Chin. Ann. Math. Ser. B
46(2), 2025, 163–180
DOI: 10.1007/s11401-025-0009-0

Chinese Annals of Mathematics, Series B

Classification of the Conformally Flat Centroaffine Hypersurfaces with Vanishing Centroaffine Shape Operator*

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Abstract Cheng-Hu-Moruz (2017) completely classified the locally strongly convex centroaffine hypersurfaces with parallel cubic form based on the Calabi product (called the type I Calabi product for short) proposed by Li-Wang (1991).

In the present paper, the authors introduce the type II Calabi product (in case $\lambda_1 = 2\lambda_2$), complementing the type I Calabi product (in case $\lambda_1 \neq 2\lambda_2$), and achieve a classification of the locally strongly convex centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ with vanishing centroaffine shape operator and Weyl curvature tensor by virtue of the types I and II Calabi product.

As a corollary, 3-dimensional complete locally strongly convex centroaffine hypersurfaces with vanishing centroaffine shape operator are completely classified, which positively answers the centroaffine Bernstein problems III and V by Li-Li-Simon (2004).

Keywords Centroaffine hypersurface, Centroaffine shape operator, Calabi product, Locally conformally flat, Calabi hypersurface
2020 MR Subject Classification 53A15, 53C24, 53C42

1 Introduction

In centroaffine differential geometry, the centroaffine normalization induces the identity as Weingarten operator, which contains no further geometric information. By calculating the variation formula of the volume with respect to the centroaffine metric, Wang [22] reasonably introduced an important self-adjoint operator $\mathcal{T} := \widehat{\nabla}T$, originally named as centroaffine shape operator also called Tchebychev operator in [15], where T and $\widehat{\nabla}$ denote the Tchebychev vector field and the Levi-Civita connection with respect to the centroaffine metric. In addition, the Euler-Lagrange equation of the volume variation with respect to the centroaffine metric is geometrically equivalent to

$$\mathrm{Tr}\mathcal{T} = \mathrm{divT} = 0. \tag{1.1}$$

Manuscript received January 18, 2024. Revised September 1, 2024.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11871197, 12141104, 12271254).

The centroaffine hypersurfaces satisfying (1.1) are called centroaffine minimal hypersurfaces in [22], later also called centroaffine extremal hypersurfaces in [10]. For such hypersurfaces several related versions of centroaffine Bernstein problems were listed in [10, Section 5]. It is well known that the classification of centroaffine extremal hypersurfaces is very interesting and important but it is rather difficult. This partially motivates us to consider the centroaffine hypersurfaces with vanishing centroaffine shape operator, i.e.,

$$\mathcal{T} = \widehat{\nabla} T = 0. \tag{1.2}$$

Noting that Liu-Wang [15] classified the non-degenerate centroaffine surfaces with vanishing centroaffine shape operator. The classification of locally strongly convex flat centroaffine hypersurfaces with vanishing centroaffine shape operator is equivalent to the classification of the canonical centroaffine hypersurfaces, which has been investigated in [3, 13]. Here, a centroaffine hypersurface is called canonical if its centroaffine metric is flat and its difference tensor is parallel with respect to its centroaffine metric.

Theorem 1.1 (see [3]) *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex canonical centroaffine hypersurface. Then it is locally centroaffinely equivalent to one of the following hypersurfaces*:

(i) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1$, *where either* $\alpha_i > 0$ $(1 \le i \le n+1)$, *or* $\alpha_j > 0$ $(2 \le j \le n+1)$ *and* $\sum\limits_{j=1}^{n+1} \alpha_j < 0$;

(ii) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} (x_n^2 + x_{n+1}^2)^{\alpha_n} \exp\left(\alpha_{n+1} \arctan \frac{x_n}{x_{n+1}}\right) = 1$, *where* $\alpha_i < 0$ $(1 \le i \le n-1)$ *and* $2\alpha_n + \sum\limits_{i=1}^{n-1} \alpha_i > 0$;

(iii) $x_{n+1} = \frac{1}{2x_1}(x_2^2 + \cdots + x_{v-1}^2) - x_1(-\ln x_1 + \alpha_v \ln x_v + \cdots + \alpha_n \ln x_n)$, *where* $2 \le v \le n+1$, $\alpha_i > 0$ $(v \le i \le n)$ *are real numbers and* $\sum\limits_{i=v}^{n} \alpha_i < 1$.

Recently, Cheng-Hu-Xing [6] classified the locally strongly convex centroaffine hypersurfaces with vanishing centroaffine shape operator and constant sectional curvature. More researches on the centroaffine shape operator, we refer the readers to [1, 5, 7, 11, 14, 16, 21] etc.

The current paper concerns the locally strongly convex centroaffine hypersurfaces with vanishing centroaffine shape operator and Weyl curvature tensor. To state the main result, we need to recall some facts about Calabi hypersurfaces from [2, 9, 19]. Let $\varphi : M_1 \to \mathbb{R}^{n+1}$ be a locally strongly convex hypersurface immersion of a smooth, connected manifold into real affine space $\mathbb{R}^{n+1}$. Assume that $Y_0 = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ is a relative normalization of the hypersurface M_1, which is called the Calabi affine normalization. In the following, such immersion equipped with the Calabi affine normalization is called Calabi hypersurface. Additionally, the main result of this paper depends heavily on the (generalized) Calabi product shown in [3, 10, 13], and the new type of (generalized) Calabi product presented in Section 3. In order to distinguish two types of Calabi products, we call the Calabi products defined by (1.3)–(1.4) as the types I and II Calabi product, respectively. Firstly, let $\psi : M_2 \to \mathbb{R}^n$ be a locally strongly convex centroaffine hypersurface of dimension $n-1$. Then, for a constant $\lambda \neq 0, -1$, the type I Calabi product of a point and M_2 is defined by

$$\widetilde{\psi}(t, p) := (\mathrm{e}^t \psi(p), \mathrm{e}^{-\lambda t}) \in \mathbb{R}^{n+1}, \quad p \in M_2, \ t \in \mathbb{R}. \tag{1.3}$$

The hypersurface defined by (1.3) is a non-degenerate centroaffine hypersurface (see [3, Proposition 3.2]). Secondly, let $\varphi : M_1 \to \mathbb{R}^n$ be a Calabi hypersurface of dimension $n-1$ with $\varphi(p) = (\varphi_1(p), \cdots, \varphi_{n-1}(p), \varphi_n(p))$ for $p \in M_1$. Then the type II Calabi product of a point and M_1 is defined by

$$x(t, p) := \mathrm{e}^t (1, \varphi_1(p), \cdots, \varphi_{n-1}(p), \varphi_n(p) + t) \in \mathbb{R}^{n+1}, \quad p \in M_1, \ t \in \mathbb{R}. \tag{1.4}$$

The hypersurface defined by (1.4) is a locally strongly convex centroaffine hypersurface of elliptic type (see Proposition 3.1 below). If we focus only on the locally strongly convex centroaffine hypersurfaces, then the type II Calabi product complements the type I Calabi product, see Remark 3.2 below for the precise details.

The main result of this paper is as follows.

Theorem 1.2 *Let $M^n (n \geq 3)$ be a locally strongly convex centroaffine hypersurface with vanishing centroaffine shape operator and Weyl curvature tensor. Then one of the following cases occurs*:

(i) *In case $T = 0$, then M^n is an open part of a proper affine hypersphere centered at origin with vanishing Weyl curvature tensor.*

(ii) *In case $T \neq 0$, then*

(ii-1) *M^n is an open part of a canonical centroaffine hypersurface*; *or*

(ii-2) *$n = 3$, M^3 is obtained as either the type* I *Calabi product* ($\lambda < -1$; *resp.* $\lambda > -1, \lambda \neq 0, 3$) *of a point and a non-flat locally strongly convex proper* (*elliptic*; *resp. hyperbolic*) *affine* 2-*sphere centered at origin, or the type* II *Calabi product of a point and a non-flat locally strongly convex improper affine* 2-*sphere*; *or*

(ii-3) *$n \geq 4$, M^n is locally centroaffinely equivalent to the hypersurfaces*

$$(x_1^2 + \cdots + x_n^2)^\lambda x_{n+1}^2 = 1, \quad \lambda < -1;$$

or

$$(x_1^2 - x_2^2 - \cdots - x_n^2)^\lambda x_{n+1}^2 = 1, \quad \lambda > -1, \ \lambda \neq 0, n.$$

Remark 1.1 It is an open problem that how to classify all locally strongly convex affine hyperspheres with affine metrics being locally conformally flat. There appeared some partial results recently (see [4, 8, 23]).

Obviously, Theorem 1.2 generalizes [6, Theorem 1.1]. Moreover, the fact Weyl curvature tensor vanishes automatically for the dimension $n = 3$ implies Theorem 1.2 completely classifies the 3-dimensional locally strongly convex centroaffine hypersurfaces with vanishing centroaffine shape operator, which partially generalizes [15, Theorem 4.1]. From the proof of Theorem 1.2, one can get the following consequence.

Corollary 1.1 *Let M^3 be a locally conformally flat, locally strongly convex centroaffine hypersurface with vanishing centroaffine shape operator. Then M^3 is locally centroaffinely e-quivalent to*

(i) *a locally conformally flat proper affine hypersphere centered at origin*; *or*

(ii) *a locally strongly convex canonical centroaffine hypersurface with $T \neq 0$*; *or*

(iii) *the hypersurface $(x_1^2 + x_2^2 + x_3^2)^\lambda x_4^2 = 1$ with $\lambda < -1$*; *or*

(iv) *the hypersurface $(x_1^2 - x_2^2 - x_3^2)^\lambda x_4^2 = 1$ with $\lambda > -1$ and $\lambda \neq 0, 3$.*

As a corollary of Theorem 1.2, the classification of 3-dimensional centroaffine hypersurfaces with vanishing centroaffine shape operator and complete centroaffine metric is obtained.

Corollary 1.2 *Let M^3 be a complete locally strongly convex centroaffine hypersurface with vanishing centroaffine shape operator. Then*

(i) *M^3 is a complete proper affine hypersphere centered at origin*; *or*

(ii) *M^3 is a complete canonical centroaffine hypersurface with $T \neq 0$, namely, it is centroaffinely equivalent to one of the following hypersurfaces*:

(ii-1) $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} = 1$, *where either $\alpha_i > 0$ and $\alpha_i \neq 1$ $(1 \leq i \leq 4)$, or $\alpha_j > 0$ $(2 \leq j \leq 4)$ and $\sum_{j=1}^{4} \alpha_j < 0$*; *or*

(ii-2) $x_4 = \frac{1}{2x_1}(x_2^2 + x_3^2) + x_1 \ln x_1$; *or* $x_4 = \frac{x_2^2}{2x_1} + x_1(\ln x_1 - \alpha_3 \ln x_3)$, $0 < \alpha_3 < 1$; *or* $x_4 = x_1(\ln x_1 - \alpha_2 \ln x_2 - \alpha_3 \ln x_3), \alpha_i > 0$ $(2 \leq i \leq 3)$ *and* $\alpha_2 + \alpha_3 < 1$; *or*

(iii) *M^3 is centroaffinely equivalent to the hypersurface $(x_1^2 + x_2^2 + x_3^2)^\lambda x_4^2 = 1$ with $\lambda < -1$*; *or*

(iv) *M^3 is obtained as the type* I *Calabi product $(\lambda > -1, \lambda \neq 0, 3)$ of a point and a complete non-flat locally strongly convex hyperbolic affine* 2*-sphere centered at origin.*

Remark 1.2 Apart from hyperellipsoids, cases (ii-2) and (iii), parts of cases (ii-1) and (iv) shown above are complete centroaffine extremal hypersurfaces of elliptic type. Corollary 1.2 positively answers the centroaffine Bernstein problems III and V in [10]. In particular, the hypersurface $x_4 = \frac{1}{2x_1}(x_2^2 + x_3^2) + x_1 \ln x_1$ has a Euclidean boundary point $(0, 0, 0, 0)$, which implies that it cannot be represented as graph over $\mathbb{R}^3$.

The remainder of this paper is organized as follows. Firstly, Section 2 presents some basic facts of the centroaffine geometry and warped product manifold. In Section 3, a new type of (generalized) Calabi product is introduced in centroaffine geometry and a decomposition theorem is proved in terms of their centroaffine invariants, which will play a critical role in the proof of main results. Finally, using the nice property (see Lemma 4.1 below) of the locally strongly convex centroaffine hypersurface with vanishing centroaffine shape operator and Weyl curvature tensor, we complete the proofs of main results in Section 4.

2 Preliminaries

In this section, we shall show some basic facts of the centroaffine geometry. For more details see [17, 20]. Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional affine space equipped with the standard flat connection D. For an immersion $x : M^n \to \mathbb{R}^{n+1}$ of an n-dimensional smooth manifold M^n, if the position vector $x(p)$ (from the origin O) is transversal to $x_*(T_pM^n)$ at each point $p \in M^n$, then $x : M^n \to \mathbb{R}^{n+1}$ defines a centroaffine hypersurface and the position vector defines the so-called centroaffine normalization modulo orientation.

For any vector fields X and Y tangent to M^n, the centroaffine formula of Gauss reads

$$D_X x_*(Y) = x_*(\nabla_X Y) + h(X,Y)(-\varepsilon x), \tag{2.1}$$

where $\varepsilon = \pm 1$. Associated with (2.1), we call $-\varepsilon x$, ∇ and h the centroaffine normal, the induced connection and the centroaffine metric induced by $-\varepsilon x$, respectively. Moreover, the centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is called non-degenerate if the centroaffine metric h, defined by (2.1), remains non-degenerate. In this paper, we always assume that $x : M^n \to \mathbb{R}^{n+1}$ is a locally strongly convex centroaffine hypersurface, i.e., the centroaffine metric h induced by $-\varepsilon x$ is positive definite. More precisely, if $\varepsilon = 1$, the centroaffine hypersurface is elliptic and if $\varepsilon = -1$, the centroaffine hypersurface is hyperbolic.

For a given centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, the difference tensor is defined by

$$K(X,Y) := K_X Y := \nabla_X Y - \widehat{\nabla}_X Y, \tag{2.2}$$

where $\widehat{\nabla}$ is the Levi-Civita connection with respect to the centroaffine metric h. It follows from both connections ∇ and $\widehat{\nabla}$ are torsion free that K is symmetric. According to (2.2), the Tchebychev vector field T is given by

$$h(T,X) = \frac{1}{n}\mathrm{Tr}(K_X). \tag{2.3}$$

Noting that if $T = 0$, which is equivalent to $\mathrm{Tr}(K_X) = 0$ for any vector field X, then M^n is reduced to be the so-called proper (equi-)affine hypersphere centered at the origin of $\mathbb{R}^{n+1}$ (see [12, p. 279]). Denote by $\widehat{R}$ the Riemannian curvature tensor of the centroaffine metric h. Then the integrability conditions read

$$\widehat{R}(X,Y)Z = \varepsilon(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z, \tag{2.4}$$

$$(\widehat{\nabla}_Z K)(X,Y) = (\widehat{\nabla}_X K)(Z,Y). \tag{2.5}$$

By choosing an h-orthonormal tangential frame field $\{e_1, \cdots, e_n\}$ on M^n, we have $T = \sum\limits_i T^i e_i = \frac{1}{n}\sum\limits_{i,j} K^i_{jj} e_i$, the Gauss equation and Codazzi equation

$$R_{ijkl} = \varepsilon(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_m (K^m_{jk}K^m_{il} - K^m_{ik}K^m_{jl}), \tag{2.6}$$

$$K_{ij,l}^{k}=K_{il,j}^{k}. \tag{2.7}$$

From (2.6) the components of Ricci tensor are

$$R_{ij}=\varepsilon(n-1)\delta_{ij}+\sum_{m,k}K_{jk}^{m}K_{ik}^{m}-n\sum_{m}T^{m}K_{ij}^{m}. \tag{2.8}$$

Thus, the scalar curvature is given by

$$R=n(n-1)(J+\varepsilon)-n^{2}|T|^{2}, \tag{2.9}$$

where $J:=\frac{1}{n(n-1)}\Sigma(K_{ij}^{k})^{2}$ is called the centroaffine Pick invariant. Additionally, a useful formula can be concluded from (2.6)–(2.7) and the Ricci identity.

Lemma 2.1 (see [6, Lemma 4.2]) *For a locally strongly convex centroaffine hypersurface of dimension n, the following formula holds*

$$\begin{aligned}\frac{n(n-1)}{2}\Delta J=&\sum_{i,j,k,l}(K_{ij,l}^{k})^{2}+\sum_{i,j,k,l}(R_{ijkl})^{2}+\sum_{i,j}(R_{ij})^{2}-\varepsilon(n+1)R\\&+n\sum_{i,j,k}K_{ij}^{k}R_{ij}T^{k}+n\sum_{i,j,k}K_{ij}^{k}T_{,ij}^{k}.\end{aligned} \tag{2.10}$$

From (2.9)–(2.10), it is obvious that for a flat centroaffine hypersurface, the parallelism of Tchebychev vector field is equivalent to the parallelism of difference tensor.

Let (M_1,g_1) and (M_2,g_2) be two Riemannian manifolds and f be a positive smooth function defined on M_1. The warped product $M:=M_1\times_f M_2$ is the product manifold $M_1\times M_2$ equipped with the Riemannian metric $g=g_1\oplus f^2g_2$. The function f is called the warping function of the warped product. If X and Y are two linear independent vector fields on M_2, then, by [18, Chapter 7, Proposition 42], the sectional curvature of M satisfies

$$\mathcal{K}^{M}(X,Y)=f^{-2}(\mathcal{K}^{M_2}(X,Y)-g(\widehat{\nabla}f,\widehat{\nabla}f)),$$

where $\widehat{\nabla}$ is the Levi-Civita connection of (M,g). Particularly, if f is a constant, then

$$\mathcal{K}^{M}(X,Y)=f^{-2}\mathcal{K}^{M_2}(X,Y). \tag{2.11}$$

3 Generalized Calabi Product and Decomposition Theorem

The purpose of this section is to introduce a new type of (generalized) Calabi product and prove a decomposition theorem in centroaffine differential geometry. Firstly, by some elementary calculations on the type II Calabi product, Proposition 3.1 is formulated. Then, considering the converse of this proposition, we obtain Theorem 3.1.

Let $\varphi:M_1\to\mathbb{R}^n$ be a Calabi hypersurface relative to the Calabi affine normalization $Y_0=(0,\cdots,0,1)^t\in\mathbb{R}^n$. Denote by G the Calabi metric of $\varphi(M_1)$ and by $\{u_1,\cdots,u_{n-1}\}$ the local coordinates for M_1. Then, the type II Calabi product of a point and the Calabi hypersurface M_1

$$x:M^n=\mathbb{R}\times M_1\to\mathbb{R}^{n+1}$$

is defined by

$$x(t,p) := \mathrm{e}^t(1, \varphi_1(p), \cdots, \varphi_{n-1}(p), \varphi_n(p)+t), \quad p \in M_1,\ t \in \mathbb{R}. \tag{3.1}$$

Claim The hypersurface $x(M^n)$ defined by (3.1) is a centroaffine hypersurface.

In fact, direct calculations show that

$$\begin{aligned} \frac{\partial x}{\partial t} &= \mathrm{e}^t(1, \varphi_1, \cdots, \varphi_{n-1}, \varphi_n + t + 1), \\ \frac{\partial x}{\partial u_i} &= \mathrm{e}^t\Big(0, \frac{\partial \varphi_1}{\partial u_i}, \cdots, \frac{\partial \varphi_n}{\partial u_i}\Big), \quad 1 \le i \le n-1. \end{aligned}$$

It follows from $\varphi(M_1)$ being a Calabi hypersurface relative to the Calabi affine normalization Y_0 that

$$\det\Big(\frac{\partial \varphi}{\partial u_1}, \cdots, \frac{\partial \varphi}{\partial u_{n-1}}, Y_0\Big) \ne 0.$$

Thus

$$\det\Big(x, \frac{\partial x}{\partial t}, \frac{\partial x}{\partial u_1}, \cdots, \frac{\partial x}{\partial u_{n-1}}\Big) \ne 0,$$

which indicates that $x(M^n)$ is a centroaffine hypersurface. Hence, the Claim is demonstrated. More precisely, the following result can be verified easily.

Proposition 3.1 *The type* II *Calabi product of a point and the Calabi hypersurface M_1 is a locally strongly convex elliptic centroaffine hypersurface, and the centroaffine metric h induced by $-x$ is expressed as*

$$h = \mathrm{d}t^2 \oplus G. \tag{3.2}$$

The difference tensor K of $x(M^n)$ takes the following form:

$$K(\widetilde{T}, \widetilde{T}) = 2\widetilde{T}, \quad K\Big(\widetilde{T}, \frac{\partial x}{\partial u_i}\Big) = \frac{\partial x}{\partial u_i}, \quad 1 \le i \le n-1, \tag{3.3}$$

and the Tchebychev vector field of $x(M^n)$ satisfies

$$T = \frac{n+1}{n}\widetilde{T} + \frac{n-1}{n} \cdot \mathrm{e}^t(0, T^{M_1}), \tag{3.4}$$

where $\widetilde{T} := \frac{\partial x}{\partial t}$ is a unit vector field and T^{M_1} denotes the Tchebychev vector field of $\varphi(M_1)$. Moreover, $x(M^n)$ is flat (*resp. of parallel difference tensor*) *if and only if $\varphi(M_1)$ is flat* (*resp. of parallel Fubini-Pick tensor*).

Remark 3.1 It follows from (3.4) that the unit vector field $\widetilde{T}$ of M^n is not necessarily parallel to its Tchebychev vector field T. If it happens, then the immersion $\varphi : M_1 \to \mathbb{R}^n$ reduces to the locally strongly convex improper affine hypersphere.

From (3.1), one can obtain a locally strongly convex elliptic centroaffine hypersurface by a point and a lower dimensional Calabi hypersurface.

Example 3.1 Let us write

$$x = (x_1, \cdots, x_{n+1}) = (\mathrm{e}^t, \mathrm{e}^t y_1, \cdots, \mathrm{e}^t y_{n-1}, \mathrm{e}^t y_n + t\mathrm{e}^t).$$

For $n = 3$, if $\varphi(M_1)$ in (3.1) is chosen to be, respectively, the canonical Calabi surfaces $y_3 = -c_1 \ln y_1 + \frac{1}{2} y_2^2$ $(c_1 > 0)$, $y_3 = -c_1 \ln y_1 - c_2 \ln y_2$ $(c_1, c_2 > 0)$ and $y_3 = \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2$, then the centroaffine hypersurface, obtained as the type II Calabi product of a point and $\varphi(M_1)$, is locally centroaffinely equivalent to the hypersurfaces

$$x_4 = \frac{x_2^2}{2x_1} + x_1 \Big(\ln x_1 - \frac{c_1}{1+c_1} \ln x_3 \Big), \tag{3.5}$$

$$x_4 = x_1 \Big(\ln x_1 - \frac{c_1}{1+c_1+c_2} \ln x_2 - \frac{c_2}{1+c_1+c_2} \ln x_3 \Big), \tag{3.6}$$

$$x_4 = \frac{1}{2x_1}(x_2^2 + x_3^2) + x_1 \ln x_1, \tag{3.7}$$

respectively.

In fact, the centroaffine hypersurfaces defined by (3.5)–(3.7) are exactly the case of $n = 3$ for the canonical centroaffine hypersurfaces $M^{(3)}_{\alpha(v)}$, presented in [6, (3.9)], which can be obtained as the type II Calabi product of a point and the canonical Calabi hypersurfaces $Q(c_1, \cdots, c_r; n-1), 1 \le r \le n-2$, $Q(c_1, \cdots, c_{n-1}; n-1)$ and elliptic paraboloid, respectively. Here, the definition of Calabi hypersurfaces $Q(c_1, \cdots, c_r; n-1),\ 1 \le r \le n-1$ is given by [25, Example 3.1].

Next, the following decomposition theorem is the converse of Proposition 3.1.

Theorem 3.1 *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex elliptic centroaffine hypersurface. Assume that there exist orthogonal distributions $\mathcal{D}_1$ (of dimension 1, spanned by a unit vector field $\widetilde{T}$), $\mathcal{D}_2$ (of dimension $n-1$) with respect to the positive definite centroaffine metric h induced by $-x$ such that*

(i) *the unit vector field $\widetilde{T}$ is parallel with respect to the Levi-Civita connection of the centroaffine metric h;*

(ii) *the difference tensor takes the following form:*

$$K(\widetilde{T}, \widetilde{T}) = 2\widetilde{T}, \quad K(\widetilde{T}, V) = V, \quad \forall V \in \mathcal{D}_2.$$

Then $x : M^n \to \mathbb{R}^{n+1}$ can be locally decomposed as the type II *Calabi product of a point and a Calabi hypersurface $\varphi : M_1^{n-1} \to \mathbb{R}^n$ with Calabi metric $G = h|_{\mathcal{D}_2}$.*

Proof Firstly, for any vector $X \in TM$ and $V \in \mathcal{D}_2$, the item (i) indicates

$$\widehat{\nabla}_X \widetilde{T} = 0, \quad \widehat{\nabla}_X V \in \mathcal{D}_2. \tag{3.8}$$

It follows from de Rham decomposition theorem that (M^n, h) is locally isometric to $\mathbb{R} \times M_1^{n-1}$ such that $\widetilde{T}$ is tangent to $\mathbb{R}$ and $\mathcal{D}_2$ is tangent to M_1^{n-1}.

Secondly, assume that $\widetilde{T} = \frac{\partial}{\partial t}$ and $U_0 \in \mathbb{R}^{n+1}$ is a constant vector. Let

$$\varphi := (1+t)\mathrm{e}^{-t}x - t\mathrm{e}^{-t}\widetilde{T} - U_0, \quad \psi := \mathrm{e}^{-t}(\widetilde{T} - x). \tag{3.9}$$

It follows that

$$\begin{aligned} D_{\widetilde{T}}\varphi &= -t\mathrm{e}^{-t}x + (1+t)\mathrm{e}^{-t}\widetilde{T} - (1-t)\mathrm{e}^{-t}\widetilde{T} - t\mathrm{e}^{-t}D_{\widetilde{T}}\widetilde{T} \\ &= \mathrm{e}^{-t}(-tx + 2t\widetilde{T} - t(2\widetilde{T} - x)) = 0. \end{aligned} \tag{3.10}$$

Similarly,

$$\begin{aligned} D_{\widetilde{T}}\psi &= D_V\psi = 0, \\ \mathrm{d}\varphi(V) &= D_V\varphi = \mathrm{e}^{-t}V. \end{aligned} \tag{3.11}$$

The above relations imply that φ reduces to a map of M_1^{n-1} in $\mathbb{R}^{n+1}$ and ψ is a constant vector in $\mathbb{R}^{n+1}$. Moreover, denoting by ∇^1 the $\mathcal{D}_2$ component of induced connection ∇ and for any $V, \widetilde{V} \in \mathcal{D}_2$, we find that

$$\begin{aligned} D_V\mathrm{d}\varphi(\widetilde{V}) &= \mathrm{e}^{-t}D_V\widetilde{V} = \mathrm{e}^{-t}\Big(\nabla_V^1\widetilde{V} + h(\nabla_V\widetilde{V}, \widetilde{T})\widetilde{T} - h(V, \widetilde{V})x\Big) \\ &= \mathrm{d}\varphi(\nabla_V^1\widetilde{V}) + h(V, \widetilde{V})\mathrm{e}^{-t}(\widetilde{T} - x) \\ &= \mathrm{d}\varphi(\nabla_V^1\widetilde{V}) + h(V, \widetilde{V})\psi. \end{aligned} \tag{3.12}$$

Hence, the constant vector ψ is a relative normalization of the hypersurface M_1^{n-1} contained in an n-dimensional vector subspace of $\mathbb{R}^{n+1}$ with induced connection ∇^1 and positive definite relative metric

$$G(V, \widetilde{V}) = h(V, \widetilde{V}), \quad \forall V,\ \widetilde{V} \in \mathcal{D}_2. \tag{3.13}$$

Solving (3.9) for the immersion x, we have

$$x = \mathrm{e}^t\varphi + t\mathrm{e}^t\psi + \mathrm{e}^tU_0. \tag{3.14}$$

As the constant vector ψ is the relative normalization of φ, up to a centroaffine transformation, one may assume that φ lies in the space spanned by the last n coordinates of $\mathbb{R}^{n+1}$, whereas ψ lies in the direction of $(n+1)$-th coordinate with $\psi = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$. Thus, φ can be interpreted as an $(n-1)$-dimensional Calabi hypersurface equipped with the Calabi affine normalization $(0, \cdots, 0, 1) \in \mathbb{R}^n$. Since M^n is non-degenerate, x lies full in $\mathbb{R}^{n+1}$. Suppose that $U_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, then x can be written as

$$x = (\mathrm{e}^t, \mathrm{e}^t\varphi) + t\mathrm{e}^t\psi,$$

namely,

$$x = \mathrm{e}^t(1, \varphi_1, \cdots, \varphi_{n-1}, \varphi_n + t). \tag{3.15}$$

It completes the proof of Theorem 3.1.

We end this section by recalling some useful facts of the type I Calabi product given [3, Section 3, 13] and stating the relationship between the types I and II Calabi product.

Let $\widetilde{\psi} : M^n \to \mathbb{R}^{n+1}$ be the non-degenerate centroaffine hypersurface obtained as the type I Calabi product ($\lambda \neq 0, -1$) of a point and a locally strongly convex centroaffine hypersurface $\psi : M_2 \to \mathbb{R}^n$. Then, it follows from [3, Proposition 3.2] that the Tchebychev vector field of $\widetilde{\psi}(M^n)$ takes the following form:

$$T = \frac{\mathrm{sgn}\lambda \cdot (n-\lambda)}{n\sqrt{|\lambda|}}\widehat{T} + \frac{(n-1)(\lambda+1)}{n\lambda} \cdot \mathrm{e}^t (T^{M_2}, 0), \tag{3.16}$$

where $\widehat{T} := \frac{1}{\sqrt{|\lambda|}}\frac{\partial \widetilde{\psi}}{\partial t}$ is a unit vector field with respect to the centroaffine metric $\widetilde{h}$ shown in [3, (3.7)], and T^{M_2} is the Tchebychev vector field of $\psi(M_2)$. Moreover, we know from [3, (3.8)] that, by suitably selecting the constant λ in (1.3), the type I Calabi product centroaffine hypersurface is locally strongly convex. Conversely, if the type I Calabi product centroaffine hypersurface $\widetilde{\psi}(M^n)$ is locally strongly convex, then $\lambda < 0$ (resp. $\lambda > 0$) as $\widetilde{\psi}(M^n)$ is of elliptic (resp. hyperbolic) type. It follows that $-\mathrm{sgn}\lambda = \varepsilon$ holds in [3, (3.10)], where ε is chosen so that the centroaffine metric induced by $-\varepsilon\widetilde{\psi}$ is positive definite.

Thus, from the analysis above, we can get:

Remark 3.2 If a type I Calabi product centroaffine hypersurface is locally strongly convex, then the constants λ_1 and λ_2 in [3, (3.9)] satisfy $\lambda_1\lambda_2 - \lambda_2^2 = \varepsilon$ and $\lambda_1 \neq 2\lambda_2$. While the type II Calabi product centroaffine hypersurface is the case of $\lambda_1\lambda_2 - \lambda_2^2 = \varepsilon$ and $\lambda_1 = 2\lambda_2$, namely, $\frac{1}{2}\lambda_1 = \lambda_2 = \varepsilon = 1$ (see (3.3)). Hence, the type II Calabi product can be viewed as the complementation of the type I Calabi product.

4 Proofs of the Main Results

Here and after, if there is no additional explanation, we shall use the following indices' convention:

$$2 \leq i, j, k, \cdots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

Denote by $\{E_1, \cdots, E_n\}$ the local orthonormal frame field of a locally strongly convex centroaffine hypersurface (M^n, h) with $\widehat{\nabla} T = 0$ and vanishing Weyl curvature tensor. If $T \neq 0$, then we choose $E_1 = \frac{T}{|T|}$. It follows that $\widehat{\nabla} E_1 = 0$ and

$$R_{1\alpha\beta\gamma} = 0. \tag{4.1}$$

On the other hand, for any smooth tangent vector fields X, Y, Z, Weyl curvature tensor

$$W(X,Y)Z = \widehat{R}(X,Y)Z - \{\langle Y,Z\rangle P(X) - \langle X,Z\rangle P(Y) + \langle P(Y),Z\rangle X - \langle P(X),Z\rangle Y\}$$

vanishes identically on M^n means that the Riemannian curvature tensor can be expressed as

$$\widehat{R}(X,Y)Z = \langle Y,Z\rangle P(X) - \langle X,Z\rangle P(Y) + \langle P(Y),Z\rangle X - \langle P(X),Z\rangle Y, \tag{4.2}$$

where P is the Schouten tensor of $(1,1)$ type and $\langle\cdot,\cdot\rangle := h(\cdot,\cdot)$. For any i, it concludes from (4.1)–(4.2) that

$$0 = \widehat{R}(E_i, E_j)E_1 = \langle P(E_j), E_1\rangle E_i - \langle P(E_i), E_1\rangle E_j, \quad j \neq i,$$

which implies

$$\langle P(E_i), E_1\rangle = 0, \quad \forall i. \tag{4.3}$$

Similarly, for any i and j, one can get

$$0 = \widehat{R}(E_1, E_i)E_j = \delta_{ij}P(E_1) + \langle P(E_i), E_j\rangle E_1 - \langle P(E_1), E_j\rangle E_i. \tag{4.4}$$

Taking $i \neq j$ in (4.4), we obtain

$$\langle P(E_i), E_j\rangle = \langle P(E_1), E_j\rangle = 0.$$

Thus $\{E_1, \cdots, E_n\}$ as above is the eigenvector field of the self-adjoint operator P. Without loss of generality, suppose that

$$P(E_\alpha) = \mu_\alpha E_\alpha,$$

and denote by $\mu_1, \cdots, \mu_s$ the distinct eigenvalues for the tensor P of multiplicity $n_1, \cdots, n_s$, respectively. It follows from taking $i = j$ in (4.4) that

$$0 = \mu_1 + \mu_i,$$

which means

$$\mu_2 = \cdots = \mu_n = -\mu_1.$$

In fact, it is easy to see from Lemma 2.1 that the following lemma holds.

Lemma 4.1 *Let $M^n (n \geq 3)$ be a locally strongly convex centroaffine hypersurface with vanishing centroaffine shape operator and Weyl curvature tensor. If $T \neq 0$, then the number of distinct eigenvalues of the Schouten operator P is at most 2, namely, $s \leq 2$. More precisely,*

(i) *if $s = 1$, then 0 is the only eigenvalue of P and M^n is flat. It follows that M^n is an open part of a canonical centroaffine hypersurface*;

(ii) *if $s = 2$, let μ_1, μ_2 be the two distinct eigenvalues for P of multiplicity 1 and $n-1$, respectively, then $\mu_2 = -\mu_1 \neq 0$.*

Proof of Theorem 1.2 The proof of this Theorem is divided into two steps.

Step 1 From Lemma 4.1, for an n-dimensional locally strongly convex centroaffine hypersurface (M^n, h) with $\widehat{\nabla}T = 0$ $(T \neq 0)$ and vanishing Weyl curvature tensor, one only needs to deal with two cases, namely, $s = 1$ and $s = 2$. Obviously, case (ii-1) in Theorem 1.2 occurs as $s = 1$.

In the following, we are going to discuss $s = 2$. For this case, it is easy to conclude from (4.2) that

$$\widehat{R}(E_i, E_j)E_k = 2\mu_2(\delta_{jk}E_i - \delta_{ik}E_j),$$

i.e.,

$$R_{ijkl} = 2\mu_2(\delta_{jl}\delta_{ik} - \delta_{il}\delta_{jk}). \tag{4.5}$$

Direct computations get

$$R_{1\alpha} = 0, \quad R_{ij} = 2(n-2)\mu_2\delta_{ij} \tag{4.6}$$

and

$$R = 2(n-1)(n-2)\mu_2 \neq 0. \tag{4.7}$$

Obviously, the hypersurfaces occurring in this case are non-flat.

On the other hand, the Ricci identity and (4.1) indicate

$$K_{\alpha\beta\gamma,\delta 1} - K_{\alpha\beta\gamma,1\delta} = \Sigma K_{\zeta\beta\gamma}R_{\zeta\alpha\delta 1} + \Sigma K_{\alpha\zeta\gamma}R_{\zeta\beta\delta 1} + \Sigma K_{\alpha\beta\zeta}R_{\zeta\gamma\delta 1} = 0.$$

It follows from the Codazzi equation (2.7) that $K_{1\alpha\beta,\gamma\delta}$ are totally symmetric for all indices. For any j, by taking $i \neq j$, we have

$$0 = K_{11i,ij} - K_{11i,ji} = 2\Sigma K_{1i\alpha}R_{\alpha 1ij} + \Sigma K_{11\alpha}R_{\alpha iij} = K_{11j}R_{jiij}.$$

From (4.5), one can get

$$K_{11j} = 0. \tag{4.8}$$

Similarly, for any $i \neq j$,

$$0 = K_{1ii,ij} - K_{1ii,ji} = 2\Sigma K_{1i\alpha}R_{\alpha iij} + \Sigma K_{ii\alpha}R_{\alpha 1ij} = 2K_{1ij}R_{jiij}.$$

Then

$$K_{1ij} = 0. \tag{4.9}$$

Finally, for any $i \neq j$,

$$0 = K_{1ij,ij} - K_{1ij,ji} = (K_{1ii} - K_{1jj})R_{ijij},$$

which shows

$$K_{1ii} = K_{1jj}. \tag{4.10}$$

Accordingly, the following result can be obtained from (4.8)–(4.10).

Lemma 4.2 *With respect to the local orthonormal frame field $\{E_1, \cdots, E_n\}$ as above, the difference tensor takes the following form*

$$K(E_1, E_1) = \lambda_1 E_1, \quad K(E_1, E_i) = \lambda_2 E_i. \tag{4.11}$$

In addition, the fact $T = |T|E_1$ shows that

$$\lambda_1 + (n-1)\lambda_2 = n|T| = \text{const.} > 0. \tag{4.12}$$

A straightforward computation shows

$$R_{11} = (n-1)(\varepsilon + \lambda_2^2 - \lambda_1\lambda_2), \tag{4.13}$$

where $\varepsilon = \pm 1$. It follows from (4.6) that

$$\varepsilon + \lambda_2^2 - \lambda_1\lambda_2 = 0, \tag{4.14}$$

which implies

$$\lambda_1 \neq \lambda_2, \quad \lambda_2 \neq 0. \tag{4.15}$$

On the other hand, note that

$$(\widehat{\nabla} K)(E_i, E_1, E_1) = \widehat{\nabla}_{E_i} K(E_1, E_1) - 2K(\widehat{\nabla}_{E_i} E_1, E_1) = E_i(\lambda_1)E_1 \tag{4.16}$$

and

$$(\widehat{\nabla} K)(E_1, E_i, E_1) = \widehat{\nabla}_{E_1} K(E_i, E_1) - K(\widehat{\nabla}_{E_1} E_i, E_1) = E_1(\lambda_2)E_i. \tag{4.17}$$

Taking inner product of the right-hand sides of (4.16) and (4.17) with E_1 (resp. E_i) and using the Codazzi equation, we obtain

$$E_1(\lambda_2) = E_i(\lambda_1) = 0. \tag{4.18}$$

It follows from (4.14) and (4.18) that

$$E_1(\lambda_1) = E_1\Big(\lambda_2 + \frac{\varepsilon}{\lambda_2}\Big) = 0, \tag{4.19}$$

which shows that both λ_1 and λ_2 are constants. Associated with (4.14), we have if $\lambda_1 = 2\lambda_2$ then, up to a direction of E_1, $\lambda_1 = 2$ and $\lambda_2 = \varepsilon = 1$. Thus, one can get either $\frac{1}{2}\lambda_1 = \lambda_2 = \varepsilon = 1$ or $\lambda_1 \neq 2\lambda_2$, $\varepsilon + \lambda_2^2 - \lambda_1\lambda_2 = 0$.

Case $\mathfrak{C}_1$ $\lambda_2(\lambda_1 - \lambda_2) = \varepsilon$ and $\lambda_1 = 2\lambda_2$, i.e., $\frac{1}{2}\lambda_1 = \lambda_2 = \varepsilon = 1$.

For this case, Theorem 3.1 and Lemma 4.2 imply (M^n, h) can be locally decomposed as the type II Calabi product of a point and an $(n-1)$-dimensional Calabi hypersurface N^{n-1} with Calabi metric $G = h|_{\mathcal{D}_2}$, where $\mathcal{D}_2 = \mathrm{span}\{E_2, \cdots, E_n\}$. More precisely, Remark 3.1 indicates N^{n-1} reduces to a locally strongly convex improper affine hypersphere. Proposition 3.1 and the fact (M^n, h) is non-flat imply (N^{n-1}, G) is also not flat.

Case $\mathfrak{C}_2$ $\lambda_2(\lambda_1 - \lambda_2) = \varepsilon$ and $\lambda_1 \neq 2\lambda_2$.

Similarly, by the fact λ_1, λ_2 in Lemma 4.2 are constants and [3, Theorem 3.3], (M^n, h) can be locally decomposed as the type I Calabi product of a point and an $(n-1)$-dimensional locally strongly convex centroaffine hypersurface $\psi : \overline{N}^{n-1} \to \mathbb{R}^n$ with centroaffine metric

$$\overline{h} = \lambda_2(2\lambda_2 - \lambda_1)h|_{\mathcal{D}_2} \tag{4.20}$$

induced by the position vector ψ. It follows from (3.16) that $\overline{N}^{n-1}$ reduces to a locally strongly convex proper affine hypersphere centered at origin. [3, Proposition 3.2] and the fact (M^n, h) is non-flat imply $(\overline{N}^{n-1}, \overline{h})$ is also not flat.

Recall that, for the locally strongly convex centroaffine hypersurface $(\overline{N}^{n-1}, \overline{h})$, one can choose $\varepsilon = \pm 1$ such that the centroaffine metric $\overline{h}$ is positive definite. Namely, if $\lambda_2(2\lambda_2 - \lambda_1)$ is negative (resp. positive) then we choose $\varepsilon = 1$ (resp. $\varepsilon = -1$) and say $\overline{N}^{n-1}$ is of elliptic (resp. hyperbolic). To ensure the centroaffine hypersurface obtained as the type I Calabi product of a point and $\overline{N}^{n-1}$ is locally strongly convex, one shall restrict the constant λ in (1.3) as following:

(i) if $\overline{N}^{n-1}$ is elliptic, then $\lambda < -1$;

(ii) if $\overline{N}^{n-1}$ is hyperbolic, then $\lambda > -1$ and $\lambda \neq 0$,

where [3, (3.8)] is used by taking $n_1 = n - 1$. Additionally, it follows from (3.16) and $T \neq 0$ that $\lambda \neq n$ holds in this case.

Thus, taking account what we have discussed above, we get the following result.

Proposition 4.1 *Let $M^n (n \geq 3)$ be a locally strongly convex centroaffine hypersurface with $\widehat{\nabla} T = 0$ and vanishing Weyl curvature tensor. If $T \neq 0$, then*

(i) *M^n is an open part of a canonical centroaffine hypersurface*; *or*

(ii) *M^n is obtained as the type* II *Calabi product of a point and an $(n-1)$-dimensional non-flat locally strongly convex improper affine hypersphere*; *or*

(iii) *M^n is obtained as the type* I *Calabi product* ($\lambda < -1$; *resp.* $\lambda > -1, \lambda \neq 0, n$) *of a point and an $(n-1)$-dimensional non-flat locally strongly convex proper* (*elliptic*; *resp. hyperbolic*) *affine hypersphere centered at origin.*

Step 2 Furthermore, we proceed to discuss cases (ii)–(iii) of Proposition 4.1 as $n \geq 4$ in this step.

Firstly, from case (ii) of Proposition 4.1, (M^n, h) can be obtained as the type II Calabi product of a point and an $(n-1)$-dimensional non-flat Calabi hypersurface N^{n-1} with vanishing Tchebychev vector field. In this case, by employing (2.11), (3.13) and (4.5), we have the sectional curvature of N^{n-1} $(n \geq 4)$ is constant. Thus, the following equation [24, (2.10)]

$$(n-1)(n-2)J^N = R^N$$

shows J^N is also a constant. Here and after, denote by $\star^N$ the geometric invariants of Calabi hypersurface N^{n-1}. It follows from [24, (2.9)] that

$$R^N_{ij} = \sum_{k,l} A^N_{ikl} A^N_{jkl}.$$

Then, for the Calabi hypersurface N^{n-1}, [24, (2.12)] becomes

$$\frac{(n-1)(n-2)}{2} \Delta^N J^N = \Sigma (A^N_{ijk,l})^2 + \Sigma (R^N_{ijkl})^2 + \Sigma (R^N_{ij})^2 > 0,$$

where we have used N^{n-1} is non-flat in the last step. This is a contradiction to $J^N = \text{const}$.

Secondly, by case (iii) of Proposition 4.1, (M^n, h) can be obtained as the type I Calabi product of a point and a non-flat locally strongly convex proper affine hypersphere $\overline{N}^{n-1}$ centered at origin. Similarly, combining (2.11), (4.5) and (4.20), we have the sectional curvature of $\overline{N}^{n-1}$ $(n \geq 4)$ is a non-zero constant. It follows from [12, Theorem 3.11] that $\overline{N}^{n-1}$ must be contained in either the hyperellipsoid

$$y_1^2 + \cdots + y_n^2 = c^2$$

or the hyperboloid

$$y_1^2 + \cdots + y_{n-1}^2 - y_n^2 = -c^2,$$

where $c > 0$. Hence, by the type I Calabi product defined by (1.3), M^n is locally centroaffinely equivalent to the locally strongly convex centroaffine hypersurfaces

$$(x_1^2 + \cdots + x_n^2)^\lambda x_{n+1}^2 = 1, \quad \lambda < -1 \tag{4.21}$$

or

$$(x_n^2 - x_1^2 - \cdots - x_{n-1}^2)^\lambda x_{n+1}^2 = 1, \quad \lambda > -1,\ \lambda \neq 0,\ n. \tag{4.22}$$

The proof of Theorem 1.2 is complete finished.

Proof of Corollary 1.1 As is well known, the Weyl curvature tensor vanishes automatically on 3-dimensional Riemannian manifolds. Accordingly, Proposition 4.1 still holds and we only need to consider the last two cases. In the following, we are going to prove the Gauss curvatures of the non-flat locally strongly convex improper affine sphere N^2 and the non-flat locally strongly convex proper affine sphere $\overline{N}^2$ centered at origin are constants, which is equivalent to the eigenvalues of the Schouten tensor are constants.

Denote by $\{\omega_\alpha^\beta\}$ the Levi-Civita connection forms with respect to the orthonormal frame field $\{E_1, E_2, E_3\}$, where $E_1 = \frac{T}{|T|}$ as before. Recall from Lemma 4.1 and suppose that

$$P_{11} = -P_{22} = -P_{33} =: \mu\ (\neq 0). \tag{4.23}$$

Noting from

$$\Sigma P_{\alpha\beta,\gamma}\omega^\gamma = \mathrm{d}(P_{\alpha\beta}) + (P_{\alpha\alpha} - P_{\beta\beta})\omega_\alpha^\beta,$$

the fact $\omega_1^\beta \equiv 0$ and (4.23) that

$$P_{\alpha\beta,\gamma} = 0, \quad \alpha \neq \beta. \tag{4.24}$$

Recall that M^3 is locally conformally flat means P is a Codazzi tensor. It follows from (4.24) that

$$-E_1(\mu) = E_1(P_{22}) = P_{22,1} = P_{12,2} = 0.$$

Similarly, for $i > 1$, one can get

$$E_i(\mu) = E_i(P_{11}) = 0.$$

Hence

$$\mu = \text{const.} \neq 0. \tag{4.25}$$

Then the Gauss curvatures of N^2 and $\overline{N}^2$ are non-zero constants, and the remaining process closely follows Step 2 of Theorem 1.2.

Thus, the proof of Corollary 1.1 is finished.

Proof of Corollary 1.2 Let M^3 be a complete locally strongly convex centroaffine hypersurface with vanishing centroaffine shape operator. As $|T|$ is a constant, we consider the following two subcases.

Case $\mathfrak{C}_1$ $|T| = 0$.

In this case, the completeness of M^3 shows it is either a hyperellipsoid centered at origin, or a complete hyperbolic affine hypersphere centered at origin. This proves case (i) of Corollary 1.2.

Case $\mathfrak{C}_2$ $|T| = \text{const.} > 0$.

In this case, Proposition 4.1 is still available.

Firstly, case (i) of Proposition 4.1 indicates M^3 is a locally strongly convex canonical centroaffine hypersurface with $T \neq 0$. Obviously, such hypersurfaces are included in the three types of hypersurfaces given in Theorem 1.1. In the following, we are going to discuss the completeness of the canonical centroaffine hypersurfaces shown in Theorem 1.1 as $n = 3$.

(i) For the hypersurfaces in case (i) of Theorem 1.1, by taking $x_i = \mathrm{e}^{u_i}$ $(2 \leq i \leq 4)$, we get the components of flat centroaffine metric are constants in terms of the new coordinates (u_2, u_3, u_4), where $-\infty < u_i < +\infty$, $2 \leq i \leq 4$, see the proof of [6, Claim 3.1] for details. Therefore, the hypersurfaces in case (i) of Theorem 1.1 are complete with respect to its centroaffine metric. Additionally, noting that $x_1x_2x_3x_4 = 1$ is a hyperbolic affine hypersphere included in case (i) of Corollary 1.2. Thus, case (ii-1) of Corollary 1.2 is demonstrated.

(ii) For the hypersurfaces in case (ii) of Theorem 1.1, by taking $x_2 = \mathrm{e}^{u_2}$, $x_3 = \mathrm{e}^{u_3} \sin u_4$ and $x_4 = \mathrm{e}^{u_3} \cos u_4$, we have the components of flat centroaffine metric $h = h_{ij}\mathrm{d}u_i\mathrm{d}u_j$ are also constants in terms of the new coordinates (u_2, u_3, u_4), where $-\infty < u_i < +\infty$ $(2 \leq i \leq 3)$ and $k\pi - \frac{\pi}{2} < u_4 < k\pi + \frac{\pi}{2}$, $k \in N$. For more details see the proof of [6, Claim 3.2]. Consider the curve

$$u_2(t) = u_3(t) = 0, \quad u_4(t) = t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

whose centroaffine arc length is given by

$$l = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{h_{44}(t)}\mathrm{d}t < \infty.$$

Thus, the hypersurfaces in case (ii) of Theorem 1.1 are not complete with respect to its centroaffine metric.

(iii) Noting that the centroaffine hypersurfaces in case (iii) of Theorem 1.1 can be obtained as the type II Calabi product of a point and a canonical Calabi surface, see Example 3.1. Note that elliptic paraboloid, the Calabi surfaces $Q(c_1;2)$ and $Q(c_1,c_2;2)$ are all Calabi complete, for details see [25], which indicates that the centroaffine hypersurfaces in case (iii) of Theorem 1.1 are centroaffine complete. This proves case (ii-2) of Corollary 1.2.

Secondly, case (ii) of Proposition 4.1 shows that (M^3,h) can be obtained as the type II Calabi product of a point and a non-flat Calabi surface N^2 with vanishing Tchebychev vector field. It follows from the relationship of metrics shown in Theorem 3.1 that N^2 is complete with respect to its Calabi metric G. The fact complete Calabi surface with vanishing Tchebychev vector field is an elliptic paraboloid shows that N^2 is flat, which is a contradiction.

Finally, it follows from case (iii) of Proposition 4.1 and the relationship of metrics shown in (4.20) that $\overline{N}^2$ is a complete non-flat locally strongly convex proper affine 2-sphere centered at origin. Then, the last two cases (iii) and (iv) of Corollary 1.2 are demonstrated.

Thus, the proof of Corollary 1.2 is finished.

Acknowledgements The authors would like to express their heartfelt thanks to Prof. Zejun Hu for his helpful discussions and suggestions during the preparation of this paper. We also appreciate the time and effort invested by the referees in reviewing our work.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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