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Lifting Theorem for the Virtual Pure Braid Groups*

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Abstract In this article the authors prove theorem on Lifting for the set of virtual pure braid groups. This theorem says that if they know presentation of virtual pure braid group VP_4, then they can find presentation of VP_n for arbitrary $n > 4$. Using this theorem they find the set of generators and defining relations for simplicial group T_* which was defined in [Bardakov, V. G. and Wu, J., On virtual cabling and structure of 4-strand virtual pure braid group, *J. Knot Theory and Ram.*, **29**(10), 2020, 1–32]. They find a decomposition of the Artin pure braid group P_n in semi-direct product of free groups in the cabled generators.

Keywords Virtual braid group, Pure braid group, Simplicial group, Virtual cabling
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1 Introduction

The operation cabling for classical braids was studied in [7]. Generators of virtual pure braid group VP_n have geometric interpretation (see [1]). Using this interpretation, in [4] we constructed cabling generators for VP_n. It was proved that for $n \geq 3$, the group VP_n is generated by the n-strand virtual braids obtained by taking (k, l)-cabling on the standard generators $\lambda_{1,2}$ and $\lambda_{2,1}$ of VP_2 together with adding trivial strands $n - k - l$ to the end for $1 \leq k \leq n - 1$ and $2 \leq k + l \leq n$, where a (k, l)-cabling on a 2-strand virtual braid means to take k-cabling on the first strand and l-cabling on the second strand.

Different from the classical situation (see [7]) that the n-strand braids cabled from the standard generator $A_{1,2}$ for P_2 generates a free group of rank $n - 1$, the subgroup of VP_n generated by n-strand virtual braids cabled from $\lambda_{1,2}$ and $\lambda_{2,1}$, which is denoted by T_{n-1}, is no longer free for $n \geq 3$.

For the first nontrivial case that $n = 3$, a presentation of T_2 has been explored with producing a decomposition theorem for VP_3 using cabled generators (see [3]).

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In the present article we continue to study VP_n in cabled generators, which we started in [4]. We find some sufficient condition under which a simplicial group G_* is contractible. In particular, we prove that the simplicial group $VAP_* = \{VP_i\}_{i=1,2,\cdots}$ is contractible. Also, we prove the lifting theorem for the virtual pure braid groups. From this theorem, it follows that if we know the structure of VP_4, T_3 or P_4, then using degeneracy maps we can find the structure of VP_n, T_n or P_n for all bigger n. On the other side, we prove that if we know a presentation of VP_n, $n \geq 4$, then conjugate it by elements ρ_n, $\rho_n\rho_{n-1}$, $\cdots$, $\rho_n\rho_{n-1}\cdots\rho_1 \in VB_{n+1}$, we can find the presentation of VP_{n+1}.

The article is organized as follows. In Section 2, we give a review on braid groups and virtual braid groups. The simplicial structure on virtual pure braid groups will be discussed in Section 3. In Subsection 4.1 we prove the lifting theorem. In Section 6, we discuss the cabling operation on classical pure braid group P_n as subgroup of VP_n. We know two types of decompositions of P_n as semi-direct products (see [1]). In Section 6 we construct new decomposition of this type in terms of the cabled generators. In the last Section 7 we formulate some questions for further research.

2 Braid and Virtual Braid Groups

2.1 Braid group

The braid group B_n on n strings is generated by σ_1, $\sigma_2, \cdots, \sigma_{n-1}$ and is defined by relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad i = 1, 2, \cdots, n-2,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1.$$

Let S_n, $n \geq 1$ be the symmetric group which is generated by $\rho_1, \rho_2, \cdots, \rho_{n-1}$ and is defined by relations

$$\rho_i^2 = 1, \quad i = 1, 2, \cdots, n-1,$$
$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}, \quad i = 1, 2, \cdots, n-2,$$
$$\rho_i\rho_j = \rho_j\rho_i, \quad |i-j| > 1.$$

There is a homomorphism $B_n \to S_n$, which sends σ_i to ρ_i. Its kernel is the pure braid group P_n. This group is generated by elements $A_{i,j}$, $1 \leq i < j \leq n$, where

$$A_{i,i+1} = \sigma_i^2,$$

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \quad i+1 < j \leq n,$$

and is defined by relations (where $\varepsilon = \pm 1$)

$$A_{ik}^{-\varepsilon}A_{kj}A_{ik}^{\varepsilon} = (A_{ij}A_{kj})^{\varepsilon}A_{kj}(A_{ij}A_{kj})^{-\varepsilon},$$
$$A_{km}^{-\varepsilon}A_{kj}A_{km}^{\varepsilon} = (A_{kj}A_{mj})^{\varepsilon}A_{kj}(A_{kj}A_{mj})^{-\varepsilon}, \quad m < j,$$

$$A_{im}^{-\varepsilon} A_{kj} A_{im}^{\varepsilon} = [A_{ij}^{-\varepsilon}, A_{mj}^{-\varepsilon}]^{\varepsilon} A_{kj} [A_{ij}^{-\varepsilon}, A_{mj}^{-\varepsilon}]^{-\varepsilon}, \quad i < k < m,$$
$$A_{im}^{-\varepsilon} A_{kj} A_{im}^{\varepsilon} = A_{kj}, \quad k < i, \ m < j \text{ or } m < k.$$

Here and further $[a, b] = a^{-1} b^{-1} ab$ is the commutator of a and b.

There is an epimorphism of P_n to P_{n-1} what is removing of the n-th string. Its kernel $U_n = \langle A_{1n}, A_{2n}, \cdots, A_{n-1,n} \rangle$ is a free group of rank $n-1$ and $P_n = U_n \rtimes P_{n-1}$ is a semi-direct product of U_n and P_{n-1}. Hence,

$$P_n = U_n \rtimes (U_{n-1} \rtimes (\cdots \rtimes (U_3 \rtimes U_2)) \cdots)$$

is a semi-direct product of free groups and $U_2 = \langle A_{12} \rangle$ is the infinite cyclic group.

2.2 Virtual braid group

The virtual braid group VB_n is generated by elements

$$\sigma_1, \sigma_2, \cdots, \sigma_{n-1}, \rho_1, \rho_2, \cdots, \rho_{n-1},$$

where $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ generate the classical braid group B_n and the elements ρ_1, ρ_2, $\cdots$, ρ_{n-1} generate the symmetric group S_n. Hence, VB_n is defined by relations of B_n, relations of S_n and mixed relation

$$\sigma_i \rho_j = \rho_j \sigma_i, \quad |i - j| > 1,$$
$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \cdots, n-2.$$

As for the classical braid groups there exists the canonical epimorphism of VB_n onto the symmetric group $VB_n \to S_n$ with the kernel called the virtual pure braid group VP_n. So we have a short exact sequence

$$1 \to VP_n \to VB_n \to S_n \to 1.$$

Define the following elements in VP_n:

$$\lambda_{i,i+1} = \rho_i \, \sigma_i^{-1}, \quad \lambda_{i+1,i} = \rho_i \, \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i, \quad i = 1, 2, \cdots, n-1,$$

$$\lambda_{ij} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \, \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1},$$

$$\lambda_{ji} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \, \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \quad 1 \le i < j - 1 \le n - 1.$$

It is shown in [1] that the group VP_n, $n \ge 2$ admits a presentation with the generators λ_{ij}, $1 \le i \ne j \le n$ and the following relations

$$\lambda_{ij} \lambda_{kl} = \lambda_{kl} \lambda_{ij}, \tag{2.1}$$

$$\lambda_{ki} \lambda_{kj} \lambda_{ij} = \lambda_{ij} \lambda_{kj} \lambda_{ki}, \tag{2.2}$$

where distinct letters stand for distinct indices.

Like the classical pure braid groups, groups VP_n admit a semi-direct product decompositions (see [1]): For $n \ge 2$, the n-th virtual pure braid group can be decomposed as

$$VP_n = V_{n-1}^* \rtimes VP_{n-1}, \quad n \ge 2, \tag{2.3}$$

where V_{n-1}^* is a subgroup of VP_n, $V_1^* = F_2$, VP_1 is supposed to be the trivial group.

3 Simplicial Groups

3.1 Simplicial sets and simplicial groups

Recall the definition of simplicial groups (see [8, p.300, 5]). A sequence of sets $X_* = \{X_n\}_{n\geq 0}$ is called a simplicial set if there are face maps

$$d_i : X_n \to X_{n-1} \quad \text{for } 0 \leq i \leq n$$

and degeneracy maps

$$s_i : X_n \to X_{n+1} \quad \text{for } 0 \leq i \leq n,$$

that satisfy the following simplicial identities:

(1) $d_i d_j = d_{j-1} d_i$ if $i < j$,
(2) $s_i s_j = s_{j+1} s_i$ if $i \leq j$,
(3) $d_i s_j = s_{j-1} d_i$ if $i < j$,
(4) $d_j s_j = \mathrm{id} = d_{j+1} s_j$,
(5) $d_i s_j = s_j d_{i-1}$ if $i > j+1$.

Here X_n can be geometrically viewed as the set of n-simplices including all possible degenerate simplices.

A simplicial group is a simplicial set X_* such that each X_n is a group and all face and degeneracy operations are group homomorphism. Let G_* be a simplicial group. The Moore cycles $\mathrm{Z}_n(G_*) \leq G_n$ is defined by

$$\mathrm{Z}_n(G_*) = \bigcap_{i=0}^{n} \mathrm{Ker}(d_i \colon G_n \to G_{n-1})$$

and the Moore boundaries $\mathcal{B}_n(G_*) \leq G_n$ is defined by

$$\mathcal{B}_n(G_*) = d_0\Big(\bigcap_{i=1}^{n+1} \mathrm{Ker}(d_i \colon G_{n+1} \to G_n)\Big).$$

Simplicial identities guarantee that $\mathcal{B}_n(G_*)$ is a (normal) subgroup of $\mathrm{Z}_n(G_*)$. The Moore homotopy group $\pi_n(G_*)$ is defined by

$$\pi_n(G_*) = \mathrm{Z}_n(G_*)/\mathcal{B}_n(G_*).$$

It is a classical result due to Moore [9] that $\pi_n(G_*)$ is isomorphic to the n-th homotopy group of the geometric realization of G_*.

3.2 Simplicial group on virtual pure braid groups

By using the same ideas in the work [5, 7] on the classical braids, in [4] we introduced a simplcial group

$$\mathrm{VAP}_* : \quad \cdots \; \overset{\longrightarrow}{\underset{\longleftarrow}{\vdots}} \; VP_4 \; \overset{\longrightarrow}{\underset{\longleftarrow}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}}} \; VP_3 \; \overset{\longrightarrow}{\underset{\longleftarrow}{\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}}} \; VP_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} \; VP_1$$

on pure virtual braid groups with $\mathrm{VAP}_n = VP_{n+1}$, the face homomorphism

$$d_i : \mathrm{VAP}_n = VP_{n+1} \to \mathrm{VAP}_{n-1} = VP_n$$

given by deleting $(i+1)$th strand for $0 \le i \le n$, and the degeneracy homomorphism

$$s_i : \mathrm{VAP}_n = VP_{n+1} \to \mathrm{VAP}_{n+1} = VP_{n+2}$$

given by doubling the $(i+1)$th strand for $0 \le i \le n$.

Let $\iota_n \colon VP_n \to VP_{n+1}$ be the inclusion. Geometrically ι_n is the group homomorphism by adding a trivial strand on the end. From geometric information, we have the following formulae

$$s_j \iota_n = \iota_{n+1} s_j \colon VP_n \to VP_{n+1} \quad \text{for } \ 0 \le j \le n-1, \tag{3.1}$$

$$d_j \iota_n = \begin{cases} \iota_{n-1} d_j, & \text{if } j < n, \\ \mathrm{id}, & \text{if } j = n. \end{cases} \tag{3.2}$$

From the above formulae, the inclusion $\iota_n \colon VP_n \to VP_{n+1}$ gives an extra operation on the simplicial group VAP_* so that the simplicial identities still hold by regarding ι_n as extra degeneracy

$$s_n = \iota_n \colon \mathrm{VAP}_{n-1} = VP_n \to \mathrm{VAP}_n = VP_{n+1}.$$

Motivated from this example, a simplicial group G_* is called conic if there exists an extra degeneracy homomorphism $s_n \colon G_{n-1} \to G_n$ so that simplicial identities (including formulae involving s_n) hold.

Proposition 3.1 *Any conic simplicial group G_* is contractible.*

Proof Let $x \in \mathrm{Z}_n(G_*)$ be a Moore cycle, that is $x \in G_n$ with $d_j x = 1$ for $0 \le j \le n$. Note that we have the extra operation $s_{n+1} \colon G_n \to G_{n+1}$. Let $y = s_{n+1} x \in G_{n+1}$. Then

$$d_j y = d_j s_{n+1} x = s_n d_j x = s_n(1) = 1$$

for $0 \le j \le n$ and

$$d_{n+1} y = d_{n+1} s_{n+1} x = x.$$

It follows that x is a Moore boundary. Thus $\pi_n(G_*) = 0$ for all n, and so G_* is contractible.

Proposition 3.2 *Let G_* be a conic simplicial subgroup of* VAP_* *such that* $G_1 = \mathrm{VAP}_1 = VP_2$. *Then* $G_* = \mathrm{VAP}_*$.

Proof The proof is given by induction on the dimension n of G_n. From the hypothesis, $G_1 = \mathrm{VAP}_1$. Suppose that $G_{n-1} = \mathrm{VAP}_{n-1} = VP_n$. In [4, Proposition 3.2], it was proved that $VP_{n+1} = \langle \iota_n(VP_n), s_0(VP_n), \cdots, s_{n-1}(VP_n) \rangle$. From this equality we see that $G_n = \mathrm{VAP}_n = VP_{n+1}$ and hence the result holds.

The main point for introducing the new notion of conic simplicial group is to give a new presentation of VP_n using degeneracy operations (including the extra degeneracies). From the above proposition, the new generators for VP_n with $n \ge 2$ are given by

$$s_{k_{n-2}} s_{k_{n-3}} \cdots s_{k_1} \lambda_{1,2} \quad \text{and} \quad s_{k_{n-2}} s_{k_{n-3}} \cdots s_{k_1} \lambda_{2,1}$$

for $0 \leq k_1 < k_2 < \cdots < k_{n-2} \leq n-1$. Let

$$\mu_{i,j}^{k,l} = s_{n-1}s_{n-2}\cdots \widehat{s}_{l-1}\cdots \widehat{s}_{k-1}\cdots s_0\lambda_{i,j}$$

for $(i,j) = (1,2)$ or $(2,1)$ and $1 < k < l \leq n$. Here the notation $\widehat{s}_l$ means that s_l is removed. Then

$$VP_n = \langle \mu_{1,2}^{k,l}, \mu_{2,1}^{k,l}, 1 \leq k < l \leq n\rangle.$$

The relations with $a_{i,j}$ and $b_{i,j}$, which were defined in [4] (in this paper we are also given their geometric interpretation) are given by

$$a_{k,l-k} = \mu_{1,2}^{k,l} \quad \text{and} \quad b_{k,l-k} = \mu_{2,1}^{k,l}. \tag{3.3}$$

By direct computations, we have the degeneracy formulae

$$s_t(\mu_{i,j}^{k,l}) = \begin{cases} \mu_{i,j}^{k,l}, & \text{if } t \geq l, \\ \mu_{i,j}^{k,l+1}, & \text{if } k \leq t < l, \\ \mu_{i,j}^{k+1,l+1}, & \text{if } 0 \leq t < k. \end{cases} \tag{3.4}$$

By writing it in terms of $a_{i,j}$ and $b_{i,j}$, we have

$$s_k a_{i,j} = \begin{cases} a_{i,j}, & \text{if } k \geq i+j, \\ a_{i,j+1}, & \text{if } i \leq k < i+j, \\ a_{i+1,j+1}, & \text{if } 0 \leq k < i, \end{cases} \qquad s_k b_{i,j} = \begin{cases} b_{i,j}, & \text{if } k \geq i+j, \\ b_{i,j+1}, & \text{if } i \leq k < i+j, \\ b_{i+1,j+1}, & \text{if } 0 \leq k < i. \end{cases} \tag{3.5}$$

For obtaining a new presentation of VP_n on generators $\mu_{1,2}^{k,l}$ and $\mu_{2,1}^{k,l}$, we need to rewrite the relations

$$s_{k_{n-3}}s_{k_{n-4}}\cdots s_{k_1}(\lambda_{ki}\lambda_{kj}\lambda_{ij}) = s_{k_{n-3}}s_{k_{n-4}}\cdots s_{k_1}(\lambda_{ij}\lambda_{kj}\lambda_{ki}) \tag{3.6}$$

for distinct $1 \leq i,j,k \leq 3$ and $0 \leq k_1 < k_2 < \cdots < k_{n-3} \leq n-1$, and

$$\begin{aligned} &s_{k_{n-4}}s_{k_{n-5}}\cdots s_{k_1}(\lambda_{i,j})s_{k_{n-4}}s_{k_{n-5}}\cdots s_{k_1}(\lambda_{k,l}) \\ &= s_{k_{n-4}}s_{k_{n-5}}\cdots s_{k_1}(\lambda_{k,l})s_{k_{n-4}}s_{k_{n-5}}\cdots s_{k_1}(\lambda_{i,j}) \end{aligned} \tag{3.7}$$

for distinct $1 \leq i,j,k,l \leq 4$ and $0 \leq k_1 < k_2 < \cdots < k_{n-4} \leq n-1$ in terms of $\mu_{i,j}^{k,l}$.

4 Lifting Defining Relations of VP_{n-1} to VP_n

Let $n \geq 4$. Let $\mathcal{R}^V(n)$ denote the defining relations (2.1)–(2.2) of VP_n. By applying the degeneracy homomorphism $s_t\colon VP_n \to VP_{n+1}$ to $\mathcal{R}^V(n)$, we have the following equations

$$s_t(\lambda_{ij})s_t(\lambda_{kl}) = s_t(\lambda_{kl})s_t(\lambda_{ij}), \tag{4.1}$$

$$s_t(\lambda_{ki})s_t(\lambda_{kj})s_t(\lambda_{ij}) = s_t(\lambda_{ij})s_t(\lambda_{kj})s_t(\lambda_{ki}) \tag{4.2}$$

in VP_{n+1} for $1 \leq i,j,k,l \leq n$ with distinct letters standing for distinct indices, which is denoted as $s_t(\mathcal{R}^V(n))$.

The main aim of the present section is the proof of the following.

Theorem 4.1 *Let $n \geq 4$. Consider VP_n as a subgroup of VP_{n+1} by adding a trivial strand in the end. Then*

$$\mathcal{R}^V(n) \cup \bigcup_{i=0}^{n-1} s_i(\mathcal{R}^V(n))$$

gives the full set of the defining relations for VP_{n+1}.

We will use the following proposition.

Proposition 4.1 *The degeneracy map $s_j : VP_n \to VP_{n+1}$, $j = 0, 1, \cdots, n-1$ acts on the generators $\lambda_{k,l}$ and $\lambda_{l,k}$, $1 \leq k < l \leq n$ of VP_n by the rules*

$$s_{i-1}(\lambda_{k,l}) = \begin{cases} \lambda_{k+1,l+1} & \text{for } i < k, \\ \lambda_{k,l+1}\lambda_{k+1,l+1} & \text{for } i = k, \\ \lambda_{k,l+1} & \text{for } k < i < l, \\ \lambda_{k,l+1}\lambda_{k,l} & \text{for } i = l, \\ \lambda_{k,l} & \text{for } i > l, \end{cases}$$

$$s_{i-1}(\lambda_{l,k}) = \begin{cases} \lambda_{l+1,k+1} & \text{for } i < k, \\ \lambda_{l+1,k+1}\lambda_{l+1,k} & \text{for } i = k, \\ \lambda_{l+1,k} & \text{for } k < i < l, \\ \lambda_{l,k}\lambda_{l+1,k} & \text{for } i = l, \\ \lambda_{l,k} & \text{for } i > l. \end{cases}$$

4.1 Lifting defining relations of VP_3 to VP_4

In the group VP_3 we have 6 relations:

$$\lambda_{12}\lambda_{13}\lambda_{23} = \lambda_{23}\lambda_{13}\lambda_{12}, \quad \lambda_{21}\lambda_{23}\lambda_{13} = \lambda_{13}\lambda_{23}\lambda_{21}, \quad \lambda_{13}\lambda_{12}\lambda_{32} = \lambda_{32}\lambda_{12}\lambda_{13},$$

$$\lambda_{31}\lambda_{32}\lambda_{12} = \lambda_{12}\lambda_{32}\lambda_{31}, \quad \lambda_{23}\lambda_{21}\lambda_{31} = \lambda_{31}\lambda_{21}\lambda_{23}, \quad \lambda_{32}\lambda_{31}\lambda_{21} = \lambda_{21}\lambda_{31}\lambda_{32}.$$

Acting on these relations by degeneracy map s_2 we get 6 relations in VP_4. Let us analyze these relations.

(1) The image of the first relation has the form

$$\lambda_{12} \cdot \lambda_{14}(\lambda_{13} \cdot \lambda_{24})\lambda_{23} = \lambda_{24}(\lambda_{23} \cdot \lambda_{14})\lambda_{13} \cdot \lambda_{12}.$$

Using the commutativity relation

$$\lambda_{13}\lambda_{24} = \lambda_{24}\lambda_{13}, \quad \lambda_{23}\lambda_{14} = \lambda_{14}\lambda_{23},$$

we get

$$\lambda_{12}\lambda_{14}\lambda_{24} \cdot \lambda_{13}\lambda_{23} = \lambda_{24}\lambda_{14}(\lambda_{23}\lambda_{13}\lambda_{12}).$$

Using the following relation of VP_3:

$$\lambda_{23}\lambda_{13}\lambda_{12} = \lambda_{12}\lambda_{13}\lambda_{23},$$

we get

$$\lambda_{12}\lambda_{14}\lambda_{24} = \lambda_{24}\lambda_{14}\lambda_{12},$$

that is the long relation in VP_4.

(2) The image of the second relation has the form

$$\lambda_{21} \cdot \lambda_{24}(\lambda_{23} \cdot \lambda_{14})\lambda_{13} = \lambda_{14}(\lambda_{13} \cdot \lambda_{24})\lambda_{23} \cdot \lambda_{21}.$$

Using the commutativity relations

$$\lambda_{23}\lambda_{14} = \lambda_{14}\lambda_{23}, \quad \lambda_{13}\lambda_{24} = \lambda_{24}\lambda_{13},$$

we get

$$\lambda_{21}\lambda_{24}\lambda_{14}\lambda_{23}\lambda_{13} = \lambda_{14}\lambda_{24}(\lambda_{13}\lambda_{23}\lambda_{21}).$$

From the relation of VP_3:

$$\lambda_{13}\lambda_{23}\lambda_{21} = \lambda_{24}\lambda_{14}\lambda_{12},$$

we get

$$\lambda_{21}\lambda_{24}\lambda_{14} = \lambda_{14}\lambda_{24}\lambda_{21},$$

i.e., the long relation in VP_4.

(3) The image of the third relation has the form

$$\lambda_{14}(\lambda_{13} \cdot \lambda_{12} \cdot \lambda_{32})\lambda_{42} = \lambda_{32}\lambda_{42} \cdot \lambda_{12} \cdot \lambda_{14}\lambda_{13}.$$

Using the following relation from VP_3:

$$\lambda_{13}\lambda_{12}\lambda_{32} = \lambda_{32}\lambda_{12}\lambda_{13},$$

we get

$$(\lambda_{14}\lambda_{32})\lambda_{12}(\lambda_{13}\lambda_{42}) = \lambda_{32}\lambda_{42}\lambda_{12}\lambda_{14}\lambda_{13}.$$

Using the commutativity relations

$$\lambda_{14}\lambda_{32} = \lambda_{32}\lambda_{14}, \quad \lambda_{13}\lambda_{42} = \lambda_{42}\lambda_{13},$$

we have

$$\lambda_{32}\lambda_{14}\lambda_{12}\lambda_{42}\lambda_{13} = \lambda_{32}\lambda_{42}\lambda_{12}\lambda_{14}\lambda_{13}.$$

After cancellation we get

$$\lambda_{14}\lambda_{12}\lambda_{42} = \lambda_{42}\lambda_{12}\lambda_{14},$$

i.e., the long relation in VP_4.

(4) The image of the fourth relation has the form

$$\lambda_{31}(\lambda_{41} \cdot \lambda_{32})\lambda_{42} \cdot \lambda_{12} = \lambda_{12} \cdot \lambda_{32}(\lambda_{42} \cdot \lambda_{31})\lambda_{41}.$$

Using the commutativity relations

$$\lambda_{41}\lambda_{32} = \lambda_{32}\lambda_{41}, \quad \lambda_{42} \cdot \lambda_{31} = \lambda_{31} \cdot \lambda_{42},$$

we get

$$\lambda_{31}\lambda_{32}\lambda_{41}\lambda_{42}\lambda_{12} = (\lambda_{12}\lambda_{32}\lambda_{31})\lambda_{42}\lambda_{41}.$$

Using the following relation from VP_3:

$$\lambda_{12}\lambda_{32}\lambda_{31} = \lambda_{31}\lambda_{32}\lambda_{12},$$

after cancelations we get

$$\lambda_{41}\lambda_{42}\lambda_{12} = \lambda_{12}\lambda_{42}\lambda_{41},$$

i.e., the long relation in VP_4.

(5) The image of the fifth relation has the form

$$\lambda_{24}(\lambda_{23} \cdot \lambda_{21} \cdot \lambda_{31})\lambda_{41} = \lambda_{31}\lambda_{41} \cdot \lambda_{21} \cdot \lambda_{24}\lambda_{23}.$$

Using the following relation from VP_3:

$$\lambda_{23}\lambda_{21}\lambda_{31} = \lambda_{31}\lambda_{21}\lambda_{23},$$

and the commutativity relations

$$\lambda_{24}\lambda_{13} = \lambda_{13}\lambda_{24}, \quad \lambda_{23}\lambda_{41} = \lambda_{41}\lambda_{23},$$

we get

$$\lambda_{24}\lambda_{21}\lambda_{41} = \lambda_{41}\lambda_{21}\lambda_{24},$$

i.e., the long relation in VP_4.

(6) The image of the sixth relation has the form

$$\lambda_{32}(\lambda_{42} \cdot \lambda_{31})\lambda_{41} \cdot \lambda_{21} = \lambda_{21} \cdot \lambda_{31}(\lambda_{41} \cdot \lambda_{32})\lambda_{42}.$$

Using the commutativity relations

$$\lambda_{42}\lambda_{31} = \lambda_{31}\lambda_{42}, \quad \lambda_{41}\lambda_{32} = \lambda_{32}\lambda_{41},$$

we get

$$\lambda_{32}\lambda_{31}\lambda_{42}\lambda_{41}\lambda_{21} = (\lambda_{21}\lambda_{31}\lambda_{32})\lambda_{41}\lambda_{42}.$$

Using the following relation from VP_3:

$$\lambda_{21}\lambda_{31}\lambda_{32} = \lambda_{32}\lambda_{31}\lambda_{21},$$

we get

$$\lambda_{42}\lambda_{41}\lambda_{21} = \lambda_{21}\lambda_{41}\lambda_{42},$$

i.e., the long relation in VP_4. Hence, we proved the following lemma.

Lemma 4.1 *From relations $\mathcal{R}^V(3)$, relations $s_2(\mathcal{R}^V(3))$ and the commutativity relations in $\mathcal{R}^V(4)$, it follows the next set of relations in $\mathcal{R}^V(4)$:*

$$\lambda_{12}\lambda_{14}\lambda_{24} = \lambda_{24}\lambda_{14}\lambda_{12}, \quad \lambda_{21}\lambda_{24}\lambda_{14} = \lambda_{14}\lambda_{24}\lambda_{21}, \quad \lambda_{14}\lambda_{12}\lambda_{42} = \lambda_{42}\lambda_{12}\lambda_{14},$$

$$\lambda_{41}\lambda_{42}\lambda_{12} = \lambda_{12}\lambda_{42}\lambda_{41}, \quad \lambda_{24}\lambda_{21}\lambda_{41} = \lambda_{41}\lambda_{21}\lambda_{24}, \quad \lambda_{42}\lambda_{41}\lambda_{21} = \lambda_{21}\lambda_{41}\lambda_{42},$$

i.e., the set of relations where the indices of the generators lie in the set $\{1, 2, 4\}$.

Considering the set $s_1(\mathcal{R}^V(3))$, we can prove the following lemma.

Lemma 4.2 *From relations* $\mathcal{R}^V(3)$, *relations* $s_1(\mathcal{R}^V(3))$, $s_2(\mathcal{R}^V(3))$ *and commutativity relations in* $\mathcal{R}^V(4)$, *it follows the next set of relations in* $\mathcal{R}^V(4)$:

$$\lambda_{13}\lambda_{14}\lambda_{34} = \lambda_{34}\lambda_{14}\lambda_{13}, \quad \lambda_{31}\lambda_{34}\lambda_{14} = \lambda_{14}\lambda_{34}\lambda_{31}, \quad \lambda_{14}\lambda_{13}\lambda_{43} = \lambda_{43}\lambda_{13}\lambda_{14},$$

$$\lambda_{41}\lambda_{43}\lambda_{13} = \lambda_{13}\lambda_{43}\lambda_{41}, \quad \lambda_{34}\lambda_{31}\lambda_{41} = \lambda_{41}\lambda_{31}\lambda_{34}, \quad \lambda_{43}\lambda_{41}\lambda_{31} = \lambda_{31}\lambda_{41}\lambda_{43},$$

i.e., the set of relations where the indices of the generators lie in the set $\{1, 3, 4\}$.

Considering the set of relations $s_0(\mathcal{R}^V(3))$, we can prove the following lemma.

Lemma 4.3 *From relations* $\mathcal{R}^V(3)$, *relations* $s_i(\mathcal{R}^V(3))$, $i = 0, 1, 2$ *and commutativity relations in* $\mathcal{R}^V(4)$, *it follows the next set relations in* $\mathcal{R}^V(4)$:

$$\lambda_{23}\lambda_{24}\lambda_{34} = \lambda_{34}\lambda_{24}\lambda_{23}, \quad \lambda_{32}\lambda_{34}\lambda_{24} = \lambda_{24}\lambda_{34}\lambda_{32}, \quad \lambda_{24}\lambda_{23}\lambda_{43} = \lambda_{43}\lambda_{23}\lambda_{24},$$

$$\lambda_{42}\lambda_{43}\lambda_{23} = \lambda_{23}\lambda_{43}\lambda_{42}, \quad \lambda_{34}\lambda_{32}\lambda_{42} = \lambda_{42}\lambda_{32}\lambda_{34}, \quad \lambda_{43}\lambda_{42}\lambda_{32} = \lambda_{32}\lambda_{42}\lambda_{43},$$

i.e., the set of relations where the indices of the generators lie in the set $\{2, 3, 4\}$.

4.2 Lifting the commutativity relations from $\mathcal{R}^V(4)$ into $\mathcal{R}^V(5)$

We have to show that $\mathcal{R}^V(5) = \langle \mathcal{R}^V(4), s_i(\mathcal{R}^V(4)), i = 0, 1, 2, 3\rangle$. At first, we consider the commutativity relations

$$[\lambda_{i4}^*, \lambda_{kl}^*], \quad 1 \leq i \leq 3, \ 1 \leq k < l \leq 3$$

in $\mathcal{R}^V(4)$. We divide them into four groups:

1-st group: $[\lambda_{34}, \lambda_{12}] = [\lambda_{24}, \lambda_{13}] = [\lambda_{14}, \lambda_{23}] = 1$;

2-nd group: $[\lambda_{34}, \lambda_{21}] = [\lambda_{24}, \lambda_{31}] = [\lambda_{14}, \lambda_{32}] = 1$;

3-d group: $[\lambda_{43}, \lambda_{21}] = [\lambda_{42}, \lambda_{31}] = [\lambda_{41}, \lambda_{32}] = 1$;

4-th group: $[\lambda_{43}, \lambda_{12}] = [\lambda_{42}, \lambda_{13}] = [\lambda_{41}, \lambda_{23}] = 1$.

Taking the third relation from the 1-st group and acting on it by s_i, $i = 0, 1, 2, 3$, we get the following relations

$$[\lambda_{15}\lambda_{25}, \lambda_{34}] = [\lambda_{15}, \lambda_{24}\lambda_{34}] = [\lambda_{15}, \lambda_{24}\lambda_{23}] = [\lambda_{15}\lambda_{14}, \lambda_{23}] = 1.$$

Using the commutativity relation

$$\lambda_{14}\lambda_{23} = \lambda_{23}\lambda_{14},$$

which holds in VP_4, and from the last relation, we have

$$[\lambda_{15}, \lambda_{23}] = 1. \tag{4.3}$$

With considering (4.3) we get

$$[\lambda_{15}, \lambda_{24}] = 1.$$

Then the second relation follows relation $[\lambda_{15}, \lambda_{34}] = 1$ and the first relation follows $[\lambda_{25}, \lambda_{34}] = 1$. Hence, we have proved the following lemma.

Lemma 4.4 *From the lifting s_i, $i = 0, 1, 2, 3$ of the relation $[\lambda_{14}, \lambda_{23}] = 1$ and the commutativity relations in $\mathcal{R}^V(4)$, it follows the commutativity relations*

$$[\lambda_{15}, \lambda_{23}] = [\lambda_{15}, \lambda_{24}] = [\lambda_{15}, \lambda_{34}] = [\lambda_{25}, \lambda_{34}] = 1$$

from $\mathcal{R}^V(5)$.

Taking the second relation in the 1-st group and acting on it by s_i, $i = 0, 1, 2, 3$, we get the following relations

$$[\lambda_{35}, \lambda_{14}\lambda_{24}] = [\lambda_{25}\lambda_{35}, \lambda_{14}] = [\lambda_{25}, \lambda_{14}\lambda_{13}] = [\lambda_{25}\lambda_{24}, \lambda_{13}] = 1.$$

Using the commutativity relation

$$\lambda_{24}\lambda_{13} = \lambda_{13}\lambda_{24},$$

which holds in VP_4, and from the last relation, we have

$$[\lambda_{25}, \lambda_{14}] = 1. \tag{4.4}$$

Then the third relation follows $[\lambda_{25}, \lambda_{13}] = 1$, the second relation follows $[\lambda_{35}, \lambda_{14}] = 1$ and the first relation follows $[\lambda_{35}, \lambda_{24}] = 1$. Hence, we have proved the following lemma.

Lemma 4.5 *From the lifting s_i, $i = 0, 1, 2, 3$, of the relation $[\lambda_{24}, \lambda_{13}] = 1$ and the commutativity relations in $\mathcal{R}^V(4)$, it follows the commutativity relations*

$$[\lambda_{25}, \lambda_{14}] = [\lambda_{25}, \lambda_{13}] = [\lambda_{35}, \lambda_{14}] = [\lambda_{35}, \lambda_{24}] = 1$$

from $\mathcal{R}^V(5)$.

Taking the first relation in the 1-st group and acting on it by s_i, $i = 0, 1, 2, 3$, we can prove the following lemma.

Lemma 4.6 *From the lifting s_i, $i = 0, 1, 2, 3$ of the relation $[\lambda_{34}, \lambda_{12}] = 1$, the commutativity relations in $\mathcal{R}^V(4)$ and relations from Lemma 4.5, it follows the commutativity relations*

$$[\lambda_{35}, \lambda_{12}] = [\lambda_{45}, \lambda_{12}] = [\lambda_{45}, \lambda_{13}] = [\lambda_{45}, \lambda_{23}] = 1$$

from VP_5.

Considering the 2-nd group of commutativity relations we can prove the following lemma.

Lemma 4.7 *Acting by lifting s_i, $i = 0, 1, 2, 3$ and using the commutativity relations of VP_4, it is possible to get*

(1) *from the commutativity relation* $[\lambda_{14}, \lambda_{32}] = 1$ *relations that*

$$[\lambda_{15}, \lambda_{32}] = [\lambda_{15}, \lambda_{42}] = [\lambda_{15}, \lambda_{43}] = [\lambda_{25}, \lambda_{43}] = 1,$$

(2) *from the commutativity relation* $[\lambda_{24}, \lambda_{31}] = 1$ *relations that*

$$[\lambda_{25}, \lambda_{31}] = [\lambda_{25}, \lambda_{41}] = [\lambda_{35}, \lambda_{41}] = [\lambda_{35}, \lambda_{42}] = 1,$$

(3) *from the commutativity relation* $[\lambda_{34}, \lambda_{21}] = 1$ *relations that*

$$[\lambda_{35}, \lambda_{21}] = [\lambda_{45}, \lambda_{21}] = [\lambda_{45}, \lambda_{31}] = [\lambda_{45}, \lambda_{32}] = 1.$$

In VP_5 we have 24 commutativity relations of the form $[\lambda_{i5}, \lambda_{kl}^*] = 1$, $\lambda_{kl}^* \in \{\lambda_{kl}, \lambda_{lk}\}$, where $1 \leq i < 5$, $1 \leq k < l \leq 4$,

$$[\lambda_{45}, \lambda_{12}^*] = [\lambda_{45}, \lambda_{13}^*] = [\lambda_{45}, \lambda_{23}^*] = [\lambda_{35}, \lambda_{12}^*] = [\lambda_{35}, \lambda_{14}^*] = [\lambda_{35}, \lambda_{24}^*] = 1,$$

$$[\lambda_{25}, \lambda_{13}^*] = [\lambda_{25}, \lambda_{14}^*] = [\lambda_{25}, \lambda_{34}^*] = [\lambda_{15}, \lambda_{23}^*] = [\lambda_{15}, \lambda_{24}^*] = [\lambda_{15}, \lambda_{34}^*] = 1.$$

These relations follow from the 1-st and the 2-nd groups of commutativity relations in $\mathcal{R}^V(4)$. The other commutativity relations from $\mathcal{R}^V(5) \setminus \mathcal{R}^V(4)$ follow by the same way from the 3-d and the 4-th groups of relations.

4.3 Lifting the commutativity relations from $\mathcal{R}^V(n)$ to $\mathcal{R}^V(n+1)$, $n \geq 5$

We have to show that $\mathcal{R}^V(n+1) = \langle \mathcal{R}^V(n), s_i(\mathcal{R}^V(n)), i = 0, 1, \cdots, n-1 \rangle$. At first, we consider the commutativity relations

$$[\lambda_{mn}^*, \lambda_{kl}^*], \quad 1 \leq m < n,\ 1 \leq k < l < n$$

in VP_n, which are not commutativity relations in VP_{n-1}. We divide them into four groups:

1-st group: $[\lambda_{mn}, \lambda_{kl}] = 1$;

2-nd group: $[\lambda_{mn}, \lambda_{lk}] = 1$;

3-d group: $[\lambda_{nm}, \lambda_{lk}] = 1$;

4-th group: $[\lambda_{nm}, \lambda_{kl}] = 1$.

Consider the relations from the 1-st group and divide them into some subgroups.

(1) Suppose that $m < k < l < n$.

Acting on the relation $[\lambda_{mn}, \lambda_{kl}] = 1$ by s_{n-1} and using Proposition 4.1, we get the following relations.

$$[\lambda_{m,n+1}\lambda_{mn}, \lambda_{kl}] = 1.$$

Since $[\lambda_{mn}, \lambda_{kl}] = 1$ and this relation is a relation in VP_n, we have relation in VP_{n+1}:

$$[\lambda_{m,n+1}, \lambda_{kl}] = 1. \tag{4.5}$$

Let i be such that $m < k < l < i < n$. Acting by s_{i-1} on the relation $[\lambda_{mn}, \lambda_{kl}] = 1$, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{kl}] = 1,$$

that is a relation in VP_{n+1}.

Let $i = l$, then

$$s_{l-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{k,l+1}\lambda_{kl}] = 1.$$

Using the commutativity relations in VP_n and relation (4.5), we have

$$[\lambda_{m,n+1}, \lambda_{k,l+1}] = 1, \tag{4.6}$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $m < k < i < l < n$. Acting by s_{i-1}, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{k,l+1}] = 1,$$

that is a relation in VP_{n+1}.

Let $i = k$. Acting by s_{k-1}, we get

$$s_{k-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{k,l+1}\lambda_{k+1,l+1}] = 1.$$

Using relation (4.6), we have

$$[\lambda_{m,n+1}, \lambda_{k+1,l+1}] = 1, \tag{4.7}$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $m < i < k < l < n$. Acting by s_{i-1}, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{k+1,l+1}] = 1,$$

that is a relation in VP_{n+1}.

Let $i = m$. Acting by s_{m-1}, we get

$$s_{m-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1.$$

Using relation (4.7), we have

$$[\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1,$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $i < m < k < l < n$. Acting by s_{i-1}, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1,$$

that is a relation in VP_{n+1}.

(2) Suppose that $k < m < l < n$.

Acting on the relation $[\lambda_{mn}, \lambda_{kl}] = 1$ by s_{n-1}, we get the relation

$$[\lambda_{m,n+1}\lambda_{mn}, \lambda_{kl}] = 1.$$

Since $[\lambda_{mn}, \lambda_{kl}] = 1$ that follows from the relations in VP_n, we have relation

$$[\lambda_{m,n+1}, \lambda_{kl}] = 1.$$

Let i be such that $k < m < l < i < n$. Acting by s_{i-1} on the relation $[\lambda_{mn}, \lambda_{kl}] = 1$, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{kl}] = 1, \tag{4.8}$$

that is a relation in VP_{n+1}.

Let $i = l$, then

$$s_{l-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{k,l+1}\lambda_{kl}] = 1.$$

Using the commutativity relations in VP_n and relation (4.8), we have

$$[\lambda_{m,n+1}, \lambda_{k,l+1}] = 1,$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $k < m < i < l < n$. Acting by s_{i-1}, we get

$$s_i([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{k,l+1}] = 1, \tag{4.9}$$

that is a relation in VP_{n+1}.

Let $i = m$. Acting by s_{m-1}, we get

$$s_m([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}\lambda_{m+1,n+1}, \lambda_{k,l+1}] = 1. \tag{4.10}$$

Using relation (4.9), we have

$$[\lambda_{m+1,n+1}, \lambda_{k,l+1}] = 1,$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $k < i < m < l < n$. Acting by s_{i-1}, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k,l+1}] = 1, \tag{4.11}$$

that is a relation in VP_{n+1}.

Let $i = k$. Acting by s_{k-1} we get

$$s_{k-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k,l+1}\lambda_{k+1,l+1}] = 1. \tag{4.12}$$

Using relation (4.11), we have

$$[\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1,$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $i < k < m < l < n$. Acting by s_{i-1}, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1, \tag{4.13}$$

that is a relation in VP_{n+1}.

(3) Suppose that $k < l < m < n$.

Acting on $[\lambda_{mn}, \lambda_{kl}] = 1$ by s_{n-1}, we get the relation

$$s_{n-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}\lambda_{mn}, \lambda_{kl}] = 1.$$

Using commutativity relations in VP_n and the commutativity relations in VP_{n+1}, which were proved in (2), from our relation it follows

$$[\lambda_{m,n+1}, \lambda_{kl}] = 1. \tag{4.14}$$

Let i be such that $k < l < m < i < n$. Acting by s_{i-1} on the relation $[\lambda_{mn}, \lambda_{kl}] = 1$, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}, \lambda_{kl}] = 1,$$

that is a relation in VP_{n+1}.

Let $i = m$, then

$$s_{m-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m,n+1}\lambda_{m+1,n+1}, \lambda_{kl}] = 1.$$

Using relation (4.14), we have

$$[\lambda_{m+1,n+1}, \lambda_{k,l}] = 1, \tag{4.15}$$

i.e., a commutativity relation in VP_{n+1}.

Let i satisfy the inequality $k < l < i < m < n$. Acting by s_{i-1}, we get

$$s_i([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,l+1}, \lambda_{kl}] = 1,$$

that is a relation in VP_{n+1}.

Let $i = l$. Acting by s_{l-1}, we get

$$s_{l-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k,l+1}\lambda_{kl}] = 1.$$

Using relation (4.15), we get

$$[\lambda_{m+1,n+1}, \lambda_{k,l+1}] = 1. \tag{4.16}$$

Let i satisfy the inequality $k < i < l < m < n$. Acting by s_{i-1}, we get

$$s_{i-1}([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k,l+1}] = 1, \tag{4.17}$$

that is a relation in VP_{n+1}.

Let $i = k$. Acting by s_{k-1}, we get

$$s_k([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k,l+1}\lambda_{k+1,l+1}] = 1.$$

Using relation (4.16), we have

$$[\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1.$$

Let i satisfy the inequality $i < k < l < m < n$. Acting by s_{i-1}, we get

$$s_i([\lambda_{mn}, \lambda_{kl}]) = [\lambda_{m+1,n+1}, \lambda_{k+1,l+1}] = 1.$$

We considered only the 1-st group of relations. The proof for the other groups is similar.

4.4 Lifting the long relations from $\mathcal{R}^V(n)$ to $\mathcal{R}^V(n+1)$, $n \geq 4$

Denote by R_{ijk} the following set of long relations:

$$\lambda_{ij}\lambda_{ik}\lambda_{jk} = \lambda_{jk}\lambda_{ik}\lambda_{ij}, \quad \lambda_{ji}\lambda_{jk}\lambda_{ik} = \lambda_{ik}\lambda_{jk}\lambda_{ji},$$

$$\lambda_{ik}\lambda_{ij}\lambda_{kj} = \lambda_{kj}\lambda_{ij}\lambda_{ik}, \quad \lambda_{ki}\lambda_{kj}\lambda_{ij} = \lambda_{ij}\lambda_{kj}\lambda_{ki},$$

$$\lambda_{jk}\lambda_{ji}\lambda_{ki} = \lambda_{ki}\lambda_{ji}\lambda_{jk}, \quad \lambda_{kj}\lambda_{ki}\lambda_{ji} = \lambda_{ji}\lambda_{ki}\lambda_{kj},$$

i.e., relations which contain the generators with indices from the set $\{i, j, k\}$.

We have to prove that relations $R_{i,j,n+1}$ follow from relations of $\mathcal{R}^V(n)$, $s_l(\mathcal{R}^V(n))$, $l = 0, 1, \cdots, n-1$ and commutativity relations of $\mathcal{R}^V(n+1)$.

Theorem 4.1 *The long relations $R_{i,j,n+1}$ in $\mathcal{R}^V(n+1)$ follow from the relations of $\mathcal{R}^V(n)$, $s_l(\mathcal{R}^V(n))$, $l = 0, 1, \cdots, n-1$ and commutativity relations of $\mathcal{R}^V(n+1)$.*

To prove this theorem we start with the following lemma.

Lemma 4.8 *Let $n \geq 4$ and for the set of integer numbers $\{i, j, n+1\}$, $1 \leq i < j \leq n+1$, one of the following conditions holds*

(1) $i \geq 3$;

(2) $j - i \geq 3$;

(3) $n + 1 - j \geq 3$.

Then there is an integer k, $1 \leq k \leq n$, such that the relations $R_{i,j,n+1} \subseteq \mathcal{R}^V(n+1)$ follow from the relations $s_{k-1}(\mathcal{R}^V(n))$.

Proof (1) Suppose that the condition (1) holds. Put $k = 1$ and consider the relations $R_{i-1,j-1,n}$ in $\mathcal{R}^V(n)$. It is not difficult to see that $s_0(R_{i-1,j-1,n}) = R_{i,j,n+1}$.

(2) Suppose that the condition (2) holds. Put $k = i + 1$ and consider the relations $R_{i,j-1,n}$ in $\mathcal{R}^V(n)$. It is not difficult to see that $s_i(R_{i,j-1,n}) = R_{i,j,n+1}$.

(3) Suppose that the condition(3) holds. Put $k = j + 1$ and consider the relations $R_{i,j,n}$ in $\mathcal{R}^V(n)$. It is not difficult to see that $s_j(R_{i,j,n}) = R_{i,j,n+1}$.

Now suppose that $i = 2$ and for the set $\{i, j, n+1\}$, none of the conditions of the lemma is satisfied. Take the set of relations $R_{1,j-1,n}$ and find $s_0(R_{1,j-1,n})$. The first relation in $R_{1,j-1,n}$ has the form

$$\lambda_{1,j-1}\lambda_{1n}\lambda_{j-1,n} = \lambda_{j-1,n}\lambda_{1n}\lambda_{1,j-1}.$$

Acting by s_0 we get the relation

$$(\lambda_{1,j}\lambda_{2,j})(\lambda_{1,n+1}\lambda_{2,n+1})\lambda_{j,n+1} = \lambda_{j,n+1}(\lambda_{1,n+1}\lambda_{2,n+1})(\lambda_{1,j}\lambda_{2,j}).$$

Since $\lambda_{2,j}\lambda_{1,n+1} = \lambda_{1,n+1}\lambda_{2,j}$ and $\lambda_{2,n+1}\lambda_{1,j} = \lambda_{1,j}\lambda_{2,n+1}$, we rewrite the last relation in the form

$$\lambda_{1,j}\lambda_{1,n+1}\lambda_{2,j}\lambda_{2,n+1}\lambda_{j,n+1} = (\lambda_{j,n+1}\lambda_{1,n+1}\lambda_{1,j})\lambda_{2,n+1}\lambda_{2,j}. \tag{4.18}$$

Take the set $\{1, j, n+1\}$. Since $n \geq 4$, for this set condition (2) or condition (3) of Lemma (4.8) holds. Then the set of relation $R_{1,j,n+1}$ comes from relations of VP_n. In particular, the relation

$$\lambda_{j,n+1}\lambda_{1,n+1}\lambda_{1,j} = \lambda_{1,j}\lambda_{1,n+1}\lambda_{j,n+1}$$

holds. Using this relation, we rewrite (4.18) as

$$\lambda_{1,j}\lambda_{1,n+1}\lambda_{2,j}\lambda_{2,n+1}\lambda_{j,n+1} = (\lambda_{1,j}\lambda_{1,n+1}\lambda_{j,n+1})\lambda_{2,n+1}\lambda_{2,j}.$$

After cancelations we have

$$\lambda_{2,j}\lambda_{2,n+1}\lambda_{j,n+1} = \lambda_{j,n+1}\lambda_{2,n+1}\lambda_{2,j}.$$

It is the first relation from $R_{2,j,n+1}$.

The second relation in $R_{1,j-1,n}$ has the form

$$\lambda_{j-1,1}\lambda_{j-1,n}\lambda_{1,n} = \lambda_{1,n}\lambda_{j-1,n}\lambda_{j-1,1}.$$

Acting by s_0, we get the relation

$$(\lambda_{j2}\lambda_{j1})\lambda_{j,n+1}(\lambda_{1,n+1}\lambda_{2,n+1}) = (\lambda_{1,n+1}\lambda_{2,n+1})\lambda_{j,n+1}(\lambda_{j2}\lambda_{j1}).$$

As we saw before, the set of relation $R_{1,j,n+1}$ holds in VP_{n+1}. Using the relation

$$\lambda_{j1}\lambda_{j,n+1}\lambda_{1,n+1} = \lambda_{1,n+1}\lambda_{j,n+1}\lambda_{j1},$$

we rewrite our relation in the form

$$\lambda_{j2}(\lambda_{1,n+1}\lambda_{j,n+1}\lambda_{j1})\lambda_{2,n+1} = \lambda_{1,n+1}\lambda_{2,n+1}\lambda_{j,n+1}\lambda_{j2}\lambda_{j1}.$$

Using the commutativity relations $\lambda_{j2}\lambda_{1,n+1} = \lambda_{1,n+1}\lambda_{j2}$ and $\lambda_{j1}\lambda_{2,n+1} = \lambda_{2,n+1}\lambda_{j1}$ we have

$$(\lambda_{1,n+1}\lambda_{j2})\lambda_{j,n+1}(\lambda_{2,n+1}\lambda_{j1}) = \lambda_{1,n+1}\lambda_{2,n+1}\lambda_{j,n+1}\lambda_{j2}\lambda_{j1}.$$

After cancelations we get

$$\lambda_{j2}\lambda_{j,n+1}\lambda_{2,n+1} = \lambda_{2,n+1}\lambda_{j,n+1}\lambda_{j2}.$$

It is the second relation from $R_{1,j,n+1}$.

The third relation in $R_{1,j-1,n}$ has the form

$$\lambda_{1n}\lambda_{1,j-1}\lambda_{n,j-1} = \lambda_{n,j-1}\lambda_{1,j-1}\lambda_{1n}.$$

Acting by s_0 we get the relation

$$(\lambda_{1,n+1}\lambda_{2,n+1})(\lambda_{1j}\lambda_{2j})\lambda_{n+1,j} = \lambda_{n+1,j}(\lambda_{1j}\lambda_{2j})(\lambda_{1,n+1}\lambda_{2,n+1}).$$

Since $\lambda_{2,n+1}\lambda_{1j} = \lambda_{1j}\lambda_{2,n+1}$ and $\lambda_{2j}\lambda_{1,n+1} = \lambda_{1,n+1}\lambda_{2j}$, we rewrite the last relation in the form

$$\lambda_{1,n+1}\lambda_{1j}\lambda_{2,n+1}\lambda_{2j}\lambda_{n+1,j} = (\lambda_{n+1,j}\lambda_{1j}\lambda_{1,n+1})\lambda_{2j}\lambda_{2,n+1}. \tag{4.19}$$

As we saw, the set of relation $R_{1,j,n+1}$ comes from relations of VP_n. In particular, the relation

$$\lambda_{n+1,j}\lambda_{1j}\lambda_{1,n+1} = \lambda_{1,n+1}\lambda_{1j}\lambda_{n+1,j}$$

holds. Using this relation, we rewrite (4.19) as

$$\lambda_{1,n+1}\lambda_{1j}\lambda_{2,n+1}\lambda_{2j}\lambda_{n+1,j} = (\lambda_{1,n+1}\lambda_{1j}\lambda_{n+1,j})\lambda_{2j}\lambda_{2,n+1}.$$

After cancelations we have

$$\lambda_{2,n+1}\lambda_{2j}\lambda_{n+1,j} = \lambda_{n+1,j}\lambda_{2j}\lambda_{2,n+1}.$$

It is the third relation from $R_{1,j,n+1}$.

The fourth relation in $R_{1,j-1,n}$ has the form

$$\lambda_{n1}\lambda_{n,j-1}\lambda_{1,j-1} = \lambda_{1,j-1}\lambda_{n,j-1}\lambda_{n1}.$$

Acting by s_0, we get the relation

$$(\lambda_{n+1,2}\lambda_{n+1,1})\lambda_{n+1,j}(\lambda_{1j}\lambda_{2j}) = (\lambda_{1j}\lambda_{2j})\lambda_{n+1,j}(\lambda_{n+1,2}\lambda_{n+1,1}).$$

As we saw before, the set of relation $R_{1,j,n+1}$ holds in VP_{n+1}. Using the relation

$$\lambda_{n+1,1}\lambda_{n+1,j}\lambda_{1j} = \lambda_{1j}\lambda_{n+1,j}\lambda_{n+1,1},$$

we rewrite our relation in the form

$$\lambda_{n+1,2}(\lambda_{1j}\lambda_{n+1,j}\lambda_{n+1,1})\lambda_{2j} = \lambda_{1j}\lambda_{2j}\lambda_{n+1,j}\lambda_{n+1,2}\lambda_{n+1,1}.$$

Using the commutativity relations $\lambda_{n+1,2}\lambda_{1j} = \lambda_{1j}\lambda_{n+1,2}$ and $\lambda_{n+1,1}\lambda_{2j} = \lambda_{2j}\lambda_{n+1,1}$ we have

$$(\lambda_{1j}\lambda_{n+1,2})\lambda_{n+1,j}(\lambda_{2j}\lambda_{n+1,1}) = \lambda_{1j}\lambda_{2j}\lambda_{n+1,j}\lambda_{n+1,2}\lambda_{n+1,1}.$$

After cancelations we get

$$\lambda_{n+1,2}\lambda_{n+1,j}\lambda_{2j} = \lambda_{2j}\lambda_{n+1,j}\lambda_{n+1,2}.$$

It is the fourth relation from $R_{1,j,n+1}$.

The fifth relation in $R_{1,j-1,n}$ has the form

$$\lambda_{j-1,n}\lambda_{j-1,1}\lambda_{n1} = \lambda_{n1}\lambda_{j-1,1}\lambda_{j-1,n}.$$

Acting by s_0, we get the relation

$$\lambda_{j,n+1}(\lambda_{j2}\lambda_{j1})(\lambda_{n+1,2}\lambda_{n+1,1}) = (\lambda_{n+1,2}\lambda_{n+1,1})(\lambda_{j2}\lambda_{j1})\lambda_{j,n+1}.$$

Since $\lambda_{j1}\lambda_{n+1,2} = \lambda_{n+1,2}\lambda_{j1}$ and $\lambda_{n+1,1}\lambda_{j2} = \lambda_{j2}\lambda_{n+1,1}$, we rewrite the last relation in the form

$$\lambda_{j,n+1}\lambda_{j2}(\lambda_{n+1,2}\lambda_{j1})\lambda_{n+1,1} = \lambda_{n+1,2}(\lambda_{j2}\lambda_{n+1,1})\lambda_{j1}\lambda_{j,n+1}. \quad (4.20)$$

As we noted before, the set of relation $R_{1,j,n+1}$ comes from relations of VP_n and in particular, the relation

$$\lambda_{n+1,1}\lambda_{j1}\lambda_{j,n+1} = \lambda_{j,n+1}\lambda_{j1}\lambda_{n+1,1}$$

holds. Using this relation, we rewrite (4.20) as

$$\lambda_{j,n+1}\lambda_{j2}\lambda_{j,n+1}\lambda_{j1}\lambda_{n+1,1} = \lambda_{n+1,2}\lambda_{j2}\lambda_{j,n+1}\lambda_{j1}\lambda_{n+1,1}.$$

After cancelations we have

$$\lambda_{j,n+1}\lambda_{j2}\lambda_{j,n+1} = \lambda_{j,n+1}\lambda_{j2}\lambda_{j,n+1}.$$

It is the fifth relation from $R_{1,j,n+1}$.

The sixth relation in $R_{1,j-1,n}$ has the form

$$\lambda_{n,j-1}\lambda_{n1}\lambda_{j-1,1} = \lambda_{j-1,1}\lambda_{n1}\lambda_{n,j-1}.$$

Acting by s_0, we get the relation

$$\lambda_{n+1,j}(\lambda_{n+1,2}\lambda_{n+1,1})(\lambda_{j2}\lambda_{j1}) = (\lambda_{j2}\lambda_{j1})(\lambda_{n+1,2}\lambda_{n+1,1})\lambda_{n+1,j}.$$

Using the commutativity relations $\lambda_{n+1,1}\lambda_{j2} = \lambda_{j2}\lambda_{n+1,1}$ and $\lambda_{j1}\lambda_{n+1,2} = \lambda_{n+1,2}\lambda_{j1}$ we have

$$\lambda_{n+1,j}\lambda_{n+1,2}\lambda_{j2}\lambda_{n+1,1}\lambda_{j1} = \lambda_{j2}\lambda_{n+1,2}(\lambda_{j1}\lambda_{n+1,1}\lambda_{n+1,j}).$$

Using the relation

$$\lambda_{j1}\lambda_{n+1,1}\lambda_{n+1,j} = \lambda_{n+1,j}\lambda_{n+1,1}\lambda_{j1},$$

we rewrite our relation in the form

$$\lambda_{n+1,j}\lambda_{n+1,2}\lambda_{j2}\lambda_{n+1,1}\lambda_{j1} = \lambda_{j2}\lambda_{n+1,2}(\lambda_{n+1,j}\lambda_{n+1,1}\lambda_{j1}).$$

After cancelations we get

$$\lambda_{n+1,j}\lambda_{n+1,2}\lambda_{j2} = \lambda_{j2}\lambda_{n+1,2}\lambda_{n+1,j}.$$

It is the sixth relation from $R_{2,j,n+1}$.

Hence, we have proved the following lemma.

Lemma 4.9 *Let $n \geq 4$. Acting on the relations $R_{1,j-1,n}$ of VP_n by s_0 and using the relations, which we got in Lemma 4.8, we get relations $R_{2,j,n+1}$ in VP_{n+1}.*

Next, suppose that $i = 1$ in the set $\{i, j, n+1\}$. Since $n \geq 4$ and we can not use Lemma 4.8 for the relations $R_{i,j,n+1}$, we see that it is possible only in the case $j = 3$, $n+1 = 5$. Hence we have to prove that the relations $R_{1,3,5}$ follow from relations $s_k(\mathcal{R}^V(4))$ for some k.

Consider relations $R_{1,2,4}$ in VP_4 and act on them by s_1. The first relation in $R_{1,2,4}$ has the form

$$\lambda_{12}\lambda_{14}\lambda_{24} = \lambda_{24}\lambda_{14}\lambda_{12}.$$

Acting on it by s_1, we get

$$(\lambda_{13}\lambda_{12})\lambda_{15}(\lambda_{25}\lambda_{35}) = (\lambda_{25}\lambda_{35})\lambda_{15}(\lambda_{13}\lambda_{12}).$$

Note that relations $R_{1,2,5}$ satisfy condition (3) in Lemma 4.8. Using the first relation from this set

$$\lambda_{12}\lambda_{15}\lambda_{25} = \lambda_{25}\lambda_{15}\lambda_{12},$$

we get

$$\lambda_{13}(\lambda_{25}\lambda_{15}\lambda_{12})\lambda_{35} = \lambda_{25}\lambda_{35}\lambda_{15}\lambda_{13}\lambda_{12}.$$

Using the commutativity relations $\lambda_{13}\lambda_{25} = \lambda_{25}\lambda_{13}$ and $\lambda_{12}\lambda_{35} = \lambda_{35}\lambda_{12}$, we have

$$(\lambda_{25}\lambda_{13})\lambda_{15}(\lambda_{35}\lambda_{12}) = \lambda_{25}\lambda_{35}\lambda_{15}\lambda_{13}\lambda_{12}.$$

After cancelation we arrive to the relation

$$\lambda_{13}\lambda_{15}\lambda_{35} = \lambda_{35}\lambda_{15}\lambda_{13}.$$

This is the first relation from $R_{1,3,5}$.

The second relation in $R_{1,2,4}$ has the form

$$\lambda_{21}\lambda_{24}\lambda_{14} = \lambda_{14}\lambda_{24}\lambda_{21}.$$

Acting on it by s_1, we get

$$(\lambda_{21}\lambda_{31})(\lambda_{25}\lambda_{35})\lambda_{15} = \lambda_{15}(\lambda_{25}\lambda_{35})(\lambda_{21}\lambda_{31}).$$

Using the commutativity relation $\lambda_{31}\lambda_{25} = \lambda_{25}\lambda_{31}$ and $\lambda_{35}\lambda_{21} = \lambda_{21}\lambda_{35}$, we have

$$\lambda_{21}(\lambda_{25}\lambda_{31})\lambda_{35}\lambda_{15} = \lambda_{15}\lambda_{25}(\lambda_{21}\lambda_{35})\lambda_{31}.$$

By Lemma 4.8, we have relation

$$\lambda_{15}\lambda_{25}\lambda_{21} = \lambda_{21}\lambda_{25}\lambda_{15}.$$

Using it we get

$$\lambda_{21}\lambda_{25}\lambda_{31}\lambda_{35}\lambda_{15} = (\lambda_{21}\lambda_{25}\lambda_{15})\lambda_{35}\lambda_{31}.$$

After cancelation we arrive to the relation

$$\lambda_{31}\lambda_{35}\lambda_{15} = \lambda_{15}\lambda_{35}\lambda_{31}.$$

This is the second relation from $R_{1,3,5}$.

Using the third relation in the set $R_{1,2,4}$

$$\lambda_{14}\lambda_{12}\lambda_{42} = \lambda_{42}\lambda_{12}\lambda_{14}$$

and acting by s_1, we get

$$\lambda_{15}(\lambda_{13}\lambda_{12})(\lambda_{53}\lambda_{52}) = (\lambda_{53}\lambda_{52})(\lambda_{13}\lambda_{12})\lambda_{15}.$$

Using the commutativity relation $\lambda_{12}\lambda_{53} = \lambda_{53}\lambda_{12}$ and $\lambda_{52}\lambda_{13} = \lambda_{13}\lambda_{52}$, we have

$$\lambda_{15}\lambda_{13}(\lambda_{53}\lambda_{12})\lambda_{52} = \lambda_{53}(\lambda_{13}\lambda_{52})\lambda_{12}\lambda_{15}.$$

Using the relation

$$\lambda_{52}\lambda_{12}\lambda_{15} = \lambda_{15}\lambda_{12}\lambda_{52},$$

which we have by Lemma 4.8, we get

$$\lambda_{15}\lambda_{13}\lambda_{53}\lambda_{12}\lambda_{52} = \lambda_{53}\lambda_{13}(\lambda_{15}\lambda_{12}\lambda_{52}).$$

After cancelation we arrive to the relation

$$\lambda_{15}\lambda_{13}\lambda_{53} = \lambda_{53}\lambda_{13}\lambda_{15}.$$

This is the third relation in $R_{1,3,5}$.

The fourth relation in $R_{1,2,4}$ has the form

$$\lambda_{41}\lambda_{42}\lambda_{12} = \lambda_{12}\lambda_{42}\lambda_{41}.$$

Acting on it by s_1, we get

$$\lambda_{51}(\lambda_{53}\lambda_{52})(\lambda_{13}\lambda_{12}) = (\lambda_{13}\lambda_{12})(\lambda_{53}\lambda_{52})\lambda_{51}.$$

Using the commutativity relation $\lambda_{52}\lambda_{13} = \lambda_{13}\lambda_{52}$ and $\lambda_{12}\lambda_{53} = \lambda_{53}\lambda_{12}$, we have

$$\lambda_{51}\lambda_{53}(\lambda_{13}\lambda_{52})\lambda_{12} = \lambda_{13}(\lambda_{53}\lambda_{12})\lambda_{52}\lambda_{51}.$$

By Lemma 4.8, we have relation

$$\lambda_{12}\lambda_{52}\lambda_{51} = \lambda_{51}\lambda_{52}\lambda_{12}.$$

Using it, we get

$$\lambda_{51}\lambda_{53}\lambda_{13}\lambda_{52}\lambda_{12} = \lambda_{13}\lambda_{53}(\lambda_{51}\lambda_{52}\lambda_{12}).$$

After cancelation we arrive to the relation

$$\lambda_{51}\lambda_{53}\lambda_{13} = \lambda_{13}\lambda_{53}\lambda_{51}.$$

This is the fourth relation in $R_{1,3,5}$.

Using the fifth relation in the set $R_{1,2,4}$

$$\lambda_{24}\lambda_{21}\lambda_{41} = \lambda_{41}\lambda_{21}\lambda_{24}$$

and acting by s_1, we get

$$(\lambda_{25}\lambda_{35})(\lambda_{21}\lambda_{31})\lambda_{51} = \lambda_{51}(\lambda_{21}\lambda_{31})(\lambda_{25}\lambda_{35}).$$

Using the commutativity relation $\lambda_{35}\lambda_{21} = \lambda_{21}\lambda_{35}$ and $\lambda_{31}\lambda_{25} = \lambda_{25}\lambda_{31}$, we have

$$\lambda_{25}(\lambda_{21}\lambda_{35})\lambda_{31}\lambda_{51} = \lambda_{51}\lambda_{21}(\lambda_{25}\lambda_{31})\lambda_{35}.$$

Using the relation

$$\lambda_{51}\lambda_{21}\lambda_{25} = \lambda_{25}\lambda_{21}\lambda_{51},$$

which we have by Lemma 4.8, we get

$$\lambda_{25}\lambda_{21}\lambda_{35}\lambda_{31}\lambda_{51} = (\lambda_{25}\lambda_{21}\lambda_{51})\lambda_{31}\lambda_{35}.$$

After cancelation we arrive to the relation

$$\lambda_{35}\lambda_{31}\lambda_{51} = \lambda_{51}\lambda_{31}\lambda_{35}.$$

This is the fifth relation from $R_{1,3,5}$.

The sixth relation in $R_{1,2,4}$ has the form

$$\lambda_{42}\lambda_{41}\lambda_{21} = \lambda_{21}\lambda_{41}\lambda_{42}.$$

Acting on it by s_1, we get

$$(\lambda_{53}\lambda_{52})\lambda_{51}(\lambda_{21}\lambda_{31}) = (\lambda_{21}\lambda_{31})\lambda_{51}(\lambda_{53}\lambda_{52}).$$

By Lemma 4.8, we have relation

$$\lambda_{52}\lambda_{51}\lambda_{21} = \lambda_{21}\lambda_{51}\lambda_{52},$$

from which

$$\lambda_{53}(\lambda_{21}\lambda_{51}\lambda_{52})\lambda_{31} = \lambda_{21}\lambda_{31}\lambda_{51}\lambda_{53}\lambda_{52}.$$

Using the commutativity relation $\lambda_{53}\lambda_{21} = \lambda_{21}\lambda_{53}$ and $\lambda_{52}\lambda_{31} = \lambda_{31}\lambda_{52}$, we have

$$(\lambda_{21}\lambda_{53})\lambda_{51}(\lambda_{31}\lambda_{52}) = \lambda_{21}\lambda_{31}\lambda_{51}\lambda_{53}\lambda_{52}.$$

After cancelation we arrive to the relation

$$\lambda_{53}\lambda_{51}\lambda_{31} = \lambda_{31}\lambda_{51}\lambda_{53}.$$

This is the sixth relation from $R_{1,3,5}$.

4.5 Simplicial group T_*

The simplicial group T_* was defined in the paper [4]. In the same paper it was proved that T_3 is generated by elements

$$a_{31},\ a_{22},\ a_{13},\ b_{31},\ b_{22},\ b_{13}$$

and is defined by relations

$$[a_{31}, a_{22}]^{c_{11}^k c_{21}^m} = [a_{31}, a_{13}]^{c_{11}^k c_{21}^m} = [a_{22}, a_{13}]^{c_{11}^k c_{21}^m} = 1,$$

$$[b_{31}, b_{22}]^{c_{11}^k c_{21}^m} = [b_{31}, b_{13}]^{c_{11}^k c_{21}^m} = [b_{22}, b_{13}]^{c_{11}^k c_{21}^m} = 1$$

that can be written in the form

$$[a_{31}, a_{22}^{c_{22}^m c_{31}^{-m}}] = [a_{31}, a_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = [a_{22}^{c_{22}^m c_{31}^{-m}}, a_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = 1,$$

$$[b_{31}, b_{22}^{c_{22}^m c_{31}^{-m}}] = [b_{31}, b_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = [b_{22}^{c_{22}^m c_{31}^{-m}}, b_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = 1,$$

where $k, m \in \mathbb{Z}$.

In the general case we will prove the following theorem.

Theorem 4.2 *The group T_n, $n \geq 2$ is generated by elements*

$$a_{i,n+1-i},\ b_{i,n+1-i}, \quad i = 1, 2, \cdots, n,$$

and is defined by relations

$$[a_{i,n+1-i}, a_{j,n+1-j}]^{c_{11}^{k_1} c_{21}^{k_2} \cdots c_{n-1,1}^{k_{n-1}}},$$

$$[b_{i,n+1-i}, b_{j,n+1-j}]^{c_{11}^{k_1} c_{21}^{k_2} \cdots c_{n-1,1}^{k_{n-1}}},$$

where $1 \leq i \neq j \leq n$, $k_l \in \mathbb{Z}$.

5 VP_n as a Subgroup of VB_{n+1}

In the previous section we showed how it is possible to construct VP_n from VP_{n-1} using operation cabling. In this section we will show how it is possible to construct VP_{n+1}, using the action of the symmetric group $S_{n+1} = \langle \rho_1, \rho_2, \cdots, \rho_{n_1} \rangle$, which is a subgroup of the virtual braid group $VB_{n+1} = VP_{n+1} \rtimes S_{n+1}$. Recall that S_{n+1} acts on the generators of VP_{n+1} by the rule

$$\rho_k \lambda_{ij} \rho_k = \lambda_{\rho_k(i), \rho_k(j)}, \quad k = 1, 2, \cdots, n-1.$$

The symmetric group S_{n+1} ia s disjoint union of cosets by S_n:

$$S_{n+1} = S_n e \sqcup S_n \rho_n \sqcup S_n \rho_n \rho_{n-1} \sqcup \cdots \sqcup S_n \rho_n \rho_{n-1} \cdots \rho_1.$$

We will denote $\mathcal{X}_k$ the set of generators of VP_k, $k \geq 2$, i.e.,

$$\mathcal{X}_k = \{\lambda_{ij} \mid 1 \leq i \neq j \leq k\};$$

$\mathcal{R}_k$ will denote the set of defining relations of VP_k. In particular, $\mathcal{LR}_k$ will denote the set of long relations and $\mathcal{CR}_k$ will denote the set of commutativity relations. It is evident that

$$\mathcal{R}_k = \mathcal{LR}_k \cup \mathcal{CR}_k.$$

Since VP_3 does not contain commutativity relations, $\mathcal{R}_3 = \mathcal{LR}_3$.

Let $k > 2$ and $1 \leq i < j < l \leq k$ be three distinct integer numbers. Denote by $\mathcal{R}_k^{ijl}$ the following set of long defining relations from $\mathcal{R}_k$:

$$\lambda_{ij}\lambda_{il}\lambda_{jl} = \lambda_{jl}\lambda_{il}\lambda_{ij}, \quad \lambda_{ji}\lambda_{jl}\lambda_{il} = \lambda_{il}\lambda_{jl}\lambda_{ji},$$

$$\lambda_{il}\lambda_{ij}\lambda_{lj} = \lambda_{lj}\lambda_{ij}\lambda_{il}, \quad \lambda_{li}\lambda_{lj}\lambda_{ij} = \lambda_{ij}\lambda_{lj}\lambda_{li},$$

$$\lambda_{jl}\lambda_{ji}\lambda_{li} = \lambda_{li}\lambda_{ji}\lambda_{jl}, \quad \lambda_{lj}\lambda_{li}\lambda_{ji} = \lambda_{ji}\lambda_{li}\lambda_{lj}.$$

Then

$$\mathcal{LR}_k = \bigsqcup_{1 \leq i < j < l \leq k} \mathcal{R}_k^{ijl}.$$

In particular,

$$\mathcal{R}_3 = \mathcal{R}_3^{123}.$$

Let the integers $i, j, l, m \in \{1, 2, \cdots, k\}$ satisfy the conditions

$$i < j, \quad l < m, \quad j > m.$$

Denote

$$\mathcal{R}_k^{i,j,l,m} = \{\lambda_{ij}^* \lambda_{lm}^* = \lambda_{lm}^* \lambda_{ij}^*\}$$

the set of four commutativity relations with fixed indices, then

$$\mathcal{CR}_k = \bigsqcup_{i<j, \ l<m, \ j>m} \mathcal{R}_k^{i,j,l,m}$$

is the full set of the commutativity relations in VP_k

Taking the set of generators of VP_3:

$$\mathcal{X}_3 = \{\lambda_{12}, \lambda_{21}, \lambda_{13}, \lambda_{23}, \lambda_{31}, \lambda_{32}\}$$

and acting on it by coset representatives of S_4 by S_3, we get

$$\begin{aligned}
\mathcal{X}_3^{\rho_3} &= \{\lambda_{12}, \lambda_{21}, \lambda_{14}, \lambda_{24}, \lambda_{41}, \lambda_{42}\}, \\
\mathcal{X}_3^{\rho_3\rho_2} &= \{\lambda_{13}, \lambda_{31}, \lambda_{14}, \lambda_{34}, \lambda_{41}, \lambda_{43}\}, \\
\mathcal{X}_3^{\rho_3\rho_2\rho_1} &= \{\lambda_{23}, \lambda_{32}, \lambda_{24}, \lambda_{34}, \lambda_{42}, \lambda_{43}\}.
\end{aligned}$$

We see that

$$\mathcal{X}_4 = \mathcal{X}_3 \cup \mathcal{X}_3^{\rho_3} \cup \mathcal{X}_3^{\rho_3\rho_2}.$$

In the general case we have the similar result.

Proposition 5.1 *For $n \geq 3$ the following equality holds*

$$\mathcal{X}_{n+1} = \mathcal{X}_n \cup \mathcal{X}_n^{\rho_n} \cup \mathcal{X}_n^{\rho_n \rho_{n-1}}.$$

Proof Any generator in $\mathcal{X}_{n+1} \setminus \mathcal{X}_n$ has the form $\lambda^*_{i,n+1}$ for some i, $1 \leq i \leq n$. Taking the generator $\lambda^*_{1n} \in \mathcal{X}_n$ and acting on it by conjugation of ρ_n:

$$(\lambda^*_{1n})^{\rho_n} = \lambda^*_{1,n+1}, \ (\lambda^*_{2n})^{\rho_n} = \lambda^*_{2,n+1}, \cdots, (\lambda^*_{n-1,n})^{\rho_n} = \lambda^*_{n-1,n+1}.$$

To find the last generator $\lambda^*_{n,n+1}$, taking the generator $\lambda^*_{n-1,n}$ and acting of conjugation by $\rho_n \rho_{n-1}$, we get

$$(\lambda^*_{n-1,n})^{\rho_n \rho_{n-1}} = (\lambda^*_{n-1,n+1})^{\rho_{n-1}} = \lambda^*_{n,n+1}.$$

To find the set of defining relations in $\mathcal{R}_4$, taking the defining relations of $\mathcal{R}_3 = \mathcal{R}^{123}$ and acting by coset representatives, we get

$$\mathcal{R}_3^{\rho_3} = \mathcal{R}_4^{124}, \quad \mathcal{R}_3^{\rho_3 \rho_2} = \mathcal{R}_4^{134}, \quad \mathcal{R}_3^{\rho_3 \rho_2 \rho_1} = \mathcal{R}_4^{234}.$$

Since

$$\mathcal{LR}_4 = \mathcal{R}_4^{123} \sqcup \mathcal{R}_4^{124} \sqcup \mathcal{R}_4^{134} \sqcup \mathcal{R}_4^{234} \ \text{ and } \ \mathcal{R}_4^{123} = \mathcal{R}_3^{123} = \mathcal{R}_3,$$

we get

$$\mathcal{LR}_4 = \mathcal{R}_3 \sqcup \mathcal{R}_3^{\rho_3} \sqcup \mathcal{R}_3^{\rho_3 \rho_2} \sqcup \mathcal{R}_3^{\rho_3 \rho_2 \rho_1}.$$

In VP_3 we don't have commutativity relations, hence we have the following proposition

Proposition 5.2

$$\mathcal{R}_4 = \mathcal{R}_3 \sqcup \mathcal{R}_3^{\rho_3} \sqcup \mathcal{R}_3^{\rho_3 \rho_2} \sqcup \mathcal{R}_3^{\rho_3 \rho_2 \rho_1} \sqcup \mathcal{CR}_4.$$

In the general case we can prove the following theorem.

Theorem 5.1 *For $n \geq 4$ we have*

$$\mathcal{R}_{n+1} = \mathcal{R}_n \sqcup \mathcal{R}_n^{\rho_n} \sqcup \mathcal{R}_n^{\rho_n \rho_{n-1}} \sqcup \cdots \sqcup \mathcal{R}_n^{\rho_n \rho_{n-1} \cdots \rho_1}.$$

Proof Consider the set of long relations $\mathcal{R}_{n+1}^{i,j,n+1}$ which does not lie in $\mathcal{R}_n$. If $j \neq n$, then the relations $\mathcal{R}_n^{i,j,n}$ lie in $\mathcal{R}_n$, acting by ρ_n, we get

$$(\mathcal{R}_n^{i,j,n})^{\rho_n} = \mathcal{R}_{n+1}^{i,j,n+1}.$$

If $j = n$, but $i \neq n-1$, then

$$(\mathcal{R}_n^{i,n-1,n})^{\rho_n \rho_{n-1}} = (\mathcal{R}_{n+1}^{i,n-1,n+1})^{\rho_{n-1}} = \mathcal{R}_{n+1}^{i,n,n+1}.$$

If $j = n$, $i = n-1$, then

$$(\mathcal{R}_n^{n-2,n-1,n})^{\rho_n \rho_{n-1} \rho_{n-2}} = (\mathcal{R}_{n+1}^{n-2,n-1,n+1})^{\rho_{n-1} \rho_{n-2}} = (\mathcal{R}_{n+1}^{n-2,n,n+1})^{\rho_{n-2}} = \mathcal{R}_{n+1}^{n-1,n,n+1}.$$

Consider a set of commutativity relations

$$\mathcal{R}_{n+1}^{i,n+1,l,m} \in \mathcal{R}_{n+1} \setminus \mathcal{R}_n.$$

We will assume that $i < l < m$. Proofs for other cases are similar.

If $m \neq n$, then

$$(\mathcal{R}_n^{i,n,l,m})^{\rho_n} = \mathcal{R}_{n+1}^{i,n+1,l,m}.$$

If $m = n$, but $l \neq n-1$, then

$$(\mathcal{R}_n^{i,n,l,n-1})^{\rho_n \rho_{n-1}} = (\mathcal{R}_{n+1}^{i,n+1,l,n-1})^{\rho_{n-1}} = \mathcal{R}_{n+1}^{i,n+1,l,n}.$$

If $m = n$, $l = n-1$ but $i \neq n-2$, then

$$(\mathcal{R}_n^{i,n,n-2,n-1})^{\rho_n \rho_{n-1} \rho_{n-2}} = (\mathcal{R}_{n+1}^{i,n+1,n-2,n-1})^{\rho_{n-1} \rho_{n-2}} = (\mathcal{R}_n^{i,n+1,n-2,n})^{\rho_{n-2}} = \mathcal{R}_{n+1}^{i,n+1,n-1,n}.$$

If $m = n$, $l = n-1$ and $i = n-2$, then

$$(\mathcal{R}_n^{n-3,n,n-2,n-1})^{\rho_n \rho_{n-1} \rho_{n-2} \rho_{n-3}} = (\mathcal{R}_{n+1}^{n-3,n+1,n-2,n-1})^{\rho_{n-1} \rho_{n-2} \rho_{n-3}}$$

$$= (\mathcal{R}_n^{n-3,n+1,n-2,n})^{\rho_{n-2} \rho_{n-3}} = (\mathcal{R}_{n+1}^{n-3,n+1,n-1,n})^{\rho_{n-3}} = \mathcal{R}_{n+1}^{n-2,n+1,n-1,n}.$$

6 Cabling of the Artin Pure Braid Group

In the paper [7] it was defined a cabling on the set of pure braid groups $\{P_n\}_{n=2,3,\cdots}$. It was proved that in fact all generators of P_n come from the unique generator A_{12} of U_2, using cabling. In this section we find a set of defining relation of P_4 in these generators.

In the previous section we define elements $c_{ij} = b_{ij} a_{ij}$. Put

$$T_k^c = \langle c_{ij} \mid i + j = k + 1 \rangle, \quad k = 1, 2, \cdots, n-1.$$

Any group T_k^c for $k > 1$ is getting from T_{k-1}^c using cabling, i.e.,

$$T_k^c = \langle s_0(T_{k-1}^c), s_1(T_{k-1}^c), \cdots, s_{k-2}(T_{k-1}^c) \rangle.$$

Then $P_n = \langle T_1^c, T_2^c, \cdots, T_{n-1}^c \rangle$.

In the paper [4] it was found the set of defining relations of P_4 in the cabled generators c_{ij}, which was more precisely proved.

Proposition 6.1 *The group P_4 is generated by elements*

$$c_{11}, \ c_{21}, \ c_{12}, \ c_{31}, \ c_{22}, \ c_{13}$$

and is defined by relations (where $\varepsilon = \pm 1$)

$$c_{21}^{c_{11}^{\varepsilon}} = c_{21}, \quad c_{12}^{c_{11}^{\varepsilon}} = c_{12}^{c_{21}^{-\varepsilon}}, \quad c_{31}^{c_{11}^{\varepsilon}} = c_{31}, \quad c_{22}^{c_{11}^{\varepsilon}} = c_{22}, \quad c_{13}^{c_{11}^{\varepsilon}} = c_{13}^{c_{22}^{-\varepsilon}},$$
$$c_{31}^{c_{21}^{\varepsilon}} = c_{31}, \quad c_{22}^{c_{21}^{\varepsilon}} = c_{22}^{c_{31}^{-\varepsilon}}, \quad c_{13}^{c_{21}^{\varepsilon}} = c_{13}^{c_{22}^{\varepsilon} c_{31}^{-\varepsilon}},$$

$$c_{31}^{c_{12}^{\varepsilon}} = c_{31}, \quad c_{13}^{c_{12}^{\varepsilon}} = c_{13}^{c_{31}^{-\varepsilon}}.$$
$$c_{22}^{c_{12}^{-1}} = [c_{31}, c_{13}^{-1}]\,[c_{13}^{-1}, c_{22}]\,c_{22}\,[c_{21}^{2}, c_{12}^{-1}] = c_{13}^{c_{31}}\,c_{13}^{-c_{22}}c_{22}[c_{21}^{2}, c_{12}^{-1}],$$
$$c_{22}^{c_{12}} = [c_{12}, c_{21}^{-2}]\,c_{22}\,[c_{22}^{-3}, c_{13}]\,[c_{13}, c_{31}^{-1}] = [c_{12}, c_{21}^{-2}]\,c_{13}^{-c_{22}^{-2}}\,c_{22}\,c_{13}^{c_{31}^{-1}}.$$

Define the following subgroups of P_4:

$$V_1 = \langle c_{11}, c_{12}, c_{13}\rangle, \quad V_2 = \langle c_{21}, c_{22}\rangle, \quad V_3 = \langle c_{31}\rangle.$$

Then we have the following theorem.

Theorem 6.1 $P_4 = V_1 \rtimes (V_2 \rtimes V_3)$.

Proof At first we prove that $\langle V_2, V_3\rangle = V_2 \rtimes V_3$. Indeed, this group is defined by relations

$$[c_{31}, c_{21}] = 1, \quad c_{22}^{c_{21}} = c_{22}^{c_{31}^{-1}}.$$

Since the first relation we can write in the form

$$c_{21}^{c_{31}} = c_{21},$$

we have the need decomposition.

From the defining relations of P_4, we find the following formulas of conjugation by c_{31}:

$$c_{11}^{c_{31}} = c_{11}, \quad c_{12}^{c_{31}} = c_{12}, \quad c_{13}^{c_{31}} = c_{13}^{c_{12}^{-1}}.$$

Hence

$$P_4 = \langle V_1, V_2\rangle \rtimes V_3.$$

Find the formulas of conjugations by c_{21}:

$$c_{11}^{c_{21}} = c_{11}, \quad c_{12}^{c_{21}} = c_{12}^{c_{11}^{-1}}, \quad c_{13}^{c_{21}} = c_{13}^{c_{12}c_{11}^{-1}}.$$

Also we have two formulas of conjugation by c_{22}:

$$c_{11}^{c_{22}} = c_{11}, \quad c_{13}^{c_{22}} = c_{13}^{c_{11}^{-1}}.$$

To finish the proof we need to find a formula for the conjugation $c_{12}^{c_{22}}$ and $c_{12}^{c_{22}^{-1}}$.

In the proof of the previous theorem, we have found relation

$$c_{21}c_{22}^{-1}c_{13}c_{12}^{-1} = c_{21}^{-1}c_{12}^{-1}c_{21}^{2}c_{22}^{-1}(c_{22}^{-1}c_{13}c_{22}).$$

Multiplying both sides on c_{21}^{-1} to the left and using relation

$$c_{22}^{-1}c_{13}c_{22} = c_{11}c_{13}c_{11}^{-1},$$

we get

$$c_{22}^{-1}c_{13}c_{12}^{-1} = (c_{21}^{-2}c_{12}^{-1}c_{21}^{2})(c_{11}c_{13}c_{11}^{-1})^{c_{22}}c_{22}^{-1}.$$

Using the conjugation formulas

$$c_{21}^{-2}c_{12}^{-1}c_{21}^{2}=c_{11}^{2}c_{12}^{-1}c_{11}^{-2},\quad (c_{11}c_{13}c_{11}^{-1})^{c_{22}}=c_{11}^{2}c_{13}c_{11}^{-2},$$

we get

$$(c_{13}c_{12}^{-1})^{c_{22}}=c_{11}^{2}c_{12}^{-1}c_{13}c_{11}^{-2}.$$

Using the conjugation formula

$$c_{13}^{c_{22}}=c_{13}^{c_{11}^{-1}},$$

we have

$$c_{13}c_{11}^{-1}c_{12}^{-c_{22}}=c_{11}c_{12}^{-1}c_{13}c_{11}^{-2}.$$

From this relation we get the need formula

$$c_{12}^{c_{22}}=c_{11}^{2}c_{13}^{-1}c_{12}c_{11}^{-1}c_{13}c_{11}^{-1}.$$

Conjugating both sides by c_{22}^{-1}, we find

$$c_{12}^{c_{22}^{-1}}=c_{11}^{-1}c_{13}c_{11}^{-1}c_{12}c_{13}^{-1}c_{11}^{2}.$$

In this theorem we used full set of defining relations for P_4. Let us consider the group P_3. It has the following presentation

$$P_3=\langle c_{11},c_{21},c_{12}\mid c_{11}^{c_{21}}=c_{11},\ c_{12}^{c_{21}}=c_{12}^{c_{11}^{-1}}\rangle.$$

Using degeneracy maps s_0,s_1,s_2, we construct the following subgroups of P_4:

$$s_0(P_3)=\langle c_{21},c_{31},c_{22}\mid c_{21}^{c_{31}}=c_{21},\ c_{22}^{c_{31}}=c_{22}^{c_{21}^{-1}}\rangle,$$

$$s_1(P_3)=\langle c_{12},c_{31},c_{13}\mid c_{12}^{c_{31}}=c_{12},\ c_{13}^{c_{31}}=c_{13}^{c_{12}^{-1}}\rangle,$$

$$s_2(P_3)=\langle c_{11},c_{22},c_{13}\mid c_{11}^{c_{22}}=c_{11},\ c_{13}^{c_{22}}=c_{13}^{c_{11}^{-1}}\rangle.$$

From the list of relations in P_3, $s_i(P_3)$, $i=0,1,2$, we see that it is not the full list of relations for P_4. To have a full list we can add the relations

$$c_{11}^{c_{31}}=c_{11},\quad c_{13}^{c_{21}}=c_{13}^{c_{12}c_{11}^{-1}},\quad c_{12}^{c_{22}}=c_{11}^{2}\,c_{13}^{-1}\,c_{12}\,c_{11}^{-1}\,c_{13}\,c_{11}^{-1}.$$

But as follows from Theorem 4.1, for $n\geq 5$ the full list of relations for P_n comes from relations of P_{n-1}, $s_i(P_{n-1})$, $i=0,1,\cdots,n-2$. Using induction by n, we can find relations of P_n. We get the following relations:

– conjugations by $c_{n-1,1}$,

$$c_{n-k,k}^{c_{n-1,1}}=c_{n-k,k}^{c_{n-k,k-1}^{-1}},\quad k=2,3,\cdots,n-1;\quad c_{ij}^{c_{n-1,1}}=c_{ij},\quad \text{if } i+j<n;$$

– conjugations by $c_{n-2,2}$,

$$c_{n-k,k}^{c_{n-2,2}}=c_{n-k,k}^{c_{n-k,k-2}^{-1}},\quad k=2,3,\cdots,n-1;\quad c_{ij}^{c_{n-2,2}}=c_{11}^{2}\,c_{13}^{-1}\,c_{ij}\,c_{11}^{-1}\,c_{13}\,c_{11}^{-1},\quad i+j<n;$$

$$c_{lm}^{c_{n-2,}} = c_{lm} \quad \text{in all other cases.}$$

In the general case we prove the following theorem, which gives a new semi-direct product decompositon of the pure braid groups.

Theorem 6.2 *For $n \geq 3$ the pure braid group P_n is the semi-direct product of free groups:*

$$P_n = V_1 \rtimes (V_2 \rtimes (\cdots (V_{n-2} \rtimes V_{n-1}) \cdots)),$$

where

$$\begin{aligned} V_{n-1} &= \langle c_{n-1,1} \rangle, \\ V_{n-2} &= \langle c_{c_{n-2,1},n-2,2} \rangle, \\ &\vdots \\ V_1 &= \langle c_{11}, c_{12}, \cdots, c_{1,n-1} \rangle. \end{aligned}$$

Proof The theorem is true for $n = 4$. We prove that $P_n = V_1 \rtimes P_{n-1}$ for $n > 4$. By the lifting theorem, the set of defining relations for P_n comes from the set of defining relations for P_{n-1} by degeneracy maps. Using this fact, let us prove that V_1 is normal in P_n.

7 Directions for Further Research

We know some generalizations of the Artin braid group B_n, for example, welded braid group, singular braid groups and others (see [1]). In these groups it is possible to define pure subgroups. It is interesting to study presentations of these subgroups in cabled generators, define analogs of simplicial group T_* and find its homotopy type.

For example, the welded braid group WB_n contains the group of basis conjugating automorphisms Cb_n.

Question 1 The group of basic conjugating automorphisms Cb_2 is generated by two automorphisms ε_{21} and ε_{12} which generate a free group of rank 2. Using operation cabling can we find a presentation of Cb_n in the cable generators?

Question 2 Let $\varphi : VP_n \to Cb_n$ be a homomorphism which sends λ_{ij} to ε_{ij}. Is it true that T_{n-1} is isomorphic to its image $\varphi(T_{n-1})$?

We know Artin and Gassner representations of P_n (see [6, Chapter 3]).

Question 3 Find analogs of Artin and Gassner representations of P_n, using decomposition from Section 6. Are they equivalent to the classical representations?

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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