Chin. Ann. Math. Ser. B
46(2), 2025, 271–286
DOI: 10.1007/s11401-025-0015-2

Chinese Annals of Mathematics, Series B

Convexity and Uniform Monotone Approximation of Differentiable Function in Banach Spaces*

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Abstract In this paper, the author proves that if the dual X^* of X is weakly locally uniformly convex and the convex function f is continuous on X, then there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of continuous functions on X^{**} such that (1) $f_n(x) \leq f_{n+1}(x) \leq f(x) \leq g_{n+1}(x) \leq g_n(x)$ whenever $x \in X$; (2) the two convex functions f_n and g_n are Gâteaux differentiable on X; (3) $f_n \to f$ and $g_n \to f$ uniformly on X. Moreover, if the function f is coercive on X, then (1) f_n and g_n are two w^*-lower semicontinuous convex functions on X^{**}; (2) $\mathrm{epi} f_n = \overline{\mathrm{epi}\, f_n \cap (X \times R)}^{w^*}$ and $\mathrm{epi}\, g_n = \overline{\mathrm{epi}\, g_n \cap (X \times R)}^{w^*}$.

Keywords Uniform monotone approximation, Gâteaux differentiable, Weakly locally uniformly convex space, w^*-Lower semicontinuous convex functions
2020 MR Subject Classification 46B20

1 Introduction

In this paper, let $(X, \|\cdot\|)$ denote a real Banach space and X^* denote the dual space of Banach space X. Let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$. Moreover, let $x_n \xrightarrow{w} x$ denote that $\{x_n\}_{n=1}^{\infty}$ weakly converges to x and $x_n^* \xrightarrow{w^*} x^*$ denote that $\{x_n^*\}_{n=1}^{\infty}$ is weakly* convergent to x^*.

Definition 1.1 (see [8]) *Suppose that D is an open subset of Banach space X, a continuous function f is called Gâteaux (Frechet) differentiable at $x \in D$ if there exists a functional $d_G f(x) \in X^*$ ($d_F f(x) \in X^*$) such that*

$$\lim_{t\to 0}\Big[\frac{f(x+ty)-f(x)}{t} - \langle d_G f(x), y\rangle\Big] = 0$$

$$\Big(\lim_{t\to 0}\sup_{y\in B(X)}\Big[\frac{f(x+ty)-f(x)}{t} - \langle d_F f(x), y\rangle\Big] = 0\Big).$$

In 1979, Ekeland and Lebourg [6] proved that if a Banach space X is a strongly smooth space, then X is an Asplund space. In 1990, Preiss, Phelps and Namioka [10] proved that if a Banach space X is a smooth space, then X is a weak Asplund space. Converses of previous theorems fail in general. It is well known that continuous convex functions are not necessarily differentiable in

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*This work was supported by the National Natural Science Foundation of China (No. 12271121).

the definition domain, for example $y = |x|, x \in R$. Therefore, we naturally ask what conditions can ensure that every continuous convex functions can be uniformly approximated by a sequence of differentiable convex functions. In 2002, Cheng and Ruan studied uniform approximation of Lipschitzian convex functions in locally uniformly convex space (see [4]). In 2015, Azagra and Mudarra [1] proved that if the dual X^* of X is locally uniformly convex and f is a continuous convex function on X, then there exists a continuous convex function sequence $\{f_n\}_{n=1}^{\infty}$ on X such that f_n is Frechet differentiable on X and $f_n \to f$ uniformly on X. In 2019, Shang [11] proved that if X^* is a smooth space and convex function f is Lipschitzian on X^*, then there exist two w^*-lower semicontinuous convex function sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ such that

(1) $f_n(x^*) \le f_{n+1}(x^*) \le f(x^*) \le g_{n+1}(x^*) \le g_n(x^*)$ whenever $x^* \in X^*$;

(2) $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are Gâteaux differentiable on X^*;

(3) both function sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ converge uniformly to f on X^*.

In 2020, Shang [12] proved that if the dual X^* of X is a strictly convex space and the convex function f is coercive, bounded on every bounded subset of X, then f can be uniformly approximated by Gâteaux differentiable, continuous convex functions. We refer to [2, 4–5 7, 9, 14–15] for further details on the differentiability of functions.

From the above description, the geometric properties of Banach space play an essential role in studying the approximation via differentiable functions. In this paper, we continue to study the approximation properties of differentiable functions by using the geometric properties of Banach spaces. We prove that if the dual X^* of X is weakly locally uniformly convex and the convex function f is continuous on X, then there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of continuous functions on X^{**} such that

(1) $f_n(x) \le f_{n+1}(x) \le f(x) \le g_{n+1}(x) \le g_n(x)$ whenever $x \in X$;

(2) the two convex functions f_n and g_n are Gâteaux differentiable on X;

(3) $f_n \to f$ and $g_n \to f$ uniformly on X.

Moreover, if the convex function f is coercive on X, then (1) f_n and g_n are two w^*-lower semicontinuous convex functions on X^{**}; (2) $\operatorname{epi} f_n = \overline{\operatorname{epi} f_n \cap (X \times R)}^{w^*}$ and $\operatorname{epi} g_n = \overline{\operatorname{epi} g_n \cap (X \times R)}^{w^*}$. For the convenience of readers, we first recall some definitions and lemmas needed in this paper.

Definition 1.2 (see [3]) *A Banach space X is called weakly locally uniformly convex if $x_n \xrightarrow{w} x$ whenever $x \in S(X)$, $\{x_n\}_{n=1}^{\infty} \subset S(X)$ and $\|x_n + x\| \to 2$.*

Definition 1.3 (see [3]) *A Banach space X is called locally uniformly convex if $x_n \to x$ whenever $x \in S(X)$, $\{x_n\}_{n=1}^{\infty} \subset S(X)$ and $\|x_n + x\| \to 2$.*

It is well known that if X is a locally uniformly convex space, then X is weakly locally uniformly convex and the converse does not hold. It is well known that if the dual X^* of X is a locally uniformly convex space, then X is not necessarily reflexive.

Definition 1.4 (see [13]) *A point $x_0 \in C$ is said to be an exposed point of C if there exists* *a functional $x^* \in X^*$ such that $x^*(x) > x^*(y)$ whenever $y \in C \setminus \{x\}$.*

Definition 1.5 (see [13]) *A point $x_0 \in C$ is said to be a weakly exposed point of C if there exists a functional $x^* \in X^*$ such that $x_n \xrightarrow{w} x$ whenever $\{x_n\}_{n=1}^{\infty} \subset C$ and $x^*(x_n) \to \sigma_C(x^*)$, where $\sigma_C(x^*) = \sup\{x^*(x) : x \in C\}$.*

Lemma 1.1 (see [13]) *Suppose that C is a bounded closed convex subset of X and $x \in C$. Then σ_C is Gâteaux differentiable at point x^* and $d_G\sigma_C(x^*) = x$ if and only if the point x is a weakly exposed point of C and exposed by x^*.*

Definition 1.6 (see [8]) *Suppose that f is a continuous convex function on X. The subdifferential of f, denoted by ∂f, is the set-valued mapping given by $\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le f(y) - f(x)$ for each $y \in X\}$.*

It is well known that f is Gâteaux differentiable at x if and only the set $\partial f(x)$ is a singleton. Moreover, if the function f is a real-valued function, its epigraph is defined by

$$\operatorname{epi} f = \{(x, r) \in X \times R : f(x) \le r\}.$$

It is well known that if f is convex on X, then f is lower semi-continuous on X if and only if $\operatorname{epi} f$ is a closed subset of $X \times R$ and if f is convex on X^*, then f is w^*-lower semi-continuous on X^* if and only if $\operatorname{epi} f$ is a w^*-closed subset of $X^* \times R$. Moreover, a convex function f is said to be coercive if $\lim\limits_{\|x\|\to\infty} f(x) = +\infty$.

Lemma 1.2 (see [12]) *Suppose that $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of functions on X converging uniformly to f and $g_n = f_n + 2\sup\{f(x) - f_n(x) : x \in X\}$. Then there exists a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that $\{g_{n_k}\}_{k=1}^{\infty}$ is a decreasing sequence converging uniformly to f.*

2 Uniform Monotone Approximation of Convex Function in Banach Spaces

Theorem 2.1 *Let the dual X^* of X be a weakly locally uniformly convex space and the convex function f be continuous and coercive on X. Then there exist two sequences of w^*-lower semicontinuous convex functions on X^{**}, namaly $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$, such that*

(1) *$f_n(x) \le f_{n+1}(x) \le f(x) \le g_{n+1}(x) \le g_n(x)$ whenever $x \in X$;*

(2) *the two functions f_n and g_n are Gâteaux differentiable on X;*

(3) $\operatorname{epi} f_n = \overline{\operatorname{epi} f_n \cap (X \times R)}^{w^*}$ *and* $\operatorname{epi} g_n = \overline{\operatorname{epi} g_n \cap (X \times R)}^{w^*}$;

(4) *$f_n \to f$ and $g_n \to f$ uniformly on X.*

In order to prove the theorem, we give some lemmas.

Lemma 2.1 *Suppose that X is a Banach space such that its dual X^* is weakly locally uniformly convex and $C \subset X$ containing the origin. For $r > 0$, define sets $D = C + B(0, r)$,*

$$D^* = \{x^* \in X^* : \langle x^*, x\rangle \le 1, x \in D = C + B(0, r)\}$$

and

$$D^{**} = \{x^{**} \in X^{**} : \langle x^{**}, x^* \rangle \le 1, x^* \in D^*\},$$

respectively. If $x_0^{**} \in D^{**}$ *and* $x_0^* \in D^*$ *with* $\langle x_0^{**}, x_0^* \rangle = 1$*, then* x_0^* *is a weakly exposed point of* D^* *and is exposed by* x_0^{**}*.*

Proof Since $0 \in \operatorname{int} D$, we get that D^* is a bounded set. Let $x_0^{**} \in D^{**}$, $x_0^* \in D^*$, $\{x_n^*\}_{n=1}^\infty \subset D^*$ and $x_0^{**}(x_n^*) \to x_0^{**}(x_0^*) = 1$ as $n \to \infty$. Since D^* is a bounded set, we obtain that $\{x_n^*\}_{n=1}^\infty$ is a bounded sequence. Since the set $B(X^{***})$ is a weak* compact subset of X^{***}, there exists a subnet $\{x_\alpha^*, \alpha \in \Delta\}$ of $\{x_n^*\}_{n=1}^\infty$ such that $x_\alpha^* \xrightarrow{w^*} x_0^{***} \in X^{***}$ and the net $\{x_\alpha^*, \alpha \in \Delta\}$ contains an infinite number of terms in $\{x_n^*\}_{n=1}^\infty$. Since $x_0^{**}(x_n^*) \to x_0^{**}(x_0^*) = 1$, by $\{x_\alpha^*, \alpha \in \Delta\} \subset \{x_n^*\}_{n=1}^\infty$ and $x_\alpha^* \xrightarrow{w^*} x_0^{***} \in X^{***}$, we get that $x_0^{***}(x_0^{**}) = 1$. We will complete the proof by the following two steps.

Step 1 We will prove that $x_0^{***} = x_0^*$. In fact, suppose that $x_0^{***} \neq x_0^*$. Since the set $\overline{\text{co}}^{w^*}(C) + \overline{B(0,r)}^{w^*}$ is a w^*-closed subset of X^{**}, by the formula $\text{co}(C + B(0,r)) = \text{co}(C) + B(0,r)$, we get that

$$D^{**} = \overline{\text{co}}^{w^*}(C + B(0,r)) = \overline{\text{co}(C) + B(0,r)}^{w^*} = \overline{\text{co}}^{w^*}(C) + \overline{B(0,r)}^{w^*} \subset X^{**}. \tag{2.1}$$

Moreover, since $x_0^{**} \in D^{**}$, by (2.1), there exists a point $y_0^{**} \in \overline{\text{co}}^{w^*}(C)$ such that

$$x_0^{**} \in y_0^{**} + \overline{B(0,r)}^{w^*} \subset \overline{\text{co}}^{w^*}(C) + \overline{B(0,r)}^{w^*} = D^{**} \subset X^{**}. \tag{2.2}$$

Hence we get that $\|x_0^{**} - y_0^{**}\| \le r$. Therefore, by (2.2) and $x_0^{***}(x_0^{**}) = 1$, we get that

$$\begin{aligned} x_0^{***}(x_0^{**}) &= \sup\{x_0^{***}(x^{**}) : x^{**} \in D^{**}\} \\ &= \sup\{x_0^{***}(x^{**}) : x^{**} \in y_0^{**} + \overline{B(0,r)}^{w^*}\} \\ &= x_0^{***}(y_0^{**}) + \sup\{x_0^{***}(x^{**}) : x^{**} \in \overline{B(0,r)}^{w^*}\}. \end{aligned}$$

Therefore, by the above equations, we get that

$$\langle x_0^{***}, x_0^{**} - y_0^{**} \rangle = \sup\{\langle x_0^{***}, x^{**} \rangle : x^{**} \in \overline{B(0,r)}^{w^*}\} = r\|x_0^{***}\|. \tag{2.3}$$

Moreover, since $x_0^* \in D^*$ and $x_0^{**}(x_0^*) = 1$, as in the previous proof, we get that

$$\langle x_0^{**} - y_0^{**}, x_0^* \rangle = \sup\{\langle x^{**}, x_0^* \rangle : x^{**} \in \overline{B(0,r)}^{w^*}\} = r\|x_0^*\|. \tag{2.4}$$

Since $\|x_0^{***}\| > 0$ and $\|x_0^*\| > 0$, by (2.3)–(2.4), there exists a real number $k \in (0, +\infty)$ such that $\|kx_0^{***}\| = \|x_0^*\|$. Hence we get that

$$\langle kx_0^{***}, x_0^{**} - y_0^{**} \rangle = r\|kx_0^{***}\| = r\|x_0^*\| = \langle x_0^{**} - y_0^{**}, x_0^* \rangle. \tag{2.5}$$

Since $x_0^{***} \neq x_0^*$, there exists a weak* neighbourhood V of origin in X^{***} such that $(x_0^{***}+V)\cap(x_0^*+V)=\emptyset$. Therefore, by the formula $x_\alpha^* \xrightarrow{w^*} x_0^{***} \in X^{***}$, we may assume without loss of generality that $\{x_\alpha^*, \alpha\in\Delta\} \subset x_0^{***}+V$. This implies that

$$\{x_\alpha^*, \alpha\in\Delta\}\cap(x_0^*+V)=\emptyset. \tag{2.6}$$

Moreover, since $x_\alpha^* \xrightarrow{w^*} x_0^{***} \in X^{***}$ and $x_0^{***}(x_0^{**})=1$, we have the following formulas

$$\lim_{\alpha\in\Delta} x_0^{**}(x_\alpha^*) = x_0^{***}(x_0^{**}) = 1 \quad \text{and} \quad \lim_{\alpha\in\Delta}\langle x_0^{**}-y_0^{**}, x_\alpha^*\rangle = \langle x_0^{***}, x_0^{**}-y_0^{**}\rangle. \tag{2.7}$$

Since $x_\alpha^* \xrightarrow{w^*} x_0^{***} \in X^{***}$, by (2.5)–(2.7), we obtain that

$$\lim_{\alpha\in\Delta}\langle x_0^{**}-y_0^{**}, kx_\alpha^*\rangle = r\|kx_0^{***}\| = r\|x_0^*\| = \langle x_0^{**}-y_0^{**}, x_0^*\rangle. \tag{2.8}$$

Since $y_0^{**}+\overline{B(0,r)}^{w^*} \subset D^{**}$ and $\{x_\alpha^*, \alpha\in\Delta\}\subset D^*$, by (2.7), we get that

$$\sup\{x^{**}(x_\alpha^*) : x^{**}\in y_0^{**}+\overline{B(0,r)}^{w^*}\} \le 1 = x_0^{***}(x_0^{**}).$$

Therefore, by (2.7) and the above inequalities, we get that

$$\begin{aligned}
0 &= \lim_{\alpha\in\Delta}[x_0^{**}(x_\alpha^*) - x_0^{***}(x_0^{**})] \\
&\le \limsup_{\alpha\in\Delta}[x_0^{**}(x_\alpha^*) - \sup\{x^{**}(x_\alpha^*) : x^{**}\in y_0^{**}+\overline{B(0,r)}^{w^*}\}] \\
&= \limsup_{\alpha\in\Delta}[\langle x_0^{**}-y_0^{**}, x_\alpha^*\rangle - \sup\{x^{**}(x_\alpha^*) : x^{**}\in \overline{B(0,r)}^{w^*}\}] \\
&= \limsup_{\alpha\in\Delta}[\langle x_0^{**}-y_0^{**}, x_\alpha^*\rangle - r\|x_\alpha^*\|].
\end{aligned}$$

Therefore, by $\|x_0^{**}-y_0^{**}\|\le r$ and the above inequalities, we have $\langle x_0^{**}-y_0^{**}, kx_\alpha^*\rangle - kr\|x_\alpha^*\| \to 0$. Therefore, by (2.8) and $k>0$, we get that $\|kx_\alpha^*\|\to\|x_0^*\|$. Moreover, by formula (2.8) and $\|x_0^{**}-y_0^{**}\|\le r$, we obtain that

$$\begin{aligned}
\lim_{\alpha\in\Delta}[r\|kx_\alpha^*\| + r\|x_0^*\|] &\ge \limsup_{\alpha\in\Delta} r\|kx_\alpha^*+x_0^*\| \\
&\ge \limsup_{\alpha\in\Delta}[\|kx_\alpha^*+x_0^*\|\cdot\|x_0^{**}-y_0^{**}\|] \\
&\ge \limsup_{\alpha\in\Delta}\langle x_0^{**}-y_0^{**}, kx_\alpha^*+x_0^*\rangle \\
&= \lim_{\alpha\in\Delta}\langle x_0^{**}-y_0^{**}, kx_\alpha^*\rangle + \langle x_0^{**}-y_0^{**}, x_0^*\rangle \\
&= 2r\|x_0^*\|.
\end{aligned}$$

Therefore, by the above inequalities and $\|kx_\alpha^*\|\to\|x_0^*\|$, we have the following equations

$$\lim_{\alpha\in\Delta}\|kx_\alpha^*+x_0^*\| = \|x_0^*\| + \lim_{\alpha\in\Delta}\|kx_\alpha^*\| = 2\lim_{\alpha\in\Delta}\|kx_\alpha^*\| = 2\|x_0^*\|.$$

Since $\{x_\alpha^*, \alpha\in\Delta\}\subset\{x_n^*\}_{n=1}^\infty$, by the above equations, there exists a subsequence $\{x_{n_i}^*\}_{i=1}^\infty$ of $\{x_n^*\}_{n=1}^\infty$ such that

$$\{x_{n_i}^*\}_{i=1}^\infty \subset \{x_\alpha^*\}_{\alpha\in\Delta} \quad \text{and} \quad \lim_{i\to\infty}\|kx_{n_i}^*+x_0^*\| = \lim_{i\to\infty}2\|kx_{n_i}^*\| = 2\|x_0^*\|.$$

Since the space X^* is weakly locally uniformly convex, we obtain that $kx^*_{n_i} \xrightarrow{w} x^*_0$ as $i \to \infty$. Hence we obtain that $kx^*_{n_i}(x^{**}_0) \to x^*_0(x^{**}_0) = 1$ as $i \to \infty$. Therefore, by $x^*_n(x^{**}_0) \to x^*_0(x^{**}_0) = 1$, we obtain that $x^*_{n_i} \xrightarrow{w} x^*_0$ as $i \to \infty$. Hence we can assume without loss of generality that $x^*_{n_i} \in x^*_0 + V$ for every $i \in N$, which contradicts (2.6). Hence we obtain that $x^{***}_0 = x^*_0$.

Step 2 We next will prove that $x^*_n \xrightarrow{w} x^*_0$ as $n \to \infty$. In fact, suppose that there exists a subsequence $\{x^*_{n_l}\}_{l=1}^{\infty}$ of $\{x^*_n\}_{n=1}^{\infty}$ a weak neighbourhood V of origin in X^* such that $\{x^*_{n_l}\}_{l=1}^{\infty} \cap (x^*_0 + V) = \emptyset$. Since $x^{**}_0(x^*_{n_l}) \to x^{**}_0(x^*_0) = 1$, there exists a net $\{x^*_\beta\}_{\beta\in\Delta} \subset \{x^*_{n_l}\}_{l=1}^{\infty}$ such that $x^*_\beta \xrightarrow{w^*} x^{***}_1 \in X^{***}$. Therefore, from the previous proof, we get that $x^{***}_1 = x^*_0$, which contradicts $\{x^*_{n_l}\}_{l=1}^{\infty} \cap (x^*_0 + V) = \emptyset$. Hence $x^*_n \xrightarrow{w} x^*_0$ as $n \to \infty$. This implies that x^*_0 is a weakly exposed point of D^* and is exposed by x^{**}_0, which finishes the proof.

Lemma 2.2 *Suppose that f is a continuous convex function on X and*

$$f_0(x^{**}) = \inf\{r \in R : (x^{**}, r) \in \overline{\operatorname{epi} f}^{w^*}\}, \quad x^{**} \in X^{**}.$$

Then $f(x) = f_0(x)$ whenever $x \in X$.

Proof Pick a point $x_0 \in X$. Then we obtain that $f(x_0) \geq f_0(x_0)$. Suppose that $f(x_0) > f_0(x_0)$. Then there exists a real number $r \in (0, +\infty)$ such that $f(x_0) - r > f_0(x_0)$. Therefore, by the definition of f_0, there exists a net $\{(x_\alpha, r_\alpha), \alpha \in \Delta\}$ in $\operatorname{epi} f$ such that

$$(x_\alpha, r_\alpha) \xrightarrow{w} (x_0, f(x_0) - r) \in X \times R. \tag{2.9}$$

Since $(x_0, f(x_0) - r) \notin \operatorname{epi} f$, by the separation Theorem, there exist a functional $(x^*_0, t) \in X^* \times R$ and a real number $s > 0$ such that

$$\langle (x^*_0, t), (x_0, f(x_0) - r)\rangle - s > \sup\{\langle (x^*_0, -1), (x, s)\rangle : (x, s) \in \operatorname{epi} f\},$$

which contradicts (2.9). Hence we have $f(x_0) = f_0(x_0)$, which finishes the proof.

We next will prove Theorem 2.1.

Proof Let the convex function f be coercive and continuous on X. Then we may assume without loss of generality that $f(0) = -1$. Since the space X^* is weakly locally uniformly convex, we define the norm $\|(x^*, r)\| = \sqrt{\|x^*\|^2 + r^2}$ on $X^* \times R$. Then we get that $X^* \times R$ is a weakly locally uniformly convex space. Define the w^*-lower semicontinuous convex function

$$f_0(x^{**}) = \inf\{r \in R : (x^{**}, r) \in \overline{\operatorname{epi} f}^{w^*}\}, \quad x^{**} \in X^{**}.$$

Then, by Lemma 2.2, we get that $f(x) = f_0(x)$ whenever $x \in X$. For convenience, remember f_0 as f. We will complete the proof by the following two steps.

Step 1 We will construct the function f_n such that f_n satisfies the conclusions (1) and (4). Since the convex function f is w^*-lower semicontinuous on X^{**}, we get that $f|_X$ is continuous on X. Then we get that $\partial f|_X(x) \neq \emptyset$ for every $x \in X$. Since $f(0) = -1$, we get that

$$E_i = \Big\{x \in X : \sup_{x^* \in \partial f|_X(x)} \|x^*\| < 2^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\|\Big\} \neq \emptyset$$

for all natural number $i \in N$. Moreover, by the definition of E_i, we define the set

$$C_i = \{(x, r) \in X \times R : x \in E_i, r \geq f(x)\} \quad \text{for every } i \in N.$$

Since $f(0) = -1$, by the definition of E_i, we get that $(0, 0) \in C_i$ for each $i \in N$. Moreover, it is easy to see that $X = \bigcup_{i=1}^{\infty} E_i$. Define the set

$$H_{i,n} = C_i + B\Big((0,0), \frac{1}{n\Big[4^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i \Big[2^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\| + 2\Big]}\Big)$$

for all $n \in N$ and $i \in N$. Hence, for all $n \in N$ and $i \in N$, we define the function

$$h_{n,i}(x) = \inf\{r \in R : (x^{**}, r) \in H_{i,n}\}, \quad x \in E_i.$$

We assert that the following inequality

$$f(x) - h_{n,i}(x) \leq \frac{1}{n\Big[4^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i}, \quad x \in E_i \tag{2.10}$$

holds. Indeed, suppose that there exists a point $x_0 \in E_i$ such that

$$f(x_0) - h_{n,i}(x_0) > \frac{1}{n\Big[4^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i}.$$

Therefore, by the definition of $h_{n,i}$ and the above inequality, there exists a point $(x_0, d_{n,i}(x_0)) \in H_{i,n}$ such that

$$f(x_0) - d_{n,i}(x_0) \geq \frac{1}{n\Big[4^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i}. \tag{2.11}$$

Therefore, by the definition of $H_{i,n}$ and $(x_0, d_{n,i}(x_0)) \in H_{i,n}$, there exists a point $(u_0, f(u_0)) \in C_i$ such that

$$\begin{aligned} &(u_0, f(u_0)) - (x_0, d_{n,i}(x_0)) \\ &\in B\Big((0,0), \frac{1}{n\Big[4^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i \Big[2^i + \sup_{z^* \in \partial f|_X(0)} \|z^*\| + 2\Big]}\Big). \end{aligned} \tag{2.12}$$

Pick a functional $x_0^* \in \partial f|_X(x_0)$. Then, for any $y \in X$, we get that $\langle x_0^*, y - x_0\rangle \leq f(y) - f(x_0)$. Hence, for every $(y, t) \in \operatorname{epi} f$, we get that $\langle x_0^*, y - x_0\rangle \leq t - f(x_0)$. This implies that $x_0^*(x_0) - f(x_0) \geq x_0^*(y) - t$. Hence we have

$$\langle (x_0^*, -1), (x_0, f(x_0))\rangle = \sup\{\langle (x_0^*, -1), (y, t)\rangle : (y, t) \in \operatorname{epi} f\}. \tag{2.13}$$

Therefore, by $(u_0, f(u_0)) \in \operatorname{epi} f$ and (2.11), we get that

$$\langle (x_0^*, -1), (u_0, f(u_0))\rangle \leq \langle (x_0^*, -1), (x_0, f(x_0))\rangle \leq \langle (x_0^*, -1), (x_0, d_{n,i}(x_0))\rangle.$$

Hence there exists a real number $\lambda \in [0,1]$ such that

$$\langle (x_0^*, -1), \lambda(u_0, f(u_0)) + (1-\lambda)(x_0, d_{n,i}(x_0)) \rangle = \langle (x_0^*, -1), (x_0, f(x_0)) \rangle.$$

This implies that $\lambda(u_0, f(u_0)) + (1-\lambda)(x_0, d_{n,i}(x_0)) \in H(x_0, f(x_0))$, where

$$H(x_0, f(x_0)) = \{(x,r) : \langle (x_0^*, -1), (x,r) \rangle = \langle (x_0^*, -1), (x_0, f(x_0)) \rangle\}.$$

Therefore, by $\lambda(u_0, f(u_0)) + (1-\lambda)(x_0, d_{n,i}(x_0)) \in H(x_0, f(x_0))$ and (2.12), we have

$$\begin{aligned}
&\mathrm{dist}((x_0, d_{n,i}(x_0)), H(x_0, f(x_0))) \\
&\le \|\lambda(u_0, f(u_0)) + (1-\lambda)(x_0, d_{n,i}(x_0)) - (x_0, d_{n,i}(x_0))\| \\
&= \lambda \|(u_0, f(u_0)) - (x_0, d_{n,i}(x_0))\| \\
&\le \frac{\lambda}{n\Big[4^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i \Big[2^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\| + 2\Big]}. \\
&\le \frac{1}{n\Big[4^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i \Big[2^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\| + 2\Big]}.
\end{aligned} \tag{2.14}$$

Moreover, we define the hyperplane

$$H(0,0) = \{(x,r) \in X \times R : \langle (x_0^*, -1), (x,r) \rangle = 0\}$$

of $X \times R$. Since $x_0 \in E_i$, by (2.11) and the definition of E_i, we have the following inequalities

$$\begin{aligned}
&\mathrm{dist}((x_0, d_n(x_0)), H(x_0, f(x_0))) \\
&= \mathrm{dist}((x_0 - x_0, d_n(x_0) - f(x_0)), H(x_0, f(x_0)) - (x_0, f(x_0))) \\
&= \mathrm{dist}((0, d_n(x_0) - f(x_0)), H(0,0)) \\
&= \frac{1}{\sqrt{\|x_0^*\|^2 + 1}} |\langle (x_0^*, -1), (0, d_n(x_0) - f(x_0)) \rangle| \\
&= \frac{1}{\sqrt{\|x_0^*\|^2 + 1}} [f(x_0) - d_n(x_0)] \\
&\ge \frac{1}{\sqrt{\|x_0^*\|^2 + 1}} \left(\frac{1}{n\Big[4^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i} \right) \\
&> \frac{1}{\|x_0^*\| + 1} \left(\frac{1}{n\Big[4^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i} \right) \\
&\ge \frac{1}{n\Big[4^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\|\Big]^i \Big[2^i + \sup\limits_{z^* \in \partial f|_X(0)} \|z^*\| + 1\Big]},
\end{aligned}$$

which contradicts (2.14). Hence we obtain that (2.10) is true. This implies that $f(x) - h_{n,i}(x) <$

$\frac{1}{n}$ whenever $x \in E_i$. Hence, for every natural number $n \in N$, we define the function

$$h_n(x) = \begin{cases} h_{n,1}(x), & x \in E_1, \\ h_{n,2}(x), & x \in E_2 \backslash E_1, \\ \cdots & \cdots \\ h_{n,j}(x), & x \in E_j \backslash \Big(\bigcup\limits_{i=1}^{j} E_i \Big), \\ \cdots & \cdots \end{cases}$$

on X. Since $X = \bigcup\limits_{i=1}^{\infty} E_i$, by (2.10) and the definition of $h_{i,n}$, we get that $0 < f(x) - h_n(x) < \frac{2}{n}$ whenever $x \in X$. We define the w^*-lower semicontinuous convex function

$$f_n(x^{**}) = \inf\{r \in R : (x^{**}, r) \in \overline{\mathrm{co}}^{w^*}(\mathrm{epi}\, h_n)\}, \quad x^{**} \in X^{**}.$$

Then, by the definition of h_n, we get that $f(x^{**}) - f_n(x^{**}) \geq 0$ whenever $x^{**} \in X^{**}$. Since the convex function f is w^*-lower semicontinuous on X^{**}, by the inequality $h_n(x) > f(x) - \frac{2}{n}$ and the definition of f_n, we obtain that

$$\mathrm{epi}\, f_n = \overline{\mathrm{co}}^{w^*}(\mathrm{epi}\, h_n\Big) \subset \mathrm{epi}\Big(f - \frac{2}{n}\Big) \subset X^{**} \times R$$

for all $n \in N$. Hence we get that $0 \leq f(x^{**}) - f_n(x^{**}) < \frac{2}{n}$ whenever $x^{**} \in X^{**}$. Therefore, by Lemma 2.2, we obtain that $f_n \to f$ uniformly on X and $f_n(x) \leq f_{n+1}(x) \leq f(x)$ whenever $x \in X$ and $n \in N$.

Step 2 We will prove that the w^*-lower semicontinuous convex function f_n are Gâteaux differentiable at every point of X. Pick a natural number $n \in N$ and pick a point $(x_0, f_n(x_0)) \in \mathrm{epi}\, f_n$. Define the closed convex set

$$C_n^* = \{(x^*, s) \in X^* \times R : \langle (x^*, s), (x, t) \rangle \leq 1, (x, t) \in \mathrm{epi}\, f_n|_X\}. \tag{2.15}$$

Pick a functional $y_0^* \in \partial f_n|_X(x_0)$. Then we get that $y_0^*(x_0) - f_n(x_0) \geq -f_n(0) \geq 1$. Therefore, by the above formula, we get that

$$\langle (x_0^*, t), (x_0, f_n(x_0)) \rangle = \sup\{\langle (x_0^*, t), (x, r) \rangle : (x, r) \in \mathrm{epi}\, f_n|_X\} = 1, \tag{2.16}$$

where

$$x_0^* = \frac{y_0^*}{y_0^*(x_0) - f_n(x_0)} \quad \text{and} \quad t = -\frac{1}{y_0^*(x_0) - f_n(x_0)}. \tag{2.17}$$

Moreover, by $y_0^*(x_0) - f_n(x_0) \geq -f_n(0) \geq 1$, we obtain that $t < 0$. Let $\delta_n = f(x_0) - f_n(x_0)$. Then, by the definition of f_n, we obtain that $\delta_n > 0$. Moreover, by the definition of f_n, we get that $(x_0, f_n(x_0)) \in \overline{\mathrm{co}}^{w}(\mathrm{epi}\, h_n) \subset X \times R$. This implies that

$$(x_0, f_n(x_0)) \in \overline{\mathrm{co}}^{w}(\mathrm{epi}\, h_n) = \overline{\mathrm{co}}(\mathrm{epi}\, h_n) \subset X \times R.$$

Therefore, by $(x_0, f_n(x_0)) \in \overline{\mathrm{co}}(\mathrm{epi}\, h_n) \subset X \times R$, there exist two sequences $\{(z(i,j), r_0(i,j))\}_{j=1}^{\infty} \subset \mathrm{epi}\, h_n$ and $\{\lambda(i,j)\}_{j=1}^{\infty} \subset [0,1]$ such that

$$\sum_{i=1}^{k_j}[\lambda(i,j)\cdot(z(i,j), r_0(i,j))] \in \mathrm{co}(\mathrm{epi}\, h_n), \quad \sum_{i=1}^{k_j}\lambda(i,j) = 1,$$

$$f\Big(\sum_{i=1}^{k_j}\lambda(i,j)\cdot z(i,j)\Big) > f(x_0) - \frac{1}{16}\delta_n, \quad \sum_{i=1}^{k_j}[\lambda(i,j)\cdot r_0(i,j)] < f_n(x_0) + \frac{1}{16}\delta_n$$

and

$$\lim_{j\to\infty}\Big\langle (x_0^*, t), \sum_{i=1}^{k_j}\lambda(i,j)\cdot(z(i,j), r_0(i,j))\Big\rangle = \langle (x_0^*, t), (x_0, f_n(x_0))\rangle = 1.$$

Moreover, by the definition of h_n, there exists a real number $s(i,j) > 0$ such that $r_0(i,j)) \geq f(z(i,j)) - s(i,j) = h_n(z(i,j))$. Therefore, by (2.16) and $t < 0$, we get the following formulas

$$\sum_{i=1}^{k_j}[\lambda(i,j)\cdot(z(i,j), f(z(i,j)) - s(i,j))] \in \mathrm{co}(\mathrm{epi}\, h_n), \quad \sum_{i=1}^{k_j}\lambda(i,j) = 1,$$

$$\sum_{i=1}^{k_j}[\lambda(i,j)\cdot(f(z(i,j)) - s(i,j))] < f_n(x_0) + \frac{1}{16}\delta_n$$

and

$$\lim_{j\to\infty}\Big\langle (x_0^*, t), \sum_{i=1}^{k_j}\lambda(i,j)\cdot(z(i,j), f(z(i,j)) - s(i,j))\Big\rangle = \langle (x_0^*, t), (x_0, f_n(x_0)\rangle = 1.$$

Therefore, from the previous proof and the convexity of f, we have the following inequalities

$$\begin{aligned} f_n(x_0) + \frac{1}{16}\delta_n &> \sum_{i=1}^{k_j}[\lambda(i,j)\cdot(f(z(i,j)) - s(i,j))] \\ &\geq f\Big(\sum_{i=1}^{k_j}\lambda(i,j)\cdot z(i,j)\Big) - \sum_{i=1}^{k_j}[\lambda(i,j)\cdot s(i,j)] \\ &\geq f(x_0) - \frac{1}{16}\delta_n - \sum_{i=1}^{k_j}[\lambda(i,j)\cdot s(i,j)]. \end{aligned}$$

Therefore, by the above inequalities and $\delta_n = f(x_0) - f_n(x_0)$, we have the following inequalities

$$\sum_{i=1}^{k_j}[\lambda(i,j)\cdot s(i,j)] \geq [f(x_0) - f_n(x_0)] - \frac{1}{8}\delta_n = \frac{7}{8}\delta_n. \tag{2.18}$$

Moreover, by (2.10), it is easy to see that $\eta_{i,n} \to 0$ and $\rho_{i,n} \to 0$ as $i \to \infty$, where

$$\eta_{i,n} = \inf\{f(x) - h_{n,i}(x) : x \in E_i\} \leq \frac{1}{n\Big[4^i + \sup\limits_{z^*\in\partial f|_X(0)}\|z^*\|\Big]^i} = \rho_{i,n}.$$

Hence there exist $\mu_n \in (0,1)$ and $i_0 \in N$ such that $\mu_n \rho_{1,n} < \frac{\delta_n}{16}$ and $\rho_{i,n} < \frac{\delta_n}{16}$ whenever $i \geq i_0$. Since $\eta_{i,n} \leq \rho_{i,n}$, $\rho_{i+1,n} \leq \rho_{i,n}$ and $\sum_{i=1}^{k_j} \lambda(i,j) = 1$, by $\max\{\mu_n \rho_{1,n}, \rho_{i,n}\} < \frac{\delta_n}{16}$ and (2.18), there exists a natural number $k_l \in N$ with $k_l \leq k_j$ such that

$$\sum_{i=1}^{k_l} \lambda(i,j) > \mu_n > 0 \quad \text{and} \quad s(i,j) \geq \eta_{i_0,n} > 0, \quad \text{where } i \in \{1, 2, \cdots, k_l\}. \tag{2.19}$$

Since the convex function f is coercive on X, from the proof of [12, Theorem 2.1], we obtain that $\inf_{x \in X} f(x) = m > -\infty$. Therefore, by the previous proof and (2.19), it is easy to see that there exists a point $(z_j, f(z_j) - s_j)$ in $X \times R$ such that

$$\begin{gathered}(z_j, f(z_j) - s_j) \in \{(z(1,j), f(z(1,j)) - s(1,j)), \cdots, (z(k_j,j), f(z(k_j,j)) - s(k_j,j))\} \subset \operatorname{epi} h_n, \\ s_j \geq \eta_{i_0,n} > 0, \quad f(z_j) - s_j \leq \frac{4}{\mu_n}(2|f_n(x_0)| + 8 + |2m|)\end{gathered} \tag{2.20}$$

and

$$\lim_{j \to \infty} \langle (x_0^*, t), (z_j, f(z_j) - s_j) \rangle = \langle (x_0^*, t), (x_0, f_n(x_0)) \rangle = 1. \tag{2.21}$$

Therefore, by (2.10) and (2.20), we obtain that $\{s_j\}_{j=1}^\infty$ and $\{f(z_j)\}_{j=1}^\infty$ are two bounded sequences. Hence we can assume without loss of generality that $\{s_j\}_{j=1}^\infty$ and $\{f(z_j)\}_{j=1}^\infty$ are two Cauchy sequences. Since $\lim_{\|x\| \to +\infty} f(x) = +\infty$, we obtain that the sequence $\{z_j\}_{j=1}^\infty \subset X$ is a bounded sequence. Since $B(X^{**})$ is w^*-compact, by (2.21), there exists a subnet $\{(z_\alpha, f(z_\alpha) - s_\alpha)\}_{\alpha \in \Delta}$ of $\{(z_j, f(z_j) - s_j)\}_{j=1}^\infty$ such that

$$(z_\alpha, f(z_\alpha) - s_\alpha) \xrightarrow{w^*} (z_0^{**}, r_0) \in \operatorname{epi} f_n \subset X^{**} \times R \tag{2.22}$$

and

$$\langle (x_0^*, t), (x_0, f_n(x_0)) \rangle = \lim_{\alpha \in \Delta} \langle (x_0^*, t), (z_\alpha, f(z_\alpha) - s_\alpha) \rangle = 1. \tag{2.23}$$

Since $f(z_j) - s_j = h_n(z_j)$ and $s_j \geq \eta_{i_0,n} > 0$, by (2.19)–(2.21) and the definition of h_n, we may assume that there exists a natural number $k_0 \in N$ such that $\{(z_j, f(z_j) - s_j)\}_{j=1}^\infty \subset H_{k_0,n}$. Therefore, by (2.22)–(2.23), we get that

$$(z_0^{**}, r_0) \in \overline{\mathrm{co}}^{w^*}(H_{k_0,n}) \quad \text{and} \quad \langle (z_0^{**}, r_0), (x_0^*, t) \rangle = 1. \tag{2.24}$$

Moreover, by the definitions of $H_{k_0,n}$ and f_n, we have $H_{k_0,n} \subset \operatorname{epi} f_n|_X$. Therefore, by (2.16) and (2.24), we have the following inequalities

$$\begin{aligned} 1 &= \langle (z_0^{**}, r_0), (x_0^*, t) \rangle \\ &\geq \sup\{\langle (x_0^*, t), (z, r) \rangle : (z, r) \in \operatorname{epi} f_n|_X\} \\ &\geq \sup\{\langle (x_0^*, t), (z, r) \rangle : (z, r) \in H_{k_0,n}\} \\ &= \sup\{\langle (x_0^*, t), (z^{**}, r) \rangle : (z^{**}, r) \in \overline{\mathrm{co}}^{w^*}(H_{k_0,n})\} \end{aligned}$$

$$= \langle (z_0^{**}, r_0), (x_0^*, t)\rangle = 1.$$

Since $(0,0) \in \operatorname{int}(H_{k_0,n})$, we define the weak* bounded closed convex set

$$H_{k_0,n}^* = \{(x^*, l) \in X^* \times R : \langle (x^*, t), (x, l)\rangle \leq 1, (x, l) \in H_{k_0,n}\}.$$

Therefore, by formulas $H_{k_0,n} \subset \operatorname{epi} f_n|_X$ and $(x_0^*, t) \in C_n^*$, we obtain that $(x_0^*, t) \in H_{k_0,n}^*$. Moreover, we define the weak* closed convex set

$$H_{k_0,n}^{**} = \{(x^{**}, l) \in X^{**} \times R : \langle (x^*, t), (x^{**}, l)\rangle \leq 1, (x^*, t) \in H_{k_0,n}^*\}.$$

Then we get that $H_{k_0,n}^{**} = \overline{\operatorname{co}}^{w^*}(H_{k_0,n})$. Therefore, by (2.24), we obtain that $(z_0^{**}, r_0) \in H_{k_0,n}^{**}$. Therefore, by Lemma 2.1 and $\langle (z_0^{**}, r_0), (x_0^*, t)\rangle = 1$, we obtain that (x_0^*, t) is a weakly exposed point of $H_{k_0,n}^*$ and is exposed by (z_0^{**}, r_0). Since $H_{k_0,n} \subset \operatorname{epi} f_n|_X$, we obtain that $C_n^* \subset H_{k_0,n}^*$. Moreover, if there exists a sequence $\{(x_i^*, t_i)\}_{i=1}^{\infty} \subset C_n^*$ such that $\langle (z_0^{**}, r_0), (x_i^*, t_i)\rangle \to \sigma_{C_n^*}(z_0^{**}, r_0)$ as $i \to \infty$, then, by $\sigma_{C_n^*}(z_0^{**}, r_0) = \langle (z_0^{**}, r_0), (x_0^*, t)\rangle$, we get that

$$\lim_{i\to\infty} \langle (z_0^{**}, r_0), (x_i^*, t_i)\rangle = \sigma_{C_n^*}(z_0^{**}, r_0) = \langle (z_0^{**}, r_0), (x_0^*, t)\rangle = \sigma_{H_{k_0,n}^*}(z_0^{**}, r_0).$$

Since the point (x_0^*, t) is a weakly exposed point of $H_{k_0,n}^*$ and exposed by (z_0^{**}, r_0), by $\{(x_i^*, t_i)\}_{i=1}^{\infty} \subset C_n^* \subset H_{k_0,n}^*$, we get that $(x_i^*, t_i) \xrightarrow{w} (x_0^*, t)$ as $i \to \infty$. Therefore, by $(z_0^{**}, r_0) \in C_n^{**}$ and $(x_0^*, t) \in C_n^*$, we get that (x_0^*, t) is a weakly exposed point of C_n^* and is exposed by (z_0^{**}, r_0), where

$$C_n^{**} = \{(x^{**}, s) \in X^{**} \times R : \langle (x^{**}, t), (x^*, s)\rangle \leq 1, (x^*, s) \in C_n^*\}.$$

We define the weak* bounded closed set

$$C_n^{***} = \{(x^{***}, s) \in X^{***} \times R : \langle (x^{***}, s), (x^{**}, t)\rangle \leq 1, (x^{**}, t) \in C_n^{**}\}.$$

Then, by the definitions of f_n and C_n^{**}, we obtain that $\overline{\operatorname{co}}^{w^*}(\operatorname{epi} h_n) = C_n^{**}$. Hence we get that

$$f_n(x^{**}) = \inf\{t \in R : (x^{**}, t) \in C_n^{**}\}, \quad x^{**} \in X^{**}.$$

We claim that $\partial f_n(x_0) = \{y_0^*\}$. In fact, suppose that there exists a point $y_0^{***} \in X^{***}$ such that $y_0^{***} \in \partial f_n(x_0)$ and $y_0^{***} - y_0^* \neq 0$. Then, from the previous proof, we get that

$$\langle (x_0^{***}, l), (x_0, f_n(x_0))\rangle = \sup\{\langle (x_0^{***}, l), (z, r)\rangle : (z, r) \in \operatorname{epi} f_n|_X\} = 1, \tag{2.25}$$

where

$$x_0^{***} = \frac{y_0^{***}}{y_0^{***}(x_0) - f_n(x_0)} \quad \text{and} \quad l = -\frac{1}{y_0^{***}(x_0) - f_n(x_0)}.$$

Therefore, by $y_0^{***} - y_0^* \neq 0$, we get that $(x_0^*, t) \neq (x_0^{***}, l)$. Moreover, by (2.25) and $\overline{\operatorname{co}}^{w^*}(\operatorname{epi} h_n) = C_n^{**}$, we get that

$$\langle (x_0^{***}, l), (x_0, f_n(x_0))\rangle = \sup\{\langle (x_0^{***}, l), (z^{**}, r)\rangle : (z^{**}, r) \in C_n^{**}\} = 1.$$

Hence we get that $(x_0^{***}, l) \in C_n^{***}$. Then, by formulas $\overline{C_n^*}^{w^*} = C_n^{***}$ and $(x_0^{***}, l) \neq (x_0^*, t)$, there exists a weak* neighbourhood V of origin in X^{***} such that

$$((x_0^{***}, l) + V) \cap ((x_0^*, t) + V) = \emptyset. \tag{2.26}$$

Moreover, by formulas $\overline{C_n^*}^{w^*} = C_n^{***}$ and $(x_0^{***}, l) \in C_n^{***}$, there exists a sequence $\{(x_i^*, t_i)\}_{i=1}^{\infty} \subset C_n^*$ such that

$$\lim_{i\to\infty} \langle (z_0^{**}, r_0), (x_i^*, t_i)\rangle = \langle (z_0^{**}, r_0), (x_0^*, t)\rangle \quad \text{and} \quad \{(x_i^*, t_i)\}_{n=1}^{\infty} \subset (x_0^{***}, l) + V.$$

Since the point (x_0^*, t) is a weakly exposed point of C_n^* and exposed by (z_0^{**}, r_0), we can assume that $\{(x_i^*, t_i)\}_{n=1}^{\infty} \subset (x_0^*, t)+V$. Then $((x_0^{***}, l)+V)\cap((x_0^*, t)+V) \neq \emptyset$, which contradicts (2.26). Hence the function f_n is Gâteaux differentiable at point x_0. This implies that the function f_n is Gâteaux differentiable on X. Moreover, we have proved that (1) $f_n \to f$ uniformly on X; (2) $f_n(x) \leq f_{n+1}(x) \leq f(x)$ whenever $x \in X$ and $n \in N$. Define

$$g_n(x^{**}) = f_n(x^{**}) + 2\sup\{f(x^{**}) - f_n(x^{**}) : x^{**} \in X^{**}\}$$

for each $n \in N$. Since the function f_n is Gâteaux differentiable on X, by Lemma 1.2, we obtain that (1) g_n is Gâteaux differentiable on X; (2) $g_n(x) \geq g_{n+1}(x) \geq f(x)$; (3) $g_n \to g$ uniformly on X. Moreover, by the definition of f_n and g_n, it is easy to see that

$$\operatorname{epi} f_n = \overline{\operatorname{epi} f_n \cap (X \times R)}^{w^*} \quad \text{and} \quad \operatorname{epi} g_n = \overline{\operatorname{epi} g_n \cap (X \times R)}^{w^*}.$$

Hence we get that Theorem 2.1 is true, which finishes the proof.

Let f be continuous on X and f_0 be a function defined by Lemma 2.1. Then, from the proof of Theorem 2.1, we get the following theorem.

Theorem 2.2 *Let the dual space X^* be a weakly locally uniformly convex space and the convex function f be continuous and coercive on X. Then there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of w^*-lower semicontinuous convex functions on X^{**} such that*

(1) $f_n(x^{**}) \leq f_{n+1}(x^{**}) \leq f_0(x^{**}) \leq g_{n+1}(x^{**}) \leq g_n(x^{**})$ *whenever* $x^{**} \in X^{**}$;

(2) *the two functions f_n and g_n are Gâteaux differentiable on X;*

(3) $\operatorname{epi} f_n = \overline{\operatorname{epi} f_n \cap (X \times R)}^{w^*}$ *and* $\operatorname{epi} g_n = \overline{\operatorname{epi} g_n \cap (X \times R)}^{w^*}$;

(4) $f_n \to f$ *and* $g_n \to f$ *uniformly on* X^{**}.

3 Uniform Monotone Approximation of Non-convex Function in Banach Spaces

Theorem 3.1 *Let the dual space X^* be weakly locally uniformly convex and the convex function f be continuous on X. Then there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of continuous functions on X^{**} such that*

(1) $f_n(x) \le f_{n+1}(x) \le f(x) \le g_{n+1}(x) \le g_n(x)$ *whenever* $x \in X$;

(2) *the two functions* f_n *and* g_n *are Gâteaux differentiable on* X;

(3) $f_n \to f$ *and* $g_n \to f$ *uniformly on* X.

Proof Pick a point $x_0 \in S(X)$. Then, by the Hahn-Banach Theorem, there exists a functional $x_0^* \in S(X^*)$ such that $x_0^*(x_0) = 1$. Let $\{x_n^*\}_{n=1}^{\infty} \subset B(X^*)$ and $x_n^*(x_0) \to x_0^*(x_0) = 1$ as $n \to \infty$. Then we have $\langle x_n^* + x_0^*, x_0\rangle \to 2$ as $n \to \infty$. Therefore, by $x_0 \in S(X)$, we get that

$$2 = \lim_{n\to\infty} [\|x_n^*\| + \|x_0^*\|] \ge \liminf_{n\to\infty} \|x_n^* + x_0^*\| \ge \lim_{n\to\infty} \langle x_n^* + x_0^*, x_0\rangle = 2.$$

Since the space X^* is weakly locally uniformly convex, we obtain that $x_n^* \xrightarrow{w} x_0^*$ as $n \to \infty$. This implies that x_0^* is a weakly exposed point of $B(X^*)$ and exposed by x_0. Therefore, by Lemma 1.1, we obtain that x_0 is a Gâteaux differentiable point of the norm on X^{**}. Hence we get that the norm of X^{**} is Gâteaux differentiable on $X\backslash\{0\}$. Pick a point $x \in X\backslash\{0\}$ and let $x^* = d_G\|x\|$. Then, for every $y^{**} \in X^{**}$, we have the following equations

$$\begin{aligned}
\lim_{t\to 0} \frac{\|x + ty^{**}\|^2 - \|x\|^2}{t} &= \lim_{t\to 0} \Big[\frac{\|x + ty^{**}\| - \|x\|}{t}(\|x + ty^{**}\| + \|x\|)\Big] \\
&= 2\|x\| \lim_{t\to 0} \frac{\|x + ty^{**}\| - \|x\|}{t} \\
&= \langle 2\|x\| d_G\|x\|, y^{**}\rangle = \langle 2\|x\| x^*, y^{**}\rangle.
\end{aligned}$$

Moreover, if $\|x\| = 0$, then for every $y^{**} \in X^{**}$, we get that

$$\lim_{t\to 0} \frac{\|x + ty^{**}\|^2 - \|x\|^2}{t} = \lim_{t\to 0} \Big[\frac{\|ty^{**}\|^2}{t}\Big] = \lim_{t\to 0} [t\|y^{**}\|^2] = \langle 0, y^{**}\rangle.$$

Hence we obtain that square of norm of X^{**} is Gâteaux differentiable on X. Since the convex function f is continuous on X, by [8, Proposition 1.6], there exist two real numbers $\delta \in (0,1)$ and $\eta \in (0, +\infty)$ such that $|f(x)| \le \eta$ whenever $\|x\| \le \delta$. Since the function f is convex, we have the following inequalities

$$-\eta \le f\Big(\frac{\delta x}{\|x\|}\Big) = f\Big(\frac{\delta}{\|x\|} x + \frac{\|x\| - \delta}{\|x\|} 0\Big) \le \frac{\delta}{\|x\|} f(x) + \frac{\|x\| - \delta}{\|x\|} f(0)$$

for each $x \in X\backslash B(X)$. Therefore, by the above inequalities, we get that

$$f(x) \ge -\frac{\|x\| - \delta}{\delta} f(0) - \frac{\eta}{\delta}\|x\| = -\frac{\|x\|}{\delta}(f(0) - \eta) + f(0)$$

for each $x \in X\backslash B(X)$. Define the function $h(x) = f(x) + \|x\|^2$. Then we get that

$$\lim_{\|x\|\to+\infty} h(x) \ge \lim_{\|x\|\to+\infty} \Big[\|x\|^2 - \frac{\|x\|}{\delta}(f(0) - \eta) + f(0)\Big] = +\infty.$$

Therefore, by Theorem 2.1, there exists a w^*-lower semicontinuous convex function h_n on X^{**} such that (1) $h_n(x) \le h_{n+1}(x) \le h(x)$ whenever $x \in X$; (2) the function h_n is Gâteaux differentiable on X; (3) $h_n \to h$ uniformly on X. Define $f_n(x^{**}) = h_n(x^{**}) - \|x^{**}\|^2$ for every

$n \in N$. Then the function f_n is a continuous function on X and $f_n(x) \leq f_{n+1}(x) \leq f(x)$ whenever $x \in X$. Moreover, since $h_n \to h$ uniformly on X, by $f_n(x^{**}) = h_n(x^{**}) - \|x^{**}\|^2$, we obtain that $f_n \to f$ uniformly on X. Pick a point $x \in X$. Since the square of norm of X^{**} is Gâteaux differentiable on X, we get that

$$\begin{aligned}\lim_{t\to 0}\frac{f_n(x+ty^{**})-f_n(x)}{t} &= \lim_{t\to 0}\Big[\frac{h_n(x+ty^{**})-h_n(x)}{t} - \frac{\|x+ty^{**}\|^2-\|x\|^2}{t}\Big]\\ &= \lim_{t\to 0}\frac{h_n(x+ty^{**})-h_n(x)}{t} - \lim_{t\to 0}\frac{\|x+ty^{**}\|^2-\|x\|^2}{t}\\ &= \langle d_G h_n(x) - d_G\|x\|^2, y^{**}\rangle\end{aligned}$$

for every $y^{**} \in X^{**}$. Hence we obtain that the function f_n is Gâteaux differentiable on X. Moreover, by Theorem 2.1, there exists a sequence of w^*-lower semicontinuous convex functions $\{u_n\}_{n=1}^{\infty}$ such that (1) $h(x) \leq u_{n+1}(x) \leq u_n(x)$ whenever $x \in X$; (2) the function u_n is Gâteaux differentiable on X; (3) $u_n \to h$ uniformly on X. Let $g_n(x^{**}) = u_n(x^{**}) - \|x^{**}\|^2$. Then we obtain that (1) $f(x) \leq g_{n+1}(x) \leq g_n(x)$ whenever $x \in X$; (2) g_n is Gâteaux differentiable on X; (3) $g_n \to f$ uniformly on X, which finishes the proof.

Let f be continuous on X and f_0 be a function defined by Lemma 2.1. Then, from the proofs of Theorems 2.1 and 3.1, we get the following theorem.

Theorem 3.2 *Let the dual space X^* be a weakly locally uniformly convex space and the convex function f be continuous on X. Then there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of continuous functions on X^{**} such that*

(1) $f_n(x^{**}) \leq f_{n+1}(x^{**}) \leq f_0(x^{**}) \leq g_{n+1}(x^{**}) \leq g_n(x^{**})$ *whenever* $x^{**} \in X^{**}$;

(2) *the two functions f_n and g_n are Gâteaux differentiable on X;*

(3) $f_n \to f$ *and* $g_n \to f$ *uniformly on* X^{**}.

Conflicts of interest The authors declare no conflicts of interest.

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