Chin. Ann. Math. Ser. B
46(2), 2025, 181–200
DOI: 10.1007/s11401-025-0010-7

Chinese Annals of Mathematics, Series B

A Blow-up Result for an Extensible Beam with Degenerate Nonlocal Nonlinear Damping

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Abstract The results of this work deal with the existence and blow up of solutions for the following damped extensible beam with degenerate nonlocal damping and source term $u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + \|\Delta u\|^{2\alpha}|u_t|^\gamma u_t = |u|^\rho u$. It is regarded as the second part of the paper by Narciso et al. (in 2023), where global existence, uniqueness and asymptotic stability of strong solutions were obtained for regular initial data in the case $|u|^\rho u \equiv 0$.

Keywords Beam equation, Blow-up, Degenerate damping, Global and local solution
2020 MR Subject Classification 35L75, 35B35, 35B40, 35B44, 35G61

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with regular boundary $\Gamma = \partial\Omega$. This work is dedicated to the study of the existence and blow-up of solutions of the following class of extensible beams

$$\begin{cases} u_{tt} + \Delta^2 u - M(\|\nabla u(t)\|^2)\Delta u + \|\Delta u(t)\|^{2\alpha}|u_t|^\gamma u_t = |u|^\rho u \quad \text{in } \Omega \times \mathbb{R}^+, \\ u|_\Gamma = \Delta u|_\Gamma = 0, \quad u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \tag{1.1}$$

where $\alpha > 0, \gamma \geq 0$ and $\rho > 0$, the function $M(\|\nabla u\|^2) \approx -\varpi + \|\nabla u\|^{2\zeta}$, $\varpi, \zeta > 0$, corresponds to a nonlocal function of extensibility which appears in the context of extensible beams (see e.g. Woinowsky-Krieger [46] and Berger [5]), and $\|\cdot\|$ stands for the norm in $L^2(\Omega)$. The great novelty of the article is to consider the dissipative term given by the product of a nonlocal degenerate term by a nonlinear function. This type of dissipativity is connected to the class of nonlocal damping suggested by Balakrishnan-Taylor [1]. For more details on the model formulation see [36, Section 1.1].

This work is the second part of Narciso et al. [36] where we consider model (1.1) without the presence of the source term $|u|^\rho u$. Existence and uniqueness of regular global solutions and stability for regular initial data were obtained. The main result in [36] was the stability result which, due to the difficulties generated by the degenerate nonlocal term, was obtained by a contradiction method without explaining the decay rate. As a complement to the results

Manuscript received December 15, 2022. Revised October 16, 2023.
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obtained in [36], in this work, we studied the changes generated in the results by the presence of the term force $|u|^\rho u$ in the model. Our main aim is to discuss the well-posedness of problem (1.1), on a regular space $\mathcal{H}_2$. More specifically, we studied the existence and uniqueness of both global and local solutions and also blow-up of the solutions.

Works associated with wave or plate models that consider dissipations given by the product of a nonlocal degenerate term by a dissipative term are recent in the literature. A pioneering work in this sense was the paper by Cavalcanti et al. [8] who considered the following wave model

$$u_{tt} - \Delta u + \|\nabla u(t)\|^2 u_t = 0 \quad \text{in } \Omega \times \mathbb{R}^+. \tag{1.2}$$

The authors studied that the well-posedness and stability results were established through contradiction arguments for regular initial data taken in bounded sets. Afterwards, Cavalcanti et al. [7] considered the presence of a degenerate nonlocal damping for the following extensible beam model

$$u_{tt} + \Delta^2 u - M(\|\nabla u(t)\|^2)\Delta u + \|\Delta u(t)\|^2 A u_t = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ and $A = -\Delta$ or $A = I$. Also using arguments of contradiction stability results were obtained for regular initial data taken in bounded sets. The contradiction arguments were an appropriate way to show stability for problems with this class of damping because techniques that are standard in the study of stability of second order evolution equations are not applicable in this situation. For works dealing with nondegenerate nonlocal damping, see [9, 11–12, 17, 24, 26, 28, 30, 34–35, 37] and its references. A model associated with (1.3) in the case where the damping coefficient is dependent on the linear energy of the system was treated in [27].

When in (1.2) the nonlocal term $\|\nabla u(t)\|^2$ is replaced by a polynomial term of the form $|u|^r$, (1.2) is associated with the well-known polynomially-damped wave equation studied extensively in the literature. See for example [43–44]. In this context, it is important to mention the work of Barbu et al. [4] which considered the following wave model

$$u_{tt} - \Delta u + |u|^k \partial j(u_t) = |u|^{p-1} u \quad \text{in } \Omega \times \mathbb{R}^+,$$

where j is a continuous convex function defined on $\mathbb{R}$ and ∂j is its sub-differential operator (in particular, if $j(s) = \frac{1}{\gamma+2}|s|^{\gamma+2}$, then $\partial j(s) = |s|^\gamma s$). Under suitable conditions on function j and parameters k, γ and p, several results on the existence of global solutions, uniqueness, nonexistence and propagation of regularity are obtained. With further restrictions on the parameters they prove the existence and uniqueness of a global weak solution. In addition, they prove a result on the nonexistence of global weak solutions to the equation whenever the exponent p is greater than the critical value $k + m$, and the initial energy is negative. Recently, this type of dissipation $|u|^r\partial j(u_t)$ was considered for an extensible beam model by Ekinci and Pişkin [20]. They prove the nonexistence of global solutions with arbitrary positive initial energy.

This kind of (1.1) has its origin in the canonical model introduced by Woinowsky-Krieger [46] which arises in the dynamic bucking of a hinged extensible beam of the length L whose ends are attached at a fixed distance

$$\partial_{tt}u+\frac{EI}{\rho}\partial_{xxxx}u-\Big[\frac{H}{\rho}+\frac{EA}{2\rho L}\int_0^L|\partial_x u|^2\mathrm{d}x\Big]\partial_{xx}u=0, \tag{1.4}$$

where E, I, ρ, H and A denote, respectively, the Young's modulus, the cross sectional moment of inertia, the mass density, the tension in the rest position and the cross-sectional area. The modeling aspects were also discussed by Berger [5] and Eisley [19]. One of the first mathematical analysis for global existence and asymptotic behavior of these extensible beams was investigated by Ball [2–3], Dickey [16] and Medeiros [33]. Later, it was extensively studied by several researchers in different contexts (see [6, 10, 13–15, 18, 20–23, 25, 29, 31–32, 38–42, 45, 47–48]).

1.1 Organization of the paper

Our paper is organized as follows: In Section 2, we establish the existence and uniqueness of global regular solution for appropriate small initial data. In Section 3, we prove the existence and uniqueness of a local solution without restriction on the initial data. We end this work by proving that for appropriate conditions on the initial data and exponents α, γ, ρ, the solutions blow up in finite time.

2 Existence and Uniqueness of Global Solution for Small Initial Data

This section is dedicated to the existence of a unique global solution to problem (1.1) under conditions of small initial data.

2.1 Notation and statement of results

We begin by introducing some notation that will be used throughout this work. In order, with respect to the boundary condition $u=\Delta u=0$, we define $W_0=L^2(\Omega)$,

$$W_1=H_0^1(\Omega)\quad\text{and}\quad W_m=\begin{cases}H^2(\Omega)\cap H_0^1(\Omega), & \text{if } m=2,\\ \{u\in H^m(\Omega)\cap H_0^1(\Omega);\ \Delta u\in H_0^1(\Omega)\}, & \text{if } m=3,4.\end{cases} \tag{2.1}$$

Here the notation $(\,\cdot\,,\,\cdot\,)$ stands for L^2-inner product and $\|\cdot\|_p$ denotes L^p-norm. By simplicity we will denote the standard $L^2(\Omega)$ norm by $\|\cdot\|=\|\cdot\|_2$. Thus, $\|\nabla\cdot\|$ and $\|\Delta\cdot\|$ represent the norms in W_1 and W_2, respectively. Denoting by $\lambda_1>0$ the first eigenvalue of the bi-harmonic operator Δ^2 with boundary condition $(1.1)_2$ then

$$\lambda_1\|u\|^2\le\|\Delta u\|^2,\quad \lambda_1^{\frac{1}{2}}\|\nabla u\|^2\le\|\Delta u\|^2,\quad \forall\, u\in W_2. \tag{2.2}$$

We also consider the following phase spaces $\mathcal{H}_j = W_{j+2} \times W_j$, $j = 0, 1, 2$, equipped with the following standardized norms

$$\begin{aligned}
&\|(u,v)\|^2_{\mathcal{H}_0} = \|\Delta u\|^2 + \|v\|^2, \quad (u,v) \in \mathcal{H}_0 = W_2 \times W_0,\\
&\|(u,v)\|^2_{\mathcal{H}_1} = \|\nabla(\Delta u)\|^2 + \|\nabla v\|^2, \quad (u,v) \in \mathcal{H}_1 = W_3 \times W_1,\\
&\|(u,v)\|^2_{\mathcal{H}_2} = \|\Delta^2 u\|^2 + \|\Delta v\|^2, \quad (u,v) \in \mathcal{H}_2 = W_4 \times W_2.
\end{aligned}$$

From (2.2), we have

$$\|(u,v)\|^2_{\mathcal{H}_0} \leq \frac{1}{\lambda_1^{\frac{1}{2}}}\|(u,v)\|^2_{\mathcal{H}_1} \leq \frac{1}{\lambda_1}\|(u,v)\|^2_{\mathcal{H}_2}, \quad (u,v) \in \mathcal{H}_2. \tag{2.3}$$

To investigated the existence, uniqueness and regularity of solutions for the initial value problem (1.1) we assume the following hypotheses.

Assumption 2.1 (I) $M \in C^1([0,\infty))$ *with* $M(\tau) \geq -\varpi$ *for all* $\tau \geq 0$, *where* $0 \leq \varpi < \lambda_1^{\frac{1}{2}}$.
(II) *The exponents* ρ *and* γ *satisfy the following growth conditions*

$$\begin{aligned}
&\rho > 0, \quad \text{if } 1 \leq n \leq 4 \quad \text{or} \quad 0 < \rho \leq \frac{4}{n-4}, \quad \text{if } n \geq 5,\\
&\gamma \geq 0, \quad \text{if } 1 \leq n < 3 \quad \text{or} \quad 0 \leq \gamma \leq \frac{2}{n-2}, \quad \text{if } n \geq 3.
\end{aligned}$$

Note that Assumption 2.1(II) implies that $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$ and $W_1 \hookrightarrow L^{2(\gamma+1)}(\Omega)$.

The energy associated with problem (1.1) is given by

$$\mathcal{E}_U(t) = \frac{1}{2}\|U(t)\|^2_{\mathcal{H}_0} + \frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) - \frac{1}{\rho+2}\|u(t)\|^{\rho+2}_{\rho+2}, \tag{2.4}$$

where $U = (u, u_t)$ and $\widehat{M}(\tau) = \int_0^\tau M(s)\mathrm{d}s$.

2.2 Assumption on initial data

Let us consider a heuristic method in order to obtain an appropriate hypothesis on the initial data (u_0, u_1). Multiplying (1.1) by u_t and integrating over Ω we obtain the following equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U(t) + \|\Delta u^k(t)\|^{2\alpha}\|u_t(t)\|^{\gamma+2}_{\gamma+2} = 0. \tag{2.5}$$

We define the functional $J : W_2 \to \mathbb{R}$ by

$$J(\phi) := \frac{1}{2}\|\Delta\phi\|^2 + \frac{1}{2}\widehat{M}(\|\nabla\phi\|^2) - \frac{1}{\rho+2}\|\phi\|^{\rho+2}_{\rho+2}.$$

From Assumption 2.1(I) and immersion $W_2 \hookrightarrow W_1$, we get

$$\frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) = \frac{1}{2}\int_0^{\|\nabla u(t)\|^2} M(s)\mathrm{d}s \geq \frac{-\varpi}{2}\|\nabla u(t)\|^2 \geq \frac{-\varpi}{2\lambda_1^{\frac{1}{2}}}\|\Delta u(t)\|^2.$$

From immersion $W_2 \hookrightarrow L^{\rho+2}(\Omega)$, we have

$$\|u(t)\|_{\rho+2}^{\rho+2} \leq \varrho \|\Delta u(t)\|^{\rho+2}.$$

Then, taking $\omega := 1 - \frac{\varpi}{\lambda_1^{\frac{1}{2}}} > 0$, we obtain

$$J(u(t)) \geq \frac{\omega}{2}\|\Delta u(t)\|^2 - \frac{\varrho}{\rho+2}\|\Delta u(t)\|^{\rho+2} =: P(\|\Delta u(t)\|).$$

Note that, the polynomial function $P(\lambda) = \frac{\omega}{2}\lambda^2 - \frac{\varrho}{\rho+2}\lambda^{\rho+2}$ has roots in zero and $\left(\frac{\omega(\rho+2)}{2\varrho}\right)^{\frac{1}{\rho}}$. It has a minimum in zero and a maximum in $d = \left(\frac{\omega}{\varrho}\right)^{\frac{1}{\rho}}$. It is easy to see that $P(s)$ is increasing in the interval $[0, d]$ from its minimum zero to its maximum $\frac{\omega\rho}{2(\rho+2)}d^2$ assumed at d. Thus, for each $0 < \mu < \frac{\omega\rho}{2(\rho+2)}d^2$ there exists a unique $0 < \beta < d$ such that $P(\beta) = \mu$.

2.3 Global solution

We will use the following definition of a regular (strong) solution to problem (1.1).

Definition 2.1 (Regular solution) *A function $u(t) \in C([0,T], \mathcal{H}_0)$ possessing the properties $u(0) = u_0$ and $u_t(0) = u_1$ is said to be regular solution to problem* (1.1) *on the interval* $[0,T]$*, if and only if*

(1) $U = (u, u_t) \in L^\infty([0,T], \mathcal{H}_2)$, $u_{tt} \in L^\infty([0,T], W_0)$,

(2) (1.1) *is satisfied in W_0' for almost all $t \in [0,T]$.*

We are now in a position to state the following theorem of existence of global regular solution. In order, we define the open bounded set

$$\mathcal{V}_d := \Big\{U = (u, v) \in \mathcal{H}_0 \mid \mathcal{E}_U < \frac{\omega\rho}{2(\rho+2)}d^2 \text{ and } \|\Delta u\| < d\Big\}.$$

Theorem 2.1 (Global solution) *We assume the Assumption* 2.1 *holds with $(u_0, u_1) \in \mathcal{V}_d \cap \mathcal{H}_2$ with $\alpha \geq \frac{1}{2}$. Then problem* (1.1) *has a regular solution u according to the Definition* 2.1.

Proof The proof relies on the Faedo-Galerkin method, where we use compactness arguments.

Approximate problem Let us consider the spectral problem

$$(\Delta\omega_j, \Delta v) = \lambda_j(\omega_j, v) \quad \text{for all } v \in W_2 \text{ and } j = 1, 2, \cdots$$

with boundary condition $u = \Delta u = 0$. We represent by $V_k = \text{span}\{\omega_1, \cdots, \omega_k\}$ the subspace of W_2 generated by vectors $\omega_1, \cdots, \omega_k$. For every $k \in \mathbb{N}$, we can find a function

$$u^k(t) = \sum_{j=1}^{k} y_{jk}(t)\,\omega_j, \quad 0 \leq t \leq T,$$

which is a solution to the approximate ODE system

$$
\begin{aligned}
&(u_{tt}^k(t),\omega_j)+(\Delta u^k(t),\Delta\omega_j)+M(\|\nabla u^k(t)\|^2)(\nabla u^k,\nabla\omega_j)\\
&+\|\Delta u^k(t)\|^{2\alpha}(|u_t^k|^\gamma u_t^k,\omega_j)=(|u^k|^\rho u^k,\omega_j)
\end{aligned}
\tag{2.6}
$$

on $[0,t_k)$, $t_k>0$, $1\le j\le k$, with initial condition

$$
(u^k(0),u_t^k(0))=(u_{0k},u_{1k})\to(u_0,u_1) \tag{2.7}
$$

by using standard methods in ODE. We must obtain estimates to extend the solution to the interval $[0,T]$.

A priori estimates

Estimate I We first consider the approximate system (2.6) with

$$
U_0^k=(u_{0k},u_{1k})\to(u_0,u_1)=U_0 \quad \text{strongly in } \mathcal{V}_d\cap\mathcal{H}_0.
$$

Multiplying the approximate equation (2.6) by $y'_{jk}(t)$ with $1\le j\le k$ and taking the sum from $j=1$ to k, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{U^k}(t)+\|\Delta u^k(t)\|^{2\alpha}\|u_t^k(t)\|_{\gamma+2}^{\gamma+2}=0. \tag{2.8}
$$

Integrating (2.8) from 0 to $t\le t_k$, we obtain

$$
\mathcal{E}_{U^k}(t)+\int_0^t\|\Delta u^k(s)\|^{2\alpha}\|u_t^k(t)\|_{\gamma+2}^{\gamma+2}\mathrm{d}s=\mathcal{E}_{U^k}(0). \tag{2.9}
$$

Note that, from condition (2.7), if $\|\Delta u(0)\|<d$, then $\|\Delta u_{0k}\|\le d$ for large k. Let us prove, by contradiction, that it implies $\|\Delta u^k(t)\|\le d$ in $[0,t_k)$. Indeed, let $\mathcal{L}:=\{t\in[0,t_k);\ \|\Delta u^k(t)\|>d\}$. If $\mathcal{L}$ is empty, the conclusion is true. Suppose $\mathcal{L}$ is not empty and let $t^*=\inf\mathcal{L}$. Then, since $\|\Delta u^k(0)\|=\|\Delta u_{0k}\|\le d$, we have $t^*>0$. By continuity of $u^k(t)$ in $[0,t^*)$, we have $\|\Delta u^k(t^*)\|=d$ and $\|\Delta u^k(t)\|\le d$ in $[0,t^*]$. Thus, $P(\|\Delta u^k(t)\|)\ge 0$ in $[0,t^*]$. Consequently, from (2.7) and (2.9) , we obtain

$$
\begin{aligned}
P(\|\Delta u^k(t)\|)&\le\frac12\|u_t^k(t)\|^2+P(\|\Delta u^k(t)\|)\\
&\le\frac12\|u_t^k(t)\|^2+J(u^k(t))\\
&=\mathcal{E}_{U^k}(t)\le\mathcal{E}_{U^k}(0)<\frac{\omega\rho}{2(\rho+2)}d^2
\end{aligned}
$$

for all $t\in[0,t^*]$. Hence, there exists $0<\mu<\frac{\omega\rho}{2(\rho+2)}d^2$ such that

$$
P(\|\Delta u^k(t)\|)\le\mu \quad \text{on } [0,t^*].
$$

Note that $P(\beta)=\mu$ with $0<\beta<d$. Since $P(\lambda)$ is increasing in $[0,d]$, we get

$$
\|\Delta u^k(t)\|\le\beta \quad \text{in } [0,t^*].
$$

We found, by continuity of $\|\Delta u^k(t)\|$ on $[0,t_k)$, that $\|\Delta u^k(t^*)\| = d$ or $d \leq \beta$, contradiction. This implies that $\mathcal{L}$ is empty and we have

$$\|\Delta u^k(t)\| \leq d \quad \text{in } [0,t_k).$$

Since $\|u_t^k(t)\|^2 \leq \frac{\omega\rho}{2(\rho+2)}d^2$ in $[0,t_k)$, we can extend the approximated solution $u^k(t)$ to $[0,T]$ and we have the following estimate

$$\|U^k(t)\|_{\mathcal{H}_0}^2 = \|u_t^k(t)\|^2 + \|\Delta u^k(t)\|^2 \leq R_0 < \infty \quad \text{in } [0,T], \quad \forall\, k \in \mathbb{N}, \tag{2.10}$$

where $R_0 = \big(\frac{\omega\rho}{2(\rho+2)} + 1\big)d^2$. From (2.10), we have

$$U^k = (u^k, u_t^k) \rightharpoonup (u, u_t) = U \quad \text{weakly * in } L^\infty(0,T;\mathcal{H}_0). \tag{2.11}$$

Estimate II We consider the approximate problem (2.6) with

$$U_0^k \to U_0 \quad \text{strongly in } \mathcal{V}_d \cap \mathcal{H}_1.$$

We define the functional

$$\mathcal{F}_{U^k}(t) := \frac{1}{2}\|U^k(t)\|_{\mathcal{H}_1}^2 + \frac{1}{2}M(\|\nabla u^k(t)\|^2)\|\Delta u^k(t)\|^2.$$

Note that, from Assumption 2.1(I), immersion $W_1 \hookrightarrow W_0$ and using that $\omega = 1 - \frac{1}{\lambda_1^{\frac{1}{2}}}$, we get

$$\frac{\omega}{2}\|U^k(t)\|_{\mathcal{H}_1}^2 \leq \mathcal{F}_{U^k}(t) \leq \frac{1}{2}\Big(1 + \frac{M_0}{\lambda_1^{\frac{1}{2}}}\Big)\|U^k(t)\|_{\mathcal{H}_1}^2,$$

where $M_0 := \max\limits_{0\leq\tau\leq\frac{d^2}{\lambda_1^{\frac{1}{2}}}} |M(\tau)|$. Now, let us consider $\omega_j = -\Delta u_t^k$ in (2.6). Then, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{U^k}(t) + \frac{4(\gamma+1)}{(\gamma+2)^2}\|\Delta u^k(t)\|^{2\alpha}\int_\Omega [\nabla(|u_t^k|^{\frac{\gamma}{2}}u_t^k)]^2\mathrm{d}x = \sum_{i=1}^{2}\mathrm{I}_i, \tag{2.12}$$

where

$$\mathrm{I}_1 = (\rho+1)\int_\Omega |u^k|^\rho \nabla u^k \nabla u_t^k \mathrm{d}x,$$
$$\mathrm{I}_2 = M'(\|\nabla u^k(t)\|^2)\int_\Omega \nabla u^k \nabla u_t^k \mathrm{d}x \|\Delta u^k(t)\|^2.$$

In what follows we will estimate the terms I_1 and I_2. From Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$ and immersion $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$ with $\|\cdot\|_{2(\rho+1)} \leq \widehat{\varrho}\|\Delta\cdot\|$, we have

$$\mathrm{I}_1 \leq (\rho+1)\|u^k(t)\|_{2(\rho+1)}^\rho \|\nabla u^k(t)\|_{2(\rho+1)}\|\nabla u_t^k(t)\| \leq \frac{(\rho+1)\widehat{\varrho}^{\rho+1}d^\rho}{\omega}\mathcal{F}_{U^k}(t).$$

Since $M \in C^1([0,+\infty))$, taking $M_1 = \max_{0 \le \tau \le \frac{d^2}{\lambda_1^{\frac{1}{2}}}} |M'(\tau)|$, from immersion $W_1 \hookrightarrow W_0$ and estimate (2.10), we infer

$$\begin{aligned} \mathrm{I}_2 &= M'(\|\nabla u^k(t)\|^2) \int_\Omega (-\Delta u^k) u_t^k \mathrm{d}x \|\Delta u^k(t)\|^2 \\ &\le \frac{M_1}{\lambda_1^{\frac{1}{2}}} \|\Delta u^k(t)\| \|u_t^k(t)\| \|\nabla(\Delta u^k(t))\|^2 \le \frac{2M_1 d^2}{\omega \lambda_1^{\frac{1}{2}}} \mathcal{F}_{U^k}(t). \end{aligned}$$

Substituting I_1 and I_2 in (2.12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{U^k}(t) + \frac{4p}{(\gamma+2)^2}\|\Delta u^k(t)\|^{2\alpha} \int_\Omega [\nabla(|u_t^k|^{\frac{\gamma}{2}} u_t^k)]^2 \mathrm{d}x \le \widehat{R}_0 \mathcal{F}_{U^k}(t), \tag{2.13}$$

where $\widehat{R}_0 = \left(\frac{q\widehat{\varrho}^q d^\rho}{\omega} + \frac{2M_1 d^2}{\omega \lambda_1^{\frac{1}{2}}}\right)$. From Gronwall's lemma, we obtain

$$\mathcal{F}_{U^k}(t) \le \mathrm{e}^{\widehat{R}_0 t} \mathcal{F}_{U^k}(0), \quad \forall t \in [0,T],$$

which implies that

$$\|U^k(t)\|_{\mathcal{H}_1}^2 \le \omega\Big(1 + \frac{M_1}{\lambda_1^{\frac{1}{2}}}\Big)\mathrm{e}^{\widehat{R}_0 t}\|U_0^k\|_{\mathcal{H}_1}^2 \le R_1 \quad \text{in } [0,T], \quad \forall k \in \mathbb{N}. \tag{2.14}$$

From (2.14), we have

$$U^k = (u^k, u_t^k) \rightharpoonup (u, u_t) = U \quad \text{weakly * in } L^\infty(0,T;\mathcal{H}_1). \tag{2.15}$$

Estimate III Now we consider the approximate problem (2.6) with

$$U^k \to U_0 \quad \text{strongly in } \mathcal{V}_d \cap \mathcal{H}_2. \tag{2.16}$$

In order, let us define the functional

$$\mathcal{G}_{U^k}(t) := \frac{1}{2}\|U_t^k(t)\|_{\mathcal{H}_0}^2 + \frac{1}{2}M(\|\nabla u^k(t)\|^2)\|\nabla u_t^k(t)\|^2.$$

It follows from Assumption 2.1(I), embedding $W_2 \hookrightarrow W_1$, and $\omega = 1 - \frac{\varpi}{\lambda_1^{\frac{1}{2}}}$ that

$$\frac{\omega}{2}\|U_t^k(t)\|_{\mathcal{H}_0}^2 \le \mathcal{G}_{U^k}(t) \le \frac{1}{2}\Big(1 + \frac{M_0}{\lambda_1^{\frac{1}{2}}}\Big)\|U_t^k(t)\|_{\mathcal{H}_0}^2. \tag{2.17}$$

Next, deriving the approximate equation (2.6) with respect to variable t and substituting $\omega_j = u_{tt}^k$ in the resulting expression, it results

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{U^k}(t) + (\gamma+1)\|\Delta u^k(t)\|^{2\alpha}\int_\Omega |u^k|^\gamma (u_{tt}^k)^2 \mathrm{d}x = \sum_{j=1}^4 \mathrm{J}_j, \tag{2.18}$$

where

$$\begin{aligned}
\mathrm{J}_1 &= 2M'(\|\nabla u^k(t)\|^2)\int_\Omega \nabla u^k \nabla u_t^k \mathrm{d}x \int_\Omega \Delta u^k u_{tt}^k \mathrm{d}x,\\
\mathrm{J}_2 &= M'(\|\nabla u^k(t)\|^2)\int_\Omega \nabla u^k \nabla u_t^k \mathrm{d}x \|\nabla u_t^k(t)\|^2,\\
\mathrm{J}_3 &= -\alpha\|\Delta u^k(t)\|^{2(\alpha-1)}\int_\Omega \Delta u^k \Delta u_t^k \mathrm{d}x \int_\Omega |u_t^k|^\gamma u_t^k u_{tt}^k \mathrm{d}x,\\
\mathrm{J}_4 &= (\rho+1)\int_\Omega |u|^\rho u_t u_{tt}\mathrm{d}x.
\end{aligned}$$

Now let us estimate the terms on the right-hand side of (2.18). First, using immersions $W_2 \hookrightarrow W_1 \hookrightarrow W_0$, (2.10) and (2.17), we can estimate $\mathrm{J}_1 + \mathrm{J}_2$ as follows

$$\mathrm{J}_1 + \mathrm{J}_2 \le \frac{4M_1}{\omega\lambda_1^{\frac{1}{2}}}\|\Delta u^k\|^2 \mathcal{G}_{U^k}(t) + \frac{2M_1}{\omega\lambda_1^{\frac{1}{2}}}\|\Delta u^k(t)\|\|u_t^k(t)\|\mathcal{G}_{U^k}(t) \le \frac{6M_1R_0}{\omega\lambda_1^{\frac{1}{2}}}\mathcal{G}_{U^k}(t),$$

where $M_1 = \max\limits_{0\le\tau\le\frac{d^2}{\lambda_1^{\frac{1}{2}}}}|M'(\tau)|$. From Hölder inequality with $\frac{\gamma+1}{2(\gamma+1)} + \frac{1}{2} = 1$, embedding $W_1 \hookrightarrow L^{2(\gamma+1)}(\Omega)$ with $\|\cdot\|_{2(\gamma+1)} \le c_\gamma\|\nabla\cdot\|$, (2.10), (2.14) and (2.17), we can estimate the term J_3 as follows

$$\mathrm{J}_3 \le \alpha\|\Delta u^k(t)\|^{2\alpha-1}\|\Delta u_t^k(t)\|\|u_t^k(t)\|_{2(\gamma+1)}^{\gamma+1}\|u_{tt}^k(t)\| \le \frac{2\alpha c_\gamma^{\gamma+1}R_0^{\frac{\alpha-1}{2}}R_1^{\frac{\gamma+1}{2}}}{\omega}\mathcal{G}_{U^k}(t).$$

Finally, from Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$, immersion $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$ and (2.10), we get

$$\mathrm{J}_4 \le C(1 + \|u^k(t)\|_{2(\rho+1)}^\rho)\|u_t^k(t)\|_{2(\rho+1)}\|u_{tt}^k(t)\| \le C_{R_0}\mathcal{G}_{U^k}(t).$$

Returning to (2.18), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_{U^k}(t) \le C_{R_0,R_1}\mathcal{G}_{U^k}(t). \tag{2.19}$$

Applying Gronwall's lemma to (2.19), we get

$$\mathcal{G}_{U^k}(t) \le \mathrm{e}^{C_{R_0,R_1}t}\mathcal{G}_{U^k}(0).$$

To estimate the term $\mathcal{G}_{U^k}(0)$, first note that, taking $t = 0$ in the approximate equation (2.6) and substituting $\omega_j = u_{tt}^k(0)$, it results that

$$\|u_{tt}^k(0)\| \le \|\Delta^2 u_{0k}\| + |M(\|\nabla u_{0k}\|^2)|\|\Delta u_{0k}\| + \|\Delta u_{0k}\|^{2\alpha}\|u_{1k}\|_{2(\gamma+1)}^{\gamma+1} + \|u_{0k}\|_{2(\rho+1)}^{\rho+1}.$$

Thus, from the convergence (2.16), we have

$$\mathcal{G}_{U^k}(0) \le \frac{1}{2}\|U_t^k(0)\|_{\mathcal{H}_0}^2 + \frac{1}{2}|M(\|\nabla u_{0k}\|^2)|\|\nabla u_{1k}\|^2.$$

Hence, from (2.17),

$$\frac{\omega}{2}\|U_t^k(t)\|_{\mathcal{H}_0}^2 \le \mathcal{G}_{U^k}(t) \le \mathrm{e}^{C_{R_0,R_1}t}. \tag{2.20}$$

Moreover, taking $\omega_j = \Delta^2 u^k$ in (2.6), there exists also a constant $R_2 > 0$, such that

$$\|\Delta^2 u^k(t)\| \le \|u_{tt}^k(t)\| + M_0\|\nabla u^k(t)\| + \|\Delta u^k(t)\|^{2\alpha}\|u_t^k(t)\|_{2(\gamma+1)}^{\gamma+1} + \|u^k(t)\|_{2(\rho+1)}^{\rho+1} \le R_2 \tag{2.21}$$

for all $t \in [0,T)$ and $\forall k \in \mathbb{N}$. From (2.20)–(2.21), we obtain

$$\|U^k(t)\|_{\mathcal{H}_2} \equiv \|\Delta^2 u^k(t)\|^2 + \|\Delta u_t^k(t)\|^2 \le R_3, \tag{2.22}$$

where $R_3 = R_3(T, ||U_0||_{\mathcal{H}_2})$.

Therefore, from the estimates (2.11), (2.15), (2.20) and (2.22), we can pass the limit in the approximated equation (2.6) for a subsequence of (u^k), obtaining a function $u : [0,T] \to \mathbb{R}$, which is a regular solution claimed in Theorem 2.1.

2.4 Uniqueness

The uniqueness of solution for problem (1.1) under the conditions of Theorem 2.1 is an immediate consequence of Theorem 2.2 below.

Theorem 2.2 *Assume the Assumptions of Theorem* 2.1 *hold. If* $U_1 = (u, u_t)$, $U_2 = (v, v_t)$ *are regular solutions of* (1.1) *corresponding to* $U_1(0) = (u_0, u_1)$, $U_2(0) = (v_0, v_1)$, *respectively. Then*

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}_0} \le C_T\|U_1(0) - U_2(0)\|_{\mathcal{H}_0}, \quad t \in [0,T] \tag{2.23}$$

for some constant $C_T > 0$ *depending on initial data in* $\mathcal{H}_2$. *In particular, problem* (1.1) *has a unique regular solution.*

Proof Let $U^1 = (u, u_t)$ and $U^2 = (v, v_t)$ be two regular solutions of (1.1) with initial data $U_0^1 = (u_0, u_1)$ and $U_0^2 = (v_0, v_1)$, respectively. Setting $w = u - v$, the difference $U^1 - U^2 = (w, w_t) =: W$ solves the following problem in the strong (or weak) sense

$$\begin{aligned} &w_{tt} + \Delta^2 w - M(\|\nabla u(t)\|^2)\Delta w + \|\Delta u(t)\|^{2\alpha}(\,|u_t|^\gamma u_t - |v_t|^\gamma v_t) \\ &= (|u|^\rho u - |v|^\rho v) + \Delta_M \Delta v - \Delta_{\|\cdot\|^{2\alpha}}|v_t|^\gamma v_t \end{aligned} \tag{2.24}$$

with initial condition $(w(0), w_t(0)) = z_0^1 - z_0^2$, where

$$\Delta_M := M(\|\nabla u(t)\|^2) - M(\|\nabla v(t)\|^2) \quad \text{and} \quad \Delta_{\|\Delta\cdot\|^{2\alpha}} := \|\Delta u(t)\|^{2\alpha} - \|\Delta v(t)\|^{2\alpha}.$$

Let the functional

$$\mathcal{E}_W(t) = \frac{1}{2}\|W(t)\|_{\mathcal{H}_0}^2 + \frac{1}{2}M(\|\nabla u(t)\|^2)\|\nabla w(t)\|^2.$$

From Assumption 2.1(I), immersion $W_2 \hookrightarrow W_1$, we have

$$\frac{\omega}{2}\|W(t)\|_{\mathcal{H}_0}^2 \le \mathcal{E}_W(t) \le \frac{1}{2}\Big(1+\frac{M_0}{\lambda_1^{\frac{1}{2}}}\Big)\|W(t)\|_{\mathcal{H}_0}^2. \tag{2.25}$$

Multiplying (2.24) by w_t and integrating over Ω, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_W(t) + \|\Delta u(t)\|^{2\alpha}\int_\Omega (|u_t|^\gamma u_t - |v_t|^\gamma v_t)w_t \mathrm{d}x = \sum_{i=1}^4 \mathrm{L}_i, \tag{2.26}$$

where

$$\begin{aligned}
\mathrm{L}_1 &= M'(\|\nabla u(t)\|^2)\int_\Omega \nabla u \nabla u_t \mathrm{d}x \|\nabla w(t)\|^2,\\
\mathrm{L}_2 &= \Delta_M \int_\Omega \Delta v w_t \mathrm{d}x,\\
\mathrm{L}_3 &= \int_\Omega (|u|^\rho u - |v|^\rho v)w_t \mathrm{d}x,\\
\mathrm{L}_4 &= -\,\Delta_{\|\Delta\cdot\|^{2\alpha}}\int_\Omega |v_t|^\gamma v_t w_t \mathrm{d}x.
\end{aligned}$$

Firstly, using Mean Value Theorem (MVT for short), there exists $C_\gamma > 0$ such that

$$\int_\Omega (|u_t|^\gamma u_t - |v_t|^\gamma v_t)w_t \mathrm{d}x \ge C_\gamma \int_\Omega (|u_t|^\gamma + |v_t|^\gamma)w_t^2 \mathrm{d}x \ge 0.$$

Now let us estimate the right-hand side of (2.26). Using that $M \in C^1(\mathbb{R}^+)$ and immersion $W_2 \hookrightarrow W_1$, we have

$$\mathrm{L}_1 \le \max_{0\le\tau\le\frac{d^2}{\lambda_1^{\frac{1}{2}}}} |M'(\tau)|\|\Delta u(t)\|\|u_t(t)\|\|\nabla w(t)\|^2 \le \frac{M_1 R_0}{\lambda_1^{\frac{1}{2}}}\|\Delta w(t)\|^2$$

and

$$\mathrm{L}_2 \le M_1[\,\|\Delta u(t)\| + \|\Delta v(t)\|]\|\Delta w(t)\|\|w_t(t)\| \le \frac{M_1 R_0^{\frac{1}{2}}}{\lambda_1^{\frac{1}{4}}}\|W(t)\|_{\mathcal{H}_0}^2.$$

Using MVT to $\psi(s) = |s|^\rho s$, Hölder's inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$, embedding $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$ and Young's inequality, one gets

$$\begin{aligned}
\mathrm{L}_3 &\le q\int_\Omega |\theta u + (1-\theta)v|^\rho |w||w_t| \mathrm{d}x\\
&\le 2^\rho q[\|u(t)\|_{2(\rho+1)}^\rho + \|v(t)\|_{2(\rho+1)}^\rho]\|w(t)\|_{2(\rho+1)}\|w_t(t)\|\\
&\le 2^\rho q\widehat{\varrho}^q[\|\Delta u(t)\|^\rho + \|\Delta v(t)\|^\rho]\|\Delta w(t)\|\|w_t(t)\|\\
&\le 2^\rho q\widehat{\varrho}^q R_0^{\frac{\rho}{2}}\|W(t)\|_{\mathcal{H}_0}^2.
\end{aligned}$$

Finally, applying MVT to function $|s|^{2\alpha}$, $\alpha \ge \frac{1}{2}$, using Hölder inequality and immersion $W_1 \hookrightarrow$

$L^{2(\gamma+1)}(\Omega)$ with $\|\cdot\|_{2(\gamma+1)} \leq \widetilde{\varrho}\|\nabla\cdot\|$, one has

$$\begin{aligned}
\mathrm{L}_4 &\leq 2\alpha[\,\|\Delta u(t)\| + \|\Delta v(t)\|]^{2\alpha-1}\|\Delta w(t)\| \int_\Omega |v_t|^{\gamma+1}|w_t|\mathrm{d}x \\
&\leq 2\alpha[\,\|\Delta u(t)\| + \|\Delta v(t)\|]^{2\alpha-1}\|\Delta w(t)\|\|v_t(t)\|_{2(\gamma+1)}^{\gamma+1}\|w_t(t)\| \\
&\leq 2\alpha\widetilde{\varrho}^{\gamma+1}[\|\Delta u(t)\| + \|\Delta v(t)\|]^{2\alpha-1}\|\nabla v_t(t)\|^{\gamma+1}\|\Delta w(t)\|\|w_t(t)\| \\
&\leq \alpha[2R_0^{\frac{1}{2}}]^{2\alpha-1}\widetilde{\varrho}^{\gamma+1}R_{1,T}^{\gamma+1}\|W(t)\|_{\mathcal{H}_0}^2.
\end{aligned}$$

Replacing the last four estimates in (2.26) results

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_W(t) \leq R\mathcal{E}_W(t) \tag{2.27}$$

for all $t \in [0,T]$ and some constant $R = R(T, \|U_0^1\|_{\mathcal{H}_2}, \|U_0^2\|_{\mathcal{H}_2}) > 0$. Integrating (2.27) on $[0,t]$ and applying Gronwall's inequality, we arrive at

$$\mathcal{E}_W(t) \leq \mathcal{E}_W(0)\mathrm{e}^{Rt}, \quad \forall\, t \in [0,T]. \tag{2.28}$$

Thus, from (2.25), we obtain

$$\|W(t)\|_{\mathcal{H}_0}^2 \leq \frac{1}{\omega}\Big(1 + \frac{M_0}{\lambda_1^{\frac{1}{2}}}\Big)\mathrm{e}^{Rt}\|W(0)\|_{\mathcal{H}_0}^2. \tag{2.29}$$

Hence, taking $C_T = \frac{1}{\omega}\big(1 + \frac{M_0}{\lambda_1^{\frac{1}{2}}}\big)\mathrm{e}^{RT}$, from (2.29), we obtain (2.23). This shows that solutions of (1.1) depend continuously on initial data. In particular, we have uniqueness of solution by taking $U_0^1 = U_0^2$. Therefore, this completes the proof of Theorem 2.2.

Remark 2.1 The proof of the existence and uniqueness of a weak solution remains open. The difficulty arises in the estimation of the term L_4 in Theorem 2.2 above. Because the constant $R_{1,T}$ in the estimate $\mathrm{L}_4 \leq \alpha[2R_0^{\frac{1}{2}}]^{2\alpha-1}\widetilde{\varrho}^{\gamma+1}R_{1,T}^{\gamma+1}\|W(t)\|_{\mathcal{H}_0}^2$ depends on the regular initial data in $\mathcal{H}_2$. As a consequence of this, the constant R in (2.29) also depends on the strong initial data in $\mathcal{H}_2$, so it is not possible to apply density arguments to prove the existence of a weak solution.

3 Local Existence and Blow-up

In this section, we deal with the local existence and blow-up properties of problem (1.1).

3.1 Local solution

Our next result shows that for initial data $U_0 \in \mathcal{H}_2$, problem (1.1) has a local solution.

Theorem 3.1 *We assume Assumption* 2.1 *holds with* $(u_0, u_1) \in \mathcal{H}_2$ *and* $\alpha \geq \frac{1}{2}$. *Then there exists a* $T > 0$ *such that problem* (1.1) *has a unique regular solution* u.

Proof The proof can also be done through the Faedo-Galerkin method and compactness arguments. Without restriction of initial data in $(u_0, u_1) \in \mathcal{V}_d \cap \mathcal{H}_2$, the proof changes (in relation to proof of Theorem 2.1 for global solution) in Estimate I. Indeed, let approximate problem (2.6) have initial condition

$$(u^k(0), u_t^k(0)) = (u_{0k}, u_{1k}) \to (u_0, u_1) \quad \text{strongly in } \mathcal{H}_2. \tag{3.1}$$

Estimate I in this case is established as follows. Let E_{U^k} be defined by

$$E_{U^k} = \frac{1}{2}\|U^k(t)\|_{\mathcal{H}_0}^2 + \frac{1}{2}\widehat{M}(\|\nabla u^k(t)\|^2).$$

From Assumption 2.1(I), we have

$$E_{U^k}(t) \geq \frac{\omega}{2}\|\Delta u^k(t)\|^2 + \frac{1}{2}\|u_t^k(t)\|^2 \geq \frac{\omega}{2}\|U^k(t)\|_{\mathcal{H}_0}^2.$$

Substituting $\omega_j = u_t^k$ in (1.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{U^k}(t) + \|\Delta u^k(t)\|^{2\alpha}\|u_t^k(t)\|_{\gamma+2}^{\gamma+2} = \int_\Omega |u^k|^\rho u^k u_t^k \mathrm{d}x. \tag{3.2}$$

From Hölder's inequality with $\frac{1}{2} + \frac{1}{2} = 1$ and immersion $W_2 \hookrightarrow L^{2(\rho+1)}(\Omega)$, we get

$$\int_\Omega |u^k|^\rho u^k u_t^k \mathrm{d}x \leq \|u^k(t)\|_{2(\rho+1)}^{\rho+1}\|u_t^k(t)\| \leq C\|\Delta u^k(t)\|^{\rho+1}\|u_t^k(t)\| \leq \frac{2^{\frac{\rho+2}{2}}C}{\omega^{\frac{\rho+1}{2}}}[E_{U^k}(t)]^{\frac{\rho+2}{2}}.$$

Returning to (3.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{U^k}(t) \leq L[E_{U^k}(t)]^{\frac{\rho+2}{2}}, \quad \text{where } L = \frac{2^{\frac{\rho+2}{2}}C}{\omega^{\frac{\rho+1}{2}}}. \tag{3.3}$$

From (3.3),we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[E_{U^k}(t)][E_{U^k}(t)]^{-\frac{\rho+2}{2}} \leq L,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}[E_{U^k}(t)]^{-\frac{\rho}{2}} \geq -\frac{L\rho}{2}.$$

Integrating from 0 to t we get

$$[E_{U^k}(t)]^{\frac{\rho}{2}} \leq \frac{1}{[E_{U^k}(0)]^{-\frac{\rho}{2}} - \frac{L\rho}{2}t}.$$

Therefore

$$\frac{\omega}{2}\|U^k(t)\|_{\mathcal{H}_0}^2 \leq E_{U^k}(t) \leq \frac{1}{\left([E_{U^k}(0)]^{-\frac{\rho}{2}} - \frac{L\rho}{2}t\right)^{\frac{2}{\rho}}},$$

which implies that the approximate solution u^k exists locally at $[0, T]$ with $T < \frac{2}{L\rho(E_U(0))^{\frac{\rho}{2}}}$. To show the regularity of the approximate solution u^k and pass the limit on approximate problem (2.6) the arguments are the same as in Estimate II and Estimate III. The uniqueness of the solution is established analogously as in the proof of Theorem 2.2.

3.2 Blow-up

We consider the polynomial function $P(\lambda) = \frac{\omega}{2}\lambda^2 - \frac{\varrho}{\rho+2}\lambda^{\rho+2}$ defined in Subsection 2.2. We have already seen that P has a maximum in $d = \left(\frac{\omega}{\varrho}\right)^{\frac{1}{\rho}}$ and

$$P(d) = \frac{\omega}{2}d^2 - \frac{\varrho}{\rho+2}d^{\rho+2} = \frac{\omega\rho}{2(\rho+2)}d^2 > 0.$$

We set

$$\mathcal{W}_d := \Big\{U = (u, v) \in \mathcal{H}_0 \mid \mathcal{E}_U < \frac{\omega\rho}{2(\rho+2)}d^2 \text{ and } \|\Delta u\| > d\Big\}.$$

Our objective now is to study the behavior of solutions to problem (1.1) with $U_0 \in \mathcal{W}_d \cap \mathcal{H}_2$. Firstly, we observe that for data $U_0 \in \mathcal{W}_d \cap \mathcal{H}_2$, repeating the proof of Theorem 3.1 with $\|\Delta u_0\| > d$, the results are valid for $\alpha > 0$. Then we have the following statement.

Theorem 3.2 *We assume Assumption* 2.1 *holds with* $(u_0, u_1) \in \mathcal{W}_d \cap \mathcal{H}_2$. *Then there exists a* $T > 0$ *such that problem* (1.1) *has a unique regular solution* u.

Proposition 3.1 *Let us assume the hypotheses of Theorem* 3.2. *Then, the following statements are valid*:

(i) $\mathcal{E}_U(t) \leq \mathcal{E}_U(0)$ *for all* $t \in [0, T]$,

(ii) $\|\Delta u(t)\| \geq d_1$ *for all* $t \in [0, T]$ *for some* $d_1 > d$.

Proof Multiplying (1.1) by u_t and integrating over $\Omega \times [0, t]$ with $t \leq T$, we obtain

$$\mathcal{E}_U(t) + \|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+2} = \mathcal{E}_U(0), \tag{3.4}$$

which implies (i).

From definition $J(u(t))$ given in Subsection 2.2, this yields

$$\mathcal{E}_U(t) = \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) \geq P(\|\Delta u(t)\|). \tag{3.5}$$

Note that, P takes its maximum for $d = \left(\frac{\omega}{\varrho}\right)^{\frac{1}{\rho}}$ with $P(d) = \frac{\omega\rho}{2(\rho+2)}d^2$, being strictly decreasing for $\lambda \geq d$, and that $P(\lambda) \to -\infty$ as $\lambda \to \infty$. Then there exists $d_1 > d$ such that $P(d_1) = \mathcal{E}_U(0)$. From (3.5), we have

$$P(\|\Delta u_0\|) \leq \mathcal{E}_U(0) = P(d_1).$$

It follows that $d_1 \leq \|\Delta u_0\|$. Now suppose for contradiction that $\|\Delta u(t_0)\| < d_1$ for some $t_0 \in (0, T]$. By continuity of $\|\Delta u(\cdot)\|$ we can suppose that $d < \|\Delta u(t_0)\|$. But then

$$\mathcal{E}_U(t_0) \geq P(\|\Delta u(t_0)\|) > P(d_1) = \mathcal{E}_U(0).$$

But this is a contradiction with (i). Therefore (ii) is valid.

Remark 3.1 In what follows, for simplicity let us assume that

$$M(s) \equiv -\varpi + s^{\zeta}, \quad \zeta > 0 \text{ for all } s \geq 0.$$

Note that, M satisfies Assumption 2.1(I).

We are now in a position to establish the main result of this section given by Theorem 3.3 below.

Theorem 3.3 *Suppose the hypotheses of Theorem* 3.2 *are valid with M given as in Remark* 3.1*. In addition, let $\alpha \in (0,1)$ and $\rho > \max\big\{\frac{2\zeta}{d_1^2-d^2}, \frac{\gamma+2\alpha}{1-\alpha}\big\}$. If $U_0 \in \mathcal{W}_d \cap \mathcal{H}_2$, then T is necessarily finite, i.e., u can not be continued for all $t > 0$.*

Proof We assume the solution exists for all time and we arrive to a contradiction. We fix $E_1 \in \big(\mathcal{E}_U(0), \frac{\omega\rho}{2(\rho+2)}d^2\big)$ and set

$$H(t) = E_1 - \mathcal{E}_U(t).$$

Deriving $H(t)$ with respect to t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U(t) = \|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+2} \geq 0. \tag{3.6}$$

This shows that H is an increasing function, so that

$$H(t) \geq H(0) = E_1 - \mathcal{E}_U(0) > 0, \quad t \geq 0. \tag{3.7}$$

On the other hand, by using Proposition 3.1 and the definition of $\mathcal{E}_U(t)$,

$$\begin{aligned} H(t) &\leq E_1 - \frac{1}{2}(\|U(t)\|_{\mathcal{H}_0}^2 + \widehat{M}(\|\nabla u(t)\|^2)) + \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} \\ &\leq E_1 - \frac{1}{2}\|u_t(t)\|^2 - \frac{\omega}{2}\|\Delta u(t)\|^2 + \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} \\ &\leq \frac{\omega\rho}{2(\rho+2)}d_1^2 - \frac{\omega}{2}d_1^2 + \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} \\ &\leq -\frac{\omega}{\rho+2}d_1^2 + \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2}, \quad t \geq 0. \end{aligned} \tag{3.8}$$

From (3.7)–(3.8) and immersion $W_2 \hookrightarrow L^{\rho+2}(\Omega)$, we have

$$\|u(t)\|_{\rho+2}^{\rho+2} \geq \omega d_1^2 \quad \text{and} \quad \|\Delta u(t)\|^{\rho+2} \geq \frac{\omega}{\varrho}d_1^2, \quad t \geq 0. \tag{3.9}$$

Now, we define the perturbed functional

$$\Psi(t) = H^{1-\nu}(t) + \delta\int_{\Omega} u u_t \mathrm{d}x, \tag{3.10}$$

where $\delta > 0$ is small enough and will be specified later and

$$0 < \nu \leq \nu_0 = \min\Big\{\frac{2\rho(1-\alpha) - 2(\gamma+2\alpha)}{(\gamma+1)(\rho+2)^2}, \frac{\rho}{2(\rho+2)}\Big\} < \frac{1}{2}.$$

Taking the derivative of Ψ with respect to t we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = (1-\nu)H^{-\nu}(t)\frac{\mathrm{d}}{\mathrm{d}t}H(t) + \delta\|u_t(t)\|^2 + \delta\int_\Omega u u_{tt}\mathrm{d}x. \tag{3.11}$$

From (3.6), we have

$$(1-\nu)H^{-\nu}(t)\frac{\mathrm{d}}{\mathrm{d}t}H(t) = (1-\nu)H^{-\nu}(t)\|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+2} \geq 0. \tag{3.12}$$

Let

$$\chi(t) = \|\Delta u(t)\|^{2\alpha}\int_\Omega |u_t|^\gamma u_t u \mathrm{d}x.$$

Using (1.1) and adding the term $2(\zeta+1)(H(t)-E_1+\mathcal{E}_U(t))$ we find the following equality

$$\begin{aligned}\int_\Omega u u_{tt}\mathrm{d}x &= (\zeta+1)\|u_t(t)\|^2 + \zeta\|\Delta u(t)\|^2 + \varpi(\zeta+2)\|\nabla u(t)\|^2\\ &\quad + \frac{\rho-2\zeta}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} + 2(\zeta+1)H(t) - 2(\zeta+1)E_1 - \chi(t).\end{aligned} \tag{3.13}$$

Substituting (3.12)–(3.13) in (3.11), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) &\geq (1-\nu)H^{-\nu}(t)\|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+2} + \delta(\zeta+2)\|u_t(t)\|^2\\ &\quad + \delta\zeta(\omega\|\Delta u(t)\|^2 - 2E_1) + \delta\Big(\frac{\rho}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} - 2E_1\Big)\\ &\quad + \delta\zeta(1-\omega)\|\Delta u(t)\|^2 + \delta\varpi(\zeta+2)\|\nabla u(t)\|^2 + \frac{-2\delta\zeta}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2}\\ &\quad + 2\delta(\zeta+1)H(t) - \delta\chi(t).\end{aligned} \tag{3.14}$$

Using that $\frac{\|\Delta u\|^2}{d_1^2} \geq 1$ and $E_1 < \frac{\omega\rho}{2(\rho+2))}d^2$, we have

$$\begin{aligned}\delta\zeta(\omega\|\Delta u(t)\|^2 - 2E_1) &\geq \delta\zeta\Big(\omega - \frac{2E_1}{d_1^2}\Big)\|\Delta u(t)\|^2\\ &\geq \delta\zeta\omega\Big(1 - \frac{\rho d^2}{(\rho+2)d_1^2}\Big)\|\Delta u(t)\|^2\\ &= \delta\zeta\omega\Big(\frac{\rho(d_1^2-d^2)}{\rho+2} + \frac{2}{\rho+2}\Big)\|\Delta u(t)\|^2.\end{aligned}$$

Now, using that $\frac{\|u\|_{\rho+2}^{\rho+2}}{\omega d_1^2} \geq 1$ and $E_1 < \frac{\omega\rho}{2(\rho+2))}d^2$, we have

$$\begin{aligned}\delta\Big(\frac{\rho}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} - 2E_1\Big) &\geq \delta\Big(\frac{\rho}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} - \frac{2E_1}{\omega d_1^2}\Big)\|u(t)\|_{\rho+2}^{\rho+2}\\ &\geq \frac{\delta\rho(d_1^2-d^2)}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2}.\end{aligned}$$

Returning to (3.14), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) &\geq (1-\nu)H^{-\nu}(t)\|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+2} + \delta(\zeta+2)\|u_t(t)\|^2\\ &\quad + \delta\zeta\omega\Big(\frac{\rho(d_1^2-d^2)+2}{\rho+2} + \frac{1-\omega}{\omega}\Big)\|\Delta u(t)\|^2 + \delta\varpi(\zeta+2)\|\nabla u(t)\|^2\\ &\quad + \delta\Big(\frac{\rho(d_1^2-d^2)-2\zeta}{\rho+2}\Big)\|u(t)\|_{\rho+2}^{\rho+2} + 2\delta(\zeta+1)H(t) - \delta\chi(t).\end{aligned} \tag{3.15}$$

From Holder's and Young's inequalities with $\frac{\gamma+1}{\gamma+2}+\frac{1}{\gamma+2}=1$, we have

$$\begin{aligned}|\delta\chi(t)| &\leq \delta\|\Delta u(t)\|^{2\alpha}\int_\Omega |u_t|^\gamma|u_t||u|\mathrm{d}x\\ &\leq \delta\|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+1}\|u(t)\|_{\gamma+2}\\ &\leq \frac{\gamma+1}{\gamma+2}(1-\nu)H^{-\nu}(t)H'(t)+\underbrace{\frac{\delta^{\gamma+2}H^{\nu(\gamma+1)}(t)}{(1-\nu)^{\gamma+1}(\gamma+2)}\|\Delta u(t)\|^{2\alpha}\|u(t)\|_{\gamma+2}^{\gamma+2}}_{\chi_1(t)}.\end{aligned}$$

From immersion $L^{\rho+2}(\Omega)\hookrightarrow L^{\gamma+2}(\Omega)$, using $H(t)\leq\frac{\varrho}{\rho+2}\|\Delta u(t)\|^{\rho+2}$ and Young's inequality with $\frac{\rho-\gamma}{\rho+2}+\frac{\gamma+2}{\rho+2}=1$, we obtain

$$\begin{aligned}\chi_1(t) &\leq \delta^{\gamma+2}\frac{\varrho^{\nu(\gamma+1)}|\Omega|^{\frac{\rho-\gamma}{\rho+2}}}{(1-\nu)^{\gamma+1}(\gamma+2)(\rho+2)^{\nu(\gamma+1)}}\|\Delta u(t)\|^{\nu(\gamma+1)(\rho+2)+2\alpha}\|u(t)\|_{\rho+2}^{\gamma+2}\\ &\leq \delta^{\gamma+2}\mathcal{K}_0\Big(\frac{\rho-\gamma}{\rho+2}\|\Delta u(t)\|^{[\nu(\gamma+1)(\rho+2)+2\alpha]\frac{\rho+2}{\rho-\gamma}}+\frac{\gamma+2}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2}\Big),\end{aligned}$$

where $\mathcal{K}_0=\frac{\varrho^{\nu(\gamma+1)}|\Omega|^{\frac{\rho-\gamma}{\rho+2}}}{(1-\nu)^{\gamma+1}(\gamma+2)(\rho+2)^{\nu(\gamma+1)}}$. Note that, using that $\nu\leq\frac{2\rho(1-\alpha)-2(\gamma+2\alpha)}{(\gamma+1)(\rho+2)^2}$, we have

$$P:=[\nu(\gamma+1)(\rho+2)+2\alpha]\frac{\rho+2}{\rho-\gamma}\leq 2.$$

Then, substituting $-\chi(t)$ in (3.15) and using $\|\Delta u\|\geq d_1$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \geq{}& \frac{(\nu+1)}{\gamma+2}H^\nu(t)\|\Delta u(t)\|^{2\alpha}\|u_t(t)\|_{\gamma+2}^{\gamma+2}+\delta(\zeta+2)\|u_t(t)\|^2\\ &+\delta\zeta\omega\Big(\frac{\rho(d_1^2-d^2)}{\rho+2}+\frac{2}{\rho+2}\Big)\|\Delta u(t)\|^2+\delta(\zeta(1-\omega)d_1^{2-P}-\delta^{\gamma+1}\mathcal{K}_0)\|\Delta u(t)\|^P\\ &+\delta\varpi(\zeta+2)\|\nabla u(t)\|^2+\delta\Big(\frac{\rho(d_1^2-d^2)-2\zeta}{\rho+2}-\delta^{\gamma+1}\mathcal{K}_0\Big)\|u(t)\|_{\rho+2}^{\rho+2}\\ &+2\delta(\zeta+1)H(t).\end{aligned}\tag{3.16}$$

From (3.16), taking

$$\delta<\delta_0=\min\Big\{\frac{\zeta(1-\omega)d_1^{2-P}}{\mathcal{K}_0},\frac{\rho(d_1^2-d^2)-2\zeta}{\mathcal{K}_0(\rho+2)}\Big\}^{\frac{1}{\gamma+1}},$$

there exists $\mathcal{Q}_0>0$ that does not depend on δ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)\geq\delta\mathcal{Q}_0(\|u_t(t)\|^2+\|\Delta u(t)\|^2+\|u(t)\|_{\rho+2}^{\rho+2}+H(t))\geq 0.\tag{3.17}$$

Especially, (3.17) means that $\Psi(t)$ is increasing on $(0,T)$, with

$$\Psi(t)=H^{1-\nu}(t)+\delta\int_\Omega u_tu\mathrm{d}x\geq H^{1-\nu}(0)+\delta\int_\Omega u_t(0)u(0)\mathrm{d}x.$$

We further choose δ to be sufficiently small such that $\Psi(0)>0$, so $\Psi(t)\geq\Psi(0)>0$ for $t\geq 0$.

On the other hand, using $(a+b)^{\frac{1}{1-\nu}} \leq 2^{\frac{1}{1-\nu}}(a^{\frac{1}{1-\nu}}+b^{\frac{1}{1-\nu}})$, Young's inequality with $\frac{1}{2(1-\nu)}+\frac{1-2\nu}{2(1-\nu)}=1$ and immersion $W_0 \hookrightarrow W_2$, we obtain

$$\begin{aligned}\Psi^{\frac{1}{1-\nu}}(t) &\leq 2^{\frac{1}{1-\nu}}(H(t)+\delta^{\frac{1}{1-\nu}}\|u(t)\|^{\frac{1}{1-\nu}}\|u_t(t)\|^{\frac{1}{1-\nu}})\\ &\leq 2^{\frac{1}{\nu+1}}\Big(H(t)+\frac{\delta^{\frac{1}{1-\nu}}}{2(1-\nu)}\|u_t(t)\|^2+\frac{\delta^{\frac{1}{1-\nu}}(1-2\nu)}{2(1-\nu)}\|u(t)\|^{\frac{2}{1-2\nu}}\Big).\end{aligned}$$

Note that, using that $\nu \leq \frac{\rho}{2(\rho+2)}$, we have $\frac{2}{1-2\nu} \leq \rho+2$. Since $1 \leq \frac{\|u\|_{\rho+2}}{[\omega d_1^2]^{\frac{1}{\rho+2}}}$, we obtain

$$\|u(t)\|_{\rho+2}^{\frac{2}{1-2\nu}} \leq \|u(t)\|_{\rho+2}^{\frac{2}{1-2\nu}}\cdot 1 \leq \|u(t)\|_{\rho+2}^{\frac{2}{1-2\nu}}\frac{\|u(t)\|_{\rho+2}^{(\rho+2)-\frac{2}{1-2\nu}}}{[\omega d_1^2]^{1-\frac{2}{(\rho+2)(1-2\nu)}}}=\frac{1}{[\omega d_1^2]^{1-\frac{2}{(\rho+2)(1-2\nu)}}}\|u(t)\|_{\rho+2}^{\rho+2}.$$

Thus, there exists $\mathcal{Q}_1>0$ such that

$$\Psi^{\frac{1}{1-\nu}}(t) \leq \mathcal{Q}_1[\|u_t(t)\|^2+\|\Delta u(t)\|^2+\|u(t)\|_{\rho+2}^{\rho+2}+H(t)]. \tag{3.18}$$

Combining (3.17)–(3.18) we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) \geq \mathcal{Q}\Psi^{\frac{1}{1-\nu}}(t), \quad \text{where } \mathcal{Q}=\frac{\delta\mathcal{Q}_0}{\mathcal{Q}_1}.$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)\Psi^{\frac{-1}{1-\nu}}(t) \geq \mathcal{Q} \Rightarrow \frac{1-\nu}{-\nu}\frac{\mathrm{d}}{\mathrm{d}t}[\Psi(t)]^{\frac{-\nu}{1-\nu}} \geq \mathcal{Q} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(t)]^{\frac{-\nu}{1-\nu}} \leq \frac{-\nu}{1-\nu}\mathcal{Q}.$$

Integrating from 0 to t, we get

$$\frac{1}{[\Psi(t)]^{\frac{\nu}{1-\nu}}} \leq [\Psi(0)]^{\frac{-\nu}{1-\nu}}-\frac{\nu}{1-\nu}\mathcal{Q}t. \tag{3.19}$$

Therefore

$$\Psi(t) \geq \frac{1}{([\Psi(0)]^{\frac{-\nu}{1-\nu}}-\frac{\nu}{1-\nu}\mathcal{Q}t)^{\frac{1-\nu}{\nu}}}, \tag{3.20}$$

which implies that the solution blows up in a finite time T, with $T \leq \frac{1-\nu}{\nu[\Psi(0)]^{\frac{\nu}{1-\nu}}}$. But this is a contradiction to the assertion that the solution exists for all time. This completes the proof of Theorem 3.3.

Acknowledgement The authors are grateful the referees for their remarkable comments and suggestions that inspired and encouraged ourselves to reach the final version of the current paper.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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