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Well-Posedness and Asymptotic Estimate for a Diffusion Equation with Time-Fractional Derivative*

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Abstract In this paper, the authors study the well-posedness and the asymptotic estimate of solution for a mixed-order time-fractional diffusion equation in a bounded domain subject to the homogeneous Dirichlet boundary condition. Firstly, the unique existence and regularity estimates of solution to the initial-boundary value problem are considered. Then combined with some important properties, including a maximum principle for a time-fractional ordinary equation and a coercivity inequality for fractional derivatives, the energy method shows that the decay in time of the solution is dominated by the term $t^{-\alpha}$ as t goes to infinity.

Keywords Mixed-order fractional diffusion equation, Initial-boundary value problem, Asymptotic estimate, Energy method
2020 MR Subject Classification 35R11, 35B40, 26A33, 34A08, 35B50

1 Introduction

Within the last few decades, an abundance of anomalous processes was observed and confirmed by more and more experiments in several different application areas in natural sciences including biology, geological sciences, medicine (see [11, 16–17, 36, 38]). For example, to characterize these diffusion processes, an important micro statistic quantity– the mean square displacement which describes how fast particles diffuse was used. In most of the anomalous diffusion cases, one observes a fractional power-law mean square displacement (see [28]), which shows that the diffusion is slower than that in the classical diffusion case.

For the mathematical modeling of the anomalous diffusion, we refer to, for example, Roman and Alemany [32] in which some macro models in the form of fractional diffusion equations were derived by the continuous-time random walk under some suitable conditions posed on the probability density functions for the jumps length and the waiting times between two

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successive jumps. In more detail, these macro models have forms of the single or multi-term time-, space-, or time-space-fractional differential equations, which are discussed in [2, 25–26] and the references therein.

During the last two decades, the time-fractional diffusion equations have attracted great attention from many aspects mostly due to the applications in the modeling of anomalous diffusion. For example, we mention important applications on some amorphous semiconductors (see [28, 38]), the modeling of dynamic processes in polymer materials, heat conduction with memory (see [31]) and the diffusion in fluids in porous media (see [3, 9, 13]). We refer to [4, 12, 20–21, 23] for the mathematical theory of the fractional differential equations, while we refer to [8, 14, 27, 29, 34–35] for the numerical study.

The main goal of this paper is to establish the decay estimate for the solution to our mixed-order fractional diffusion equation by an energy method. As is known, the asymptotic behavior of solutions to the equations which describe some physical processes is important both by itself and as a basis for developing suitable numerical methods and analyzing inverse problems for these equations. Researches are rapidly growing on the asymptotic behavior for the time-fractional diffusion equations and we only give a brief and typical review of the existing works instead of a comprehensive list. The asymptotic behavior as $t \to \infty$ for the single or multi-term time-fractional diffusion equations in a bounded domain was studied in [20–21, 33], where one can find that the decay of the solutions is dominated by the lowest order of the fractional derivatives. The proof of this fact is based on an explicit representation formula for the solution by the Fourier expansion method. From this explicit formula, by evaluating the inversion transform of the solutions, the decay in time of the solutions can be obtained. In unbounded domains, we refer to [4, 15], in which the authors used several special functions including the H-functions, the Mittag-Leffler functions, and their properties to obtain the solution formula. It turns out that all of the above arguments heavily rely on the explicit representation of the solution. Indeed, the coefficients of the equation are required to be at least t-independent so that the Fourier method and the Laplace transform work well and thus derivation of explicit representation formula of the solution becomes possible.

In this paper, we continue the researches initiated in [20–21, 33], and consider the case of mixed-order fractional diffusion equation with t-dependent coefficients. In order to overcome the difficulty that results from the lack of explicit representation formula of the solution, we employ an energy method which has been widely used to deal with the asymptotic estimate for several types of evolution equations, see e.g., [18, 39] and the references therein.

The rest of the paper is organized as follows: In Section 2, we formulate our problem and show our main results including the well-posedness and the long-time asymptotic behavior of the solution to the initial-boundary value problem for the mixed-order time-fractional diffusion equation. The proof of the well-posedness result is given in Section 3, while the long-time asymptotic estimate is established in Section 4. Finally, the last section is devoted to the conclusions and some open problems.

2 Problem Formulation and Main Results

Let $T > 0$ and Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. We deal with the time-fractional differential equation

$$\partial_t u + q(t)\partial_t^\alpha u = -Au + c(x,t)u + f(x,t), \quad (x,t) \in \Omega \times (0,T) \tag{2.1}$$

with the initial-boundary conditions

$$\begin{cases} u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{2.2}$$

where the coefficients q, c are smooth enough, e.g., $c \in L^\infty(0,T; W^{2,\infty}(\Omega))$, $q \in L^\infty(0,T)$ and A is a symmetric uniformly elliptic operator defined by

$$Au(x) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\Big), \quad u \in D(A) := H^2(\Omega) \cap H_0^1(\Omega)$$

with $a_{ij}(x) = a_{ji}(x)$, $1 \le i,j \le d$, $x \in \overline{\Omega}$ and $a_{ij} \in C^1(\overline{\Omega})$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \forall x \in \overline{\Omega}, \ \forall \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$$

for some constant $\nu > 0$. By ∂_t^α we denote the Caputo fractional derivative of order $\alpha \in (0,1)$:

$$\partial_t^\alpha \varphi(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi(\tau)\,\mathrm{d}\tau, \quad \varphi \in W^{1,1}(0,T).$$

Here and henceforth, $L^2(\Omega)$, $H^1(0,T)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ denote the usual Lebesgue space and the Sobolev spaces, and $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. Meanwhile we write $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(0,T)}$, $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{H^{-1}(\Omega)}$ as the corresponding norms.

In this paper, we mainly discuss the unique existence and the long-time asymptotic behavior of the solution to the initial-boundary value problem (2.1)–(2.2). Our main results are presented in Theorems 2.1–2.2 formulated below and the proofs are given in Sections 3 and 4, respectively. We start with a result of the unique existence and the regularity of the solution. For arbitrarily fixed $T > 0$, we have the following theorem.

Theorem 2.1 *Let $u_0 \in L^2(\Omega)$, $f \in L^2(0,T; H^{-1}(\Omega))$. Then there exists a unique solution $u \in H^1(0,T; H^{-1}(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$ to the initial-boundary value problem* (2.1)–(2.2)*, and there exists a constant $C_1 > 0$ such that*

$$\|u\|_{H^1(0,T;H^{-1}(\Omega))} + \|u\|_{L^2(0,T;H_0^1(\Omega))} \le C_1(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;H^{-1}(\Omega))}). \tag{2.3}$$

In addition, we assume that $u_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T; L^2(\Omega))$. Then the solution u further belongs to $H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, and there exists a constant $C_2 > 0$ satisfying

$$\|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \le C_2(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}). \tag{2.4}$$

Furthermore, we assume that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in H^1(0,T; L^2(\Omega))$. Then $u \in W^{1,\infty}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, and there exists a constant $C_3 > 0$ satisfying

$$\operatorname*{ess\,sup}_{0 \le t \le T} (\|\partial_t u(t)\|_{L^2(\Omega)} + \|u(t)\|_{H^2(\Omega)}) \le C_3(\|u_0\|_{H^2(\Omega)} + \|f\|_{H^1(0,T;L^2(\Omega))}).$$

Here the constants C_1, C_2, C_3 depend on α, T, ν and some norms of the coefficients c, a_{ij} and q. Moreover, we mention that by a classical result for parabolic equations, we have actually $u \in C^1((0,T]; L^2(\Omega))$ provided that the coefficients q, c are sufficiently smooth, but we do not discuss the details here.

Next we propose the result for the long-time asymptotic behavior. For the asymptotic estimate, we further assume that q and c satisfy $q_0 \le q(t) \le q_1,\ t > 0$ for some positive constants $q_1 \ge q_0 > 0$ and $c(x,t) \le 0$, $(x,t) \in \Omega \times (0,\infty)$. Then we have the following theorem.

Theorem 2.2 *Assume that $f = 0$, $u_0 \in L^2(\Omega)$ and q, c satisfy the above conditions. Let u be the solution to the initial-boundary value problem* (2.1)–(2.2). *Then for arbitrarily fixed $t_0 > 0$, there exists a constant $C > 0$, depending only on α, q_1, ν, Ω and t_0, such that the following long-time asymptotic estimate*

$$\|u(\,\cdot\,,t)\|_{L^2(\Omega)} \le C\|u_0\|_{L^2(\Omega)} t^{-\alpha}$$

holds true for any $t \ge t_0$.

We also mention that the decay rate is the best possible. In fact, we can consider a special case where q is a positive constant, c is a nonnegative constant and $A = -\Delta$. Then by the Fourier method, we find the long-time asymptotic behavior of the solution is exactly $t^{-\alpha}$.

For a diffusion equation with time-fractional derivatives, in general, the decay rate is characterized by the lowest fractional order of the derivatives (see e.g., [21]), which suggests that $t^{-\alpha}$ may be the best possible decay rate also for the general case in (2.1).

In this paper, we are devoted to the mixed-order case of orders 1 and $\alpha \in (0,1)$, and we obtain a similar result, that is, the decay rate is never exponential, unlike the case of only the first-order time derivative, but is $t^{-\alpha}$ as Theorem 2.2 proves. Moreover, if we consider (2.1) with more than one time-fractional derivatives, then the decay rate is subject to the lowest order, which we describe as a concluding remark in Section 5.

3 Unique Existence and Regularity of Solution

In this section, we first prove the unique existence of solution to the initial-boundary value problem (2.1)–(2.2) in the space $H^{\alpha_1}(0,T; H^{-1}(\Omega))$ with arbitrarily fixed $1 > \alpha_1 > \max\{\alpha, \frac{1}{2}\}$. The proof is based on the classical unique existence of solution to parabolic equations and the Fredholm alternative. Next we also propose some improved regularity of the solution and establish the related estimates by employing the generalized Grönwall inequality.

3.1 Preliminary

Before giving the proofs of our main results, we start with some useful representations of the solution u to the initial-boundary value problem (2.1)–(2.2).

Because of the conditions imposed on the elliptic operator A, there exists a system of eigenfunctions: $\{\varphi_k\}_{k=1}^{\infty}$, $\varphi_k \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfy the relations $A\varphi_k = \lambda_k \varphi_k,\ k = 1, 2, \cdots$ and form an orthonormal basis of $L^2(\Omega)$. The corresponding eigenvalues $\lambda_k,\ k = 1, 2, \cdots$ are all positive: $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. Henceforth, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Moreover, we define the Mittag-Leffler function by

$$E_{\alpha,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z \in \mathbb{C},$$

where $\alpha, \gamma > 0$ are arbitrary constants. The following useful lemmas hold.

Lemma 3.1 *Let the constants $\alpha \in (0, 1)$ and $\mu > 0$ be given. Then the following equalities*

$$\partial_t E_{\alpha,1}(-\mu t^\alpha) = -\mu t^{\alpha-1} E_{\alpha,\alpha}(-\mu t^\alpha) \tag{3.1}$$

and

$$\partial_t^\alpha E_{\alpha,1}(-\mu t^\alpha) = -\mu E_{\alpha,1}(-\mu t^\alpha) \tag{3.2}$$

are valid for any $t > 0$.

We refer to Podlubny [30] for the proof.

Lemma 3.2 *Let $0 < \alpha < 2$ and $\gamma > 0$. We suppose that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \gamma, \mu) > 0$ such that*

$$|E_{\alpha,\gamma}(z)| \le \frac{C}{1 + |z|}, \quad \mu \le |\arg z| \le \pi. \tag{3.3}$$

The proof can be found in Gorenflo and Mainardi [7] or in Podlubny [30, p.35].

In view of $\{\lambda_k, \varphi_k\}_{k=1}^\infty$, the solution u to (2.1) and (2.2) can be rewritten as follows:

$$\begin{aligned} u(t) = \mathrm{e}^{-tA} u_0 &+ \int_0^t \mathrm{e}^{-(t-s)A} f(s) \mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{-(t-s)A} (c(s)u(s) - q(s)\partial_s^\alpha u(s)) \mathrm{d}s, \end{aligned} \tag{3.4}$$

where the operator e^{-tA}, $t \ge 0$ is defined by

$$\mathrm{e}^{-tA} g := \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k t} \langle g, \varphi_k \rangle \varphi_k, \quad g \in H^{-1}(\Omega). \tag{3.5}$$

We denote

$$F(t) := \mathrm{e}^{-tA} u_0 + \int_0^t \mathrm{e}^{-(t-s)A} f(s) \mathrm{d}s$$

and

$$Ku(t) := \int_0^t \mathrm{e}^{-(t-s)A} (c(s)u(s) - q(s)\partial_s^\alpha u(s)) \mathrm{d}s, \quad u \in D(K), \tag{3.6}$$

where $D(K) := H^{\alpha_1}(0, T; H^{-1}(\Omega))$ with $1 > \alpha_1 > \max\left\{\alpha, \frac{1}{2}\right\}$. From (3.4), we obtain

$$u(t) = F(t) + Ku(t). \tag{3.7}$$

Here and henceforth, the domain of the Caputo fractional derivative ∂_t^α is extended from $W^{1,1}(0, T)$ to the fractional Sobolev space $H^\alpha(0, T)$ in order to justify the calculation with weak time regularity. For the detailed descriptions, we refer to Gorenflo, Luchko and Yamamoto [6] and Kubica and Yamamoto [19].

3.2 Unique existence

In this subsection, we shall prove the unique existence of solution in $H^{\alpha_1}(0,T;V)$ with $V = H^{-1}(\Omega)$. For the case where $V = L^2(\Omega)$, we can apply a similar argument as follows. Thus, for the sake of simplicity, we omit the proof of the case $V = L^2(\Omega)$ in this paper.

According to the regularity assumptions on the coefficients c, q, we see that $cu - q\partial_t^\alpha u \in L^2(0,T;H^{-1}(\Omega))$ provided $u \in D(K)$. It is readily to check that Ku is the solution to the following parabolic equation

$$\begin{cases} \partial_t Ku + AKu = cu - q\partial_t^\alpha u & \text{in } \Omega \times (0,T), \\ Ku = 0 & \text{on } \partial\Omega \times (0,T), \\ Ku(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.8}$$

and then by the well-known regularity for parabolic equations (e.g., [22, Section 4.7.1, p.243]), we have $Ku \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. By [22, Theorem 16.2, Chapter 1] and [37, Theorem 2.1], we find that $H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ is compact in $H^{\alpha_1}(0,T;H^{-1}(\Omega))$, which implies $K : H^{\alpha_1}(0,T;H^{-1}(\Omega)) \to H^{\alpha_1}(0,T;H^{-1}(\Omega))$ is a compact operator. By the Fredholm alternative, (3.7) admits a unique solution in $H^{\alpha_1}(0,T;H^{-1}(\Omega))$ as long as

(i) $F \in H^{\alpha_1}(0,T;H^{-1}(\Omega))$,

(ii) $I - K$ is one-to-one on $H^{\alpha_1}(0,T;H^{-1}(\Omega))$, that is,

$$(I-K)v = 0 \ \text{ implies } \ v = 0$$

are valid. Here I denotes the identity operator. Noting that F is the solution to

$$\begin{cases} \partial_t F(x,t) + AF(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ F(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ F(x,0) = u_0, & x \in \Omega \end{cases} \tag{3.9}$$

by the regularity assumptions $u_0 \in L^2(\Omega), f \in L^2(0,T;H^{-1}(\Omega))$, we have $F \in H^1(0,T;H^{-1}(\Omega)) \subset H^{\alpha_1}(0,T;H^{-1}(\Omega))$. Thus, (i) is verified.

Next we check (ii). In other words, we show the following uniqueness result.

Lemma 3.3 *Assume $v \in D(K)$ satisfies the following integral equation*

$$v = Kv,$$

where the operator K is defined in (3.6). *Then*

$$v = 0.$$

To prove this result, we need several lemmas.

Lemma 3.4 *Let $0 \le \beta < 1$, $0 \le s \le t$. Then*

$$\int_s^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} \mathrm{e}^{-\lambda(\tau-s)} \mathrm{d}\tau = (t-s)^{1-\beta} E_{1,2-\beta}(-\lambda(t-s)). \tag{3.10}$$

Proof From the series expansion of the exponential function, by a direct calculation, we find

$$\begin{aligned}\int_s^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}\mathrm{e}^{-\lambda(\tau-s)}\mathrm{d}\tau &= \int_s^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}\sum_{n=0}^{\infty}\frac{(-\lambda(\tau-s))^n}{n!}\mathrm{d}\tau\\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!\Gamma(1-\beta)}\int_s^t (t-\tau)^{-\beta}(\tau-s)^n\mathrm{d}\tau\\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!\Gamma(1-\beta)}\int_0^{t-s} (t-s-\tau)^{-\beta}\tau^n\mathrm{d}\tau\\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!\Gamma(1-\beta)}(t-s)^{n+1-\beta}B(1-\beta,n+1),\end{aligned}$$

where $B(a,b)$ denotes the beta function. Moreover, noting the relation between the beta function and the gamma function: $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(n+1)=n!$, from the definition of the Mittag-Leffler function we obtain (3.10).

On the basis of the above lemma, we further have the following lemma.

Lemma 3.5 *Let $0\le\beta<1$. Then there exists a constant $C=C(\beta)>0$ such that the following inequality*

$$\Big\|\partial_t^\beta\int_0^t \mathrm{e}^{-(t-s)A}w(s)\mathrm{d}s\Big\|_{H^{-1}(\Omega)}\le C\int_0^t (t-s)^{-\beta}\|w(s)\|_{H^{-1}(\Omega)}\mathrm{d}s \tag{3.11}$$

holds true for any $w\in L^2(0,T;H^{-1}(\Omega))$.

Proof By (3.5), we divide $\partial_t^\beta\int_0^t \mathrm{e}^{-(t-s)A}w(s)\mathrm{d}s$ into two parts:

$$\begin{aligned}I_1 &:= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}w(s)\mathrm{d}s,\\ I_2 &:= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}\int_0^\tau \partial_\tau\Big(\sum_{k=1}^{\infty}\mathrm{e}^{-\lambda_k(\tau-s)}\langle w(s),\varphi_k\rangle\varphi_k\Big)\mathrm{d}s\mathrm{d}\tau.\end{aligned}$$

For any $\psi\in H_0^1(\Omega)$,

$$\begin{aligned}|\langle I_1,\psi\rangle| &= \Big|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\langle w(s),\psi\rangle\mathrm{d}s\Big|\\ &\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|\langle w(s),\psi\rangle|\mathrm{d}s.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|I_1\|_{H^{-1}(\Omega)} &= \sup_{\|\psi\|_{H_0^1(\Omega)}=1}|\langle I_1,\psi\rangle|\\ &\le \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\|w(s)\|_{H^{-1}(\Omega)}\mathrm{d}s.\end{aligned}$$

On the other hand, by Fubini's theorem, noting the identity (3.10), we calculate

$$I_2=\sum_{k=1}^{\infty}-\lambda_k\int_0^t\langle w(s),\varphi_k\rangle\varphi_k\int_s^t\frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}\mathrm{e}^{-\lambda_k(\tau-s)}\mathrm{d}\tau\mathrm{d}s$$

$$= \sum_{k=1}^{\infty} -\lambda_k \int_0^t (t-s)^{1-\beta} E_{1,2-\beta}(-\lambda_k(t-s)) \langle w(s), \varphi_k \rangle \varphi_k \mathrm{d}s.$$

Consequently, for any $\psi \in H_0^1(\Omega)$, we use the estimate (3.3) for the Mittag-Leffler functions to derive

$$\begin{aligned} |\langle I_2, \psi \rangle| &\le \sum_{k=1}^{\infty} \int_0^t \lambda_k (t-s)^{1-\beta} |E_{1,2-\beta}(-\lambda_k(t-s))| |\langle w(s), \varphi_k \rangle| |\langle \varphi_k, \psi \rangle| \mathrm{d}s \\ &\le C \sum_{k=1}^{\infty} \int_0^t (t-s)^{-\beta} \frac{\lambda_k (t-s)}{1+\lambda_k(t-s)} |\lambda_k^{-\frac{1}{2}} \langle w(s), \varphi_k \rangle| |\lambda_k^{\frac{1}{2}} \langle \varphi_k, \psi \rangle| \mathrm{d}s, \end{aligned}$$

which combined with Hölder's inequality implies

$$\begin{aligned} |\langle I_2, \psi \rangle| &\le C \int_0^t (t-s)^{-\beta} \Big(\sum_{k=1}^{\infty} \lambda_k^{-1} |\langle w(s), \varphi_k \rangle|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} \lambda_k |\langle \varphi_k, \psi \rangle|^2 \Big)^{\frac{1}{2}} \mathrm{d}s \\ &= C \|\psi\|_{H_0^1(\Omega)} \int_0^t (t-s)^{-\beta} \|w(s)\|_{H^{-1}(\Omega)} \mathrm{d}s. \end{aligned}$$

Finally, we have

$$\|I_2\|_{H^{-1}(\Omega)} \le C \int_0^t (t-s)^{-\beta} \|w(s)\|_{H^{-1}(\Omega)} \mathrm{d}s.$$

Thus, we complete the proof by the triangle inequality for the norm.

Proof of Lemma 3.3 According to the equation $v = Kv$, we find

$$\begin{aligned} \|\partial_t^\beta v(t)\|_{H^{-1}(\Omega)} &= \|\partial_t^\beta K v(t)\|_{H^{-1}(\Omega)} \\ &= \Big\| \partial_t^\beta \int_0^t \mathrm{e}^{-(t-s)A} (c(s)v(s) - q(s)\partial_s^\alpha v(s)) \mathrm{d}s \Big\|_{H^{-1}(\Omega)} \end{aligned}$$

for $0 \le \beta < 1$. By taking $\beta = \alpha_1$, $\beta = 0$ in the estimate (3.11) separately, and noting that $c \in L^\infty(0,T; W^{2,\infty}(\Omega))$, $q \in L^\infty(0,T)$, we obtain

$$\begin{aligned} \|\partial_t^{\alpha_1} v(t)\|_{H^{-1}(\Omega)} \le\, & C \int_0^t (t-s)^{-\alpha_1} \|v(s)\|_{H^{-1}(\Omega)} \mathrm{d}s \\ & + C \int_0^t (t-s)^{-\alpha_1} \|\partial_s^\alpha v(s)\|_{H^{-1}(\Omega)} \mathrm{d}s \end{aligned}$$

and

$$\|v(t)\|_{H^{-1}(\Omega)} \le C \int_0^t \|v(s)\|_{H^{-1}(\Omega)} \mathrm{d}s + C \int_0^t \|\partial_s^\alpha v(s)\|_{H^{-1}(\Omega)} \mathrm{d}s.$$

Moreover, by noting the semigroup property $J^{\gamma_1+\gamma_2} = J^{\gamma_1} J^{\gamma_2}$, $\gamma_1, \gamma_2 > 0$ of the Riemann-Liouville fractional integral operator which is defined by

$$J^\gamma g(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} g(\tau) \mathrm{d}\tau, \quad \gamma > 0,$$

we see that

$$\|\partial_s^\alpha v(s)\|_{H^{-1}(\Omega)} \le C J^{\alpha_1 - \alpha} \|\partial_t^{\alpha_1} v(s)\|_{H^{-1}(\Omega)},$$

from which we further obtain that

$$\begin{aligned}\int_0^t \|\partial_s^\alpha v(s)\|_{H^{-1}(\Omega)}\mathrm{d}s &\le CJ^{1+\alpha_1-\alpha}\|\partial_t^{\alpha_1} v(t)\|_{H^{-1}(\Omega)}\\ &\le C\int_0^t (t-s)^{\alpha_1-\alpha}\|\partial_t^{\alpha_1} v(s)\|_{H^{-1}(\Omega)}\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\int_0^t (t-s)^{-\alpha_1}\|\partial_s^\alpha v(s)\|_{H^{-1}(\Omega)}\mathrm{d}s &\le CJ^{1-\alpha}\|\partial_t^{\alpha_1} v(t)\|_{H^{-1}(\Omega)}\\ &\le C\int_0^t (t-s)^{-\alpha}\|\partial_t^{\alpha_1} v(s)\|_{H^{-1}(\Omega)}\mathrm{d}s.\end{aligned}$$

Finally, since $1, (t-s)^{-\alpha}, (t-s)^{\alpha_1-\alpha} \le C(t-s)^{-\alpha_1}$ for $\alpha_1 > \alpha$, we obtain

$$\begin{aligned}&\|v(t)\|_{H^{-1}(\Omega)} + \|\partial_t^{\alpha_1} v(t)\|_{H^{-1}(\Omega)}\\ &\le C\int_0^t (t-s)^{-\alpha_1}(\|v(s)\|_{H^{-1}(\Omega)} + \|\partial_s^{\alpha_1} v(s)\|_{H^{-1}(\Omega)})\mathrm{d}s\end{aligned}$$

with a generic constant $C > 0$ which depends also on T. Therefore, the generalized Grönwall inequality (see e.g., Henry [10, Lemma 7.1.1]) implies $v = 0$. We finish the proof of the lemma.

By the Fredholm alternative, we proved that the initial-boundary value problem (2.1)–(2.2) admits a unique solution in $H^{\alpha_1}(0,T;H^{-1}(\Omega))$ with $1 > \alpha_1 > \max\{\alpha, \frac{1}{2}\}$.

3.3 Improved regularity

Next we show the improved regularity and some estimates by using the integral form (3.4).

Recalling that we rewrite (3.4) by (3.7), it is sufficient to discuss the regularity for F and Ku, respectively. Since F is the solution to the parabolic equation (3.9), under the assumptions that $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;H^{-1}(\Omega))$, we obtain by the classical regularity for parabolic equations that $F \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ (e.g., [22, Example 4.7.1, Chapter 3]). Similarly, Ku is the solution to the parabolic equation (3.8) and we have the same regularity for Ku. Therefore, we find the improved regularity

$$u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)).$$

In order to establish the estimate for the solution u, we need the following lemmas.

Lemma 3.6 *There exists a constant $C = C(\alpha,c,q,T) > 0$ such that*

$$\|\partial_t Ku(t)\|_{H^{-1}(\Omega)} \le C\|u_0\|_{L^2(\Omega)} + C\int_0^t (t-\tau)^{-\alpha}(\|u(\tau)\|_{H_0^1(\Omega)} + \|\partial_\tau u(\tau)\|_{H^{-1}(\Omega)})\mathrm{d}\tau.$$

Proof We divide $\partial_t Ku(t)$ into three parts:

$$\begin{aligned}I_0 &:= c(t)u(t) - q(t)\partial_t^\alpha u(t),\\ I_1 &:= \int_0^t \sum_{k=1}^\infty \partial_t(\mathrm{e}^{-\lambda_k(t-s)}\langle c(s)u(s),\varphi_k\rangle\varphi_k)\mathrm{d}s,\end{aligned}$$

$$I_2 := \int_0^t \sum_{k=1}^{\infty} \partial_t (\mathrm{e}^{-\lambda_k (t-s)} \langle q(s) \partial_s^\alpha u(s), \varphi_k \rangle \varphi_k) \mathrm{d}s.$$

Since $c \in L^\infty(0,T;W^{2,\infty}(\Omega))$ and $q \in L^\infty(0,T)$, it follows that

$$\begin{aligned}\|I_0\|_{H^{-1}(\Omega)} &\le C\Big\|u_0 + \int_0^t \partial_\tau u(\tau)\mathrm{d}\tau\Big\|_{H^{-1}(\Omega)} + C\Big\|\int_0^t (t-\tau)^{-\alpha}\partial_\tau u(\tau)\mathrm{d}\tau\Big\|_{H^{-1}(\Omega)}\\ &\le C\Big(\|u_0\|_{L^2(\Omega)} + \int_0^t (t-\tau)^{-\alpha}\|\partial_\tau u(\tau)\|_{H^{-1}(\Omega)}\mathrm{d}\tau\Big).\end{aligned}$$

Here we used the triangle inequality and $u(t) = u_0 + \int_0^t \partial_\tau u(\tau)\mathrm{d}\tau$. Next we derive estimates for I_1 and I_2. In fact, for any $\psi \in H_0^1(\Omega)$, we conclude from Hölder's inequality that

$$\begin{aligned}|\langle I_1, \psi\rangle| &\le \sum_{k=1}^\infty \int_0^t \lambda_k \mathrm{e}^{-\lambda_k(t-s)}|\langle c(s)u(s), \varphi_k\rangle||\langle \varphi_k, \psi\rangle|\mathrm{d}s\\ &\le C\int_0^t \Big(\sum_{k=1}^\infty \lambda_k |\langle\varphi_k,\psi\rangle|^2\mathrm{d}s\Big)^{\frac12}\Big(\sum_{k=1}^\infty \lambda_k \mathrm{e}^{-2\lambda_k(t-s)}|\langle c(s)u(s),\varphi_k\rangle|^2\Big)^{\frac12}\mathrm{d}s\\ &\le C\|\psi\|_{H_0^1(\Omega)}\int_0^t \Big(\sum_{k=1}^\infty \lambda_k \mathrm{e}^{-2\lambda_k(t-s)}|\langle c(s)u(s),\varphi_k\rangle|^2\Big)^{\frac12}\mathrm{d}s.\end{aligned}$$

Moreover, noting that $\mathrm{e}^{-2\lambda_k s} \le 1$ for $s > 0$, we see that

$$\begin{aligned}|\langle I_1,\psi\rangle| &\le C\|\psi\|_{H_0^1(\Omega)}\int_0^t \Big(\sum_{k=1}^\infty \lambda_k|\langle c(s)u(s),\varphi_k\rangle|^2\Big)^{\frac12}\mathrm{d}s\\ &\le C\|\psi\|_{H_0^1(\Omega)}\int_0^t \|c(s)u(s)\|_{H_0^1(\Omega)}\mathrm{d}s,\end{aligned}$$

which combined with the assumption that $c \in L^\infty(0,T;W^{2,\infty}(\Omega))$ implies

$$|\langle I_1,\psi\rangle| \le C\|\psi\|_{H_0^1(\Omega)}\int_0^t \|u(s)\|_{H_0^1(\Omega)}\mathrm{d}s,$$

that is,

$$\|I_1\|_{H^{-1}(\Omega)} \le C\int_0^t \|u(s)\|_{H_0^1(\Omega)}\mathrm{d}s.$$

On the other hand, for any $\psi \in H_0^1(\Omega)$, we have

$$\begin{aligned}|\langle I_2,\psi\rangle| &= \Big| -\sum_{k=1}^\infty \int_0^t \lambda_k \mathrm{e}^{-\lambda_k(t-s)} q(s)\langle \partial_s^\alpha u(s),\varphi_k\rangle\langle\varphi_k,\psi\rangle\mathrm{d}s\Big|\\ &\le \sum_{k=1}^\infty\int_0^t \lambda_k\mathrm{e}^{-\lambda_k(t-s)}|\langle\varphi_k,\psi\rangle||q(s)|\int_0^s \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)}|\langle\partial_\tau u(\tau),\varphi_k\rangle|\mathrm{d}\tau\mathrm{d}s\\ &\le C\sum_{k=1}^\infty \lambda_k|\langle\varphi_k,\psi\rangle|\int_0^t |\langle\partial_\tau u(\tau),\varphi_k\rangle|\int_\tau^t \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\mathrm{e}^{-\lambda_k(t-s)}\mathrm{d}s\mathrm{d}\tau.\end{aligned}$$

Here the last equality is due to Fubini's theorem. Similarly to the proof of Lemma 3.5, we obtain

$$\|I_2\|_{H^{-1}(\Omega)} \le C\int_0^t (t-\tau)^{-\alpha}\|\partial_\tau u(\tau)\|_{H^{-1}(\Omega)}\mathrm{d}\tau.$$

Collecting all the above estimates and noting that

$$1 \le T^\alpha t^{-\alpha} \le C t^{-\alpha}, \quad 0 < t \le T,$$

we finish the proof of the lemma.

In a similar way, we can prove the following lemma.

Lemma 3.7 *There exists a constant $C = C(\alpha, c, q, T) > 0$ such that*

$$\|Ku(t)\|_{H_0^1(\Omega)} \le C \int_0^t (t-\tau)^{-\alpha} (\|u(\tau)\|_{H_0^1(\Omega)} + \|\partial_\tau u(\tau)\|_{H^{-1}(\Omega)}) \mathrm{d}\tau.$$

Then by Lemmas 3.6–3.7 and (3.7), we obtain

$$v(t) \le a(t) + C \int_0^t (t-s)^{-\alpha} v(s) \mathrm{d}s,$$

where

$$\begin{aligned} v(t) &= \|\partial_t u(t)\|_{H^{-1}(\Omega)} + \|u(t)\|_{H_0^1(\Omega)}, \\ a(t) &= C\|u_0\|_{L^2(\Omega)} + \|\partial_t F(t)\|_{H^{-1}(\Omega)} + \|F(t)\|_{H_0^1(\Omega)}. \end{aligned}$$

Here the generic constant $C > 0$ is independent of t, but may depend on α and T as well. Finally, we employ the following generalized Grönwall inequality.

Lemma 3.8 (see [10, Lemma 7.1.1]) *Suppose $b \ge 0, \beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \le t < T$, and suppose $v(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$v(t) \le a(t) + b \int_0^t (t-s)^{\beta-1} v(s) \mathrm{d}s$$

on this interval. Then

$$v(t) \le a(t) + b\Gamma(\beta) \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(b\Gamma(\beta)(t-s)^\beta) a(s) \mathrm{d}s, \quad 0 \le t < T.$$

In particular, there exists a constant $C = C(b, \beta, T) > 0$ such that

$$v(t) \le a(t) + C \int_0^t (t-s)^{\beta-1} a(s) \mathrm{d}s, \quad 0 \le t < T.$$

Now we are ready to establish the estimates in Theorem 2.1. By Lemma 3.8, we have

$$\begin{aligned} v(t) \le\ & C\|u_0\|_{L^2(\Omega)} + \|\partial_t F(t)\|_{H^{-1}(\Omega)} + \|F(t)\|_{H_0^1(\Omega)} \\ & + C \int_0^t (t-s)^{-\alpha} (\|u_0\|_{L^2(\Omega)} + \|\partial_s F(s)\|_{H^{-1}(\Omega)} + \|F(s)\|_{H_0^1(\Omega)}) \mathrm{d}s \end{aligned}$$

with a generic constant $C > 0$. We take the L^2-norm over $t \in (0, T)$ on both sides and by Young's convolution inequality, we obtain

$$\|v\|_{L^2(0,T)} \le C(\|u_0\|_{L^2(\Omega)} + \|\partial_t F\|_{L^2(0,T;H^{-1}(\Omega))} + \|F\|_{L^2(0,T;H_0^1(\Omega))}).$$

We complete the first statement (2.3) of Theorem 2.1 by noting that the following regularity estimate

$$\|\partial_t F\|_{L^2(0,T;H^{-1}(\Omega))} + \|F\|_{L^2(0,T;H_0^1(\Omega))} \le C(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;H^{-1}(\Omega))})$$

is valid since F is the solution to parabolic problem (3.9) with $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$.

For the second statement (2.4), recall that we assume $u_0 \in H_0^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and $c \in L^\infty(0,T;W^{2,\infty}(\Omega))$. By the well-known regularity for parabolic equations (e.g., Evans [5, Chapter 7]), it is readily to see that

$$u = Ku + F \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T];H_0^1(\Omega)).$$

In a similar way, the second regularity estimate (2.4) follows immediately from

$$\|\partial_t F\|_{L^2(0,T;L^2(\Omega))} + \|F\|_{L^2(0,T;H^2(\Omega))} \le C(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}),$$

the generalized Grönwall inequality (Lemma 3.8) and the next lemma.

Lemma 3.9 *There exists a constant $C = C(\alpha, c, q, T) > 0$ such that*

$$\begin{aligned}\|\partial_t Kv(t)\|_{L^2(\Omega)}^2 &\le C\|v(0)\|_{L^2(\Omega)}^2 + C\int_0^t \|v(\tau)\|_{H^2(\Omega)}^2 \mathrm{d}\tau \\ &\quad + C\int_0^t (t-\tau)^{-\alpha}\|\partial_\tau v(\tau)\|_{L^2(\Omega)}^2 \mathrm{d}\tau\end{aligned}$$

and

$$\|Kv(t)\|_{H^2(\Omega)}^2 \le C\int_0^t \|v(\tau)\|_{H^2(\Omega)}^2 \mathrm{d}\tau + C\int_0^t (t-\tau)^{-\alpha}\|\partial_\tau v(\tau)\|_{L^2(\Omega)}^2 \mathrm{d}\tau$$

for all $v \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$.

Here we omit the proof of the above lemma since it is similar to those of Lemmas 3.6–3.7, while we note the equivalence of norms

$$\|w\|_{H^2(\Omega)}^2 \sim \|w\|_{H^2(\Omega)\cap H_0^1(\Omega)}^2 := \sum_{k=1}^\infty \lambda_k^2 |(w, \varphi_k)|^2$$

for $w \in H^2(\Omega) \cap H_0^1(\Omega)$.

Finally, we further assume $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$ and prove the third statement of Theorem 2.1. By the repeated applications of (3.7), we obtain

$$u = K^N u + \sum_{j=0}^{N-1} K^j F,$$

where $N \ge \frac{1}{1-\alpha}$. In order to argue the regularity of solution u, it is sufficient to deal with $K^N u$ and $K^j F$, $j = 0, 1, \cdots, N-1$, respectively. In terms of Lemma 3.9, we obtain

$$\|\partial_t K^j u(t)\|_{L^2(\Omega)}^2 \le C\|K^{j-1}u(0)\|_{L^2(\Omega)}^2 + C\int_0^t \|K^{j-1}u(\tau)\|_{H^2(\Omega)}^2 \mathrm{d}\tau$$

$$+ C\int_0^t (t-\tau)^{-\alpha}\|\partial_\tau K^{j-1}u(\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau \tag{3.12}$$

and

$$\begin{aligned}\|K^j u(t)\|^2_{H^2(\Omega)} \le{}& C\int_0^t \|K^{j-1}u(\tau)\|^2_{H^2(\Omega)}\mathrm{d}\tau\\ &+ C\int_0^t (t-\tau)^{-\alpha}\|\partial_\tau K^{j-1}u(\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau\end{aligned} \tag{3.13}$$

for $j = 1, 2, \cdots, N$. By the definition of operator K, it is readily to see that $K^{j-1}u(0) = 0$, $j = 2, \cdots, N$. Then by (3.12)–(3.13) and $(t-\tau)^\alpha \le T^\alpha$, we obtain

$$\begin{aligned}&\|\partial_t K^j u(t)\|^2_{L^2(\Omega)} + \|K^j u(t)\|^2_{H^2(\Omega)}\\ &\le C\int_0^t (t-\tau)^{-\alpha}(\|\partial_\tau K^{j-1}u(\tau)\|^2_{L^2(\Omega)} + \|K^{j-1}u(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau\end{aligned} \tag{3.14}$$

for all $j = 2, 3, \cdots, N$. Recall that $C > 0$ denotes a generic constant, which means that C can change values in different lines. By using (3.14) with $j = N, N-1$ and by a direct calculation, we arrive at the following estimate

$$\begin{aligned}&\|\partial_t K^N u(t)\|^2_{L^2(\Omega)} + \|K^N u(t)\|^2_{H^2(\Omega)}\\ &\le C\int_0^t (t-\tau)^{-\alpha}(\|\partial_\tau K^{N-1}u(\tau)\|^2_{L^2(\Omega)} + \|K^{N-1}u(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-\alpha}\int_0^\tau (\tau-s)^{-\alpha}(\|\partial_s K^{N-2}u(s)\|^2_{L^2(\Omega)} + \|K^{N-2}u(s)\|^2_{H^2(\Omega)})\mathrm{d}s\mathrm{d}\tau.\end{aligned}$$

Moreover, by Fubini's theorem, we see that

$$\begin{aligned}&\|\partial_t K^N u(t)\|^2_{L^2(\Omega)} + \|K^N u(t)\|^2_{H^2(\Omega)}\\ &\le C\int_0^t (\|\partial_s K^{N-2}u(s)\|^2_{L^2(\Omega)} + \|K^{N-2}u(s)\|^2_{H^2(\Omega)})\int_s^t (t-\tau)^{-\alpha}(\tau-s)^{-\alpha}\mathrm{d}\tau\mathrm{d}s\\ &\le C\int_0^t (t-s)^{1-2\alpha}(\|\partial_s K^{N-2}u(s)\|^2_{L^2(\Omega)} + \|K^{N-2}u(s)\|^2_{H^2(\Omega)})\mathrm{d}s.\end{aligned}$$

We calculate by iterations and obtain

$$\begin{aligned}&\|\partial_t K^N u(t)\|^2_{L^2(\Omega)} + \|K^N u(t)\|^2_{H^2(\Omega)}\\ &\le C\int_0^t (t-s)^{-\alpha+(N-2)(1-\alpha)}(\|\partial_s K u(s)\|^2_{L^2(\Omega)} + \|K u(s)\|^2_{H^2(\Omega)})\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\alpha+(N-2)(1-\alpha)}\int_0^s (s-\tau)^{-\alpha}(\|\partial_\tau u(\tau)\|^2_{L^2(\Omega)} + \|u(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau\mathrm{d}s\\ &\quad + C\int_0^t (t-s)^{-\alpha+(N-2)(1-\alpha)}\|u_0\|^2_{L^2(\Omega)}\mathrm{d}s\\ &\le C\|u_0\|^2_{L^2(\Omega)} + C\int_0^t (t-\tau)^{N(1-\alpha)-1}(\|\partial_\tau u(\tau)\|^2_{L^2(\Omega)} + \|u(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau.\end{aligned}$$

Since $N \ge \frac{1}{1-\alpha}$ implies $N(1-\alpha) - 1 \ge 0$, the above inequalities yield

$$\|\partial_t K^N u(t)\|^2_{L^2(\Omega)} + \|K^N u(t)\|^2_{H^2(\Omega)}$$

$$\leq C\|u_0\|^2_{L^2(\Omega)} + C\int_0^t (\|\partial_\tau u(\tau)\|^2_{L^2(\Omega)} + \|u(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau. \tag{3.15}$$

Noting that we have proved $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, and for $t \in (0,T)$, the right-hand side of (3.15) is finite, which leads to

$$K^N u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)).$$

In the same way, we can prove

$$\begin{aligned}&\|\partial_t K^j F(t)\|^2_{L^2(\Omega)} + \|K^j F(t)\|^2_{H^2(\Omega)}\\ &\leq C\|F(0)\|^2_{L^2(\Omega)} + C\int_0^t (t-\tau)^{j(1-\alpha)-1}(\|\partial_\tau F(\tau)\|^2_{L^2(\Omega)} + \|F(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau\end{aligned}$$

for all $j=1, 2, \cdots, N-1$. Moreover, under the assumptions

$$u_0 \in H^2(\Omega)\cap H_0^1(\Omega) \quad \text{and} \quad f \in H^1(0,T;L^2(\Omega)),$$

the improved regularity for parabolic equations (e.g., [5]) yields that $F \in W^{1,\infty}(0,T;\, L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and the regularity estimate

$$\|\partial_t F(t)\|^2_{L^2(\Omega)} + \|F(t)\|^2_{H^2(\Omega)} \leq C(\|u_0\|^2_{H^2(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))})$$

holds true for $t \in (0,T)$, which further implies that

$$\begin{aligned}&\|\partial_t K^j F(t)\|^2_{L^2(\Omega)} + \|K^j F(t)\|^2_{H^2(\Omega)}\\ &\leq C\|u_0\|^2_{L^2(\Omega)} + C\int_0^t (t-\tau)^{-\alpha}(\|\partial_\tau F(\tau)\|^2_{L^2(\Omega)} + \|F(\tau)\|^2_{H^2(\Omega)})\mathrm{d}\tau\\ &\leq C(1+t^{1-\alpha})(\|u_0\|^2_{H^2(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))}).\end{aligned}$$

Here in the first inequality we used $F(0) = u_0$ from (3.9) and the estimate that $(t-\tau)^{(j-1)(1-\alpha)} \leq T^{(j-1)(1-\alpha)} \leq T^{(N-2)(1-\alpha)}$ for $j = 1, 2, \cdots, N-1$. Hence we obtain

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\leq t\leq T}(\|\partial_t K^j F(t)\|^2_{L^2(\Omega)} + \|K^j F(t)\|^2_{H^2(\Omega)})\\ &\leq C(\|u_0\|^2_{H^2(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))}), \quad j = 0, 1, \cdots, N-1.\end{aligned} \tag{3.16}$$

In the end, collecting the above estimates (3.15)–(3.16) and recalling (2.4), we conclude that $u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ with the estimate

$$\operatorname*{ess\,sup}_{0\leq t\leq T}(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{H^2(\Omega)}) \leq C(\|u_0\|^2_{H^2(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))}).$$

We finish the last part of Theorem 2.1.

4 Long-Time Asymptotics

In this section, we establish the long-time asymptotic estimate for the solution to the initial-boundary value problem (2.1)–(2.2). The proof relies on a suitable energy estimate and the use of the asymptotic behavior for a related ordinary fractional differential equation.

To start with, some important auxiliary results as follows are established. Henceforth, $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. We have the following coercivity inequality for the Caputo derivative.

Lemma 4.1 *Let* $y \in H^1(0,T;L^2(\Omega))$. *Then*

$$(y(t), \partial_t^\alpha y(t)) \geq \|y(t)\|_{L^2(\Omega)}\, \partial_t^\alpha \|y(t)\|_{L^2(\Omega)}$$

holds true for $0 < t < T$.

Proof The proof is done by direct calculations. For simplicity, we set $g_\alpha(t) := \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ and its derivative $g'_\alpha(t) = -\frac{\alpha t^{-\alpha-1}}{\Gamma(1-\alpha)}$, $0 < t < T$, and we denote

$$I(t) := (y(t), \partial_t^\alpha y(t)) - \|y(t)\|_{L^2(\Omega)}\, \partial_t^\alpha \|y(t)\|_{L^2(\Omega)}.$$

Then it is sufficient to prove $I \geq 0$. For this, we divide $I(t)$ into two parts: $I(t) = I_1(t) + I_2(t)$ with

$$\begin{aligned} I_1(t) &:= (y(t), \partial_t^\alpha y(t)) - \frac{1}{2}\partial_t^\alpha \|y(t)\|^2_{L^2(\Omega)}, \\ I_2(t) &:= \frac{1}{2}\partial_t^\alpha \|y(t)\|^2_{L^2(\Omega)} - \|y(t)\|_{L^2(\Omega)}\, \partial_t^\alpha \|y(t)\|_{L^2(\Omega)}. \end{aligned}$$

By Fubini's theorem and the definition of Caputo fractional derivative, we find

$$\begin{aligned} I_1(t) &= \int_0^t g_\alpha(t-\tau)(y(t), \partial_\tau y(\tau))\mathrm{d}\tau - \int_0^t g_\alpha(t-\tau)(y(\tau), \partial_\tau y(\tau))\mathrm{d}\tau \\ &= \int_0^t g_\alpha(t-\tau)(y(t)-y(\tau), \partial_\tau y(\tau))\mathrm{d}\tau \\ &= -\frac{1}{2}\int_0^t g_\alpha(t-\tau)\partial_\tau \|y(t)-y(\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau. \end{aligned}$$

Then integration by parts yields

$$I_1(t) = -\frac{1}{2} g_\alpha(t-\tau)\|y(t)-y(\tau)\|^2_{L^2(\Omega)}\Big|_{\tau=0}^{\tau=t} - \frac{1}{2}\int_0^t g'_\alpha(t-\tau)\|y(t)-y(\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau.$$

Moreover, we claim that

$$\lim_{\tau \to t-} g_\alpha(t-\tau)\|y(t)-y(\tau)\|^2_{L^2(\Omega)} = 0. \tag{4.1}$$

Indeed, by noting that

$$\begin{aligned} &g_\alpha(t-\tau)\|y(t)-y(\tau)\|^2_{L^2(\Omega)} \\ &\leq C(t-\tau)^{-\alpha}\Big\| \int_\tau^t |\partial_s y(s)|\mathrm{d}s \Big\|^2_{L^2(\Omega)} \\ &\leq C(t-\tau)^{-\alpha}\int_\tau^t \|\partial_s y(s)\|^2_{L^2(\Omega)}\mathrm{d}s \int_\tau^t 1^2 \mathrm{d}s \leq C(t-\tau)^{1-\alpha}\|y\|^2_{H^1(0,T;L^2(\Omega))}, \end{aligned}$$

where in the last line we used Hölder's inequality and Fubini's theorem. Thus the claim (4.1) is true and we see that

$$I_1(t) = \frac{1}{2} g_\alpha(t)\|y(t)-y(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\int_0^t g'_\alpha(t-\tau)\|y(t)-y(\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau.$$

For I_2, we note that the triangle inequality $\|y(t)\|_{L^2(\Omega)} - \|y(\tau)\|_{L^2(\Omega)} \le \|y(t)-y(\tau)\|_{L^2(\Omega)}$, so that $\lim\limits_{\tau\to t-} g_\alpha(t-\tau)(\|y(t)\|_{L^2(\Omega)} - \|y(\tau)\|_{L^2(\Omega)})^2 = 0$, then by an argument similar to the calculation for I_1, we find

$$\begin{aligned} I_2(t) &= \int_0^t g_\alpha(t-\tau)(\|y(\tau)\|_{L^2(\Omega)} - \|y(t)\|_{L^2(\Omega)})\partial_\tau\|y(\tau)\|_{L^2(\Omega)}\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^t g_\alpha(t-\tau)\partial_\tau(\|y(t)\|_{L^2(\Omega)} - \|y(\tau)\|_{L^2(\Omega)})^2\mathrm{d}\tau \\ &= -\frac{1}{2}g_\alpha(t)(\|y(t)\|_{L^2(\Omega)} - \|y(0)\|_{L^2(\Omega)})^2 \\ &\quad + \frac{1}{2}\int_0^t g_\alpha'(t-\tau)(\|y(t)\|_{L^2(\Omega)} - \|y(\tau)\|_{L^2(\Omega)})^2\mathrm{d}\tau. \end{aligned}$$

Therefore, by noting

$$\|y(t)-y(\tau)\|^2_{L^2(\Omega)} = \|y(t)\|^2_{L^2(\Omega)} + \|y(\tau)\|^2_{L^2(\Omega)} - 2(y(t), y(\tau))$$

for $0 \le \tau \le t$, we obtain

$$\begin{aligned} I(t) &= g_\alpha(t)(\|y(t)\|_{L^2(\Omega)}\|y(0)\|_{L^2(\Omega)} - (y(t), y(0))) \\ &\quad - \int_0^t g_\alpha'(t-\tau)(\|y(t)\|_{L^2(\Omega)}\|y(\tau)\|_{L^2(\Omega)} - (y(t), y(\tau)))\mathrm{d}\tau. \end{aligned}$$

Finally, Hölder's inequality and $g_\alpha > 0$, $g_\alpha' < 0$ in $(0,T)$ imply $I(t) \ge 0$ for $0 < t < T$, which completes the proof of the lemma.

Lemma 4.2 *Let $\lambda > 0$ and $p_0 \ge 0$ be constants. Assume that $w \in H^1(0,T)$ satisfies*

$$\begin{cases} \partial_t w(t) + p_0\partial_t^\alpha w(t) + \lambda w(t) \le 0, \quad 0 < t < T, \\ w(0) \le 0. \end{cases} \tag{4.2}$$

Then $w(t) \le 0$ for $0 < t < T$.

Proof We start the proof in the case of $w \in C^1[0,T]$. By continuity, we find (4.2) holds true for $t \in [0,T]$. We prove the lemma by contradiction. Assume that w is positive at some point in $(0,T]$. Then w attains its positive maximum in $(0,T]$, that is, there exists $t_0 \in (0,T]$ such that $w(t_0) > 0$ and $w(t_0) \ge w(t)$ for $t \in [0,T]$. Immediately, we have $\partial_t w(t_0) \ge 0$. With reference to Luchko [24, Theorem 1], we find $\partial_t^\alpha w(t_0) \ge 0$. Thus, we obtain

$$\partial_t w(t_0) + p_0\partial_t^\alpha w(t_0) + \lambda w(t_0) > 0,$$

which is a contradiction to (4.2).

Next, we assume $w \in H^1(0,T)$. For any nonnegative function $\varphi \in C^1[0,T]$, we denote $\overline{w} := w * \varphi := \int_0^t w(t-\tau)\varphi(\tau)\mathrm{d}\tau$. It is not difficult to see that $\overline{w} \in C^1[0,T]$ and satisfies

$$\begin{cases} \partial_t \overline{w}(t) + p_0\partial_t^\alpha \overline{w}(t) + \lambda \overline{w}(t) \le 0, \quad 0 < t < T, \\ \overline{w}(0) = 0. \end{cases}$$

Therefore, from the above argument, it follows that $\overline{w}(t) \le 0$ for any $t \in (0,T)$, that is,

$$\int_0^t w(t-\tau)\varphi(\tau)\mathrm{d}\tau \le 0, \quad 0 < t < T.$$

Since $\varphi \in C^1[0,T]$ is nonnegative and can be arbitrarily chosen, we must have $w \le 0$ in $(0,T)$. Indeed, noting that $w \in H^1(0,T) \subset C[0,T]$, if $w \le 0$ fails in $(0,T)$, then we can choose $t_0 \in (0,T)$ and a sufficiently small constant $\varepsilon > 0$ such that $w(t) > 0$ for any $t \in [t_0-\varepsilon, t_0+\varepsilon]$. Then we can construct $\varphi \in C^1[0,T]$ satisfying

$$\varphi(t) = \begin{cases} 1, & \text{if } t \in \Big[t_0 - \dfrac{\varepsilon}{2}, t_0 + \dfrac{\varepsilon}{2}\Big], \\ 0, & \text{if } t \in (0, t_0-\varepsilon) \cup (t_0+\varepsilon, T). \end{cases}$$

In this case, we calculate the convolution $w * \varphi$ and find that

$$\int_0^t w(t-\tau)\varphi(\tau)\mathrm{d}\tau \ge \int_{t_0-\frac{\varepsilon}{2}}^{t_0+\frac{\varepsilon}{2}} w(t-\tau)\mathrm{d}\tau \ge \varepsilon \inf_{(t_0-\frac{\varepsilon}{2}, t_0+\frac{\varepsilon}{2})} w(t) > 0,$$

which is a contradiction. This completes the proof of the lemma.

Lemma 4.3 *Let $\lambda > 0$, $p \in L^\infty(0,T)$ and $p_0 \le p(t) \le p_1$, $t \in (0,T)$ for some positive constants $p_0, p_1 > 0$. Assume that $z \in W^{1,\infty}(0,T)$ satisfies $z(t) \ge 0$ and*

$$\partial_t z(t) + p(t)\partial_t^\alpha z(t) + \lambda z(t) \le 0, \quad 0 < t < T. \tag{4.3}$$

Then $\partial_t^\alpha z(t) \le 0$ for $0 < t < T$.

Proof Step 1 We first assume $z \in C^2[0,T]$ and we find $\partial_t^\alpha z(0) = 0$, which can be easily verified by the following estimate

$$|\partial_t^\alpha z(t)| \le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} |\partial_s z(s)| \mathrm{d}s \le \frac{\|z\|_{C^1[0,T]}}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad t \in [0,T].$$

Now we set $\widetilde{z} := \partial_t^\alpha z$ and find that $J^\alpha \widetilde{z} = z - z(0)$, where J^α denotes the Riemann-Liouville integral operator. By (4.3), we obtain

$$\begin{cases} \partial_t J^\alpha \widetilde{z}(t) + p(t)\widetilde{z}(t) \le -\lambda z(t) \le 0, \quad 0 < t < T, \\ \widetilde{z}(0) = \partial_t^\alpha z(0) = 0. \end{cases} \tag{4.4}$$

Then we claim that $\widetilde{z}(t) \le 0$ for any $t \in (0,T)$. Otherwise, there exists $t_1 \in (0,T]$ such that $\widetilde{z}$ attains its positive maximum $\widetilde{z}(t_1) > 0$ at point t_1. By [1, Theorem 2.1], we have the Riemann-Liouville fractional derivative at $t = t_1$ satisfies

$$\partial_t J^\alpha \widetilde{z}(t_1) \ge \frac{t_1^{\alpha-1}}{\Gamma(\alpha)} \widetilde{z}(t_1) > 0,$$

and hence $\partial_t J^\alpha \widetilde{z} > 0$ in a neighborhood of t_1, which yields a contraction to (4.4). Thus, $\widetilde{z}$ is non-positive and then we have $\partial_t^\alpha z(t) \le 0$ for $0 < t < T$.

Step 2 We assume $z \in W^{1,\infty}(0,T)$. We denote $z_\mu(t) := z(t) + \mu^{-1} E_{\alpha,1}(-\mu t^\alpha)$ with $\mu > 0$. Then we see that z_μ is positive and satisfies the following equation

$$(\partial_t + p(t)\partial_t^\alpha + \lambda) z_\mu = (\partial_t + p(t)\partial_t^\alpha + \lambda) z + R_\mu(t), \tag{4.5}$$

where $R_\mu(t) := -t^{\alpha-1} E_{\alpha,\alpha}(-\mu t^\alpha) - p(t) E_{\alpha,1}(-\mu t^\alpha) + \frac{\lambda}{\mu} E_{\alpha,1}(-\mu t^\alpha)$ can be easily derived by differential properties (3.1) and (3.2) of the Mittag-Leffler function. Moreover, by the useful estimate (3.3) for the Mittag-Leffler functions, we can see that there exists a constant $\delta_\mu > 0$ such that the following inequality

$$-t^{\alpha-1}E_{\alpha,\alpha}(-\mu t^\alpha) - p(t)E_{\alpha,1}(-\mu t^\alpha) + \frac{\lambda}{\mu}E_{\alpha,1}(-\mu t^\alpha) < -\delta_\mu$$

is valid for any $t \in (0,T)$ and sufficiently large $\mu > 0$, which combined with (4.5) and the assumption (4.3) implies

$$(\partial_t + p(t)\partial_t^\alpha + \lambda)z_\mu \le -\delta_\mu, \quad 0 < t < T.$$

Then for any $\varepsilon > 0$, we can choose $z_{\mu,\varepsilon} \in C^2[0,T]$ such that $z_{\mu,\varepsilon} \ge 0$ and

$$\|z_{\mu,\varepsilon} - z_\mu\|_{W^{1,\infty}(0,T)} \le \varepsilon.$$

By a direct calculation, we see that

$$\begin{aligned}(\partial_t + p(t)\partial_t^\alpha + \lambda)z_{\mu,\varepsilon} &= (\partial_t + p(t)\partial_t^\alpha + \lambda)z_\mu + (\partial_t + p(t)\partial_t^\alpha + \lambda)(z_{\mu,\varepsilon} - z_\mu) \\ &\le -\delta_\mu + \Big(1 + \frac{p_1 T^{1-\alpha}}{\Gamma(2-\alpha)} + \lambda\Big)\varepsilon.\end{aligned}$$

Consequently, for sufficiently small $\varepsilon > 0$, we see that

$$(\partial_t + p(t)\partial_t^\alpha + \lambda)z_{\mu,\varepsilon} \le 0.$$

Now by Step 1, it follows that $\partial_t^\alpha z_{\mu,\varepsilon} \le 0$ for any sufficiently small $\varepsilon > 0$. Letting $\varepsilon \to 0$ and we have $\partial_t^\alpha z_\mu \le 0$ for any sufficiently large μ. Finally, again from the estimate (3.3) for the Mittag-Leffler functions, we see that $E_{\alpha,1}(-\mu t^\alpha)$ tends to 0 as $\mu \to \infty$, hence that $\partial_t^\alpha z \le 0$ by letting $\mu \to \infty$. We then finish the proof of the lemma.

Equipped with the above lemmas, we prove our main result by applying an energy estimate.

Proof of Theorem 2.2 According to the result of the forward problem (Theorem 2.1), we note that $u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ provided that $u_0 \in L^2(\Omega)$. Since this regularity is not enough to guarantee the above lemmas that we will use in the proof, we need to introduce the approximate solutions $\{u_N\}_{N=1}^\infty$ which solve

$$\begin{cases} \partial_t u_N + q(t)\partial_t^\alpha u_N = -Au_N + c(x,t)u_N, & (x,t) \in \Omega \times (0,T), \\ u_N(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_N(x,0) = \displaystyle\sum_{k=1}^N (u_0,\varphi_k)_{L^2(\Omega)}\varphi_k, & x \in \Omega \end{cases} \tag{4.6}$$

for each $N \in \mathbb{N}$. Here we recall that $\{\varphi_k\}_{k=1}^\infty \subset H^2(\Omega) \cap H_0^1(\Omega)$ is the set of the eigenfunctions of A with the homogeneous Dirichlet boundary condition and forms an orthonormal basis of $L^2(\Omega)$. By the third part of Theorem 2.1, we see that $u_N \in W^{1,\infty}(0,T;L^2(\Omega))$, which guarantees the regularity when we apply Lemmas 4.1–4.3 in the following context.

Now we multiply u_N on both sides of the first equation of (4.6) and integrate over Ω. Integration by parts yields

$$(\partial_t u_N, u_N) + q(t)(\partial_t^\alpha u_N, u_N) + \sum_{i,j=1}^d (a_{ij}\partial_{x_i}u_N, \partial_{x_j}u_N) - (c(t)u_N, u_N) = 0 \tag{4.7}$$

for $0 < t < T$. Next we estimate the left-hand side of (4.7) from below.

The ellipticity of the operator A and the Poincaré inequality imply

$$\sum_{i,j=1}^{d}(a_{ij}\partial_{x_i}u_N(t), \partial_{x_j}u_N(t)) \ge \nu\|\nabla u_N(t)\|^2_{L^2(\Omega)} \ge \lambda\|u_N(t)\|^2_{L^2(\Omega)} \tag{4.8}$$

with some positive constant $\lambda > 0$, which depends only on ν and Ω. By Lemma 4.1, we have

$$(\partial_t^\alpha u_N(t), u_N(t)) \ge \|u_N(t)\|_{L^2(\Omega)}\,\partial_t^\alpha\|u_N(t)\|_{L^2(\Omega)} \tag{4.9}$$

for $0 < t < T$. Since $c \le 0$ in $\Omega \times (0, T)$ and

$$(\partial_t u_N(t), u_N(t)) = \frac{1}{2}\partial_t\|u_N(t)\|^2_{L^2(\Omega)} = \|u_N(t)\|_{L^2(\Omega)}\,\partial_t\|u_N(t)\|_{L^2(\Omega)}, \tag{4.10}$$

we insert (4.8)–(4.10) into (4.7) and obtain

$$\|u_N(t)\|_{L^2(\Omega)}(\partial_t\|u_N(t)\|_{L^2(\Omega)} + q(t)\partial_t^\alpha\|u_N(t)\|_{L^2(\Omega)} + \lambda\|u_N(t)\|_{L^2(\Omega)}) \le 0 \tag{4.11}$$

for $0 < t < T$. We assert that

$$\partial_t\|u_N(t)\|_{L^2(\Omega)} + q(t)\partial_t^\alpha\|u_N(t)\|_{L^2(\Omega)} + \lambda\|u_N(t)\|_{L^2(\Omega)} \le 0 \tag{4.12}$$

for $0 < t < T$. If (4.12) does not hold, then there exist $t_2 \in (0, T)$ and a small constant $\varepsilon > 0$ such that

$$\partial_t\|u_N(t)\|_{L^2(\Omega)} + q(t)\partial_t^\alpha\|u_N(t)\|_{L^2(\Omega)} + \lambda\|u_N(t)\|_{L^2(\Omega)} > 0$$

for $t \in (t_2 - \varepsilon, t_2 + \varepsilon)$. Then by (4.11), we see that $\|u_N(t_2)\|_{L^2(\Omega)} = 0$, which indicates that $\|u_N(t)\|_{L^2(\Omega)}$ attains its minimum at $t = t_2$. Immediately we have $\partial_t\|u_N(t_2)\|_{L^2(\Omega)} \le 0$ and $\partial_t^\alpha\|u_N(t_2)\|_{L^2(\Omega)} \le 0$ from [24, Theorem 1]. This yields a contradiction.

Next we estimate $\|u_N(t)\|_{L^2(\Omega)}$ by some function from above. We introduce an auxiliary function v which solves the following fractional ordinary differential equation

$$\begin{cases} \partial_t v(t) + q_1\partial_t^\alpha v(t) + \lambda v(t) = 0, \quad t > 0, \\ v(0) = \|u_0\|_{L^2(\Omega)}. \end{cases} \tag{4.13}$$

Here we recall that q_1 is a positive constant and $q(t) \le q_1$ for $t > 0$. Let $w_N(t) = \|u_N(t)\|_{L^2(\Omega)} - v(t)$. Since $T > 0$ is arbitrary, by (2.2), (4.12)–(4.13) and

$$\|u_N(0)\|_{L^2(\Omega)} = \Big(\sum_{k=1}^{N}(u_0, \varphi_k)^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \le \|u_0\|_{L^2(\Omega)} = v(0),$$

we obtain

$$\begin{cases} \partial_t w_N + q_1\partial_t^\alpha w_N + \lambda w_N \le (q_1 - q(t))\partial_t^\alpha\|u_N(t)\|_{L^2(\Omega)}, \quad 0 < t < T, \\ w_N(0) \le 0. \end{cases} \tag{4.14}$$

From (4.12), applying Lemma 4.3, we can see that $\partial_t^\alpha\|u_N(t)\|_{L^2(\Omega)} \le 0$, which means the right-hand side of (4.14) is not positive. Now we can apply Lemma 4.2 for (4.14) to obtain $w_N(t) \le 0$ for $0 < t < T$, that is,

$$\|u_N(t)\|_{L^2(\Omega)} \le v(t) \quad \text{for } 0 < t < T.$$

By Theorem 2.1, we find that for arbitrarily fixed $T > 0$, the sequence $u_N(t)$ converges to $u(t)$ in $L^2(\Omega)$ for any $0 < t < T$. Moreover, we note that v is the solution to (4.13), which is independent of N. Thus, we have

$$\|u(t)\|_{L^2(\Omega)} \le v(t) \quad \text{for } 0 < t < T.$$

Finally, since $T > 0$ can be arbitrarily fixed and v is also independent of T, it remains to discuss the long-time asymptotic behavior of v.

By applying the Laplace transform to the fractional ordinary differential equation (4.13), we can derive

$$|v(t)| \le C\|u_0\|_{L^2(\Omega)} t^{-\alpha}, \quad t \ge t_0$$

for arbitrarily fixed $t_0 > 0$. We put the details in Lemma A.1 in the appendix. This completes the proof of Theorem 2.2.

5 Conclusions and Open Problems

In this paper, we considered the diffusion equation with fractional derivative on a bounded multi-dimensional domain subject to the homogeneous Dirichlet boundary condition. Firstly, by regarding the fractional term as a source, we transferred the differential equation to an equivalent integral form, and we used the Fredholm alternative for the compact operator to show the well-posedness for the forward problem, which is essential for numerically analyzing this type of problems and for dealing with the inverse problems for the fractional diffusion equation. On the basis of the forward problem, the energy estimate and maximum principle allow us to obtain the asymptotic decay in time for the solution to the initial-boundary value problem (2.1)–(2.2).

For the sake of simplicity, we consider the case of only one fractional derivative in this paper. As one can see from the proof, we can similarly prove Theorem 2.1 for a multi-term time-fractional diffusion equation

$$\partial_t u + \sum_{j=1}^{\ell} q_j(t)\partial_t^{\alpha_j} u = -Au + c(x,t)u + f(x,t), \quad (x,t) \in \Omega \times (0,T),$$

where $\ell \in \mathbb{N}$ is given and we assume $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1$. Moreover, if we further assume $f = 0$, $c \le 0$ and $q_j(t) = q_j \ge 0$, $j = 2, \cdots, \ell$, $p_0 \le q_1(t) \le p_1$ with some positive constants $p_1 \ge p_0 > 0$, then by following the proof in Section 4 and Appendix, Theorem 2.2 can be immediately generalized in the multi-term case with the following long-time asymptotic estimate

$$\|u(\,\cdot\,,t)\|_{L^2(\Omega)} \le C\|u_0\|_{L^2(\Omega)} t^{-\alpha_1},$$

which indicates that the asymptotic behavior of the solution depends on the lowest order of the fractional derivatives. The assumption that q_j, $j = 2, \cdots, \ell$ are constants may be relaxed by modifying the argument we used in this paper but here we do not discuss more details.

As for the open problems related to the initial-boundary value problems for the fractional diffusion equations, let us mention the following ones: Our proof for the well-posedness of the problem (2.1)–(2.2) heavily relies on the eigenfunction expansion and the properties of the Mittag-Leffler functions. If the coefficients in the elliptic operator A are also t-varying, one cannot directly use the above argument to prove the well-posedness. However, the asymptotics of the solution in the case when the elliptic operator is also t-dependent can be obtained by following the strategy in Section 4, provided that the solution is sufficiently smooth. On the other hand, in the proof of the asymptotics, we need the assumption that q is independent of x, which is necessary for deriving Lemma 4.1. It would be interesting to investigate what happens with the asymptotic properties of the solution if this assumption is relaxed. Another interesting direction of the research would be that whether the estimate is valid for the fractional diffusion equation with nonlinearity. It still remains open and should be investigated.

A Asymptotics for a Fractional Ordinary Differential Equation

In this part, we will follow the argument used in Gorenflo and Mainardi [7, Section 4] to give the proof for the long-time asymptotic behavior of the solution v to the following fractional ordinary differential equation

$$\begin{cases} \partial_t v(t) + q_1 \partial_t^\alpha v(t) + \lambda v(t) = 0, \quad t > 0, \\ v(0) = v_0. \end{cases} \tag{A.1}$$

Lemma A.1 *Assume $q_1 > 0$, $\lambda > 0$ and $v_0 \neq 0$ are given constants. Then the solution v to the problem* (A.1) *admits the following long-time asymptotic estimate*

$$|v(t)| \le C|v_0| t^{-\alpha}, \quad t \ge t_0$$

for any $t_0 > 0$. Here the order α is sharp and the constant C depends only on q_1, α, λ and t_0.

Proof By applying the Laplace transform to the ordinary fractional differential equation (A.1), we find that

$$\mathcal{L}[v](s) = \frac{1 + q_1 s^{\alpha-1}}{s + q_1 s^\alpha + \lambda} v_0, \quad s > 0,$$

where $\mathcal{L}[v]$ denotes the Laplace transform of the function v. Then we get $v(t)$ by the Fourier-Mellin transform of $\mathcal{L}[v](s)$. Since it is readily to see that $s + q_1 s^\alpha + \lambda$ has no zero in the main sheet of the Riemann surface including the negative real axis, we can deform the original Bromwich path into the Hankel path $Ha(\varepsilon)$ and obtain

$$v(t) = \frac{1}{2\pi \mathrm{i}} v_0 \int_{Ha(\varepsilon)} \mathrm{e}^{st} \frac{1 + q_1 s^{\alpha-1}}{s + q_1 s^\alpha + \lambda} \mathrm{d}s.$$

Here the Hankel path $Ha(\varepsilon)$ is the loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|s| = \varepsilon$ and ends at $-\infty$ along the upper side of the negative real axis. Letting $\varepsilon \to 0$ yields

$$v(t) = v_0 \int_0^\infty \mathrm{e}^{-rt} H_{\alpha,0}^{(1)}(r; q_1, \lambda) \mathrm{d}r$$

with

$$\begin{aligned} H_{\alpha,0}^{(1)}(r; q_1, \lambda) &= -\frac{1}{\pi} \Im \Big\{ \frac{1 + q_1 s^{\alpha-1}}{s + q_1 s^\alpha + \lambda} \,|_{s = r\mathrm{e}^{\mathrm{i}\pi}} \Big\} \\ &= \frac{1}{\pi} \frac{\lambda q_1 r^{\alpha-1} \sin(\alpha\pi)}{(\lambda - r)^2 + q_1^2 r^{2\alpha} + 2(\lambda - r) q_1 r^\alpha \cos(\alpha\pi)}, \end{aligned}$$

where $\Im z$ denotes the imaginary part of $z \in \mathbb{C}$. We break the above integral into two parts as follows

$$v(t) = v_0 \int_0^\delta \mathrm{e}^{-rt} H^{(1)}_{\alpha,0}(r; q_1, \lambda)\mathrm{d}r + v_0 \int_\delta^\infty \mathrm{e}^{-rt} H^{(1)}_{\alpha,0}(r; q_1, \lambda)\mathrm{d}r =: I_1 + I_2,$$

where $0 < \delta \le \lambda$ will be chosen later. We will estimate I_1 and I_2 separately. For I_1, in view of the inequality that

$$\begin{aligned}&(\lambda - r)^2 + q_1^2 r^{2\alpha} + 2(\lambda - r)q_1 r^\alpha \cos(\alpha\pi)\\ \ge\,&(\lambda - r)^2 + q_1^2 r^{2\alpha} - 2(\lambda - r)q_1 r^\alpha = (\lambda - r - q_1 r^\alpha)^2, \quad 0 < r < \delta,\end{aligned}$$

we can choose $\delta > 0$ sufficiently small such that $\lambda - r - q_1 r^\alpha \ge \frac{\lambda}{2}$ for any $0 < r < \delta$. Consequently, we arrive at the following inequalities

$$\begin{aligned}|I_1(t)| &\le \frac{2|v_0|q_1 \sin(\alpha\pi)}{\pi} \int_0^\delta \mathrm{e}^{-rt} r^{\alpha-1}\mathrm{d}r\\ &\le \frac{2}{\pi} q_1 \sin(\alpha\pi)\Gamma(\alpha)|v_0|t^{-\alpha}, \quad t > 0.\end{aligned}$$

Next, we estimate I_2. Firstly, for any $s = r\mathrm{e}^{\mathrm{i}\pi}$ with $r > 0$, a direct calculation yields

$$|s + q_1 s^\alpha + \lambda| \ge \Im s + q_1 \Im s^\alpha = |s|\sin\pi + q_1|s|^\alpha \sin(\alpha\pi) = q_1 \sin(\alpha\pi) r^\alpha > 0.$$

Hence we see that

$$|H^{(1)}_{\alpha,0}(r; q_1, \lambda)| \le \frac{1}{\pi}\Big|\frac{1 + q_1 s^{\alpha-1}}{s + q_1 s^\alpha + \lambda}\Big|_{s = r\mathrm{e}^{\mathrm{i}\pi}}\Big| \le \frac{1 + q_1 r^{\alpha-1}}{\pi q_1 \sin(\alpha\pi) r^\alpha} \le \frac{1 + q_1 r^{\alpha-1}}{\pi q_1 \sin(\alpha\pi)\delta^\alpha}$$

holds true for any $r \ge \delta$. Therefore, we obtain

$$\begin{aligned}|I_2(t)| &\le \frac{|v_0|}{\pi q_1 \sin(\alpha\pi)\delta^\alpha} \int_\delta^\infty \mathrm{e}^{-rt}(1 + q_1 r^{\alpha-1})\mathrm{d}r\\ &\le \frac{|v_0|}{\pi q_1 \sin(\alpha\pi)\delta^\alpha}\Big(\frac{1}{t} + q_1\Gamma(\alpha)t^{-\alpha}\Big), \quad t > 0.\end{aligned}$$

Finally, collecting all the above estimates for I_1 and I_2, we arrive at the inequality

$$|v(t)| \le C|v_0|(t^{-1} + t^{-\alpha}), \quad t > 0,$$

and thus, by noting $t^{-1} = t^{\alpha-1}t^{-\alpha} \le t_0^{\alpha-1}t^{-\alpha}$ for $t \ge t_0$, we have

$$|v(t)| \le C|v_0|t^{-\alpha}, \quad t \ge t_0,$$

where the constant $C > 0$ depends only on q_1, λ, α and t_0. Moreover, for any $r > 0$, we have $H^{(1)}_{\alpha,0}(r; q_1, \lambda) > 0$ and for any $0 \le r \le 1$, we have the inequality

$$(\lambda - r)^2 + q_1^2 r^{2\alpha} + 2(\lambda - r)q_1 r^\alpha \cos(\alpha\pi) \le (|\lambda - r| + q_1 r^\alpha)^2 \le (\lambda + q_1)^2,$$

which implies

$$|v(t)| \ge |v_0| \int_0^1 \mathrm{e}^{-rt} H^{(1)}_{\alpha,0}(r; q_1, \lambda)\mathrm{d}r$$

$$\begin{aligned}
&\geq |v_0| \frac{\lambda q_1 \sin(\alpha\pi)}{\pi(\lambda+q_1)^2} \int_0^1 \mathrm{e}^{-rt} r^{\alpha-1} \mathrm{d}r \\
&= |v_0| t^{-\alpha} \frac{\lambda q_1 \sin(\alpha\pi)}{\pi(\lambda+q_1)^2} \int_0^t \mathrm{e}^{-r} r^{\alpha-1} \mathrm{d}r \\
&\geq |v_0| t^{-\alpha} \frac{\lambda q_1 \sin(\alpha\pi)}{\pi(\lambda+q_1)^2} \int_0^{t_0} \mathrm{e}^{-r} r^{\alpha-1} \mathrm{d}r, \quad t \geq t_0.
\end{aligned}$$

Thus, we find that the decay rate $t^{-\alpha}$ is sharp and we finish the proof of the lemma.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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