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On the Plus Parts of the Class Numbers of Cyclotomic Fields*

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Abstract The authors exhibit some new families of cyclotomic fields which have non-trivial plus parts of their class numbers. They also prove the 3-divisibility of the plus part of the class number of another family consisting of infinitely many cyclotomic fields. At the end, they provide some numerical examples supporting our results.

Keywords Class numbers, Maximal real subfield of cyclotomic fields, Real quadratic fields
2020 MR Subject Classification 11R29, 11R18, 11R80

1 Introduction

Let ζ_m be a primitive m-th root of unity for a positive integer m, then the field $K_m^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ is the maximal real subfield of the cyclotomic field $K_m = \mathbb{Q}(\zeta_m)$. Let $\mathcal{H}^+(m)$ denote the class-number of K_m^+ and $h(m)$ be that of $k_m = \mathbb{Q}(\sqrt{m})$. The class number $\mathcal{H}(m)$ of K_m can be written as $\mathcal{H}(m) = \mathcal{H}^+(m)\mathcal{H}^-(m)$. The factor $\mathcal{H}^-(m)$, so called relative class number, is well understood and is usually rather large. This factor can be determined in terms of Bernoulli numbers using the complex analytic class number formula (for details see [10, pp. 79–84]). Earlier in 1850, Kummer [8–9] computed $\mathcal{H}^-(m)$ for all primes m up to 97. One can find $\mathcal{H}^-(m)$ from the tables given in [19] by Schrutka von Rechtenstamm for any positive integer m satisfying $\phi(m) < 256$, where ϕ stands for Euler's phi function. In 1998, Schoof [18] computed $\mathcal{H}^-(m)$ for all odd primes $m < 509$ and in fact, he also gave the structure of the corresponding class groups.

On the other hand, the factor $\mathcal{H}^+(m)$ is not well understood and is notoriously hard to compute explicitly. In this case, the complex analytic class number formula is not very useful, since it requires the units of K_m^+ to be known. Till the date, there is no useful method to compute $\mathcal{H}^+(m)$, not even for relatively small m. The number $\mathcal{H}^+(m)$ is known only for all primes up to 151 and extended up to 241 under the assumption of GRH (generalized Riemann hypothesis). More precisely, Miller [13] proved that $\mathcal{H}^+(p) = 1$ for all prime $p \leq 151$ unconditionally. In the same paper, he also proved that $\mathcal{H}^+(p) = 1$ for all primes $p \leq 241$ except $p = 163, 181, 229$ for

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which $\mathcal{H}^+(p)$ is 4, 11, 3, respectively (again assuming GRH). For these primes, the Kummer-Vandiver conjecture, which says that m does not divide $\mathcal{H}^+(m)$ if m is a prime, holds. Recently, Buhler and Harvey [2] confirmed this conjecture for all primes less than 163577856.

On the other hand, Ankeny, Chowla and Hasse [1] proved that $\mathcal{H}^+(p) > 1$ if $p = (2nq)^2 + 1$ is a prime, where q is a prime and $n > 1$ is an integer. Subsequently, Lang [11] proved that $\mathcal{H}^+(p) > 1$ for any prime of the form $p = \{(2n+1)q\}^2 + 4$, where q is a prime and $n \geq 1$ is an integer. In 1987, Osada [15] generalized both the results. More precisely, he proved that $\mathcal{H}^+(m) > 1$ if $m = (2nq)^2 + 1$ is a square-free integer, where q is a prime and n is a positive integer such that $n \neq 1, q$. In the same paper, he also proved if $m = \{(2n+1)q\}^2+4$ is a square-free integer, where q is a prime and n is a positive integer such that $n \neq q$, then $\mathcal{H}^+(m) > 1$. All these results have been obtained in the case $m \equiv 1 (\bmod 4)$.

Furthermore, Takeuchi [20] established similar results for certain primes of the form $p \equiv 3 (\bmod 4)$. He proved that if both $12m + 7$ and $p = \{3(8m + 5)\}^2 - 2$ are primes, where $m \geq 0$ is an integer, then $\mathcal{H}^+(4p) > 1$. In the same paper, he also proved that if both $12m + 11$ and $p = \{3(8m + 7)\}^2 - 2$ are primes, where $m \geq 0$ is an integer, then $\mathcal{H}^+(4p) > 1$. Recently, Hoque and Saikia [6] generalized both the results. More precisely, they proved that if $m = \{3(8g+5)\}^2 - 2$ is a square-free integer, where g is a positive integer, then $\mathcal{H}^+(4m) > 1$. In the same paper, they also obtained a similar result for any square-free integer $m = \{3(8g+7)\}^2 - 2$, where g is a positive integer. Along the same line, we [4] produced some interesting families of cyclotomic fields whose maximal real subfields have class numbers bigger than one. All these results have been obtained in case when $m \equiv 3 (\bmod 4)$. Thus it would be interesting to try exhibiting similar families depending on m where $m \equiv 2 (\bmod 4)$. Here we find some families of cyclotomic fields K_{4m} whose maximal real subfields K_{4m}^+ have non-trivial class number when $m \equiv 2 (\bmod 4)$.

We discuss the results in three sections. In §2, we produce some families of cyclotomic fields K_{4m} with maximal real subfields K_{4m}^+ having non-trivial class numbers whenever $m \equiv 2 (\bmod 4)$. In §3, we discuss the divisibility of the class numbers of maximal real subfields of a class of cyclotomic fields. More precisely, we produce a family of cyclotomic fields whose maximal real subfields have class numbers a multiple of 3. In the concluding section, we provide some numerical evidence of our results. We have used [17, PARI 2.9.1] for these computations.

2 Non-triviality of $\mathcal{H}^+(m)$

We prove some results concerning the non-triviality of class numbers of certain maximal real subfields of cyclotomic fields. The proofs use elementary techniques on dealing with solutions of Diophantine equations, and basic properties of quadratic and cyclotomic fields. We begin with a family of real quadratic fields and show that they have non-trivial class numbers.

Proposition 2.1 *Let $m = \{14(2n+1)\}^2 + 2$ be square-free with n a positive integer. Then $h(m) > 1$.*

Proof We observe that

$$m = \{14(2n + 1)\}^2 + 2 \equiv 2 (\bmod 7).$$

Therefore the residue symbol, $\left(\frac{m}{7}\right) = \left(\frac{2}{7}\right) = 1$. Thus 7 splits completely in $k_m = \mathbb{Q}(\sqrt{m})$ as a product of a prime ideal $\mathfrak{A} \subset \mathcal{O}_{k_m}$ and its conjugate $\mathfrak{A}'$ with absolute norm $N_{k_m}(\mathfrak{A}) = 7$.

Let us assume $h(m) = 1$. Then $\mathfrak{A}$ is principal and thus, since $m \equiv 2(\bmod 4)$, we can write

$$\mathfrak{A} = (a + b\sqrt{m})$$

with $a, b \in \mathbb{Z}$. Therefore, we have (using $|N_{k_m}(\mathfrak{A})| = 7$)

$$a^2 - mb^2 = \pm 7.$$

We prove that such a Diophantine equation doesn't have any rational integer solution. Let us consider

$$a^2 - mb^2 = 7 \tag{2.1}$$

and let $t = 392n^2 + 392n + 99$. Then $t \equiv 1(\bmod 7)$ and $m = 2t$. Thus (2.1) implies that

$$a^2 \equiv 7(\bmod t). \tag{2.2}$$

But, quadratic reciprocity law gives

$$\left(\frac{7}{t}\right) = (-1)^{3(196n^2+196n+49)}\left(\frac{1}{7}\right) = -1,$$

which contradicts (2.2).

We now look at the other case, i.e.,

$$a^2 - mb^2 = -7.$$

Let a_0 be any integer and b_0 be the least positive integer such that

$$a_0^2 - mb_0^2 = -7. \tag{2.3}$$

Writing (2.3) in norm form, we have

$$N_{k_m}(\pm|a_0| + b_0\sqrt{m}) = -7. \tag{2.4}$$

We now multiply (2.4) with the norm of the unit,

$$\epsilon_m = \mp(s^2+1) + s\sqrt{m},$$

where $s = 14(2n+1)$, in the field k_m. Then we get

$$(-|a_0|(s^2+1) + b_0 sm)^2 - (\pm b_0(s^2+1) \mp |a_0|s)^2 m = -7.$$

Using now the minimality of b_0, we can write

$$|b_0(s^2+1) - |a_0|s| \geq b_0.$$

If $b_0(s^2+1) - |a_0|s \geq b_0$, then $|a_0| \leq b_0 s$, and therefore (2.3) gives that $2b_0^2 < 7$ and that would imply that $b_0 = 1$.

Now if $b_0 = 1$, the relation (2.3) implies that $a_0^2 \equiv 3 (\mathrm{mod}\, 4)$. This is an absurd.

In the other case, $|a_0|s - b_0(s^2 + 1) \geq b_0$ would give $|a_0|s > b_0 m$ and then (2.3) leads to $2mb_0^2 < -7s^2$. This is not possible as $m > 0$.

The following Lemma which can be derived from a result (see [16, main theorem]) of Osada, will be of our use.

Lemma 2.1 *Let m be a square-free positive integer. Then the ideal class group of $K^+_{\sigma_m^2 m}$ has a subgroup which is isomorphic to $\mathcal{C}(k_m)^2$, where $\mathcal{C}(k_m)$ is the ideal class group of $k_m = \mathbb{Q}(\sqrt{m})$, and*

$$\sigma_m = \begin{cases} 1, & \text{if } m \equiv 1 \pmod 4, \\ 2, & \text{if } m \equiv 2, 3 \pmod 4). \end{cases}$$

We thus obtain the first main result by applying together Proposition 2.1 and Lemma 2.1.

Theorem 2.1 *Let $m = \{14(2n + 1)\}^2 + 2$ be square-free with n a positive integer. Then $\mathcal{H}^+(4m) > 1$.*

We exhibit another family of cyclotomic fields with non-trivial plus parts in their class groups.

Proposition 2.2 *Let $m = (3(2n + 1))^2 + 1$ be a square-free integer with $n \geq 1$. Then $h(m) > 1$.*

Proof Let $m = (3(2n + 1))^2 + 1$. Then

$$m \equiv 1 (\mathrm{mod}\, 3)$$

and thus

$$\left(\frac{m}{3}\right) = 1.$$

Thus we can write

$$(3) = \mathfrak{B}\mathfrak{B}', \quad \mathfrak{B} \neq \mathfrak{B}',$$

where $\mathfrak{B}$ and $\mathfrak{B}'$ are prime (conjugates) ideals in $\mathcal{O}_{k_m}$ with $N_{k_m}(\mathfrak{B}) = 3$.

Let us assume $h(m) = 1$. Then $\mathfrak{B}$ is principal and thus since $m \equiv 2 (\mathrm{mod}\, 4)$, $\mathfrak{B}$ can be expressed as

$$\mathfrak{B} = (a + b\sqrt{m}) \quad \text{with } a, b \in \mathbb{Z}.$$

Therefore, we have

$$a^2 - mb^2 = \pm 3.$$

Clearly, $b \neq 0$. Let us assume that a_0 is an integer, and let b_0 be the least positive integer such that

$$a_0^2 - mb_0^2 = \pm 3. \tag{2.5}$$

Then $N_{k_m}(\alpha) = \pm 3$ for some integer $\alpha = a_0 - b_0\sqrt{m}$.

Let us suppose $r = 3(2n+1)$. Then the fundamental unit ϵ_m in k_m is given by

$$\epsilon_m = r + \sqrt{m}.$$

We now have $N_{k_m}(\alpha\epsilon_m) = \pm 3$ which implies that

$$(a_0 r - b_0 m)^2 - (a_0 - b_0 r)^2 m = \pm 3.$$

Employing the minimality of b_0, we have

$$|a_0 - b_0 r| \geq b_0.$$

If $a_0 - b_0 r \geq b_0$, then $a_0 \geq b_0(r+1)$ and thus (2.5) gives that $2rb_0^2 \leq \pm 3$. This is not possible as $r \geq 3$.

Again, if $b_0 r - a_0 \geq b_0$, then $a_0 \leq b_0(r-1)$. Thus from (2.5) we observe that $b_0^2(r-1)^2 - b_0^2 m \geq \pm 3$ which implies that $-2rb_0^2 \geq \pm 3$. This once again leads to an impossibility as $r \geq 3$. Thus we complete the proof.

We now use Proposition 2.2 and Lemma 2.1 to obtain the following result.

Theorem 2.2 *Let $m = (3(2n+1))^2 + 1$ with n a positive integer. Then $\mathcal{H}^+(4m) > 1$.*

We provide another similar family of maximal real subfields of certain cyclotomic fields each with class number bigger than 1.

Proposition 2.3 *Let $m = \{6(2n+1)\}^2 - 2$ with $n \geq 1$ an integer. Then $h(m) > 1$.*

Proof We observe that

$$m = \{6(2n+1)\}^2 - 2 \equiv 1 (\operatorname{mod} 3).$$

Therefore $\left(\frac{m}{3}\right) = \left(\frac{1}{3}\right) = 1$, and thus we have

$$(3) = \mathfrak{C}\mathfrak{C}'$$

with $\mathfrak{C} \neq \mathfrak{C}'$, where $\mathfrak{C}$ and $\mathfrak{C}'$ are prime ideals in $\mathcal{O}_{k_m}$ with absolute norm $N_{k_m}(\mathfrak{C}) = 3$.

Now if $h(m) = 1$, then $\mathfrak{C}$ is principal. And since $m \equiv 2 (\operatorname{mod} 4)$, we can write

$$\mathfrak{C} = (a + b\sqrt{m}),$$

where $a, b \in \mathbb{Z}$. Therefore, we have

$$a^2 - mb^2 = \pm 3.$$

We first look at the following equation

$$a^2 - mb^2 = 3. \tag{2.6}$$

Let us assume $t = 2n+1$. Then (2.6) can be written as

$$a^2 \equiv 3 (\operatorname{mod} 18t^2 - 1). \tag{2.7}$$

However, by quadratic reciprocity law, we see that

$$\Big(\frac{3}{18t^2-1}\Big) = (-1)^{\frac{18t^2-2}{2}}\Big(-\frac{1}{3}\Big) = -1.$$

This contradicts (2.7).

We now look at the following

$$a^2 - mb^2 = -3. \tag{2.8}$$

Clearly, $b \neq 0$. Let us suppose that (2.8) has a solution in integers and without loss of generality, let us assume that (a_0, b_0) is an integer solution, with $b_0 > 0$ the least one. Then

$$a_0^2 - mb_0^2 = -3. \tag{2.9}$$

In the norm form (2.9) can be written as $N_{k_m}(\alpha) = -3$ with $\alpha = a_0 - b_0\sqrt{m}$. Let $l = 6(2n+1)$. Then the fundamental unit ϵ_m in k_m is given by

$$\epsilon_m = (l^2-1) + l\sqrt{m}.$$

Thus $N_{k_m}(\alpha\epsilon_m) = -3$ and this implies that

$$(a_0(l^2-1) - b_0lm)^2 - (b_0(l^2-1) - a_0l)^2m = \pm 3.$$

By the minimality of b_0, we obtain

$$|b_0(l^2-1) - a_0l| \geq b_0.$$

If $-b_0(l^2-1) + a_0l \geq b_0$, then $a_0 \geq b_0l$ and thus (2.9) implies that $(l^2-m)b_0^2 < -3$. This is not possible.

Finally, $b_0(l^2-1) - a_0l \geq b_0$ implies $b_0m > a_0l$ and hence (2.9) gives $2mb_0^2 < 3l^2$ which implies $b_0 = 1$. Thus (2.9) implies $a_0^2 \equiv 3 \pmod 4$ which is not true. This completes the proof.

Applying Proposition 2.3 and Lemma 2.1, we obtain the following result.

Theorem 2.3 *Let $m = \{6(2n+1)\}^2 - 2$ with n a positive integer. Then $\mathcal{H}^+(4m) > 1$.*

3 Divisibility of $\mathcal{H}^+(m)$

In this section, we prove a result concerning the divisibility of the plus part $\mathcal{H}^+(m)$ of the class numbers of certain cyclotomic fields. We first fix some notations. For a number field K, we denote the discriminant, the norm map and trace map of K over $\mathbb{Q}$ by D_K, N_K and T_K, respectively. For an integer n and a prime p, by $v_p(n)$ we mean the greatest exponent μ of p such that $p^\mu \mid n$.

Let us assume that α is an algebraic integer in K such that $N_K(\alpha)$ is a cube in $\mathbb{Z}$. For such an α, define the cubic polynomial $f_\alpha(X)$ by

$$f_\alpha(X) := X^3 - 3(N_K(\alpha))^{\frac{1}{3}}X - T_K(\alpha).$$

[5, Lemma 2.1] (or [3, Lemma 2.2]) and [7, Proposition 6.5] together give the following proposition which is one of the main ingredient in the proof of the next theorem.

Proposition 3.1 *Let $d = -3d'$ for some square-free integer $d'(\neq 1, -3)$. Let α be an algebraic integer in $K' = \mathbb{Q}(\sqrt{d'})$ whose norm is a cube in $\mathbb{Z}$. Then the polynomial $f_\alpha(X)$ is irreducible over $\mathbb{Q}$ if and only if α is not a cube in K'. Moreover, if $f_\alpha(X)$ is irreducible over $\mathbb{Q}$, then the splitting field of $f_\alpha(X)$ over $\mathbb{Q}$ is an S_3-field containing $K = \mathbb{Q}(\sqrt{d})$ which is a cyclic cubic extension of K unramified outside 3 and contains a cubic subfield L with $v_3(D_L) \neq 5$.*

We extract the following result from [12, Theorem 1] which talks about ramification at $p = 3$.

Proposition 3.2 *Let us suppose that*

$$g(X) := X^3 - aX - b \in \mathbb{Z}[X]$$

is irreducible over $\mathbb{Q}$ and that either $v_3(a) < 2$ or $v_3(b) < 3$ holds. Set $K := \mathbb{Q}(\theta)$ for a root θ of $g(X)$. Then 3 is totally ramified in $K/\mathbb{Q}$ if and only if one of the following conditions holds:

(LN-1) $1 \leq v_3(b) \leq v_3(a)$;

(LN-2) $3 \mid a,\ a \not\equiv 3 \pmod 9,\ 3 \nmid b$ *and* $b^2 \not\equiv a + 1 \pmod 9$;

(LN-3) $a \equiv 3 \pmod 9,\ 3 \nmid b$ *and* $b^2 \not\equiv a + 1 \pmod{27}$.

Now we can proceed to our next result.

Theorem 3.1 *For a positive integer n satisfying $n \equiv 0 \pmod 3$, the class number of $K = \mathbb{Q}(\sqrt{3(4 \times 3^n - 1)})$ is divisible by 3. In fact, there are infinitely many such real quadratic fields with class number divisible by 3.*

Proof Let $d = 3(4 \times 3^n - 1)$ and $d' = 1 - 4.3^n$. Also let $K' = \mathbb{Q}(\sqrt{d'})$. Suppose $\alpha \in K'$ is defined by

$$\alpha = \frac{1 + \sqrt{d'}}{2}.$$

Then $T_{k'}(\alpha) = 1$ and $N_{K'}(\alpha) = 3^{\frac{n}{3}}$.

The cubic polynomial corresponding to α is

$$\begin{aligned} f_\alpha(X) &= X^3 - 3(N_{K'}(\alpha))^{\frac{1}{3}} X - T_{K'}(\alpha) \\ &= X^3 - 3^{\frac{n+3}{3}} X - 1 \\ &\equiv X^3 - X - 1 \pmod 2. \end{aligned}$$

Thus the polynomial $f_\alpha(X)$ is irreducible over $\mathbb{Z}_2$ and hence it is irreducible over $\mathbb{Q}$ too. Therefore by Proposition 3.1, the splitting field of $f_\alpha(X)$ over $\mathbb{Q}$ is a cyclic cubic extension of K which is unramified outside 3.

We now claim that the splitting field of $f_\alpha(X)$ is unramified over K at 3 too. One can easily see that the polynomial $f_\alpha(X)$ does not satisfy the conditions (LN-1), (LN-2) and (LN-3). Therefore, by Proposition 3.2, we prove the claim. Finally, by Hilbert class field theory the class number of K is divisible by 3.

The following result is an immediate implication of Theorem 3.1 and Lemma 2.1.

Theorem 3.2 *Let $m = 3(4 \times 3^n - 1)$ be square-free, where n is a positive integer satisfying $n \equiv 0 (\bmod 3)$. Then $3 \mid \mathcal{H}^+(m)$.*

4 Numerical Examples

In this section, we provide some numerical examples corroborating our results in §2 and in §3. It is sufficient to compute the class numbers of each of the families of underlying real quadratic fields, i.e., $h(m)$'s. We compute these class numbers for small values of m and list them in the tables below.

Table 1 Numerical examples of Theorem 2.1.

n	$m = \{14(2n+1)\}^2 + 2$	$h(m)$	n	$m = \{14(2n+1)\}^2 + 2$	$h(m)$
1	1766	5	2	4902	8
3	9606	18	4	15878	8
5	23718	12	6	33126	28
7	44102	15	8	56646	10
9	70758	5	10	86438	18
11	103686	32	12	122502	28
13	142886	33	14	164838	28
15	188358	44	16	213446	36
17	240102	7	18	268326	20
19	298118	21	20	329478	44
21	362406	4	22	396902	36
23	432966	66	24	470598	44
25	509798	55	26	550566	44
27	592902	21	28	636806	75
29	682278	68	30	729318	58
31	777926	57	32	828102	60

Table 2 Numerical examples of Theorem 2.2.

n	$m = \{3(2n+1)\}^2 + 1$	$h(m)$	n	$m = \{3(2n+1)\}^2 + 1$	$h(m)$
1	82	4	2	226	8
3	442	8	4	730	12
5	1090	12	6	1522	12
7	2026	14	8	2602	10
9	3250	4	10	3970	20
11	4762	22	12	5626	28
13	6502	16	14	7570	20
15	8650	6	16	9802	2
17	11026	44	18	12322	20
19	13690	2	20	15130	32
21	16642	28	22	18226	36
23	19882	34	24	21610	48
25	23410	52	26	25282	32
27	27226	58	28	29242	38
29	31330	40	30	33490	48

Table 3 Numerical examples of Theorem 2.3.

n	$m = \{6(2n+1)\}^2 - 2$	$h(m)$	n	$m = \{6(2n+1)\}^2 - 2$	$h(m)$
1	322	4	2	898	6
3	1762	4	4	2914	12
5	4354	16	6	6082	6
7	8098	12	8	10402	12
9	12994	24	10	15874	26
11	19042	24	12	22498	16
13	26242	12	14	30274	26
15	34594	4	16	39202	28
17	44098	16	18	49282	28
19	54754	24	20	60514	40
21	66562	24	22	72898	24
23	79522	48	24	86434	36
25	93634	56	26	101122	32
27	108898	40	28	116962	34
29	125314	44	30	133954	54
31	142882	40	32	152098	44
33	161602	8	34	171394	2
35	181474	60	36	191842	40
37	202498	52	38	213442	42
39	224674	60	40	236194	72

Table 4 Numerical examples of Theorem 3.1.

n	$m = 3(4 \times 3^n - 1)$	$h(m)$	n	$m = 3(4 \times 3^n - 1)$	$h(m)$
3	321	3	6	8745	12
9	236193	36	12	6377289	36
15	172186881	837	18	4649045865	36
21	125524238433	11232	24	3389154437769	36

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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