{T}})^{\mathrm{T}} \in \widetilde{z}$ with $\nu(t_0) \neq 0$. For any $(y, \widetilde{g}) \in H_0$ with $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$, one has $y(t) = 0$ for $t \in [a, t_0]$ by Proposition 4.1, (4.7), $(\mathbf{A}_1)$ and $W(t) = 0$ for $t \leq t_0 - 1$. Then

$$\|y - z\|^2{}_W \geq \nu(t_0)^* W_2(t_0)\nu(t_0) > 0.$$

Therefore, H_0 is non-densely defined in $L^2_W(\mathcal{I})$. On the other hand, if $W_2(t_0) = 0$, then $W_1(t_0) \neq 0$. Take $\widetilde{z} \in L^2_W(\mathcal{I})$ and $z = (\eta^{\mathrm{T}}, \nu^{\mathrm{T}})^{\mathrm{T}} \in \widetilde{z}$ with $\eta(t_0+1) \neq 0$. For any $(y, \widetilde{g}) \in H_0$ with $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$, one has $y(t) = 0$ for $t \in [a, t_0]$ and $x(t_0+1) = 0$ by Proposition 4.1, (4.7), $(\mathbf{A}_1)$, $W_2(t_0) = 0$ and $W(t) = 0$ for $t \leq t_0 - 1$. Then

$$\|y - z\|^2{}_W \geq \eta(t_0+1)^* W_1(t_0)\eta(t_0+1) > 0,$$

which implies that H_0 is non-densely defined in $L^2_W(\mathcal{I})$. This completes the proof.

4.2 Essential numerical ranges of singular discrete linear Hamiltonian systems

In this subsection, we apply some results for abstract relations to the minimal relation H_0 and consider various essential numerical ranges of H_0. By Propositions 2.1–2.2, 3.1, 3.4 and Theorem 3.2, if $D(H_0) \subset H_0(\widetilde{0})^{\perp}$, then we get

$$\begin{aligned} &\sigma_p(H_0) = \sigma_p(H_{0,s}), \quad \sigma(H_0) = \widetilde{\sigma}(H_{0,s}), \\ &W(H_0) = W(H_{0,s}), \quad W_e(H_0) = W_e(H_{0,s}) \end{aligned} \tag{4.10}$$

and

$$W_{e1}(H_0) \subset W_{e4}(H_0) \subset W_{e2}(H_0) = W_{e3}(H_0) = W_e(H_0), \tag{4.11}$$

where $H_{0,s}$ is the operator part of H_0. Furthermore, for H_0 being an operator, we get that $D(H_0) \subset H_0(\widetilde{0})^{\perp}$ holds since $H_0(\widetilde{0}) = \{\widetilde{0}\}$, and thus (4.10)–(4.11) hold. This implies that Theorem 4.1 gives sufficient conditions for (4.10)–(4.11). Next, we shall present some sufficient conditions for general cases. Before giving them, we introduce the following notations:

$$\begin{aligned} &\mathcal{I}_1 := \{t \in \mathcal{I} : \ W_1(t) > 0, \ W_2(t) = 0\}, \\ &\mathcal{I}_2 := \{t \in \mathcal{I} : \ W_1(t) = 0, \ W_2(t) > 0\}, \\ &\mathcal{I}_3 := \{t \in \mathcal{I} : \ W_1(t) = 0, \ W_2(t) = 0\} \end{aligned}$$

and $\mathcal{I}_4 := \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3) = \{t \in \mathcal{I} : \ W_1(t) > 0, \ W_2(t) > 0\}$.

Theorem 4.2 *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold. If* $B(t)$ *is Hermitian for* $t \in \mathcal{I}_1 \cup \mathcal{I}_3$, $C(t)$ *is Hermitian for* $t \in \mathcal{I}_2 \cup \mathcal{I}_3$ *and* $A(t) = D^*(t)$ *for* $t \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$, *then* (4.10)–(4.11) *hold.*

Proof By the above discussions, we only need to prove $D(H_0) \subset H_0(\widetilde{0})^{\perp}$. If $H_0(\widetilde{0}) \subset H_0^*(\widetilde{0})$, then we get $D(H_0) \subset H_0(\widetilde{0})^{\perp}$ since $D(H_0) \subset H_0^*(\widetilde{0})^{\perp}$ by Lemma 2.1. Hence, it suffices to show $H_0(\widetilde{0}) \subset H_0^*(\widetilde{0})$. Let $(y, \widetilde{g}) \in H_0$ and $y \in \widetilde{0}$. We shall show $(y, \widetilde{g}) \in H_{00}^* = H_0^*$. For $(z, \widetilde{f}) \in H_{00}$, we have

$$\begin{aligned}\langle \widetilde{y}, \widetilde{f} \rangle_W - \langle \widetilde{g}, \widetilde{z} \rangle_W &= \sum_{t \in \mathcal{I}} R(f)^*(t)W(t)R(y)(t) - R(z)^*(t)W(t)R(g)(t) \\ &= \sum_{t \in \mathcal{I}} [\mathcal{L}(z)^*(t)R(y)(t) - R(z)^*(t)\mathcal{L}(y)(t)] \\ &= \sum_{t \in \mathcal{I}} R(z)^*(t)(P - P^*)(t)R(y)(t) = \sum_{i=1}^{4} \sum_{t \in \mathcal{I}_i} \mathcal{K}(z, P, y)(t), \end{aligned} \tag{4.12}$$

where $\mathcal{K}(z, P, y)(t) := R(z)^*(t)(P - P^*)(t)R(y)(t)$. In addition, we have

$$x(t+1) = 0, \quad t \in \mathcal{I}_1; \quad u(t) = 0, \quad t \in \mathcal{I}_2; \quad R(y)(t) = 0, \quad t \in \mathcal{I}_4 \tag{4.13}$$

by $y \in \widetilde{0}$, i.e., $\sum\limits_{t \in \mathcal{I}} R(y)^*(t)W(t)R(y)(t) = 0$. Clearly, by the third relation of (4.13), one has $\sum\limits_{t \in \mathcal{I}_4} \mathcal{K}(z, P, y)(t) = 0$. Let $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$ and $z = (\eta^{\mathrm{T}}, \nu^{\mathrm{T}})^{\mathrm{T}}$. Then, by the first two relations of (4.13), one has

$$\begin{aligned}\sum_{t \in \mathcal{I}_1} \mathcal{K}(z, P, y)(t) &= \sum_{t \in \mathcal{I}_1} (\eta^*(t+1)(D - A^*)(t) + \nu^*(t)(B - B^*)(t))u(t), \\ \sum_{t \in \mathcal{I}_2} \mathcal{K}(z, P, y)(t) &= \sum_{t \in \mathcal{I}_2} (\eta^*(t+1)(-C + C^*)(t) + \nu^*(t)(A - D^*)(t))x(t+1).\end{aligned}$$

Therefore, $\sum\limits_{t \in \mathcal{I}_1} \mathcal{K}(z, P, y)(t) = \sum\limits_{t \in \mathcal{I}_2} \mathcal{K}(z, P, y)(t) = 0$ by the assumptions. On the other hand, we have $P(t) = P^*(t)$ for $t \in \mathcal{I}_3$ since $D(t) = A^*(t)$, $B(t) = B^*(t)$ and $C(t) = C^*(t)$ for $t \in \mathcal{I}_3$, which yields $\sum\limits_{t \in \mathcal{I}_3} \mathcal{K}(z, P, y)(t) = 0$. Then $\langle \widetilde{y}, \widetilde{f} \rangle_W - \langle \widetilde{g}, \widetilde{z} \rangle_W = 0$ by (4.12). This implies $(\widetilde{0}, \widetilde{g}) \in H_{00}^*$. Hence, $H_0(\widetilde{0}) \subset H_0^*(\widetilde{0})$ since $H_{00}^* = H_0^*$. This completes the proof.

Remark 4.2 If system (4.1) is symmetric, i.e., $P^*(t) = P(t)$ for $t \in \mathcal{I}$, then the conditions in Theorem 4.2 hold, and hence (4.10)–(4.11) hold.

Now, we give another sufficient condition.

Theorem 4.3 *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold and* $A(t), B(t), C(t)$ *and* $D(t)$ *satisfy the conditions of Theorem* 4.2 *except the point* a. *If* $C(a)$ *is Hermitian or* $W_1(a) > 0$, *then* (4.10)–(4.11) *hold.*

Proof Similarly, we shall prove $H_0(\widetilde{0}) \subset H_0^*(\widetilde{0})$. Let $(y, \widetilde{g}) \in H_0$ and $y \in \widetilde{0}$. For $(z, \widetilde{f}) \in H_{00}$, we get from (4.12) and the assumptions that

$$\langle \widetilde{y}, \widetilde{f} \rangle_W - \langle \widetilde{g}, \widetilde{z} \rangle_W = \mathcal{K}(z, P, y)(a) + \sum_{i=1}^{4} \sum_{t \in \mathcal{I}_i \setminus \{a\}} \mathcal{K}(z, P, y)(t) = \mathcal{K}(z, P, y)(a).$$

Let $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}}$ and $z = (\eta^{\mathrm{T}}, \nu^{\mathrm{T}})^{\mathrm{T}}$. Then $u(a) = 0$ by Proposition 4.1, which together with $\nu(a) = 0$, one has

$$\mathcal{K}(z, P, y)(a) = \eta^*(a+1)(-C + C^*)(a)x(a+1).$$

If $C(a)$ is Hermitian, then $\mathcal{K}(z, P, y)(a) = 0$. On the other hand, if $W_1(a) > 0$, then $x(a+1) = 0$ by $y \in \widetilde{0}$, which yields $\mathcal{K}(z, P, y)(a) = 0$. Hence, $\langle \widetilde{y}, \widetilde{f}\rangle_W - \langle \widetilde{g}, \widetilde{z}\rangle_W = 0$. This implies $(\widetilde{0}, \widetilde{g}) \in H_{00}^*$, i.e., $H_0(\widetilde{0}) \subset H_0^*(\widetilde{0})$. This completes the proof.

Finally, we can get the following result when $\mathcal{I}$ is replaced by $\mathcal{I}' = \{t\}_{t=-\infty}^{+\infty}$.

Theorem 4.4 *Assume that* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *and* $P^*(t) = P(t)$ *hold for* $t \in \mathcal{I}'$. *If there exist* $k_1, k_2 \in \mathcal{I}'$ *with* $k_1 < k_2$ *such that one of the following conditions holds*:

(1) $W(t) > 0$ *for* $t \le k_1$ *and* $t \ge k_2$,

(2) $W_2(t) = 0$ *for* $t \in \mathcal{I}'$ *and* $W_1(t) > 0$, $B(t)$ *is invertible for* $t \le k_1$ *and* $t \ge k_2$,

(3) $W_1(t) = 0$ *for* $t \in \mathcal{I}'$ *and* $W_2(t) > 0$, $C(t)$ *is invertible for* $t \le k_1$ *and* $t \ge k_2$,

then

$$W_{ei}(H_0) = W_e(H_0), \quad i = 1, 2, 3, 4. \tag{4.14}$$

Proof Since $P(t) = P^*(t)$ for $t \in \mathcal{I}'$, we get that H_0 is Hermitian and $H_{00}^* = H_0^* = H$ by [16, Theorem 3.1]. Further, if $\overline{D(H_0)} = H_0(0)^{\perp}$, then we get $\overline{D(H_0) \cap D(H_0^*)} = \overline{D(H_0)} = H_0(0)^{\perp}$, and thus (4.14) holds by (3) of Theorem 3.2. Hence, by Corollary 2.1, it suffices to show $H_0^*(\widetilde{0}) = H_0(\widetilde{0})$. It follows from [17, Theorem 3.2] that

$$H_0 = \{(z, \widetilde{f}) \in H : \lim_{t\to-\infty} \eta^* J z(t) = \lim_{t\to+\infty} \eta^* J z(t) = 0 \text{ for all } \eta \in D(H)\}. \tag{4.15}$$

Note that $H_0(\widetilde{0}) \subset H(\widetilde{0}) = H_0^*(\widetilde{0})$. We shall show that $H_0^*(\widetilde{0}) \subset H_0(\widetilde{0})$. Let $(\widetilde{0}, \widetilde{g}) \in H_0^* = H$. Then it implies that there exists a unique $y = (x^{\mathrm{T}}, u^{\mathrm{T}})^{\mathrm{T}} \in \widetilde{0}$ such that $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for any $g \in \widetilde{g}$ and $t \in \mathcal{I}'$. If condition (1) holds, then we have $R(y)(t) = 0$ by $W(t) > 0$ for $t \le k_1$ and $t \ge k_2$. Then, together with (4.15), we have $(y, \widetilde{g}) \in H_0$, and hence $\widetilde{g} \in H_0(\widetilde{0})$. Therefore, $H_0^*(\widetilde{0}) \subset H_0(\widetilde{0})$, and hence $H_0^*(\widetilde{0}) = H_0(\widetilde{0})$. On the other hand, if condition (2) holds, then $x(t+1) = 0$ for $t \le k_1$ and $t \ge k_2$. It is noted that $W_2(t) = 0$ for $t \in \mathcal{I}'$, $x(t+1) = 0$ and $B(t)$ is invertible for $t \le k_1$ and $t \ge k_2$. Then, by the first relation of (4.7), we have $u(t) = 0$ for $t \le k_1$ and $t \ge k_2 + 1$. Therefore, $y(t) = 0$ for $t \le k_1$ and $t \ge k_2 + 1$, which, together with (4.15), yields that $(y, \widetilde{g}) \in H_0$. Hence $H_0^*(\widetilde{0}) = H_0(\widetilde{0})$. In addition, if condition (3) holds, then $H_0^*(\widetilde{0}) = H_0(\widetilde{0})$ can be proved similarly. This completes the proof.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Arens, R., Operational calculus of linear relations, *Pacific J. Math*, **11**, 1961, 9–23.

[2] Barraa, M. and Müller, V., On the essential numerical range, *Acta Sci. Math.* (*Szeged*), **71**, 2005, 285–298.

[3] Bögli, S. and Marletta, M., Essential numerical ranges for linear operator pencils, *IMA J. Numer. Anal.* **40**, 2020, 2256–2308.

[4] Bögli, S., Marletta, M. and Tretter, C., The essential numerical range for unbounded linear operators, *J. Funct. Anal.*, **279**(1), 2020, 108509.

[5] Coddington, E. A., Extension theory of formally normal and symmetric subspaces, *Mem. Amer. Math. Soc.*, **134**, 1973.

[6] Coddington, E. A., Self-adjoint subspace extensions of nondensely defined symmetric operators, *Adv. Math.*, **14**, 1974, 309–332.

[7] Cross, R. W., Multivalued Linear Operators, Marcel Dekker, New York, 1998.

[8] Descloux, J., Essential numerical range of an operator with respect to a coercive form and the approximation of its spectrum by the Galerkin method, *SIAM J. Numer. Anal.*, **18**, 1981, 1128–1133.

[9] Edmunds, D. E. and Evans, W. D., Spectral Theory and Differential Operators, Clarendon Press, Oxford, 1987.

[10] Fillmore, P. A., Stampfli, J. G. and Williams, J. P., On the essential numerical range, the essential spectrum, and a problem of Halmos, *Acta Sci. Math.* (*Szeged*), **33**, 1972, 179–192.

[11] Hassi, S., de Snoo, H. S. V. and Szafraniec, F. H., Componentwise and Cartesian decompositions of linear relations, *Dissertationes Math.*, **465**, 2009, 4-58.

[12] Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1984.

[13] Lesch, M. and Malamud, M., On the deficiency indices and self-adjointness of symmetric Hamiltonian systems, *J. Differential Equations*, **189**, 2003, 556–615.

[14] Muhati, L. N., Bonyo, J. O. and Agure, J. O., Some properties of the essential numerical range on Banach spaces, *Pure Mathematical Sciences*, **6**, 2017, 105–111.

[15] Reed, M. and Simon, B., Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, London, 1972.

[16] Ren, G. and Shi, Y., Defect indices and definiteness conditions for a class of discrete linear Hamiltonian systems, *Appl. Math. Comput.*, **218**, 2011, 3414–3429.

[17] Ren, G. and Shi, Y., Self-adjoint extensions for discrete linear Hamiltonian systems, *Linear Algebra Appl.*, **454**, 2014, 1–48.

[18] Rofe-Beketov, F. S., The numerical range of a linear relation and maximum relations, *J. Math. Sci.*, **48**, 1990, 329–336.

[19] Salinas, N., Operators with essentially disconnected spectrum, *Acta Sci. Math.* (*Szeged*), **33**, 1972, 193–205.

[20] Shi, Y., The Glazman-Krein-Naimark theory for Hermitian subspaces, *J. Operator Theory*, **68**, 2012, 241–256.

[21] Shi, Y., Shao, C. and Liu, Y., Resolvent convergence and spectral approximations of sequences of self adjoint subspaces, *J. Math. Anal. Appl.*, **409**, 2014, 1005–1020.

[22] Shi, Y., Shao, C. and Ren, G., Spectral properties of self-adjoint subspaces, *Linear Algebra Appl.*, **438**, 2013, 191–218.

[23] Shi, Y., Xu, G. and Ren, G., Boundedness and closedness of linear relations, *Linear Multilinear Algebra*, **66**, 2018, 309–333.

[24] Stampfli, J. G. and Williams, J. P., Growth conditions and the numerical range in a Banach algebra, *Tôhoku Math. J.*, **20**, 1968, 417–424.

[25] Sun, H. and Shi, Y., Spectral properties of singular discrete linear Hamiltonian systems, *J. Difference Equ. Appl.*, **20**, 2014, 379–405.

[26] von Neumann, J., Functional Operator II: The Geometry of Orthogonal Spaces, *Ann. of Math. Stud.*, **22**, Princeton University Press, Princeton, NJ, 1950.

[27] Weidmann, J., Linear Operators in Hilbert Spaces, Springer-Verlag, New York, 1980.

[28] Wilcox, D., Essential spectra of linear relations, *Linear Algebra Appl.*, **462**, 2014, 110–125.