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New Molecular Characterization of Musielak-Orlicz Hardy Spaces on Spaces of Homogeneous Type and Its Applications*

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Abstract Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, in the sense of Coifman and Weiss, and $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ satisfy that, for almost every $x \in \mathcal{X}$, $\varphi(x, \cdot)$ is an Orlicz function and that $\varphi(\cdot, t)$ is a Muckenhoupt $\mathbb{A}_\infty(\mathcal{X})$ weight uniformly in $t \in [0, \infty)$. In this article, the authors first establish a new molecular characterization, associated with admissible sequences of balls on $\mathcal{X}$, of the Musielak-Orlicz Hardy space $H^\varphi(\mathcal{X})$. As an application, the authors also obtain the boundedness of Calderón-Zygmund operators from $H^\varphi(\mathcal{X})$ to $H^\varphi(\mathcal{X})$ or to the Musielak-Orlicz space $L^\varphi(\mathcal{X})$. The main novelty of these results is that, in the proof of the boundedness of Calderón-Zygmund operators on $H^\varphi(\mathcal{X})$, the authors get rid of the dependence on the reverse doubling property of μ by using this new molecular characterization of $H^\varphi(\mathcal{X})$.

Keywords Space of homogeneous type, Musielak-Orlicz function, Hardy space, Molecule, Calderón-Zygmund operator
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1 Introduction

It is well known that the real-variable theory of Hardy-type spaces on $\mathbb{R}^n$, including various equivalent characterizations and the boundedness of singular integral operators, plays a fundamental role in harmonic analysis and partial differential equations (see, for instance, [56, 61]). Recall that the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ was originally introduced by Stein and Weiss [62] which initiated the study of the real-variable theory of $H^p(\mathbb{R}^n)$. Fefferman and Stein [21] characterized $H^p(\mathbb{R}^n)$ via several maximal functions and proved that the dual space of $H^1(\mathbb{R}^n)$ is just the space $\mathrm{BMO}(\mathbb{R}^n)$ of bounded mean oscillation functions, which was introduced by John and Nirenberg in [41]; Taibleson and Weiss [64] further established the molecular characterization of $H^p(\mathbb{R}^n)$. Moreover, when $p \in (0, 1]$, $H^p(\mathbb{R}^n)$ proves a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ in the study on the boundedness of operators. For instance,

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when $p \in (0,1]$, the Calderón-Zygmund operators, including Hilbert and Riesz transforms, are bounded on $H^p(\mathbb{R}^n)$, but they are not bounded on $L^p(\mathbb{R}^n)$. Up to now, many new variants of classical Hardy spaces have sprung up and their real-variable theories have been well developed in order to meet the increasing demand from harmonic analysis, partial differential equations, and geometric analysis (see, for instance, [1, 14, 32–33, 57, 59, 63, 68]).

The bilinear decomposition of the product of Hardy spaces and their dual spaces plays key roles in improving the estimates of many nonlinear quantities such as div-curl products (see, for instance, [4, 71]), weak Jacobians (see, for instance, [15, 38]), and commutators (see, for instance, [42, 44, 54]). Bonami et al. [6] showed that, for any given $f \in H^1(\mathbb{R}^n)$, there exist two bounded linear operators $S_f :\ \mathrm{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T_f :\ \mathrm{BMO}(\mathbb{R}^n) \to H^{\Phi}_w(\mathbb{R}^n)$ such that, for any $g \in \mathrm{BMO}(\mathbb{R}^n)$, $f \times g = S_f g + T_f g$, where $H^{\Phi}_w(\mathbb{R}^n)$ denotes the weighted Orlicz-Hardy space associated to the weight function $w(x) := \frac{1}{\log(\mathrm{e}+|x|)}$ for any $x \in \mathbb{R}^n$ and to the Orlicz function

$$\Phi(t) := \frac{t}{\log(\mathrm{e}+t)}, \quad \forall\, t \in [0,\infty).$$

This result was essentially improved by Bonami et al. [5], where they further proved the following bilinear decomposition

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n),$$

where $H^{\log}(\mathbb{R}^n)$ denotes the Musielak-Orlicz Hardy space related to the Musielak-Orlicz function

$$\varphi(x,t) := \frac{t}{\log(\mathrm{e}+t)+\log(\mathrm{e}+|x|)}, \quad \forall\, x \in \mathbb{R}^n,\ \forall\, t \in (0,\infty).$$

Bonami et al. [5] also concluded that $H^{\log}(\mathbb{R}^n)$ is the smallest space in the dual sense. This result was generalized to $H^p(\mathbb{R}^n)$, with $p \in (0,1)$, and its dual space in [3, 8] and also to the localized Hardy space and its dual space in [12, 71, 73]. Motivated by this, Ky [43] introduced the Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ with φ being a Musielak-Orlicz function, which generalizes both the Orlicz-Hardy space of Janson [40] and the weighted Hardy space of Strömberg and Torchinsky [63], established both the grand maximal function and the atomic characterizations of $H^{\varphi}(\mathbb{R}^n)$, and obtained the boundedness of sublinear operators from $H^{\varphi}(\mathbb{R}^n)$ to quasi-Banach spaces. Since then, the real-variable theory of Musielak-Orlicz Hardy spaces has rapidly been developed. Precisely, Hou et al. [34] characterized $H^{\varphi}(\mathbb{R}^n)$ by the Lusin-area function and the molecule; Liang et al. [51] introduced the weak Musielak-Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$ via the grand maximal function and obtained the boundedness of Calderón-Zygmund operators from $H^{\varphi}(\mathbb{R}^n)$ to $WH^{\varphi}(\mathbb{R}^n)$ including the critical case. We refer the reader to [7, 11, 23, 36, 39, 48–50, 69] for more studies on the real-variable theory of $H^{\varphi}(\mathbb{R}^n)$ and to [9–10, 13, 19, 65, 70] for some recent progress on Musielak-Orlicz Hardy spaces associated with operators.

On the other hand, there has been an increasing interesting in extending the above results of Musielak-Orlicz Hardy spaces from the Euclidean space to more general underlying spaces such as the anisotropic Euclidean space (see, for instance, [45–46, 52–53]). In particular, Coifman and Weiss [16–17] originally introduced the concept of the space $\mathcal{X}$ of homogeneous type and the atomic Hardy space $H^{p,q}_{\mathrm{cw}}(\mathcal{X})$ with $p \in (0,1]$ and $q \in (p,\infty] \cap [1,\infty]$. They also proved that $H^{p,q}_{\mathrm{cw}}(\mathcal{X})$ is independent of the choice of q in [17]. From then on, spaces of homogeneous type have become the most natural and general underlying space to study the real-variable theory of function spaces and the boundedness of operators (see, for instance, [18, 28, 47, 58]). In [17], Coifman and Weiss also asked a question, that is, to what extent the geometrical condition of $\mathcal{X}$ is necessary for the validity of the radial maximal function characterization of $H^1_{\mathrm{cw}}(\mathcal{X})$. Since then, lots of efforts are made to establish various real-variable characterizations of the atomic Hardy spaces on $\mathcal{X}$ with few geometrical assumptions. More precisely, under the assumption that the equipped measure μ of $\mathcal{X}$ satisfying the Ahlfors-regularity condition, Macías and Segovia [55] characterized $H^p_{\mathrm{cw}}(\mathcal{X})$ via the grand maximal function when $p \le 1$ but near to 1; Duong and Yan [20] characterized Hardy spaces via the Lusin-area function associated with certain semigroup.

However, due to the lack of Calderón reproducing formulae on $\mathcal{X}$, many existing results of both function spaces and boundedness of operators, including those in the aforementioned articles [20, 55], require some additional geometrical assumptions on $\mathcal{X}$ such as the reverse doubling property of μ (see, for instance, [24, 60]). Recently, a breakthrough on the analysis over $\mathcal{X}$ without any additional geometrical assumptions was made by Auscher and Hytönen [2] who constructed an orthonormal wavelet basis, with exponential decay, of $L^2(\mathcal{X})$ by using the system of random dyadic cubes established by Hytönen and Kairema [37]. Later, Han et al. [25] established the wavelet reproducing formulae which hold true in the sense of both spaces of test functions and distributions. Motivated by these, He et al. [30] first introduced a kind of approximations of the identity with exponential decay and then obtained new Calderón reproducing formulae on $\mathcal{X}$. All of these provide elementary tools to further develop the real-variable theories of function spaces on $\mathcal{X}$.

Very recently, He et al. completely answered the aforementioned question of Coifman and Weiss by developing a quite complete real-variable theory, including various equivalent characterizations and the boundedness of sublinear operators, of the Hardy space and its localized version on $\mathcal{X}$, respectively, in [29] and [31]. Fu et al. [22] further generalized the corresponding results in [29] to Musielak-Orlicz Hardy spaces $H^{\varphi}(\mathcal{X})$. In particular, Fu et al. obtained the boundedness of Calderón-Zygmund operators on $H^{\varphi}(\mathcal{X})$ in [22, Theorem 9.2]. Indeed, let ω be the upper dimension of $\mathcal{X}$, $s \in (0,1)$, and T be an s-Calderón-Zygmund operator. Assume that φ is a growth function, with uniformly upper type 1 and uniformly lower type $p \in (0,1]$, satisfying that

$$\frac{p}{q(\varphi)} \in \Big(\frac{\omega}{\omega+s}, 1\Big] \tag{1.1}$$

and $q \in (q(\varphi),\infty)$, where $q(\varphi)$ is the critical weight index of φ. Fu et al. proved [22, Theorem

9.2] by two steps. In the first step, they showed that T is bounded from the finite atomic Musielak-Orlicz Hardy space $H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})$ to $H^{\varphi}(\mathcal{X})$ by using the molecular characterization of $H^{\varphi}(\mathcal{X})$ (see [22, Theorem 6.8]) and proving that $T(a)$ is a (φ, r, s)-molecule for any (φ, q)-atom a and some $r \in (q(\varphi), q)$. In the second step, they further proved that T can uniquely be extended to a bounded linear operator on $H^{\varphi}(\mathcal{X})$ via the criterion on the boundedness of sublinear operators from $H^{\varphi,q}_{\mathrm{fin}}(\mathcal{X})$ to quasi-Banach spaces (see [22, Theorem 7.7]). However, there exist two gaps in the proof of [22, Theorem 9.2]. Indeed, from (1.1), it follows that $s > \omega\left[\frac{q(\varphi)}{p} - 1\right]$. The first gap is that [22, Theorem 6.8] can not be applied to prove [22, Theorem 9.2] because [22, Theorem 6.8] requires

$$s \in \left(\max\left\{\omega, \omega\frac{q(\varphi)}{p}\right\}, \infty\right)$$

and s in (1.1) does not satisfy this requirement. This is essentially caused by the absence of the reverse doubling condition of the equipped measure μ on $\mathcal{X}$. Moreover, observe that $H^{\varphi}(\mathcal{X})$ is a p-quasi-Banach space now, where $p \in (0, 1]$. The second gap is that [22, Theorem 7.7] can also not be applied to prove [22, Theorem 9.2] because [22, Theorem 7.7] requires that $H^{\varphi}(\mathcal{X})$ is a 1-quasi-Banach space, but $H^{\varphi}(\mathcal{X})$ is only known to be a p-quasi-Banach space with $p \in (0, 1]$.

To seal these two gaps, in this article, we first establish a new molecular characterization of $H^{\varphi}(\mathcal{X})$, associated with sequences of admissible balls on $\mathcal{X}$, and then obtain the boundedness of Calderón-Zygmund operators from $H^{\varphi}(\mathcal{X})$ to $H^{\varphi}(\mathcal{X})$ or to the Musielak-Orlicz space $L^{\varphi}(\mathcal{X})$. In particular, since a sequence of admissible balls are both doubling and reverse doubling (see Definition 3.1 below), we can use the new molecular characterization to get rid of the dependence on the reverse doubling property of μ and seal the aforementioned first gap. Moreover, via the finite atomic decomposition of $H^{\varphi}(\mathcal{X})$ (see [22, Theorem 7.5]), by a standard density argument instead of the criterion on the boundedness of sublinear operators, we then prove [22, Theorem 9.2] and hence seal the aforementioned second gap. These are the main novelties of this article.

The organization of the remainder of this article is as follows.

In Section 2, we recall some notation and concepts which are used throughout this article. More precisely, in Subsection 2.1, we recall the definition of a space $\mathcal{X}$ of homogeneous type and state some basic properties of $\mathcal{X}$. In Subsection 2.2, we introduce the concepts of the uniformly Muckenhoupt condition, the uniformly reverse Hölder condition, and the Musielak-Orlicz space $L^{\varphi}(\mathcal{X})$. Some of their basic properties are also reviewed. In Subsection 2.3, we first recall the concepts of both spaces of test functions and distributions, the system of dyadic cubes, and approximations of the identity with exponential decay on $\mathcal{X}$. Then, via the Lusin-area function S_α with $\alpha \in (0, \infty)$, we introduce the Musielak-Orlicz Hardy space $H^{\varphi}(\mathcal{X})$.

In Section 3, we first recall the concept of admissible ball sequences on $\mathcal{X}$ and introduce the admissible molecule of $H^{\varphi}(\mathcal{X})$, which differs from the classical one in that it uses admissible balls to replace the balls with radii $\{2^k\}_{k\in\mathbb{N}}$. Then we introduce the molecular Musielak-Orlicz Hardy space $\mathring{H}^{\varphi,q,\varepsilon,c}_{\mathrm{mol}}(\mathcal{X})$ and establish a new molecular characterization of $H^{\varphi}(\mathcal{X})$ (see

Theorem 3.1 below). Indeed, on the one hand, by both the fact that each (φ, q)-atom is also an admissible molecule and the atomic characterization of $H^\varphi(\mathcal{X})$ established in [22, Theorem 6.15] (see also Lemma 3.3 below), we prove $H^\varphi(\mathcal{X}) \subset \mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})$. On the other hand, we show that any admissible $(\varphi, q, \varepsilon, c)$-molecule can be divided into an infinite linear combination of (φ, q)-atoms in Lemma 3.4, which, combined with Lemma 3.3 again, further implies that $\mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X}) \subset H^\varphi(\mathcal{X})$ and hence completes the proof of Theorem 3.1.

In Section 4, we establish the boundedness of Calderón-Zygmund operators from $H^\varphi(\mathcal{X})$ to $H^\varphi(\mathcal{X})$ or to $L^\varphi(\mathcal{X})$ (see Theorems 4.1–4.2 below, respectively). Indeed, by a standard density argument, Theorem 3.1 and Lemmas 4.2–4.3, we first prove Theorem 4.1. Moreover, by an argument similar to that used in the proof of Theorem 4.1, the boundedness of the Hardy-Littlewood maximal operator M on $L^\varphi(\mathcal{X})$ (see Lemma 4.5 below), the boundedness of Calderón-Zygmund operators on weighted Lebesgue spaces, we further prove Theorem 4.2.

At the end of this section, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \cdots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote by C a positive constant which is independent of the main parameters, but may vary from line to line. We use $C_{(\alpha,\cdots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \cdots$. The symbol $f \lesssim g$ means $f \le Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g = h$ or $f \lesssim g \le h$, rather than $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$. If E is a subset of $\mathcal{X}$, we denote by $\mathbf{1}_E$ its characteristic function and by $E^\complement$ the set $\mathcal{X} \setminus E$. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, we denote by $B(x, r)$ the ball centered at x with the radius r, that is, $B(x, r) := \{y \in \mathcal{X} : \ d(x, y) < r\}$. For any ball B, we use x_B to denote its center and r_B its radius, and denote by λB for any $\lambda \in (0, \infty)$ the ball concentric with B having the radius λr_B. For any $\alpha \in \mathbb{R}$, we denote by $\lfloor \alpha \rfloor$ the largest integer not greater than α. For any index $q \in [1, \infty]$, we denote by q' its conjugate index, that is, $\frac{1}{q} + \frac{1}{q'} = 1$. For any $x, x_0, y \in \mathcal{X}$ and $r, \vartheta \in (0, \infty)$, let $V_r(x) := \mu(B(x, r))$,

$$V(x, y) := \begin{cases} \mu(B(x, d(x, y))) & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

and

$$P_\vartheta(x_0, x; r) := \frac{1}{V_r(x_0) + V(x_0, x)} \left[\frac{r}{r + d(x_0, x)}\right]^\vartheta. \tag{1.2}$$

2 Preliminaries

In this section, we first recall some basic concepts about spaces $\mathcal{X}$ of homogeneous type and Musielak-Orlicz spaces, respectively, in Subsections 2.1–2.2, which are used throughout this article. Then we introduce the Musielak-Orlicz Hardy space on $\mathcal{X}$ via the Lusin-area function in Subsection 2.3. The concepts of both spaces of test functions and distributions, the system of dyadic cubes, and approximations of the identity with exponential decay on $\mathcal{X}$ are also stated in Subsection 2.3.

2.1 Spaces of homogeneous type

In this subsection, we recall the concept of spaces of homogeneous type and some related basic estimates.

Definition 2.1 *A quasi-metric space $(\mathcal{X}, d)$ is a non-empty set $\mathcal{X}$ equipped with a quasi-metric d, namely a non-negative function defined on $\mathcal{X} \times \mathcal{X}$ satisfying that, for any $x, y, z \in \mathcal{X}$,*

(i) $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) *there exists a constant $A_0 \in [1, \infty)$, independent of x, y and z, such that*

$$d(x, z) \leq A_0[d(x, y) + d(y, z)]. \tag{2.1}$$

The ball B of $\mathcal{X}$, centered at $x_0 \in \mathcal{X}$ with radius $r \in (0, \infty)$, is defined by setting

$$B := B(x_0, r) := \{x \in \mathcal{X} : \ d(x, x_0) < r\}.$$

For any ball B and any $\tau \in (0, \infty)$, we denote $B(x_0, \tau r)$ by τB if $B := B(x_0, r)$ for some $x_0 \in \mathcal{X}$ and $r \in (0, \infty)$.

Definition 2.2 *Let $(\mathcal{X}, d)$ be a quasi-metric space and μ be a non-negative measure on $\mathcal{X}$. The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if μ satisfies the following doubling condition: There exists a constant $C_{(\mu)} \in [1, \infty)$ such that, for any ball $B \subset \mathcal{X}$,*

$$\mu(2B) \leq C_{(\mu)}\mu(B). \tag{2.2}$$

The above doubling condition implies that, for any ball $B \subset \mathcal{X}$ and any $\lambda \in [1, \infty)$,

$$\mu(\lambda B) \leq C_{(\mu)}\lambda^{\omega}\mu(B), \tag{2.3}$$

where $\omega := \log_2 C_{(\mu)}$ is called the upper dimension of $\mathcal{X}$. If $A_0 = 1$, then $(\mathcal{X}, d, \mu)$ is called a metric measure space of homogeneous type or, simply, a doubling metric measure space.

Both spaces of homogeneous type, with some additional assumptions, and function spaces on them have been extensively investigated in many articles. One special case of spaces of homogeneous type is the RD-space, originally introduced in [27] (see also [26, 72]), which is a doubling metric measure space satisfying the following additional reverse doubling condition: There exist constants $\widetilde{C}_{(\mu)} \in (0, 1]$ and $\kappa \in (0, \omega]$ such that, for any ball $B(x, r)$ with $x \in \mathcal{X}$ and $r \in \left(0, \frac{\operatorname{diam} \mathcal{X}}{2}\right)$ and for any $\lambda \in \left[1, \frac{\operatorname{diam} \mathcal{X}}{2r}\right)$,

$$\widetilde{C}_{(\mu)}\lambda^{\kappa}\mu(B(x, r)) \leq \mu(B(x, \lambda r)),$$

here and thereafter, $\operatorname{diam} \mathcal{X} := \sup_{x, y \in \mathcal{X}} d(x, y)$.

Throughout this article, according to [17, pp. 587–588], we always make the following assumptions on $(\mathcal{X}, d, \mu)$:

(i) For any $x \in \mathcal{X}$, the balls $\{B(x,r)\}_{r\in(0,\infty)}$ form a basis of open neighborhoods of x;

(ii) μ is Borel regular which means that all open sets are μ-measurable and every set $A \subset \mathcal{X}$ is contained in a Borel set E such that $\mu(A) = \mu(E)$;

(iii) for any $x \in \mathcal{X}$ and $r \in (0,\infty)$, $\mu(B(x,r)) \in (0,\infty)$;

(iv) $\operatorname{diam} \mathcal{X} = \infty$ and $(\mathcal{X}, d, \mu)$ is non-atomic, which means $\mu(\{x\}) = 0$ for any $x \in \mathcal{X}$.

Note that $\operatorname{diam} \mathcal{X} = \infty$ implies that $\mu(\mathcal{X}) = \infty$ (see [2, p. 284] or [58, Lemma 5.1]). From this, it follows that, under the above assumptions, $\mu(\mathcal{X}) = \infty$ if and only if $\operatorname{diam} \mathcal{X} = \infty$.

The following basic estimates are from [26, Lemma 2.1], which can be proved by using (2.3).

Lemma 2.1 *Let $x, y \in \mathcal{X}$ and $r \in (0,\infty)$. Then $V(x,y) \sim V(y,x)$ and*

$$\begin{aligned} V_r(x) + V_r(y) + V(x,y) &\sim V_r(x) + V(x,y) \sim V_r(y) + V(x,y) \\ &\sim \mu(B(x, r + d(x,y))). \end{aligned}$$

Moreover, if $d(x,y) \le r$, then $V_r(x) \sim V_r(y)$. Here the positive equivalence constants are independent of x, y and r.

2.2 Musielak-Orlicz spaces

Throughout this article, we always let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type with $\mu(\mathcal{X}) = \infty$. In this subsection, we recall the concept of Musielak-Orlicz spaces and state some known results.

A function $\Phi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it is non-decreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0,\infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Then Φ is said to be of upper (resp., lower) type p for some $p \in (0,\infty)$ if there exists a positive constant C such that, for any $s \in [1,\infty)$ (resp., $s \in [0,1]$) and $t \in [0,\infty)$,

$$\Phi(st) \le C s^p \Phi(t).$$

Now, we recall the concept of uniformly upper and lower types, which was introduced in [35].

Definition 2.3 *For a given function $\varphi : \mathcal{X} \times [0,\infty) \to [0,\infty)$ such that, for almost every $x \in \mathcal{X}$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of uniformly upper (resp., lower) type p for some $p \in (0,\infty)$ if there exists a positive constant $C_{(p)}$, depending on p, such that, for almost every $x \in \mathcal{X}$, $s \in [1,\infty)$ (resp., $s \in [0,1]$) and $t \in [0,\infty)$,*

$$\varphi(x, st) \le C_{(p)} s^p \varphi(x,t).$$

The function φ is said to be of uniformly upper (resp., lower) type if it is of uniformly upper (resp., lower) type p for some $p \in (0,\infty)$, and let

$$I(\varphi) := \inf\{p \in (0,\infty) : \ \varphi \text{ is of uniformly upper type } p\}$$

and

$$i(\varphi) := \sup\{p \in (0, \infty) : \ \varphi \text{ is of uniformly lower type } p\}. \tag{2.4}$$

Next, we recall the concepts of both the uniformly Muckenhoupt condition and the uniformly reverse Hölder condition from [35, Definition 2.6].

Definition 2.4 *A function $\varphi: \ \mathcal{X} \times [0, \infty) \to [0, \infty)$ is said to satisfy the uniformly Muckenhoupt condition for some $q \in [1, \infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathcal{X})$, if, when $q \in (1, \infty)$,*

$$\begin{aligned}[]
[\varphi]_{\mathbb{A}_q(\mathcal{X})} : = & \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \frac{1}{[\mu(B)]^q} \int_B \varphi(x, t) \, \mathrm{d}\mu(x) \\
& \times \left\{ \int_B [\varphi(y, t)]^{-\frac{1}{q-1}} \, \mathrm{d}\mu(y) \right\}^{q-1} \\
< & \infty
\end{aligned}$$

or

$$[\varphi]_{\mathbb{A}_1(\mathcal{X})} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \frac{1}{\mu(B)} \int_B \varphi(x, t) \, \mathrm{d}\mu(x) \left\{ \operatorname*{ess\,sup}_{y\in B} [\varphi(y, t)]^{-1} \right\} < \infty,$$

where the first suprema are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

Throughout this article, let

$$\mathbb{A}_\infty(\mathcal{X}) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathcal{X}). \tag{2.5}$$

Definition 2.5 *A function $\varphi: \ \mathcal{X} \times [0, \infty) \to [0, \infty)$ is said to satisfy the uniformly reverse Hölder condition for some $p \in (1, \infty]$, denoted by $\varphi \in \mathbb{RH}_p(\mathcal{X})$, if, when $p \in (1, \infty)$,*

$$\begin{aligned}[]
[\varphi]_{\mathbb{RH}_p(\mathcal{X})} := & \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \left\{ \frac{1}{\mu(B)} \int_B [\varphi(x, t)]^p \, \mathrm{d}\mu(x) \right\}^{\frac{1}{p}} \\
& \times \left\{ \frac{1}{\mu(B)} \int_B \varphi(y, t) \, \mathrm{d}\mu(y) \right\}^{-1} \\
< & \infty
\end{aligned}$$

or

$$[\varphi]_{\mathbb{RH}_\infty(\mathcal{X})} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \left\{ \operatorname*{ess\,sup}_{x\in B} \varphi(x, t) \right\} \left\{ \frac{1}{\mu(B)} \int_B \varphi(y, t) \, \mathrm{d}\mu(y) \right\}^{-1} < \infty,$$

where the first suprema are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

Throughout this article, for any given $p \in (0, \infty)$, a function f is said to be locally p-integrable if, for any $x \in \mathcal{X}$, there exists an $r \in (0, \infty)$ such that

$$\int_{B(x,r)} |f(y)|^p \, \mathrm{d}\mu(y) < \infty.$$

Denote by $L^p_{\rm loc}(\mathcal{X})$ the set of all the locally p-integrable functions on $\mathcal{X}$. In what follows, we always let M denote the Hardy-Littlewood maximal operator defined by setting, for any $f \in L^1_{\rm loc}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$M(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, \mathrm{d}\mu(y), \tag{2.6}$$

where the supremum is taken over all the balls B of $\mathcal{X}$ containing x.

Now, we state some basic properties of $\mathbb{A}_q(\mathcal{X})$ with $q \in [1, \infty)$ and of $\mathbb{RH}_p(\mathcal{X})$ with $p \in (1, \infty]$, which are just [22, Lemma 2.6] (see also [69, Lemma 1.1.3] for the corresponding Euclidean case).

Lemma 2.2 *The following conclusions hold true.*

(i) $\mathbb{A}_1(\mathcal{X}) \subset \mathbb{A}_p(\mathcal{X}) \subset \mathbb{A}_q(\mathcal{X})$ *for any* p, q *satisfying* $1 \le p \le q < \infty$.

(ii) $\mathbb{RH}_\infty(\mathcal{X}) \subset \mathbb{RH}_q(\mathcal{X}) \subset \mathbb{RH}_p(\mathcal{X})$ *for any* p, q *satisfying* $1 < p \le q \le \infty$.

(iii) *If* $q \in [1, \infty)$ *and* $\varphi \in \mathbb{A}_q(\mathcal{X})$, *then there exists a positive constant* C *such that, for any ball* $B \subset \mathcal{X}$, *any* μ*-measurable function* f *and any* $t \in (0, \infty)$,

$$\left[\frac{1}{\mu(B)} \int_B |f(x)| \, \mathrm{d}\mu(x)\right]^q \le C \frac{1}{\varphi(B, t)} \int_B |f(x)|^q \varphi(x, t) \, \mathrm{d}\mu(x),$$

here and thereafter, for any μ*-measurable set* $E \subset \mathcal{X}$ *and* $t \in [0, \infty)$, *let*

$$\varphi(E, t) := \int_E \varphi(x, t) \, \mathrm{d}\mu(x).$$

(iv) *If* $q \in [1, \infty)$ *and* $\varphi \in \mathbb{A}_q(\mathcal{X})$, *then there exists a positive constant* C *such that, for any ball* $B \subset \mathcal{X}$, *any* μ*-measurable set* $E \subset B$ *and any* $t \in (0, \infty)$,

$$\frac{\varphi(B, t)}{\varphi(E, t)} \le C \left[\frac{\mu(B)}{\mu(E)}\right]^q.$$

(v) $\mathbb{A}_\infty(\mathcal{X}) = \bigcup_{q \in [1, \infty)} \mathbb{A}_q(\mathcal{X}) = \bigcup_{p \in (1, \infty]} \mathbb{RH}_p(\mathcal{X})$.

The critical weight indices $q(\varphi)$ and $r(\varphi)$ of $\varphi \in \mathbb{A}_\infty(\mathcal{X})$ are defined, respectively, by setting

$$q(\varphi) := \inf\{q \in [1, \infty) : \ \varphi \in \mathbb{A}_q(\mathcal{X})\} \tag{2.7}$$

and

$$r(\varphi) := \sup\{p \in (1, \infty] : \ \varphi \in \mathbb{RH}_p(\mathcal{X})\}. \tag{2.8}$$

The following concept of growth functions was first introduced in [35, Definition 2.7].

Definition 2.6 *A function* $\varphi : \ \mathcal{X} \times [0, \infty) \to [0, \infty)$ *is called a growth function if the following conditions are satisfied*:

(i) φ *is a Musielak-Orlicz function, that is,*

(i)_1 *the function* $\varphi(x, \cdot) : \ [0, \infty) \to [0, \infty)$ *is an Orlicz function for almost every* $x \in \mathcal{X}$;

(i)$_2$ *the function* $\varphi(\cdot,t)$ *is* μ*-measurable for any* $t\in[0,\infty)$.

(ii) $\varphi\in\mathbb{A}_\infty(\mathcal{X})$.

(iii) φ *is of uniformly lower type* p *for some* $p\in(0,1]$ *and of uniformly upper type* 1.

Next, we recall the concept of Musielak-Orlicz spaces, which was first introduced in [35, Definition 2.8].

Definition 2.7 *Let* φ *be a growth function in Definition* 2.6. *The Musielak-Orlicz space* $L^\varphi(\mathcal{X})$ *is defined to be the set of all the* μ*-measurable functions* f *such that*

$$\int_{\mathcal{X}}\varphi(x,|f(x)|)\,\mathrm{d}\mu(x)<\infty,$$

equipped with the Luxemburg (*also called the Luxemburg-Nakano*)(*quasi-*)*norm*

$$\|f\|_{L^\varphi(\mathcal{X})}:=\inf\Big\{\lambda\in(0,\infty):\ \int_{\mathcal{X}}\varphi\Big(x,\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}\mu(x)\le1\Big\}.$$

Remark 2.1 By both (2.5) and Definition 2.4, we find that, for any ball $B\subset\mathcal{X}$,

$$\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}<\infty.$$

Now, we recall some basic properties of $L^\varphi(\mathcal{X})$, which were first given in [22, Lemma 2.8] (see also [69, Lemmas 1.1.6 and 1.1.10] for the corresponding Euclidean case).

Lemma 2.3 *Let* φ *be a growth function in Definition* 2.6. *Then the following conclusions hold true.*

(i) φ *is uniformly* σ*-quasi-subadditive on* $\mathcal{X}\times[0,\infty)$, *that is, there exists a positive constant* C *such that, for any* $(x,t_j)\in\mathcal{X}\times[0,\infty)$ *with* $j\in\mathbb{N}$,

$$\varphi\Big(x,\sum_{j\in\mathbb{N}}t_j\Big)\le C\sum_{j\in\mathbb{N}}\varphi(x,t_j).$$

(ii) *For any* $f\in L^\varphi(\mathcal{X})\backslash\{\mathbf{0}\}$,

$$\int_{\mathcal{X}}\varphi\Big(x,\frac{|f(x)|}{\|f\|_{L^\varphi(\mathcal{X})}}\Big)\,\mathrm{d}\mu(x)=1.$$

Let φ be a growth function in Definition 2.6. In what follows, we always let

$$m(\varphi):=\Big\lfloor\omega\Big[\frac{q(\varphi)}{i(\varphi)}-1\Big]\Big\rfloor,\tag{2.9}$$

where, for any $\alpha\in\mathbb{R}$, $\lfloor\alpha\rfloor$ denotes the largest integer not greater than α and ω, $q(\varphi)$ and $i(\varphi)$ are the same, respectively, as in (2.3), (2.7) and (2.4).

2.3 Musielak-Orlicz Hardy spaces

In this subsection, we introduce the Musielak-Orlicz Hardy space defined via the Lusin-area function. To this end, we first recall the concept of spaces of test functions on $\mathcal{X}$, the following version of which was originally introduced by Han et al. [26, Definition 2.2] (see also [27, Definition 2.8]).

Definition 2.8 *Let $x_0 \in \mathcal{X}$, $r \in (0, \infty)$, $\varrho \in (0, 1]$ and $\vartheta \in (0, \infty)$. A function f on $\mathcal{X}$ is called a test function of type $(x_0, r, \varrho, \vartheta)$, denoted by $f \in \mathcal{G}(x_0, r, \varrho, \vartheta)$, if there exists a positive constant C such that*

(T1) *for any $x \in \mathcal{X}$,*

$$|f(x)| \le C P_\vartheta(x_0, x; r), \tag{2.10}$$

here and thereafter, P_ϑ is the same as in (1.2);

(T2) *for any $x, y \in \mathcal{X}$ satisfying $d(x, y) \le \frac{r+d(x_0,x)}{2A_0}$ with A_0 the same as in* (2.1),

$$|f(x) - f(y)| \le C\Big[\frac{d(x, y)}{r + d(x_0, x)}\Big]^\varrho P_\vartheta(x_0, x; r). \tag{2.11}$$

Moreover, for any $f \in \mathcal{G}(x_0, r, \varrho, \vartheta)$, define

$$\|f\|_{\mathcal{G}(x_0,\, r,\, \varrho,\, \vartheta)} := \inf\{C :\ C \text{ satisfies } (2.10)\text{–}(2.11)\}.$$

The subspace $\mathring{\mathcal{G}}(x_0, r, \varrho, \vartheta)$ is defined by setting

$$\mathring{\mathcal{G}}(x_0, r, \varrho, \vartheta) := \Big\{f \in \mathcal{G}(x_0, r, \varrho, \vartheta) :\ \int_{\mathcal{X}} f(x)\, \mathrm{d}\mu(x) = 0\Big\}$$

equipped with the norm $\|\cdot\|_{\mathring{\mathcal{G}}(x_0,r,\varrho,\vartheta)} := \|\cdot\|_{\mathcal{G}(x_0,r,\varrho,\vartheta)}$.

Fix an $x_0 \in \mathcal{X}$. We denote $\mathring{\mathcal{G}}(x_0, 1, \varrho, \vartheta)$ simply by $\mathring{\mathcal{G}}(\varrho, \vartheta)$. Obviously, $\mathring{\mathcal{G}}(\varrho, \vartheta)$ is a Banach space. Note that, for any fixed $x \in \mathcal{X}$ and $r \in (0, \infty)$, $\mathring{\mathcal{G}}(x, r, \varrho, \vartheta) = \mathring{\mathcal{G}}(\varrho, \vartheta)$ with equivalent norms, but the positive equivalence constants may depend on both x and r.

Fix an $\varepsilon \in (0, 1]$ and $\varrho, \vartheta \in (0, \varepsilon]$. Let $\mathring{\mathcal{G}}^\varepsilon_0(\varrho, \vartheta)$ be the completion of the set $\mathring{\mathcal{G}}(\varepsilon, \varepsilon)$ in $\mathring{\mathcal{G}}(\varrho, \vartheta)$. Furthermore, the norm of $\mathring{\mathcal{G}}^\varepsilon_0(\varrho, \vartheta)$ is defined by setting $\|\cdot\|_{\mathring{\mathcal{G}}^\varepsilon_0(\varrho,\vartheta)} := \|\cdot\|_{\mathcal{G}(\varrho,\vartheta)}$. The space $\mathring{\mathcal{G}}^\varepsilon_0(\varrho, \vartheta)$ is called the space of test functions. The dual space $(\mathring{\mathcal{G}}^\varepsilon_0(\varrho, \vartheta))'$ is defined to be the set of all continuous linear functionals from $\mathring{\mathcal{G}}^\varepsilon_0(\varrho, \vartheta)$ to $\mathbb{C}$, equipped with the weak-$*$ topology. The space $(\mathring{\mathcal{G}}^\varepsilon_0(\varrho, \vartheta))'$ is called the space of distributions.

The following system of dyadic cubes of $(\mathcal{X}, d, \mu)$ was established by Hytönen and Kairema in [37, Theorem 2.2].

Lemma 2.4 *Suppose that constants $0 < c_0 \le C_0 < \infty$ and $\delta \in (0, 1)$ satisfy $12A_0^3 C_0\delta \le c_0$ with A_0 the same as in* (2.1). *Assume that a set of points, $\{z^k_\alpha :\ k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \subset \mathcal{X}$ with $\mathcal{A}_k$, for any $k \in \mathbb{Z}$, being a set of indices, has the following properties: For any $k \in \mathbb{Z}$,*

$$d(z^k_\alpha, z^k_\beta) \ge c_0\delta^k \text{ if } \alpha \ne \beta \quad \text{and} \quad \min_{\alpha\in\mathcal{A}_k} d(x, z^k_\alpha) < C_0\delta^k \text{ for any } x \in \mathcal{X}.$$

Then there exists a family of sets, $\{Q^k_\alpha :\ k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$, satisfying that

(i) *for any $k \in \mathbb{Z}$, $\bigcup_{\alpha\in\mathcal{A}_k} Q^k_\alpha = \mathcal{X}$ and $\{Q^k_\alpha :\ \alpha \in \mathcal{A}_k\}$ is disjoint;*

(ii) *if $k, l \in \mathbb{Z}$ and $k \le l$, then, for any $\alpha \in \mathcal{A}_k$ and $\beta \in \mathcal{A}_l$, either $Q^l_\beta \subset Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$;*

(iii) *for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, $B(z^k_\alpha, (3A_0^2)^{-1}c_0\delta^k) \subset Q^k_\alpha \subset B(z^k_\alpha, 2A_0C_0\delta^k)$.*

Throughout this article, for any $k \in \mathbb{Z}$, define

$$\mathcal{G}_k := \mathcal{A}_{k+1} \setminus \mathcal{A}_k \quad \text{and} \quad \mathcal{Y}^k := \left\{z_\alpha^{k+1}\right\}_{\alpha \in \mathcal{G}_k} =: \left\{y_\alpha^k\right\}_{\alpha \in \mathcal{G}_k},$$

and for any $x \in \mathcal{X}$, define

$$d(x, \mathcal{Y}^k) := \inf_{y \in \mathcal{Y}^k} d(x, y).$$

Next, we recall the concept of approximations of the identity with exponential decay which was introduced in [30, Definition 2.7].

Definition 2.9 *Let δ be the same as in Lemma 2.4. A sequence $\{Q_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an approximation of the identity with exponential decay (for short, exp-ATI) if there exist constants $C, \nu \in (0, \infty)$, $a \in (0, 1]$ and $\eta \in (0, 1)$ such that, for any $k \in \mathbb{Z}$, the kernel of the operator Q_k, a function on $\mathcal{X} \times \mathcal{X}$, which is still denoted by Q_k, has the following properties*:

(i) (*The identity condition*) $\sum_{k\in\mathbb{Z}} Q_k = I$ *in $L^2(\mathcal{X})$, where I denotes the identity operator on* $L^2(\mathcal{X})$;

(ii) (*The size condition*) *for any $x, y \in \mathcal{X}$,*

$$|Q_k(x, y)| \le C E_k(x, y),$$

here and thereafter,

$$\begin{aligned} E_k(x, y) := {} & \frac{1}{\sqrt{V_{\delta^k}(x) V_{\delta^k}(y)}} \exp\left\{ -\nu \left[\frac{d(x, y)}{\delta^k}\right]^a \right\} \\ & \times \exp\left\{ -\nu \left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a \right\}; \end{aligned}$$

(iii) (*The regularity condition*) *for any $x, x', y \in \mathcal{X}$ with $d(x, x') \le \delta^k$,*

$$|Q_k(x, y) - Q_k(x', y)| + |Q_k(y, x) - Q_k(y, x')| \le C \left[\frac{d(x, x')}{\delta^k}\right]^\eta E_k(x, y);$$

(iv) (*The second difference regularity condition*) *for any $x, x', y, y' \in \mathcal{X}$ with $d(x, x') \le \delta^k$ and $d(y, y') \le \delta^k$,*

$$\begin{aligned} & |[Q_k(x, y) - Q_k(x', y)] - [Q_k(x, y') - Q_k(x', y')]| \\ & \le C \left[\frac{d(x, x')}{\delta^k}\right]^\eta \left[\frac{d(y, y')}{\delta^k}\right]^\eta E_k(x, y); \end{aligned}$$

(v) (*The cancellation condition*) *for any $x, y \in \mathcal{X}$,*

$$\int_{\mathcal{X}} Q_k(x, y') \, d\mu(y') = 0 = \int_{\mathcal{X}} Q_k(x', y) \, d\mu(x').$$

Now, we recall the concept of the Lusin-area function (see, for instance, [29, Section 5]).

Definition 2.10 *Let δ and η be the same, respectively, as in Lemma 2.4 and Definition 2.9, and let $\varrho,\vartheta\in(0,\eta)$. Assume that $f\in(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$ and $\{Q_k\}_{k\in\mathbb{Z}}$ is an exp-ATI. For any $\alpha\in(0,\infty)$, the Lusin-area function $S_\alpha(f)$ of f with aperture α is defined by setting, for any $x\in\mathcal{X}$,*

$$S_\alpha(f)(x):=\Big\{\sum_{k\in\mathbb{Z}}\int_{B(x,\alpha\delta^k)}|Q_kf(y)|^2\,\frac{\mathrm{d}\mu(y)}{V_{\alpha\delta^k}(x)}\Big\}^{\frac12}.$$

When $\alpha:=1$, we simply write $S:=S_1$.

Next, we recall the concept of Musielak-Orlicz Hardy spaces, which was first introduced in [22, Definition 6.2].

Definition 2.11 *Let η be the same as in Definition 2.9 and φ be a growth function in Definition 2.6 with uniformly lower type $p\in(0,1]$ satisfying*

$$\frac{p}{q(\varphi)}\in\Big(\frac{\omega}{\omega+\eta},1\Big],\tag{2.12}$$

and let

$$\varrho,\vartheta\in\Big(\omega\Big[\frac{q(\varphi)}{p}-1\Big],\eta\Big),\tag{2.13}$$

where $q(\varphi)$ and ω are the same, respectively, as in (2.7) and (2.3). The Musielak-Orlicz Hardy space $H^\varphi(\mathcal{X})$ is defined by setting

$$H^\varphi(\mathcal{X}):=\{f\in(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))':\ \|S(f)\|_{L^\varphi(\mathcal{X})}<\infty\},$$

and moreover, for any $f\in H^\varphi(\mathcal{X})$, let

$$\|f\|_{H^\varphi(\mathcal{X})}:=\|S(f)\|_{L^\varphi(\mathcal{X})}.$$

Remark 2.2 (i) As it was proved in [22, Theorem 6.3], the space $H^\varphi(\mathcal{X})$ in Definition 2.11 is independent of the choices of exp-ATIs in $S(f)$.

(ii) Combining [67, Remark 3.17(iii)], [22, Theorems 5.4 and 6.15] and [22, Proposition 6.12], we conclude that the space $H^\varphi(\mathcal{X})$ in Definition 2.11 is independent of the choices of $(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$, whenever $\varrho,\vartheta\in\big(\omega\big[\frac{q(\varphi)}{p}-1\big],\eta\big)$.

3 New Molecular Characterization of $H^\varphi(\mathcal{X})$

In this section, we establish a new molecular characterization of $H^\varphi(\mathcal{X})$. To this end, we first recall the following conclusion which was obtained in [66, Lemma 6.10].

Lemma 3.1 *Let $\mathcal{X}$ be a space of homogeneous type with $\mu(\mathcal{X})=\infty$, $c\in(1,\infty)$, $x_0\in\mathcal{X}$ and $r_0\in(0,\infty)$. For any $j\in\mathbb{N}$, define*

$$r_j:=\sup\{r\in(0,\infty):\mu(B(x_0,r))\le C_{(\mu)}[cC_{(\mu)}]^j\mu(B(x_0,r_0))\},$$

where $C_{(\mu)}$ is the same as in (2.2). *Then, for any $j\in\mathbb{N}$, it holds true that $r_j\in(0,\infty)$,*

$$[cC_{(\mu)}]^j\mu(B(x_0,r_0))\le\mu(B(x_0,r_j))\le C_{(\mu)}[cC_{(\mu)}]^j\mu(B(x_0,r_0)),\tag{3.1}$$

and $\lim\limits_{j\to\infty}r_j=\infty$.

Next, we recall the concept of admissible balls on $\mathcal{X}$ (see, for instance, [66, Definition 6.11]).

Definition 3.1 *Let $\mathcal{X}$ be a space of homogeneous type with $\mu(\mathcal{X})=\infty$, $A_1,A_2\in(1,\infty)$ such that $A_1<A_2$, and $x_0\in\mathcal{X}$. A sequence $\{B_j\}_{j\in\mathbb{Z}_+}$ of balls centered at x_0 is said to be (A_1,A_2)-admissible if it satisfies that, for any $j\in\mathbb{N}$,*

$$A_1\mu(B_{j-1})<\mu(B_j)\le A_2\mu(B_{j-1}).$$

Remark 3.1 Suppose that $x_0\in\mathcal{X}$, $r_0\in(0,\infty)$, $c\in(1,\infty)$, and $C_{(\mu)}$ is the same as in (2.2). For any given $j\in\mathbb{N}$, let r_j be the same as in Lemma 3.1. By (3.1), Definition 3.1 and an elementary calculation, we easily find that the sequence $\{B(x_0,r_j)\}_{j\in\mathbb{Z}_+}$ of balls is $(c,cC_{(\mu)}^2)$-admissible.

Now, we introduce the molecule and the molecular Hardy space associated with admissible balls, which are called the admissible molecule and the admissible molecular Musielak-Orlicz Hardy space, respectively.

Definition 3.2 *Assume that η is the same as in Definition* 2.9, *φ is a growth function in Definition* 2.6 *satisfying* (2.12), *$q\in(1,\infty]$, $\varepsilon\in(0,\infty)$ and $c\in(1,\infty)$.*

(i) *Let $x_0\in\mathcal{X}$, $r_0\in(0,\infty)$, $\{r_j\}_{j\in\mathbb{N}}$ be the same as in Lemma* 3.1, *and*

$$\varepsilon_j:=[cC_{(\mu)}]^{-\frac{\omega+\varepsilon}{\omega}j},\quad\forall\, j\in\mathbb{Z}_+\tag{3.2}$$

with $C_{(\mu)}$ and ω the same, respectively, as in (2.2) *and* (2.3). *A μ-measurable function m on $\mathcal{X}$ is called an admissible $(\varphi,q,\varepsilon,c)$-molecule centered at $B^{(0)}:=B(x_0,r_0)$ if*

(i)$_1$ *for any $j\in\mathbb{Z}_+$,*

$$\|m\|_{L^q(A_j(B))}\le\varepsilon_j[\mu(B^{(j)})]^{\frac1q}\,\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}^{-1},\tag{3.3}$$

here and thereafter, $B^{(j)}:=B(x_0,r_j)$ and

$$A_j(B):=\begin{cases}B^{(0)}, & \text{if } j=0,\\ B^{(j)}\backslash B^{(j-1)}, & \text{if } j\in\mathbb{N};\end{cases}\tag{3.4}$$

(i)$_2$ $\int_{\mathcal{X}}m(x)\,\mathrm{d}\mu(x)=0$.

(ii) *Let ϱ,ϑ be the same as in* (2.13). *The admissible molecular Musielak-Orlicz Hardy space $\mathring{H}^{\varphi,q,\varepsilon,c}_{\mathrm{mol}}(\mathcal{X})$ is defined to be the set of all the $f\in(\mathring{\mathcal{G}}^\eta_0(\varrho,\vartheta))'$ satisfying that there exists a sequence $\{m_j\}_{j\in\mathbb{N}}$ of admissible $(\varphi,q,\varepsilon,c)$-molecules centered, respectively, at balls $\{B_j\}_{j\in\mathbb{N}}$ and a sequence $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ such that*

$$\sum_{j\in\mathbb{N}}\varphi\Big(B_j,\frac{|\lambda_j|}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\Big)<\infty$$

and $f=\sum_{j\in\mathbb{N}}\lambda_j m_j$ *in* $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$. *Moreover, let*

$$\Lambda_q(\{\lambda_j m_j\}_{j\in\mathbb{N}}):=\inf\Big\{\lambda\in(0,\infty):\ \sum_{j\in\mathbb{N}}\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\Big)\le 1\Big\}\tag{3.5}$$

and then define

$$\|f\|_{\mathring{H}_{\mathrm{mol}}^{\varphi,q,\varepsilon,c}(\mathcal{X})}:=\inf\Big\{\Lambda_q(\{\lambda_j m_j\}_{j\in\mathbb{N}}):\ f=\sum_{j\in\mathbb{N}}\lambda_j m_j\ \textit{in}\ (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'\Big\},$$

where the last infimum is taken over all the decompositions of f *as above.*

Next, we state the main results of this section as follows.

Theorem 3.1 *Let* η, φ, ϱ, ϑ *and* c *be the same as in Definition* 3.2. *Further assume that* $q\in(q(\varphi)[r(\varphi)]',\infty)$ *and*

$$\varepsilon\in\Big(\max\Big\{0,\omega\Big[\frac{q(\varphi)}{i(\varphi)}-1\Big]\Big\},\infty\Big),$$

where $q(\varphi)$, $r(\varphi)$ *and* $i(\varphi)$ *are the same, respectively, as in* (2.7), (2.8) *and* (2.4). *Then*

$$H^\varphi(\mathcal{X})\cap(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'=\mathring{H}_{\mathrm{mol}}^{\varphi,q,\varepsilon,c}(\mathcal{X})\cap(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$$

with equivalent quasi-norms.

To prove Theorem 3.1, we need more preparations. The following definition is a generalization of [43, Definition 2.3] on $\mathbb{R}^n$ to $\mathcal{X}$.

Definition 3.3 *Let* φ *be a growth function in Definition* 2.6 *and* $q\in[1,\infty]$. *For any* μ*-measurable subset* E *of* $\mathcal{X}$, *the space* $L^q_\varphi(E)$ *is defined to be the set of all the* μ*-measurable functions* f *on* $\mathcal{X}$, *supported in* E, *such that*

$$\|f\|_{L^q_\varphi(E)}:=\begin{cases}\sup\limits_{t\in(0,\infty)}\Big[\dfrac{1}{\varphi(E,t)}\displaystyle\int_E|f(x)|^q\varphi(x,t)\,\mathrm{d}\mu(x)\Big]^{\frac1q}<\infty, & \textit{if } q\in[1,\infty),\\ \|f\|_{L^\infty(E)}<\infty, & \textit{if } q=\infty.\end{cases}$$

Now, we recall the concept of (φ,q)-atoms (see, for instance, [22, Definition 5.2]).

Definition 3.4 *Let* φ *be a growth function in Definition* 2.6 *satisfying* (2.12) *and let* $q\in(q(\varphi),\infty]$ *with* $q(\varphi)$ *the same as in* (2.7). *A* μ*-measurable function* a *is called a* (φ,q)*-atom supported in a ball* $B\subset\mathcal{X}$ *if the following three conditions hold true*:

(i) $\operatorname{supp} a:=\{x\in\mathcal{X}:\ a(x)\neq 0\}\subset B$;

(ii) $a\in L^q_\varphi(B)$ *and* $\|a\|_{L^q_\varphi(B)}\le\|\mathbf{1}_B\|^{-1}_{L^\varphi(\mathcal{X})}$;

(iii) $\int_{\mathcal{X}}a(x)\,\mathrm{d}\mu(x)=0$.

Remark 3.2 By (2.4), (2.9) and (2.12), we conclude that

$$m(\varphi)\le\Big\lfloor\omega\Big[\frac{q(\varphi)}{p}-1\Big]\Big\rfloor\le\Big\lfloor\omega\Big(\frac{\omega+\eta}{\omega}-1\Big)\Big\rfloor=0.$$

Thus, there is no need to add the additional assumption that $m(\varphi)\le 0$ in [22, Definition 5.2].

The following atomic Musielak-Orlicz Hardy space was introduced in [22, Definition 6.9].

Definition 3.5 *Let φ be a growth function in Definition 2.6 satisfying (2.12) and let $q \in (q(\varphi), \infty]$ with $q(\varphi)$ the same as in (2.7). Assume that ϱ, ϑ and η are the same, respectively, as in (2.13) and Definition 2.9.*

(i) *The atomic Musielak-Orlicz Hardy space $\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$ is defined to be the set of all the distributions $f \in (\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$ satisfying that there exists a sequence $\{b_j\}_{j\in\mathbb{N}}$ of multiples of (φ, q)-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbb{N}}$ such that*

$$\sum_{j\in\mathbb{N}} \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty$$

and $f = \sum_{j\in\mathbb{N}} b_j$ in $(\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$. Moreover, let

$$\widetilde{\Lambda}_q(\{b_j\}_{j\in\mathbb{N}}) := \inf\left\{\lambda \in (0,\infty) : \sum_{j\in\mathbb{N}} \varphi\left(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \le 1\right\} \tag{3.6}$$

and

$$\|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})} := \inf\left\{\widetilde{\Lambda}_q(\{b_j\}_{j\in\mathbb{N}}) : f = \sum_{j\in\mathbb{N}} b_j \text{ in } (\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'\right\},$$

where the last infimum is taken over all the decompositions of f as above.

(ii) *The atomic Musielak-Orlicz Hardy space $\mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})$ is defined to be the set of all the distributions $f \in (\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$ satisfying that there exists a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_j\}_{j\in\mathbb{N}}$ of (φ, q)-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbb{N}}$ such that*

$$\sum_{j\in\mathbb{N}} \varphi\left(B_j, \frac{|\lambda_j|}{\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) < \infty$$

and $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'$. Moreover, let

$$\|f\|_{\mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})} := \inf\left\{\widehat{\Lambda}_q(\{\lambda_j a_j\}_{j\in\mathbb{N}}) : f = \sum_{j\in\mathbb{N}} \lambda_j a_j \text{ in } (\mathring{\mathcal{G}}^{\eta}_0(\varrho,\vartheta))'\right\} < \infty,$$

where the infimum is taken over all the admissible decompositions of f as above and

$$\widehat{\Lambda}_q(\{\lambda_j a_j\}_{j\in\mathbb{N}}) := \inf\left\{\lambda \in (0,\infty) : \sum_{j\in\mathbb{N}} \varphi\left(B_j, \frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^\varphi(\mathcal{X})}}\right) \le 1\right\}. \tag{3.7}$$

The following lemma is a generalization of [46, Remark 31(i)] on the anisotropic Euclidean space to $\mathcal{X}$, the proof of which is a slight modification of [46, Remark 31(i)], we omit the details here.

Lemma 3.2 *Let φ and q be the same as in Definition 3.5. Then the atomic Musielak-Orlicz Hardy spaces $\mathring{H}^{\varphi,q}_{\mathrm{at,A}}(\mathcal{X})$ and $\mathring{H}^{\varphi,q}_{\mathrm{at}}(\mathcal{X})$ coincide with equivalent (quasi-)norms.*

The following atomic characterization of $H^\varphi(\mathcal{X})$ is just [22, Theorem 6.15].

Lemma 3.3 *Let η, φ, q, ϱ and ϑ be the same as in Definition 3.5. Then*

$$H^\varphi(\mathcal{X})\cap(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'=\mathring{H}_{\rm at}^{\varphi,q}(\mathcal{X})\cap(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$$

with equivalent quasi-norms.

Remark 3.3 From Remark 2.2(ii) and Lemma 3.3, we deduce that the space $\mathring{H}_{\rm at}^{\varphi,q}(\mathcal{X})$ in Definition 3.5 is independent of the choices of $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ whenever $\varrho,\vartheta\in\big(\omega\big[\frac{q(\varphi)}{p}-1\big],\eta\big)$.

The following lemma shows that an admissible $(\varphi,q,\varepsilon,c)$-molecule can be decomposed into a sum of a sequence of (φ,q)-atoms.

Lemma 3.4 *Let η, φ, ϱ and ϑ be the same as in Definition 3.2. Further assume that $x_0\in\mathcal{X}$, $r_0\in(0,\infty)$, $c\in(1,\infty)$, $q\in(q(\varphi)[r(\varphi)]',\infty)$ and*

$$\varepsilon\in\left(\max\left\{0,\omega\left[\frac{q(\varphi)}{i(\varphi)}-1\right]\right\},\infty\right),$$

where $q(\varphi)$, $r(\varphi)$ and $i(\varphi)$ are the same, respectively, as in (2.7), (2.8) and (2.4). If m is an admissible $(\varphi,q,\varepsilon,c)$-molecule centered at $B^{(0)}:=B(x_0,r_0)$, then there exists a $\widetilde{q}\in(q(\varphi),\infty)$ such that

$$m=\sum_{j=0}^\infty M_j+\sum_{j=1}^\infty\sum_{k=j}^\infty d_{j,k}$$

in $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$, where, for any $j\in\mathbb{Z}_+$ and $k\in[j,\infty)\cap\mathbb{Z}$, both M_j and $d_{j,k}$ are multiples of $(\varphi,\widetilde{q})$-atoms supported in $B^{(j)}:=B(x_0,r_j)$ with r_j the same as in Lemma 3.1. Moreover, there exists a positive constant C, independent of both j and k, such that

$$\widetilde{\Lambda}_{\widetilde{q}}(\{M_j\}_{j\in\mathbb{Z}_+}\cup\{d_{j,k}\}_{j\in\mathbb{N},k\in[j,\infty)\cap\mathbb{Z}})\le C\Lambda_q(\{m\}).\tag{3.8}$$

Proof Let all the symbols be the same as in the present lemma. For any given $j\in\mathbb{Z}_+$, let $A_j(B)$ be the same as in (3.4),

$$m_j:=\frac{\mathbf{1}_{B^{(j)}}}{\mu(B^{(j)})}\int_{\mathcal{X}}m(y)\mathbf{1}_{A_j(B)}(y)\,d\mu(y)$$

and

$$M_j:=m\mathbf{1}_{A_j(B)}-m_j.$$

Obviously,

$$m=\sum_{j=0}^\infty M_j+\sum_{j=0}^\infty m_j\tag{3.9}$$

pointwisely.

Now, we consider the first sum of (3.9). Fix any $j\in\mathbb{Z}_+$. We claim that M_j is a multiple of a $(\varphi,\widetilde{q})$-atom, where $\widetilde{q}\in(q(\varphi),\infty)$ is determined later. Indeed, it is easy to see that

$$\operatorname{supp} M_j\subset B^{(j)}\quad\text{and}\quad\int_{\mathcal{X}}M_j(x)\,\mathrm{d}\mu(x)=0. \tag{3.10}$$

Moreover, by both the Minkowski and the Hölder inequalities and (3.3), we obtain

$$\begin{aligned}\|M_j\|_{L^q(\mathcal{X})}&\le\|m\mathbf{1}_{A_j(B)}\|_{L^q(\mathcal{X})}+[\mu(B^{(j)})]^{-\frac{1}{q'}}\|m\mathbf{1}_{A_j(B)}\|_{L^1(\mathcal{X})}\\&\le 2\|m\mathbf{1}_{A_j(B)}\|_{L^q(\mathcal{X})}\le 2\varepsilon_j[\mu(B^{(j)})]^{\frac{1}{q}}\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})}.\end{aligned} \tag{3.11}$$

Since $q>q(\varphi)[r(\varphi)]'$, it follows that there exists a $\widetilde{q}\in(q(\varphi),\infty)$ such that $q>\widetilde{q}[r(\varphi)]'$, which further implies that $\varphi\in\mathbb{RH}_{(\frac{q}{\widetilde{q}})'}(\mathcal{X})$. From this, the Hölder inequality, (3.11) and Definition 2.5, we deduce that, for any $t\in(0,\infty)$,

$$\begin{aligned}&\left\{\frac{1}{\varphi(B^{(j)},t)}\int_{B^{(j)}}|M_j(y)|^{\widetilde{q}}\,\varphi(y,t)\,\mathrm{d}\mu(y)\right\}^{\frac{1}{\widetilde{q}}}\\&\le\frac{1}{[\varphi(B^{(j)},t)]^{\frac{1}{\widetilde{q}}}}\,\|M_j\|_{L^q(\mathcal{X})}\left\{\int_{B^{(j)}}[\varphi(y,t)]^{(\frac{q}{\widetilde{q}})'}\,\mathrm{d}\mu(y)\right\}^{\frac{1}{\widetilde{q}(\frac{q}{\widetilde{q}})'}}\\&\lesssim\frac{1}{[\varphi(B^{(j)},t)]^{\frac{1}{\widetilde{q}}}}\varepsilon_j[\mu(B^{(j)})]^{\frac{1}{q}}\,\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})}\,[\mu(B^{(j)})]^{[\frac{1}{(\frac{q}{\widetilde{q}})'}-1]\frac{1}{\widetilde{q}}}[\varphi(B^{(j)},t)]^{\frac{1}{\widetilde{q}}}\\&=\varepsilon_j\,\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})},\end{aligned}$$

which, combined with Definition 3.3, further implies that there exists a positive constant C_1, independent of j, such that

$$\|M_j\|_{L^{\widetilde{q}}_\varphi(B^{(j)})}\le C_1\varepsilon_j\,\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})}. \tag{3.12}$$

Let

$$u_j:=\frac{C_1\varepsilon_j\|\mathbf{1}_{B^{(j)}}\|_{L^\varphi(\mathcal{X})}}{\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}}\quad\text{and}\quad b_j:=\frac{M_j}{u_j}.$$

Then, by both (3.10) and (3.12), we find that b_j is a $(\varphi,\widetilde{q})$-atom supported in $B^{(j)}$. Thus, M_j is a multiple of a $(\varphi,\widetilde{q})$-atom. From the Hölder inequality, (3.10)–(3.11), (3.1)–(3.2), $\varepsilon>0$ and Remark 2.1, we deduce that

$$\begin{aligned}\sum_{j=0}^\infty\|M_j\|_{L^1(\mathcal{X})}&\le\sum_{j=0}^\infty\|M_j\|_{L^q(\mathcal{X})}\,[\mu(B^{(j)})]^{\frac{1}{q'}}\lesssim\sum_{j=0}^\infty\varepsilon_j\mu(B^{(j)})\,\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})}\\&\lesssim\mu(B^{(0)})\,\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})}\sum_{j=0}^\infty[cC_{(\mu)}]^{(1-\frac{\omega+\varepsilon}{\omega})j}\\&\sim\mu(B^{(0)})\,\|\mathbf{1}_{B^{(0)}}\|^{-1}_{L^\varphi(\mathcal{X})}<\infty.\end{aligned}$$

Thus, $\sum_{j=0}^\infty M_j$ converges in $L^1(\mathcal{X})$ and hence in $(\mathring{\mathcal{G}}^\eta_0(\varrho,\vartheta))'$.

Next, we consider the second sum of (3.9). For any $j\in\mathbb{Z}_+$, let

$$\mathbf{1}_j:=\frac{\mathbf{1}_{B^{(j)}}}{\mu(B^{(j)})},\quad\widetilde{m}_j:=\int_{\mathcal{X}}m(y)\mathbf{1}_{A_j(B)}(y)\,\mathrm{d}\mu(y)\quad\text{and}\quad N_j:=\sum_{k=j}^\infty\widetilde{m}_k.$$

Then, by the cancellation of m (see Definition 3.2(i)$_2$), we obtain

$$N_0=\sum_{k=0}^{\infty}\widetilde{m}_k=\sum_{k=0}^{\infty}\int_{\mathcal{X}}m(y)\mathbf{1}_{A_k(B)}(y)\,\mathrm{d}\mu(y)=\int_{\mathcal{X}}m(y)\,\mathrm{d}\mu(y)=0,$$

which further implies that

$$\begin{aligned}\sum_{j=0}^{\infty}m_j&=\sum_{j=0}^{\infty}\mathbf{1}_j\widetilde{m}_j=\sum_{j=0}^{\infty}\mathbf{1}_j(N_j-N_{j+1})=\sum_{j=1}^{\infty}(\mathbf{1}_j-\mathbf{1}_{j-1})N_j\\&=\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}(\mathbf{1}_j-\mathbf{1}_{j-1})\widetilde{m}_k=:\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}d_{j,k}.\end{aligned}\tag{3.13}$$

Fix any $j\in\mathbb{N}$ and any $k\in[j,\infty)\cap\mathbb{Z}$. We claim that $d_{j,k}$ is a multiple of a $(\varphi,\widetilde{q})$-atom. Indeed, it is easy to show that

$$\operatorname{supp}\,d_{j,k}\subset B^{(j)}\quad\text{and}\quad\int_{\mathcal{X}}d_{j,k}(x)\,\mathrm{d}\mu(x)=0.\tag{3.14}$$

Moreover, by both the Minkowski and the Hölder inequalities, (3.3) and (3.1), we conclude that

$$\begin{aligned}\|d_{j,k}\|_{L^{\widetilde{q}}_{\varphi}(B^{(j)})}&\le\|d_{j,k}\|_{L^{\infty}(B^{(j)})}\lesssim\frac{1}{\mu(B^{(j-1)})}\|m\mathbf{1}_{A_k(B)}\|_{L^1(\mathcal{X})}\\&\le\frac{[\mu(A_k(B))]^{\frac{1}{q'}}}{\mu(B^{(j-1)})}\|m\mathbf{1}_{A_k(B)}\|_{L^q(\mathcal{X})}\le\frac{\varepsilon_k\mu(B^{(k)})}{\mu(B^{(j-1)})}\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1}\\&\le\varepsilon_k\frac{C_{(\mu)}[cC_{(\mu)}]^k\mu(B(x_0,r_0))}{[cC_{(\mu)}]^{j-1}\mu(B(x_0,r_0))}\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1}\\&\lesssim\varepsilon_k[cC_{(\mu)}]^{k-j+1}\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1},\end{aligned}\tag{3.15}$$

which further implies that there exists a positive constant C_2, independent of both j and k, such that

$$\|d_{j,k}\|_{L^{\widetilde{q}}_{\varphi}(B^{(j)})}\le C_2\varepsilon_k[cC_{(\mu)}]^{k-j+1}\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1}.\tag{3.16}$$

Define

$$v_{j,k}:=\frac{C_2\varepsilon_k[cC_{(\mu)}]^{k-j+1}\|\mathbf{1}_{B^{(j)}}\|_{L^{\varphi}(\mathcal{X})}}{\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}}\quad\text{and}\quad c_{j,k}:=\frac{d_{j,k}}{v_{j,k}}.$$

Then, by both (3.14) and (3.16), we find that $c_{j,k}$ is a $(\varphi,\widetilde{q})$-atom supported in $B^{(j)}$. Thus, $d_{j,k}$ is a multiple of a $(\varphi,\widetilde{q})$-atom. From (3.14)–(3.15), (3.2), $\varepsilon>0$ and Remark 2.1, we deduce that

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\|d_{j,k}\|_{L^{\infty}(\mathcal{X})}&\lesssim\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\varepsilon_k[cC_{(\mu)}]^{k-j+1}\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1}\\&\lesssim\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1}\sum_{j=1}^{\infty}[cC_{(\mu)}]^{-j}\sum_{k=j}^{\infty}[cC_{(\mu)}]^{(1-\frac{\omega+\varepsilon}{\omega})k}\\&\sim\|\mathbf{1}_{B^{(0)}}\|_{L^{\varphi}(\mathcal{X})}^{-1}<\infty.\end{aligned}$$

Thus, $\sum\limits_{j=1}^{\infty}\sum\limits_{k=j}^{\infty} d_{j,k}$ converges in $L^\infty(\mathcal{X})$ and hence in $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$. By the convergence of both $\sum\limits_{j=0}^{\infty} M_j$ and $\sum\limits_{j=1}^{\infty}\sum\limits_{k=j}^{\infty} d_{j,k}$, (3.9) and (3.13), we conclude that

$$m=\sum_{j=0}^{\infty} M_j+\sum_{j=1}^{\infty}\sum_{k=j}^{\infty} d_{j,k} \tag{3.17}$$

converges in $(\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$.

Moreover, by (2.7), (2.4) and

$$\varepsilon\in\left(\omega\left[\frac{q(\varphi)}{i(\varphi)}-1\right],\infty\right),$$

we find that there exists a $q_0\in(q(\varphi),\infty)$ and a $p_0\in(0,i(\varphi))$ such that $\varphi\in\mathbb{A}_{q_0}(\mathcal{X})$, φ is of uniformly lower type p_0, and

$$\varepsilon\in\left(\omega\left[\frac{q_0}{p_0}-1\right],\infty\right).$$

From these, (3.12), (3.16), (3.1), Lemma 2.2(iv), (3.2) and $\varepsilon>0$, we deduce that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}
&\sum_{j=0}^{\infty}\varphi\left(B^{(j)},\frac{\|M_j\|_{L_\varphi^{\widetilde{q}}(B^{(j)})}}{\lambda}\right)+\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\varphi\left(B^{(j)},\frac{\|d_{j,k}\|_{L_\varphi^{\widetilde{q}}(B^{(j)})}}{\lambda}\right)\\
&\lesssim\sum_{j=0}^{\infty}\varphi\left(B^{(j)},\frac{\varepsilon_j\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}^{-1}}{\lambda}\right)\\
&\quad+\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\varphi\left(B^{(j)},\frac{\varepsilon_k[cC_{(\mu)}]^{k-j+1}\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}^{-1}}{\lambda}\right)\\
&\lesssim\sum_{j=0}^{\infty}\varepsilon_j^{p_0}[cC_{(\mu)}]^{jq_0}\varphi\left(B^{(0)},\frac{1}{\lambda\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}}\right)\\
&\quad+\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\varepsilon_k^{p_0}[cC_{(\mu)}]^{p_0(k-j+1)+jq_0}\varphi\left(B^{(0)},\frac{1}{\lambda\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}}\right)\\
&\sim\left\{\sum_{j=0}^{\infty}[cC_{(\mu)}]^{j[q_0-\frac{(\omega+\varepsilon)p_0}{\omega}]}+\sum_{j=1}^{\infty}[cC_{(\mu)}]^{j(q_0-p_0)}\sum_{k=j}^{\infty}[cC_{(\mu)}]^{p_0k(1-\frac{\omega+\varepsilon}{\omega})}\right\}\\
&\quad\times\varphi\left(B^{(0)},\frac{1}{\lambda\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}}\right)\\
&\lesssim\sum_{j=0}^{\infty}[cC_{(\mu)}]^{j[q_0-\frac{(\omega+\varepsilon)p_0}{\omega}]}\varphi\left(B^{(0)},\frac{1}{\lambda\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}}\right)\\
&\sim\varphi\left(B^{(0)},\frac{1}{\lambda\|\mathbf{1}_{B^{(0)}}\|_{L^\varphi(\mathcal{X})}}\right),
\end{aligned}$$

which, combined with (3.17) and (3.5)–(3.6), further implies that

$$\widetilde{\Lambda}_{\widetilde{q}}(\{M_j\}_{j\in\mathbb{Z}_+}\cup\{d_{j,k}\}_{j\in\mathbb{N},k\in[j,\infty)\cap\mathbb{Z}})\lesssim\Lambda_q(\{m\}).$$

This finishes the proof of Lemma 3.4.

Next, we prove Theorem 3.1.

Proof of Theorem 3.1 Let all the symbols be the same as in the present theorem. It is easy to show that any (φ,∞)-atom is an admissible $(\varphi,\infty,\varepsilon,c)$-molecule. Thus, by Lemma 3.3, we obtain

$$H^{\varphi}(\mathcal{X})\subset \mathring{H}_{\rm at}^{\varphi,\infty}(\mathcal{X})\subset \mathring{H}_{\rm mol}^{\varphi,\infty,\varepsilon,c}(\mathcal{X}), \tag{3.18}$$

and for any $f\in H^{\varphi}(\mathcal{X})$,

$$\|f\|_{\mathring{H}_{\rm mol}^{\varphi,\infty,\varepsilon,c}(\mathcal{X})}\lesssim \|f\|_{\mathring{H}_{\rm at}^{\varphi,\infty}(\mathcal{X})}\lesssim \|f\|_{H^{\varphi}(\mathcal{X})}.$$

Now, we show that

$$\mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})\subset \mathring{H}_{\rm at}^{\varphi,\widetilde{q}}(\mathcal{X}), \tag{3.19}$$

where $\widetilde{q}$ is the same as in Lemma 3.4. To this end, let $f\in \mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})$. Then, by Definition 3.2(ii), we know that there exist sequences $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and $\{m_i\}_{i\in\mathbb{N}}$ of admissible $(\varphi,q,\varepsilon,c)$-molecules centered, respectively, at balls $\{B_i\}_{i\in\mathbb{N}}$ of $\mathcal{X}$ with $B_i:=B(x_i,r_i^{(0)})$ for any $i\in\mathbb{N}$ such that $f=\sum_{i\in\mathbb{N}}\lambda_i m_i$ in $(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$ and

$$\Lambda_q(\{\lambda_i m_i\}_{i\in\mathbb{N}})\lesssim \|f\|_{\mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})}, \tag{3.20}$$

where, for any $i\in\mathbb{N}$, $x_i\in\mathcal{X}$ and $r_i^{(0)}\in(0,\infty)$. For any $i\in\mathbb{N}$ and $j\in\mathbb{Z}_+$, let $r_i^{(j)}$ be defined in Lemma 3.1 with x_0 and r_0 replaced, respectively, by x_i and $r_i^{(0)}$ and let $B_i^{(j)}:=B(x_i,r_i^{(j)})$. From these and Lemma 3.4, we deduce that

$$f=\sum_{i\in\mathbb{N}}\sum_{j=0}^{\infty}\lambda_i M_{i,j}+\sum_{i\in\mathbb{N}}\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\lambda_i d_{i,j,k}$$

in $(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$, where, for any $i\in\mathbb{N}$, $j\in\mathbb{Z}_+$ and $k\in[j,\infty)\cap\mathbb{Z}$, both $M_{i,j}$ and $d_{i,j,k}$ are multiples of $(\varphi,\widetilde{q})$-atoms supported in $B_i^{(j)}$. Moreover, by Definition 3.5, (3.8) and (3.20), we obtain

$$\begin{aligned}\|f\|_{\mathring{H}_{\rm at}^{\varphi,\widetilde{q}}(\mathcal{X})}&\le \widetilde{\Lambda}_{\widetilde{q}}(\{\lambda_i M_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}_+}\cup\{\lambda_i d_{i,j,k}\}_{i\in\mathbb{N},j\in\mathbb{N},k\in[j,\infty)\cap\mathbb{Z}})\\&\lesssim \Lambda_q(\{\lambda_i m_i\}_{i\in\mathbb{N}})\lesssim \|f\|_{\mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})},\end{aligned}$$

which further implies (3.19).

Combining (3.18)–(3.19) and Lemma 3.3, we have

$$H^{\varphi}(\mathcal{X})\subset \mathring{H}_{\rm mol}^{\varphi,\infty,\varepsilon,c}(\mathcal{X})\subset \mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})\subset \mathring{H}_{\rm at}^{\varphi,\widetilde{q}}(\mathcal{X})\subset H^{\varphi}(\mathcal{X}).$$

This finishes the proof of Theorem 3.1.

Remark 3.4 From Remark 2.2(ii) and Theorem 3.1, we deduce that the space $\mathring{H}_{\rm mol}^{\varphi,q,\varepsilon,c}(\mathcal{X})$ in Theorem 3.1 is independent of the choice of $(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$ whenever $\varrho,\vartheta\in\big(\omega\big[\frac{q(\varphi)}{p}-1\big],\eta\big)$.

4 Boundedness of Calderón-Zygmund Operators

In this section, we establish the boundedness of Calderón-Zygmund operators on $H^{\varphi}(\mathcal{X})$. To this end, we first recall some concepts.

Definition 4.1 *Let $s \in (0,1]$. A function $K:\ \{\mathcal{X} \times \mathcal{X}\}\backslash\{(x,x):\ x \in \mathcal{X}\} \to \mathbb{C}$ is called an s-Calderón-Zygmund kernel if there exists a positive constant $C_{(K)}$, depending on K, such that*

(i) *for any $x, y \in \mathcal{X}$ with $x \neq y$,*

$$|K(x,y)| \leq \frac{C_{(K)}}{V(x,y)};$$

(ii) *for any $x, \widetilde{x}, y \in \mathcal{X}$ satisfying that $0 < 2A_0 d(x,\widetilde{x}) \leq d(x,y)$ with A_0 the same as in* (2.1),

$$|K(x,y) - K(\widetilde{x},y)| + |K(y,x) - K(y,\widetilde{x})| \leq C_{(K)} \left[\frac{d(x,\widetilde{x})}{d(x,y)}\right]^s \frac{1}{V(x,y)}. \tag{4.1}$$

In what follows, let $C(\mathcal{X})$ denote the space of all continuous functions on $\mathcal{X}$ and $s \in (0,1]$. Recall that the space $C^s(\mathcal{X})$ is defined by setting

$$C^s(\mathcal{X}) := \{f \in C(\mathcal{X}):\ \|f\|_{C^s(\mathcal{X})} < \infty\}$$

with for any $f \in C^s(\mathcal{X})$,

$$\|f\|_{C^s(\mathcal{X})} := \|f\|_{L^\infty(\mathcal{X})} + \sup_{x,y \in \mathcal{X},\, x \neq y} \frac{|f(x) - f(y)|}{[d(x,y)]^s}.$$

Throughout this article, denote by $C_b^s(\mathcal{X})$ the space of all functions in $C^s(\mathcal{X})$ with bounded support, equipped with the strict inductive limit topology induced by $\|\cdot\|_{C^s(\mathcal{X})}$ (see, for instance, [55, p. 273] and [27, p. 23]), and denote by $(C_b^s(\mathcal{X}))'$ the space of all continuous linear functionals on $C_b^s(\mathcal{X})$ equipped with the weak-$*$ topology.

Let $T:\ C_b^s(\mathcal{X}) \to (C_b^s(\mathcal{X}))'$ be a linear continuous operator. Then T is called an s-Calderón-Zygmund operator if T can be extended to a bounded linear operator on $L^2(\mathcal{X})$ and if there exists an s-Calderón-Zygmund kernel K such that, for any $f \in L^2(\mathcal{X})$ and $x \notin \overline{\operatorname{supp} f}$,

$$T(f)(x) := \int_{\mathcal{X}} K(x,y) f(y)\, d\mu(y). \tag{4.2}$$

Recall that, for any given Calderón-Zygmund operator T, $T^*1 = 0$ means that

$$\int_{\mathcal{X}} T(f)(x)\, d\mu(x) = 0$$

for any $f \in L^2(\mathcal{X})$ with bounded support and $\int_{\mathcal{X}} f(x)\, d\mu(x) = 0$ (see, for instance, [74, p. 250]). Then we state the first main result of this section as follows.

Theorem 4.1 *Let $s \in (0,\eta]$ with η the same as in Definition* 2.9 *and let T be an s-Calderón-Zygmund operator satisfying that $T^*1 = 0$. Suppose that φ is a growth function in Definition* 2.6, *with uniformly lower type $p \in (0,1]$ satisfying that*

$$\frac{p}{q(\varphi)} \in \Big(\frac{\omega}{\omega + s}, 1\Big], \tag{4.3}$$

where $q(\varphi)$ *and* ω *are the same, respectively, as in* (2.7) *and* (2.3). *Then there exists a positive constant* $\widetilde{C}$ *such that, for any* $f \in H^{\varphi}(\mathcal{X})$,

$$\|T(f)\|_{H^{\varphi}(\mathcal{X})} \le \widetilde{C}\|f\|_{H^{\varphi}(\mathcal{X})}.$$

To prove Theorem 4.1, we need more preparations. The following lemma shows that the Calderón-Zygmund operator T is bounded on $L^q(\mathcal{X})$ for any given $q \in (1,\infty)$, which is just [16, Chapter III, Theorem 2.4].

Lemma 4.1 *Let* $q \in (1,\infty)$, $s \in (0,\eta]$ *with* η *the same as in Definition* 2.9, *and* T *be an* s*-Calderón-Zygmund operator. Then there exists a positive constant* C *such that, for any* $f \in L^q(\mathcal{X})$,

$$\|T(f)\|_{L^q(\mathcal{X})} \le C\|f\|_{L^q(\mathcal{X})}.$$

The following conclusion is just [22, Lemma 7.4] (see [69, Lemma 1.6.5] for the corresponding Euclidean case and [52, Lemma 4.1] for the anisotropic Euclidean case).

Lemma 4.2 *If* φ *is a growth function in Definition* 2.6 *satisfying* (2.12) *and if* $\tau \in (q(\varphi)[r(\varphi)]', \infty]$, *where* $q(\varphi)$ *and* $r(\varphi)$ *are the same, respectively, as in* (2.7) *and* (2.8), *then the space*

$$L^{\tau}_{b,0}(\mathcal{X}) := \left\{ f \in L^{\tau}(\mathcal{X}) :\ f \text{ has bounded support and } \int_{\mathcal{X}} f(x)\,\mathrm{d}\mu(x) = 0 \right\}$$

is dense in $H^{\varphi}(\mathcal{X})$.

The following conclusion is a simple application of both Lemma 3.2 and [22, Theorem 7.5], we omit the details; see also [52, Theorem 4.2] for the anisotropic Euclidean case.

Lemma 4.3 *Let* φ *and* τ *be the same as in Lemma* 4.2. *Then, for any* $q \in (q(\varphi), \frac{\tau}{[r(\varphi)]'})$, *where* $q(\varphi)$ *and* $r(\varphi)$ *are the same, respectively, as in* (2.7) *and* (2.8), *and for any* $f \in L^{\tau}_{b,0}(\mathcal{X})$, *there exists a finite sequence* $\{a_j\}_{j=1}^N (N \in \mathbb{N})$ *of* (φ, q)*-atoms and* $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ *such that*

$$f = \sum_{j=1}^{N} \lambda_j a_j.$$

Moreover, there exists a positive constant C *such that, for any* $f \in L^{\tau}_{b,0}(\mathcal{X})$ *with the decomposition as above,*

$$\widehat{\Lambda}_q(\{\lambda_j a_j\}_{j=1}^N) \le C\|f\|_{H^{\varphi}(\mathcal{X})}.$$

The following lemma plays a key role in the proof of Theorem 4.1.

Lemma 4.4 *Let* $q \in (1,\infty)$ *and* $\varphi \in \mathbb{A}_q(\mathcal{X})$ *be a growth function in Definition* 2.6 *with uniformly lower type* $p \in (0,1]$. *Then there exists a positive constant* C *such that, for any ball* $B \subset \mathcal{X}$ *and any* μ*-measurable set* $E \subset B$,

$$\|\mathbf{1}_B\|_{L^{\varphi}(\mathcal{X})} \le C\left[\frac{\mu(B)}{\mu(E)}\right]^{\frac{q}{p}} \|\mathbf{1}_E\|_{L^{\varphi}(\mathcal{X})}.$$

Proof Indeed, by the assumption that φ is of uniformly lower type p and Lemmas 2.2(iv) and 2.3(ii), we conclude that

$$\varphi\Big(B,\Big[\frac{\mu(E)}{\mu(B)}\Big]^{\frac{q}{p}}\|\mathbf{1}_E\|^{-1}_{L^\varphi(\mathcal{X})}\Big)\lesssim\Big[\frac{\mu(E)}{\mu(B)}\Big]^{q}\varphi(B,\|\mathbf{1}_E\|^{-1}_{L^\varphi(\mathcal{X})})$$
$$\le\varphi(E,\|\mathbf{1}_E\|^{-1}_{L^\varphi(\mathcal{X})})\sim 1\sim\varphi(B,\|\mathbf{1}_B\|^{-1}_{L^\varphi(\mathcal{X})}),$$

which, combined with the assumption that $\varphi(x,\cdot)$ is non-decreasing for almost every $x\in\mathcal{X}$, further implies that

$$\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}\lesssim\Big[\frac{\mu(B)}{\mu(E)}\Big]^{\frac{q}{p}}\|\mathbf{1}_E\|_{L^\varphi(\mathcal{X})},$$

where the implicit positive constant is independent of both B and E. This finishes the proof of Lemma 4.4.

Now, we prove Theorem 4.1.

Proof of Theorem 4.1 Let all the symbols be the same as in the present theorem. Assume that $c\in(1,\infty)$, $\tau\in(\max\{2,q(\varphi)[r(\varphi)]'\}q(\varphi)[r(\varphi)]',\infty)$,

$$r\in\Big(\max\{2,q(\varphi)[r(\varphi)]'\},\frac{\tau}{q(\varphi)[r(\varphi)]'}\Big)\quad\text{and}\quad q\in\Big(rq(\varphi),\frac{\tau}{[r(\varphi)]'}\Big),$$

where $r(\varphi)$ is the same as in (2.8). Next, we prove this theorem by three steps.

Step 1 In this step, we prove that, for any (φ,q)-atom a supported in a ball $B:=B(x_B,r_B)$ with $x_B\in\mathcal{X}$ and $r_B\in(0,\infty)$, $T(a)$ is a harmless constant multiple of an admissible (φ,r,s,c)-molecule centered at $\widetilde{B}^{(0)}:=4A_0^2B$. First, by the Hölder inequality and $q>rq(\varphi)\ge r>2$, we obtain $a\in L^2(\mathcal{X})$. From this, Definition 3.4(iii) and $T^*1=0$, we deduce that

$$\int_{\mathcal{X}}T(a)(x)\,\mathrm{d}\mu(x)=0.\tag{4.4}$$

Second, by $\frac{q}{r}\in(q(\varphi),\infty)$, we find that $\varphi\in\mathbb{A}_{\frac{q}{r}}(\mathcal{X})$, which, combined with Lemma 2.2(iii) and Definitions 3.3 and 3.4(ii), further implies that

$$\Big\{\frac{1}{\mu(B)}\int_B|a(x)|^r\,\mathrm{d}\mu(x)\Big\}^{\frac{q}{r}}\lesssim\frac{1}{\varphi(B,1)}\int_B|a(x)|^q\varphi(x,1)\,\mathrm{d}\mu(x)$$
$$\le\|a\|^q_{L^q_\varphi(B)}\le\frac{1}{\|\mathbf{1}_B\|^q_{L^\varphi(\mathcal{X})}}.\tag{4.5}$$

Moreover, from $\varphi\in\mathbb{A}_{\frac{q}{r}}(\mathcal{X})$, Lemma 4.4 and (2.3), we deduce that

$$\|\mathbf{1}_{\widetilde{B}^{(0)}}\|_{L^\varphi(\mathcal{X})}\lesssim\Big[\frac{\mu(\widetilde{B}^{(0)})}{\mu(B)}\Big]^{\frac{q}{rp}}\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}=\Big[\frac{\mu(\widetilde{B}^{(0)})}{\mu(B)}\Big]^{\frac{q}{rp}-\frac{1}{r}}\Big[\frac{\mu(\widetilde{B}^{(0)})}{\mu(B)}\Big]^{\frac{1}{r}}\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}$$
$$\lesssim\Big[\frac{\mu(\widetilde{B}^{(0)})}{\mu(B)}\Big]^{\frac{1}{r}}\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}.\tag{4.6}$$

Combining this, Lemma 4.1 and (4.5), we obtain

$$\|T(a)\|_{L^r(\widetilde{B}^{(0)})} \le \|T(a)\|_{L^r(\mathcal{X})} \lesssim \|a\|_{L^r(\mathcal{X})} \lesssim \frac{[\mu(B)]^{\frac{1}{r}}}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \lesssim \frac{[\mu(\widetilde{B}^{(0)})]^{\frac{1}{r}}}{\|\mathbf{1}_{\widetilde{B}^{(0)}}\|_{L^\varphi(\mathcal{X})}}. \tag{4.7}$$

Finally, for any $n \in \mathbb{N}$, let $\widetilde{B}^{(n)} := B(x_B, r_n)$ and $A_n(\widetilde{B}) := \widetilde{B}^{(n)} \backslash \widetilde{B}^{(n-1)}$, where r_n is defined the same as in Lemma 3.1 with x_0 and r_0 replaced, respectively, by x_B and $4A_0^2 r_B$. Obviously, for any $y \in B$, $n \in \mathbb{N}$ and $x \in A_n(\widetilde{B})$, we have

$$d(x_B, y) < r_B \le (2A_0)^{-1} d(x_B, x).$$

From this, $q > 2$, (4.2), Definition 3.4(iii), (4.1), the Hölder inequality, (4.5) and (2.3), we deduce that, for any $n \in \mathbb{N}$ and $x \in A_n(\widetilde{B})$,

$$\begin{aligned} |T(a)(x)| &= \left| \int_B K(x,y)a(y)\,\mathrm{d}\mu(y) \right| \le \int_B |K(x,y) - K(x,x_B)|\,|a(y)|\,\mathrm{d}\mu(y) \\ &\lesssim \int_B \left[\frac{d(x_B,y)}{d(x_B,x)}\right]^s \frac{1}{V(x_B,x)}\,|a(y)|\,\mathrm{d}\mu(y) \\ &\le \left[\frac{r_B}{d(x_B,x)}\right]^s \frac{1}{V(x_B,x)} \frac{\mu(B)}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \lesssim \left[\frac{\mu(B)}{V(x_B,x)}\right]^{\frac{\omega+s}{\omega}} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}, \end{aligned} \tag{4.8}$$

which, together with both (3.1) and (4.6), further implies that

$$\begin{aligned} \|T(a)\|_{L^r(A_n(\widetilde{B}))} &\lesssim \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \left\{ \int_{A_n(\widetilde{B})} \left[\frac{\mu(B)}{V(x_B,x)}\right]^{\frac{r(\omega+s)}{\omega}} \mathrm{d}\mu(x) \right\}^{\frac{1}{r}} \\ &\lesssim \left[\frac{\mu(B)}{\mu(\widetilde{B}^{(n-1)})}\right]^{\frac{\omega+s}{\omega}} \frac{[\mu(\widetilde{B}^{(n)})]^{\frac{1}{r}}}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \\ &\lesssim [cC_{(\mu)}]^{-\frac{\omega+s}{\omega}n} [\mu(\widetilde{B}^{(n)})]^{\frac{1}{r}} \left\|\mathbf{1}_{\widetilde{B}^{(0)}}\right\|_{L^\varphi(\mathcal{X})}^{-1}. \end{aligned} \tag{4.9}$$

Combining (4.4), (4.7) and (4.9), we conclude that $T(a)$ is a harmless constant multiple of an admissible (φ, r, s, c)-molecule centered at $\widetilde{B}^{(0)}$.

Step 2 In this step, we prove the present theorem in the case when $f \in L^\tau_{b,0}(\mathcal{X})$. Let $f \in L^\tau_{b,0}(\mathcal{X})$. By both $q < \frac{\tau}{[r(\varphi)]'}$ and Lemma 4.3, we find that there exists an $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$, and a sequence $\{a_j\}_{j=1}^N$ of (φ, q)-atoms supported, respectively, in the balls $\{B_j\}_{j=1}^N$ of $\mathcal{X}$ such that $f = \sum_{j=1}^N \lambda_j a_j$ and

$$\widehat{\Lambda}_q(\{\lambda_j a_j\}_{j=1}^N) \lesssim \|f\|_{H^\varphi(\mathcal{X})}. \tag{4.10}$$

From this, the linearity of T and Step 1, we further deduce that

$$T(f) = \sum_{j=1}^N \lambda_j T(a_j), \tag{4.11}$$

where, for any $j \in [1, N] \cap \mathbb{N}$, $T(a_j)$ is a harmless constant multiple of an admissible (φ, r, s, c)-molecule centered at $\widetilde{B}_j := 4A_0^2 B_j$. Moreover, by $\varphi \in \mathbb{A}_{\frac{q}{r}}(\mathcal{X})$, Lemma 2.2(iv), (2.3) and the assumption that $\varphi(x,\cdot)$ is non-decreasing for almost every $x\in\mathcal{X}$, we find that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}\sum_{j=1}^{N}\varphi\Big(\widetilde{B}_j,\frac{|\lambda_j|}{\lambda\|\mathbf{1}_{\widetilde{B}_j}\|_{L^{\varphi}(\mathcal{X})}}\Big)&\lesssim\sum_{j=1}^{N}\Big[\frac{\mu(\widetilde{B}_j)}{\mu(B_j)}\Big]^{\frac{q}{r}}\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\mathbf{1}_{\widetilde{B}_j}\|_{L^{\varphi}(\mathcal{X})}}\Big)\\&\lesssim\sum_{j=1}^{N}\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\mathbf{1}_{\widetilde{B}_j}\|_{L^{\varphi}(\mathcal{X})}}\Big)\\&\le\sum_{j=1}^{N}\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^{\varphi}(\mathcal{X})}}\Big).\end{aligned}\tag{4.12}$$

Meanwhile, from (4.3), we deduce that

$$s>\omega\Big[\frac{q(\varphi)}{p}-1\Big]\ge\omega\Big[\frac{q(\varphi)}{i(\varphi)}-1\Big],$$

which, combined with (4.11), $r>q(\varphi)[r(\varphi)]'$, Theorem 3.1, (4.12), (3.5), (3.7) and (4.10), further implies that

$$\begin{aligned}\|T(f)\|_{H^{\varphi}(\mathcal{X})}&\sim\Big\|\sum_{j=1}^{N}\lambda_jT(a_j)\Big\|_{\mathring{H}^{\varphi,r,s,c}_{\mathrm{mol}}(\mathcal{X})}\lesssim\Lambda_r(\{\lambda_jT(a_j)\}_{j=1}^{N})\\&\lesssim\widehat{\Lambda}_q(\{\lambda_ja_j\}_{j=1}^{N})\lesssim\|f\|_{H^{\varphi}(\mathcal{X})}.\end{aligned}\tag{4.13}$$

Step 3 Let $f\in H^{\varphi}(\mathcal{X})$. Then, by Lemma 4.2, we find that there exists a Cauchy sequence $\{f_k\}_{k\in\mathbb{N}}\subset L^{\tau}_{b,0}(\mathcal{X})$ such that

$$\lim_{k\to\infty}\|f_k-f\|_{H^{\varphi}(\mathcal{X})}=0.$$

From this, the linearity of T and (4.13), it follows that, as $k_1,k_2\to\infty$,

$$\|T(f_{k_1})-T(f_{k_2})\|_{H^{\varphi}(\mathcal{X})}=\|T(f_{k_1}-f_{k_2})\|_{H^{\varphi}(\mathcal{X})}\lesssim\|f_{k_1}-f_{k_2}\|_{H^{\varphi}(\mathcal{X})}\to0,$$

which implies that $\{T(f_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^{\varphi}(\mathcal{X})$. By this and the completeness of $H^{\varphi}(\mathcal{X})$ (see, for instance, [22, Remark 4.10(ii)]), we conclude that there exists some $h\in H^{\varphi}(\mathcal{X})$ such that $h=\lim_{k\to\infty}T(f_k)$ in $H^{\varphi}(\mathcal{X})$. Then let $T(f):=h$. From this and (4.13), it follows that $T(f)$ is well defined and, moreover, for any $f\in H^{\varphi}(\mathcal{X})$,

$$\begin{aligned}\|T(f)\|_{H^{\varphi}(\mathcal{X})}&\lesssim\limsup_{k\to\infty}[\|T(f)-T(f_k)\|_{H^{\varphi}(\mathcal{X})}+\|T(f_k)\|_{H^{\varphi}(\mathcal{X})}]\\&\le\limsup_{k\to\infty}\|T(f_k)\|_{H^{\varphi}(\mathcal{X})}\lesssim\limsup_{k\to\infty}\|f_k\|_{H^{\varphi}(\mathcal{X})}=\|f\|_{H^{\varphi}(\mathcal{X})}.\end{aligned}$$

This finishes the proof of Theorem 4.1.

Next, we state the second main result of this section as follows.

Theorem 4.2 *Let $s \in (0, \eta]$ with η the same as in Definition 2.9 and let T be an s-Calderón-Zygmund operator. Suppose that φ is a growth function in Definition 2.6 with uniformly lower type $p \in (0, 1]$ satisfying*

$$\frac{p}{q(\varphi)} \in \Big(\frac{\omega}{\omega + s}, 1\Big], \tag{4.14}$$

where $q(\varphi)$ and ω are the same, respectively, as in (2.7) and (2.3). Then there exists a positive constant C such that, for any $f \in H^\varphi(\mathcal{X})$,

$$\|T(f)\|_{L^\varphi(\mathcal{X})} \le C\|f\|_{H^\varphi(\mathcal{X})}.$$

The following boundedness of the Hardy-Littlewood maximal operator M in (2.6) on $L^\varphi(\mathcal{X})$ is just [22, Theorem 4.11] (see also [69, Corollary 2.1.2] for the corresponding Euclidean case).

Lemma 4.5 *Assume that φ is a Musielak-Orlicz function of uniformly lower type p_φ^- and of uniformly upper type p_φ^+ with $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, where $q(\varphi)$ is the same as in (2.7). Then the Hardy-Littlewood maximal function M is bounded on $L^\varphi(\mathcal{X})$, and moreover, there exists a positive constant C such that, for any $f \in L^\varphi(\mathcal{X})$,*

$$\int_{\mathcal{X}} \varphi(x, M(f)(x))\,\mathrm{d}\mu(x) \le C \int_{\mathcal{X}} \varphi(x, |f(x)|)\,\mathrm{d}\mu(x).$$

Now, we prove Theorem 4.2.

Proof of Theorem 4.2 Let all the symbols be the same as in the present theorem. Assume that $\tau \in (\max\{2, q(\varphi)[r(\varphi)]'\}q(\varphi)[r(\varphi)]', \infty)$,

$$r \in \Big(\max\{2, q(\varphi)[r(\varphi)]'\}, \frac{\tau}{q(\varphi)[r(\varphi)]'}\Big) \quad \text{and} \quad q \in \Big(rq(\varphi), \frac{\tau}{[r(\varphi)]'}\Big),$$

where $r(\varphi)$ is the same as in (2.8). Next, we prove this theorem by two steps.

Step 1 Let a be a (φ, q)-atom supported in a ball $B := B(x_B, r_B)$ with $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$ and let $\widetilde{B} := 4A_0^2 B$. In this step, we prove that there exists a positive constant C, independent of a, such that, for any $\lambda \in (0, \infty)$,

$$\int_{\mathcal{X}} \varphi(x, |T(\lambda a)(x)|)\,\mathrm{d}\mu(x) \le C\varphi(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}). \tag{4.15}$$

Indeed, on the one hand, by the assumptions that $\varphi(x, \cdot)$ is non-decreasing for almost every $x \in \mathcal{X}$ and of uniformly upper type 1, the Hölder inequality, the boundedness of T on $L^q_{\varphi(\cdot,1)}(\mathcal{X})$ (a generalization of [63, p. 173, Theorem 10] on $\mathbb{R}^n$ to $\mathcal{X}$), Definitions 3.3 and 3.4(ii) and Lemma 2.2(iv), we conclude that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned} &\int_{\widetilde{B}} \varphi(x, |T(\lambda a)(x)|)\,\mathrm{d}\mu(x) \\ &\lesssim \int_{\widetilde{B}} \Big[\frac{|T(a)(x)|}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}} + 1\Big]\varphi(x, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})\,\mathrm{d}\mu(x) \end{aligned}$$

$$
\begin{aligned}
&\lesssim \varphi(\widetilde{B}, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}) + \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \Big\{ \int_{\widetilde{B}} |T(a)(x)|^q \, \varphi(x, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}) \, \mathrm{d}\mu(x) \Big\}^{\frac{1}{q}} \\
&\quad \times [\varphi(\widetilde{B}, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})]^{\frac{1}{q'}} \\
&\lesssim \varphi(\widetilde{B}, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}) + \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \Big\{ \int_{\widetilde{B}} |a(x)|^q \, \varphi(x, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}) \, \mathrm{d}\mu(x) \Big\}^{\frac{1}{q}} \\
&\quad \times [\varphi(\widetilde{B}, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1})]^{\frac{1}{q'}} \\
&\le \varphi(\widetilde{B}, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}) + \|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})} \|a\|_{L^q_\varphi(\widetilde{B})} \varphi(\widetilde{B}, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}) \\
&\lesssim \varphi(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}).
\end{aligned} \tag{4.16}
$$

On the other hand, from an argument similar to that used in the estimation of (4.8) and from (2.6), we deduce that, for any $x \in \widetilde{B}^{\complement}$,

$$
|T(a)(x)| \lesssim \Big[\frac{\mu(B)}{V(x_B, x)}\Big]^{\frac{\omega+s}{\omega}} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \le \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} [M(\mathbf{1}_B)(x)]^{\frac{\omega+s}{\omega}},
$$

which, together with both (4.14) and Lemma 4.5, further implies that, for any $\lambda \in (0, \infty)$,

$$
\begin{aligned}
&\int_{\widetilde{B}^{\complement}} \varphi(x, |T(\lambda a)(x)|) \, \mathrm{d}\mu(x) \\
&\lesssim \int_{\widetilde{B}^{\complement}} \varphi\Big(x, \frac{\lambda [M(\mathbf{1}_B)(x)]^{\frac{\omega+s}{\omega}}}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\Big) \, \mathrm{d}\mu(x) \\
&\le \int_{\mathcal{X}} \widetilde{\varphi}\Big(x, M\Big(\Big[\frac{\lambda}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\Big]^{\frac{\omega}{\omega+s}} \mathbf{1}_B\Big)(x)\Big) \, \mathrm{d}\mu(x) \\
&\lesssim \int_{\mathcal{X}} \widetilde{\varphi}\Big(x, \Big[\frac{\lambda}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}}\Big]^{\frac{\omega}{\omega+s}} \mathbf{1}_B(x)\Big) \, \mathrm{d}\mu(x) \\
&= \int_{\mathcal{X}} \varphi\Big(x, \frac{\lambda}{\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}} \mathbf{1}_B(x)\Big) \, \mathrm{d}\mu(x) = \varphi(B, \lambda\|\mathbf{1}_B\|_{L^\varphi(\mathcal{X})}^{-1}),
\end{aligned} \tag{4.17}
$$

where, for any $x \in \mathcal{X}$ and $t \in (0, \infty)$, $\widetilde{\varphi}(x, t) := \varphi(x, t^{\frac{\omega+s}{\omega}})$. Combining (4.16) and (4.17), we obtain (4.15).

Step 2 Let $f \in L^\tau_{b,0}(\mathcal{X})$. By $q < \frac{\tau}{[r(\varphi)]'}$ and Lemma 4.3, we find that there exists an $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and a sequence $\{a_j\}_{j=1}^N$ of (φ, q)-atoms supported, respectively, in the balls $\{B_j\}_{j=1}^N$ of $\mathcal{X}$ such that $f = \sum_{j=1}^N \lambda_j a_j$ and

$$
\widehat{\Lambda}_q(\{\lambda_j a_j\}_{j=1}^N) \lesssim \|f\|_{H^\varphi(\mathcal{X})}. \tag{4.18}
$$

From this and the linearity of T, we further deduce that

$$
T(f) = \sum_{j=1}^N \lambda_j T(a_j).
$$

Thus, by the assumption that $\varphi(x, \cdot)$ is non-decreasing for almost every $x \in \mathcal{X}$, Lemma 2.3(i) and (4.15), we find that, for any $\lambda \in (0, \infty)$,

$$
\int_{\mathcal{X}} \varphi\Big(x, \frac{|T(f)(x)|}{\lambda}\Big) \, \mathrm{d}\mu(x) \le \int_{\mathcal{X}} \varphi\Big(x, \frac{1}{\lambda} \sum_{j=1}^N |\lambda_j| \, |T(a_j)(x)|\Big) \, \mathrm{d}\mu(x)
$$

$$\le \sum_{j=1}^{N} \int_{\mathcal{X}} \varphi\Big(x, \frac{|\lambda_j||T(a_j)(x)|}{\lambda}\Big)\, \mathrm{d}\mu(x)$$
$$\lesssim \sum_{j=1}^{N} \varphi\Big(B_j, \frac{|\lambda_j|}{\lambda\|\mathbf{1}_{B_j}\|_{L^{\varphi}(\mathcal{X})}}\Big).$$

This, combined with Definition 2.7, (3.7) and (4.18), further implies that,

$$\|T(f)\|_{L^{\varphi}(\mathcal{X})} \lesssim \widehat{\Lambda}_q(\{\lambda_j a_j\}_{j=1}^{N}) \lesssim \|f\|_{H^{\varphi}(\mathcal{X})}. \tag{4.19}$$

Finally, using Lemma 4.2, (4.19) and an argument similar to that used in the proof of Theorem 4.1, we then complete the proof of Theorem 4.2.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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