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On the Pagenumber of 1-Planar Graphs*

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Abstract A book embedding of a graph G is a placement of its vertices along the spine of a book, and an assignment of its edges to the pages such that no two edges on the same page cross. The pagenumber of a graph is the minimum number of pages in which it can be embedded. Determining the pagenumber of a graph is NP-hard. A graph is said to be 1-planar if it can be drawn in the plane so that each edge is crossed at most once. The anthors prove that the pagenumber of 1-planar graphs is at most 10.

Keywords Book embedding, 1-Planar graph, Pagenumber, Crossing
2020 MR Subject Classification 05C10, 68R10

1 Introduction

The book embedding problem of graphs is motivated by several areas of computer science, such as direct interconnection networks (see [18]), VLSI design (see [10]), fault-tolerant processor arrays (see [24]), sorting with parallel stacks (see [13]), single-row routing (see [25]), ordered sets (see [22]) and the like.

A book embedding of a graph $G = (V, E)$ with n vertices consists of a (linear) layout L of its nodes along a line (called spine) ℓ of a book (i.e., $L : V \to \{1, 2, \cdots, n\}$) and an embedding of each edge to some half-planes sharing the spine as a common boundary (called pages) so that two edges embedded on the same page do not cross. Note that two edges (a, b) and (c, d) on the same page such that $L(a) < L(b)$ and $L(c) < L(d)$ in the layout L cross if and only if $L(a) < L(c) < L(b) < L(d)$ or $L(c) < L(a) < L(d) < L(b)$. The minimum number of pages in which a graph can be embedded is called the pagenumber or book thickness of the graph. A central goal in the study of book embedding is to find the pagenumber of a graph. Determining the pagenumber of a graph is a hard problem. It remains a difficult problem even when the layout L is fixed, since determining whether a given layout admits a k-page book embedding is NP-complete (see [28]).

The book embedding of graphs has been discussed for many graph families, see [5–6, 12, 21]. The most famous ones are the planar graphs. Bernhart and Kainen [5] firstly characterized the graphs with pagenumber one as the outerplanar graphs and the graphs with pagenumber two as the sub-Hamiltonian planar graphs (which are the subgraphs of planar Hamiltonian graphs). Deciding whether the pagenumber of general planar graphs is two is NP-hard (see [10]). Moreover, Bernhart and Kainen [5] conjectured that planar graphs have unbounded pagenumber,

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but this was disproved in [9, 16]. Buss and Shor [9] proposed a nine-page algorithm. Heath [16] reduced the number to seven. Istrail [17] found an algorithm that embeds planar graphs in six pages. Later, Yannakakis [30] showed that planar graphs admit a four-page book embedding, which can be constructed in linear-time. Yannakakis [29] claimed that four pages are necessary without giving a formal proof for this claim. Later, Dujmovic and Wood [11] conjectured that the pagenumber of planar graphs is four. Bekos, Kaufmann and Zielke [4] also posed the same conjecture. In 2020, Yannakakis [31] showed that there are planar graphs that require four pages in any book embedding. Hence, the pagenumber of planar graphs is 4.

More recently there has been a greater interest in studying non-planar graphs which extend planar graphs by restrictions on crossings (see [19]). A famous example is 1-planar graphs which can be drawn in the plane so that each edge is crossed at most once. Bekos, Bruckdorfer, Kaufmann and Raftopoulou [3] showed that the upper bound of 1-planar graphs is 39. Alam, Brandenburg and Kobourov [2] gave an algorithm of twelve-page for 3-connected 1-planar graphs and sixteen-page for general 1-planar graphs. In this paper, we obtain the conclusion as follows by improving the algorithm of Alam, Brandenburg and Kobourov [2] .

Theorem 1.1 *There is a linear time algorithm to construct a book embedding for general* 1*-planar graphs to* 10 *pages.*

The paper is organized as follows. In Section 2, we give preliminaries and make some necessary preparations. In Section 3, we study the book embedding of 3-connected 1-planar graphs. The book embedding of general 1-planar graphs is considered in Section 4.

2 Preliminaries

A graph is said to be planar (1-planar, resp.) if it can be drawn in the plane so that its edges do not cross (each edge has at most one crossing, resp.). For example, K_4 is a planar graph, and K_5 and K_6 are 1-planar graphs. Such a drawing of a planar (1-planar, resp.) graph G is called a planar (1-planar, resp.) embedding of G. There are at most $3n-6$ edges for planar graphs with n vertices, and there are at most $4n-8$ edges for 1-planar graphs with n vertices (see [23]). However, there is a major difference in the complexity of the recognition of planar and 1-planar graphs, which can be done in linear time for planar graphs (see [26]) while it is NP-hard for 1-planar graphs (see [15, 20]).

An embedding specifies the faces, which are topologically connected regions. The unbounded face is the outer face. A 1-planar embedding $\mathcal{E}(G)$ specifies the faces in a 1-planar drawing of G including the outer face. Then $\mathcal{E}(G)$ describes the pairs of crossing edges, the faces where the edges cross, and the planar edges (the edges that do not cross any edge of G).

For a given 1-planar embedding $\mathcal{E}(G)$, we obtain an embedding by adding as many edges to $\mathcal{E}(G)$ as possible such that the newly added edges are planar in $\mathcal{E}(G)$. We call such an embedding a planar-maximal embedding of G and the operation planar-maximal augmentation. Then each pair of crossing edges is augmented to a K_4. The planar skeleton $\mathcal{P}(\mathcal{E}(G))$ consists of the planar edges of a planar-maximal augmentation.

The normal form for an embedded 3-connected 1-planar graph $\mathcal{E}(G)$ is obtained by adding the four planar edges to form a K_4 for each pair of crossing edges while routing them close to the crossing edges and removing old duplicate edges if necessary. Such an embedding of a

3-connected 1-planar graph is a normal embedding of it. A normal planar-maximal augmentation for an embedded 3-connected 1-planar graph is obtained by finding a normal form of the embedding and a planar-maximal augmentation. We say that an embedded 3-connected 1-planar graph is a normal maximal 1-planar graph if a normal planar-maximal augmentation of the graph yields the same graph. Note that the parallel edges are not allowed in normal maximal 1-planar graphs.

In a normal maximal 1-planar graph, each crossing edges pair $\langle (a,c),(b,d)\rangle$ crosses each other either inside or outside the boundary of the quadrangle *abcd* of the planar edges, and these define the so-called augmented $X-$ and augmented $B-$configurations (see [1]). For a 3-connected 1-planar graph G, Alam, Brandenburg and Kobourov [1] proved the conclusions as follows.

Lemma 2.1 (see [1]) *Let G be a 3-connected 1-planar graph with a 1-planar embedding $\mathcal{E}(G)$. Then the normal planar-maximal augmentation of $\mathcal{E}(G)$ gives a planar-maximal 1-planar embedding $\mathcal{E}(G^*)$ of a supergraph G^* of G, so that $\mathcal{E}(G^*)$ contains at most one augmented B-configuration in the outer face and each augmented X-configuration in $\mathcal{E}(G^*)$ contains no vertex inside its skeleton.*

Lemma 2.2 (see [1]) *Let G be a 3-connected 1-planar graph with a planar-maximal 1-planar embedding $\mathcal{E}(G)$. Then no three crossing edge-pairs in $\mathcal{E}(G)$ share the same base edge.*

For more detailed studies on 1-planar graphs, the readers are suggested to refer to [1, 7–8, 14, 20, 23].

A connected graph that has no cut vertices is called a block. Then blocks are 2-connected graphs (at least three vertices) or an edge (two vertices). Particularly, we say that an edge is a trivial block. For a connected graph G that has cut vertices, we say that a maximal connected subgraph that has no cut vertices is a block of G. The block-cut tree $\mathcal{T}$ of a graph G is a graph whose vertices correspond to blocks of G, and there is an edge u_1u_2 in $\mathcal{T}$ if and only if there is a common vertex two blocks corresponding to u_1 and u_2 in G. The rooted block-cut tree $\mathcal{T}$ of a graph G is a block-cut tree rooted at some vertex.

3 Book Embedding of 3-Connected 1-Planar Graphs

We first discuss the book embedding of 3-connected 1-planar graphs and then describe the book embedding of general 1-planar graphs.

Clearly, if a graph can be embedded in a given number of pages, then the same is also true for its subgraphs. Therefore, we can assume without loss of generality that our input 3-connected 1-planar graph G is a normal maximal 1-planar graph. Lemma 2.1 (see [1]) implies that the planar skeleton of normal maximal 1-planar graphs contains only triangular and quadrangular faces. Moreover, if we remove exactly one crossing edge (arbitrarily) from each pair of crossing edges in a normal maximal 1-planar graph, then the resulting graph is a maximal planar graph. Let X be the set of crossing edges removed. We can first use the algorithm by Yannakakis [30] to obtain a five-page book embedding of $G \setminus X$. Next, we place the crossing edges in X, under the order of vertices in the algorithm by Yannakakis [30], to five additional pages such that two edges embedded on the same page do not cross. Hence, we first describe the order of vertices in the algorithm by Yannakakis [30].

The vertices of the normal maximal 1-planar graph G are partitioned into levels according to their distance from the outer face of the planar skeleton $\mathcal{P}(\mathcal{E}(G))$. Vertices on the outer face of G are at level 0. Vertices on the outer face of the resulting graph by deleting vertices at level 0 are at level 1. In general, vertices on the outer face of the resulting graph by deleting all vertices at levels less than t are at level t. The edges of G (including the crossing edges) are partitioned into level i edges connecting vertices at the same level i, and binding edges connecting vertices at different levels. Note that level i vertices are only adjacent to level j ($j = i-1, i, i+1$) vertices (i.e., binding edges connect only consecutive levels). The cycle composed of level t vertices and edges is called a level t cycle. Furthermore, every level i vertex lies in the interior of some level $i-1$ cycle.

Level 0 vertices in the clockwise order (cw-order) as they appear on the outer cycle and the level 0 edges are first placed. Next, level 1 vertices on each level 1 cycle in the counterclockwise order (ccw-order) and the level 1 edges and the binding edges connecting some level 0 vertex and some level 1 vertex are placed. Level 1 vertices are placed in the ccw-order and as well as its interior vertices. Therefore, we consider a 2-level subgraph H of G defined as follows. The vertices of H are the vertices on a level i cycle C_i and all the level $i+1$ vertices interior to C_i. The edges of H are all the planar and crossing edges inside the region between C_i and the outer boundaries of all the $i+1$ level components inside C_i (including the edges on C_i and the level $i+1$ boundaries). We denote this inside 2-level subgraph of H as $H(C_i)$. We assume that C_i has been embedded where the vertices of C_i are placed in the cw (or ccw, resp.) order around C_i for all even (odd, resp.) i. We then extend this embedding to a book embedding of $H(C_i)$, by placing the remaining vertices and edges. The book embedding of G is obtained by iteratively operating the book embeddings of $H(C_i)$.

To formalize the idea mentioned above, we consider an arbitrary cycle C_i whose vertices are level i. Without loss of generality, we suppose that the vertices of C_i are placed in the cw-order (for ccw-order we flip the embedding of $H(C_i)$). Let $v_1, v_2, \cdots, v_m$ be the vertices of C_i in the cw-order around C_i (i.e., $L(v_1) < L(v_2) < \cdots < L(v_m)$). We call the vertices of C_i the outer vertices and level $i+1$ vertices of $H(C_i)$ the inner vertices. For each crossing edges pair $\langle (a,b), (c,d) \rangle$, we take one edge to be in X as follows.

Case 1 If (a,b) and (c,d) are level i edges, then we choose the edge incident to the vertex farthest from v_1 in cw-order on C_i to be in X. For example, we choose (v_3, v_5) to be in X for the crossing edges pair $\langle (v_2, v_4), (v_3, v_5) \rangle$ in Figure 1.

Case 2 If (a,b) and (c,d) are binding edges, then we choose the edge incident to the vertex farthest from v_1 in cw-order on C_i to be in X. For example, we choose $(v_2, u_{1,1})$ to be in X for the crossing edges pair $\langle (v_1, u_{1,4}), (v_2, u_{1,1}) \rangle$ in Figure 1.

Case 3 If one of (a,b) and (c,d) is a level i edge, and the other is a binding edge, then we choose the binding edge to be in X. For example, we choose $(v_6, u_{2,2})$ to be in X for the crossing edges pair $\langle (v_5, v_7), (v_6, u_{2,2}) \rangle$ in Figure 1.

Case 4 If one of (a,b) and (c,d) is a level $i+1$ edge, and the other is a binding edge, then we choose the $i+1$ level edge to be in X. For example, we choose $(u_{3,4}, u_{4,3})$ to be in X for the crossing edges pair $\langle (v_7, u_{4,1}), (u_{3,4}, u_{4,3}) \rangle$ in Figure 1.

Case 5 If one of (a,b) and (c,d) is a level i edge, and the other is a level $i+1$ edge, then we choose the $i+1$ level edge to be in X. For example, we choose $(u_{6,1}, u_{6,2})$ to be in X for

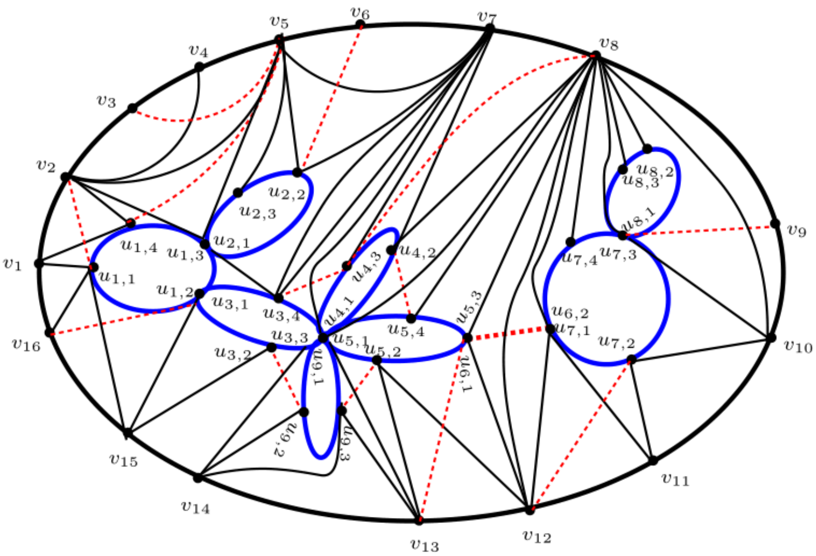

Figure 1 A 2-level subgraph $H(C_i)$ of G inside the level-i cycle $C_i = v_1, v_2, \cdots, v_{16}$, which is drawn with thick black edges. The outer boundary of the level $i+1$ component is drawn with thick blue edges. The red dashed edges are the crossing edges taken in the set X.

the crossing edges pair $\langle (v_8, v_{12}), (u_{6,1}, u_{6,2}) \rangle$ in Figure 1.

Note that G is a normal maximal 1-planar graph. We have that two crossing edges cannot be both the level $i+1$ edges by the construction of $H(C_i)$. Therefore, the above cases contain all possible pairs.

Remark 3.1 Case 5 mentioned above is different from the idea in [2], for example, we choose the edge $(u_{5,1}, u_{5,2})$ to be in X for the crossing edges pair $\langle (K, R), (u_{5,1}, u_{5,2}) \rangle$ in Figure 2 (in [2, Figure 1]). Moreover, the edge $(T, u_{4,2})$ should be in X for the crossing edges pair $\langle (S, u_{4,1}), (T, u_{4,2}) \rangle$ in Figure 2.

Denote by D the subgraph of $H(C_i)$ induced by the vertices at level $i+1$. Assume without loss of generality that D is a connected graph, since otherwise each connected component of D would be inside a different cycle induced by the vertices of C_i and these can be handled separately. By construction, each 2-connected block of D is a simple cycle. Let $B_1, B_2, \cdots, B_s$ be these blocks of D. We describe how to place all the level $i+1$ vertices as follows on $H(C_i)$.

We say that a vertex x sees an edge yz if xyz forms a triangular face in $H - X$. And an outer vertex x sees a block B_j if x sees an edge of B_j. The third node $u_{1,1}$ of the triangular face containing the edge (v_1, v_m) is called the first inner vertex. If the third node u of the triangular face containing the edge (v_1, v_m) is a level $i+1$, then we choose the level $i+1$ vertex that satisfies the third node of the triangular face containing the edge (v_{i-1}, v_i), where v_i is the closet to v_1, is called the first inner vertex. For example, if $v_1 v_m$ forms a triangle with a level i vertex v_x, and $v_2 v_1$ forms a triangle with a level $i+1$ vertex u, then we choose that u is called the first inner vertex. The block B_1 containing $u_{1,1}$ is the first block. Assume that $u_{1,1}$ lies in a unique block, otherwise, add a vertex u inside the triangular face $v_1 v_m u_{1,1}$ and edges uv_1,

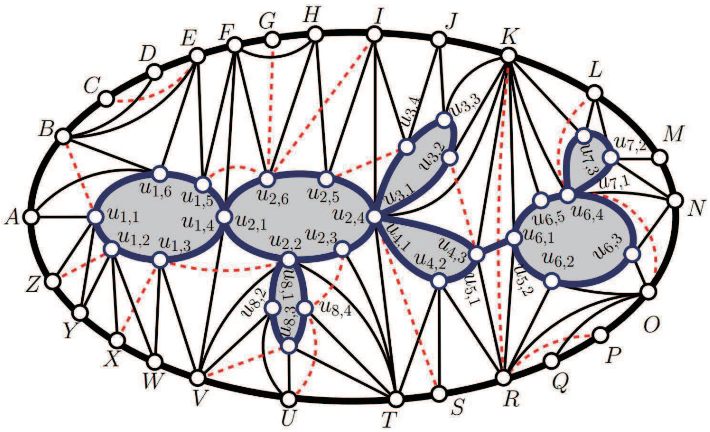

Figure 2 A 2-level subgraph $H(C_i)$ of G inside the level-i cycle $C_i = AB \cdots Z$, which is drawn with thick black edges. The outer boundary of the level $i+1$ component is drawn with thick blue edges. The red dashed edges are the crossing edges taken in the set X.

uv_m and $uu_{1,1}$. Then u is the first inner vertex of the resulting graph and belongs to only one trivial block $uu_{1,1}$ seen by v_1. Let $\mathcal{T}$ be the rooted block-cut tree rooted at B_1 of D.

The leader of a block B_j is the first vertex of B_j in any path from $u_{1,1}$ to B_j in D. Note that the leader of B_1 is $u_{1,1}$, and the leader of B_j $(j \neq 1)$ is the common vertex between B_j and its parent in $\mathcal{T}$. Although there is an inner vertex u (in particular a cut vertex of D) which may belong to more than one block, we assign u to a unique block by assigning it to the highest (i.e., closest to B_1) block that contains it in $\mathcal{T}$. Therefore, B_1 contains all its vertices, and B_j $(j \neq 1)$ contains all its vertices except its leader. The first vertex of a block B_j is the first vertex except its leader from leader of the block B_j in the ccw-order. The dominator of a block B_j is the first vertex of C_i (in the order $v_1, v_2, \cdots, v_m$) adjacent to some vertex assigned to B_j.

If an inner vertex u is assigned to a block dominated by the outer vertex v_k, then $L(v_k) < L(u) < L(v_{k+1})$. Note that there are no blocks dominated by v_m. Further, Yannakakis [30] proved the following conclusion.

Lemma 3.1 (see [30]) *The blocks dominated by the same outer vertex form a directed path in the tree $\mathcal{T}$.*

Suppose that v_k dominates a unique block B, then the vertices assigned to B are placed in the ccw-order around its boundary from its first vertex. Suppose that v_k dominates more than one block. Lemma 3.1 (see [30]) implies that these blocks form a directed path in $\mathcal{T}$. Then the vertices assigned to each block are placed consecutively in ccw-order around its boundary; the blocks are ordered one after the other in top-down order of $\mathcal{T}$: First the vertices assigned to the highest block, then the ones assigned to its child, and so on.

Let $w_1, w_2, \cdots, w_m$ be the vertices of arbitrary level $i+1$ cycle C_{i+1} in the ccw-order around C_{i+1} (i.e., $L(w_1) < L(w_2) < \cdots < L(w_m)$). Let B be the block with the cycle C_{i+1} as its outer boundary in $H(C_i)$. If B is the first block of $H(C_i)$, then there are no vertices of level $j < i+1$ between w_k and w_{k+1} $(k = 1, 2, \cdots, t)$. Otherwise, there are no vertices of level $j < i+1$ between w_k and w_{k+1} $(k = 2, 3, \cdots, t)$, and there may be some vertices of level $j < i+1$ between w_1

and w_2. In this case, we place the vertices assigned to level $i+2$ blocks dominated by w_1 in $H(C_{i+1})$ to go next to w_2. In either case, the vertices on each level $i+2$ cycles are placed in an interval with no vertices of level $j \leq i$ in between. Call this Algorithm Order-Vertices.

Lemma 3.2 (see [30]) *Let L be the vertex order for a normal maximal 1-planar graph G, obtained by Algorithm Order-Vertices. Let $C_i = v_1 \to \cdots \to v_r \to v_1$ be some level i cycle in G. Then all vertices of level $i+1$ inside C_i are placed strictly between two level i vertices v_1 and v_r, and all edges incident to a level $i+2$ vertex have no crossing to level j edges $(j \leq i)$ in L.*

Next, we describe placement of edges in X. We first assign the edges in X to the three pages c_1, c_2 and c_3 on $H(C_i)$ as following. Assume that (a,b) is in X for each crossing edges pair $\langle(a,b),(c,d)\rangle$.

Case 1 If (a,b) is a level i edge, then we assign it to page c_1.

Case 2 Assume that (a,b) is a binding edge. (a,b) is called forbidden for some block B if it connects the leader of B and v_{k+1}, where v_k is the dominator of B and is not of any child block of B in $\mathcal{T}$. If (a,b) is not forbidden for any block, then assign it to page c_1. If (a,b) is forbidden for some block B at the even (odd, resp.) level in $\mathcal{T}$, then we assign it to page c_2 (c_3, resp.).

Case 3 If (a,b) is a level $i+1$ edge, then (c,d) is a level i edge or a binding edge. Suppose (c,d) is a binding edge, where c is an outer vertex and d is an inner vertex. Let B_1 and B_2 be the two blocks of D containing a and b, respectively. Without loss of generality, we assume that the vertices of B_1 are all placed before the vertices of B_2 except for its leader. Then B_1 is the parent of B_2, or B_1 and B_2 have a common parent block B in $\mathcal{T}$ and d as their common leader, moreover, B_1 comes directly before B_2 in the cw-order from B around d. Note that G is a normal maximal 1-planar graph. We have that b is adjacent to the leader of B_2. Thus b is either the first or the last vertex (except for its leader) of B_2 in L. We say (a,b), the first (last, resp.) crossing edge of B_2 if b is either the first (last, resp.) vertex of B_2 in L.

Note that for each block, there is at most one the first crossing edge and at most one edge, which is either the last crossing edge or the forbidden binding edge. Moreover, if the edge (a,b) are both the first crossing edge of B_1 and the last crossing edge of B_2, then we say that the edge (a,b) is the last crossing edge of B_2. We have the following lemma by the fact that G is a normal maximal 1-planar graph.

Lemma 3.3 (i) *If (a,b) is the first crossing edge of B_2 ($a \in V(B_2)$ and $b \in V(B_1)$), then B_1 is the parent of B_2 in $\mathcal{T}$.*

(ii) *If (a,b) is the last crossing edge of B_2 ($a \in V(B_2)$ and $b \in V(B_1)$), then either B_1 is the parent of B_2 in $\mathcal{T}$, or B_2 and B_1 have a common leader d, moreover, B_1 comes directly before B_2 in the cw-order from their common parent block around their common leader. In either case, the vertices of B_1 are all placed before the vertices of B_2 except for its leader.*

If (a,b) is the first crossing edge of B_2, then place it to page c_1. If (a,b) is the last crossing edge of B_2, then (a,b) is assigned to page c_2 or c_3, opposite to the last crossing edge of its parent block assigned if B_2 is i-th encountered (i is even) in the cw-order from the parent block of B_2 around leader of B_2, same as the last crossing edge of its parent block assigned if B_2 is j-th encountered (j is odd) in the cw-order from the parent block of B_2 around leader of B_2. Note that if (c,d) is a level i edge, then (a,b) is a trivial block of D and it is the last crossing edge of B_2 whose leader is a (i.e., B_1 is the parent of B_2). We have that B_2 is second encountered in the cw-order from B_1 around a. Then (a, b) is assigned to page c_2 or c_3, which opposites to assign the last crossing edge of B_1. For example, Figure 3 is a 3-page book-embedding of the graph shown in Figure 1, where the last edge of the first block is assigned to page c_3.

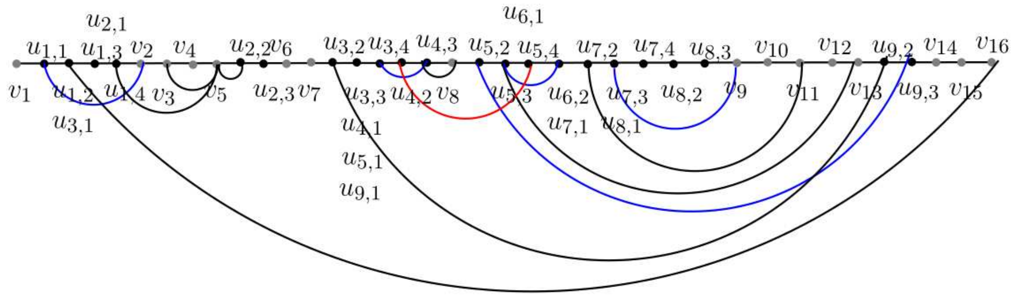

Figure 3 Book embedding of the crossing edges in X for the 2-level graph $H(C_i)$ in Figure 1 on the three pages c_1, c_2, c_3. The black edges are assigned to page c_1. The red edges are assigned to page c_2. The blue edges are assigned to page c_3.

Remark 3.2 There are two flaws in the algorithm due to Alam [2]. Firstly, a binding edge (x, v_x) and a level i edge (a trivial block) may be crossing on page c_1, where x is assigned to a block dominated by v_k $(k < r < x)$ (see Figure 4 (1)). Since $L(v_k) < L(x) < L(v_r) < L(v_x)$, (x, v_x) and (v_k, v_r) (in region II) are crossing (for example, the edges $(T, u_{4,2})$ and (K, R) in Figure 2). Secondly, there may be crossing on page c_2 for the following case. Let the edges (x_2, y_2) and (x_3, y_3) be the first crossing edges of B_{j_2} and B_{j_3}, respectively, where $x_2 \in V(B_{j_1})$, $y_2, x_3 \in V(B_{j_2})$, and $y_3 \in V(B_{j_3})$. Moreover, B_{j_1}, B_{j_2} and B_{j_3} have common leader d_B and B_{j_2} (B_{j_1}, resp.) comes directly before B_{j_3} (B_{j_2}, resp.) in the cw-order from their parent block B_j around d_B (see Figure 4(2)). Since $L(x_2) < L(x_3) < L(y_2) < L(y_3)$, (x_2, y_2) and (x_3, y_3) are crossing.

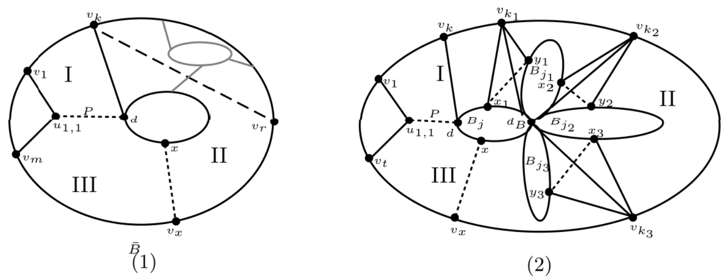

Figure 4 Illustrations for the proof of Remark 3.2.

Clearly, there are no edges incident with v_1 on page c_i $(i = 1, 2, 3)$. Next, we will show that the edges assigned to page c_i $(i = 1, 2, 3)$ do not corss each other. We firstly prove the case of the page c_1.

Lemma 3.4 *There is no crossing between edges assigned to page c_1.*

Proof The edges assigned to c_1 consist of the level i edges, the binding edges not forbidden for any block in X, and the first crossing edges in X. We show that there is no crossing for any two of them as follows. Note that the vertices of C are placed in the cw-order of its boundary, and there is no crossing between edges of X in the embedding of $H(C_i)$. Therefore, there is no crossing between two level i edges in X.

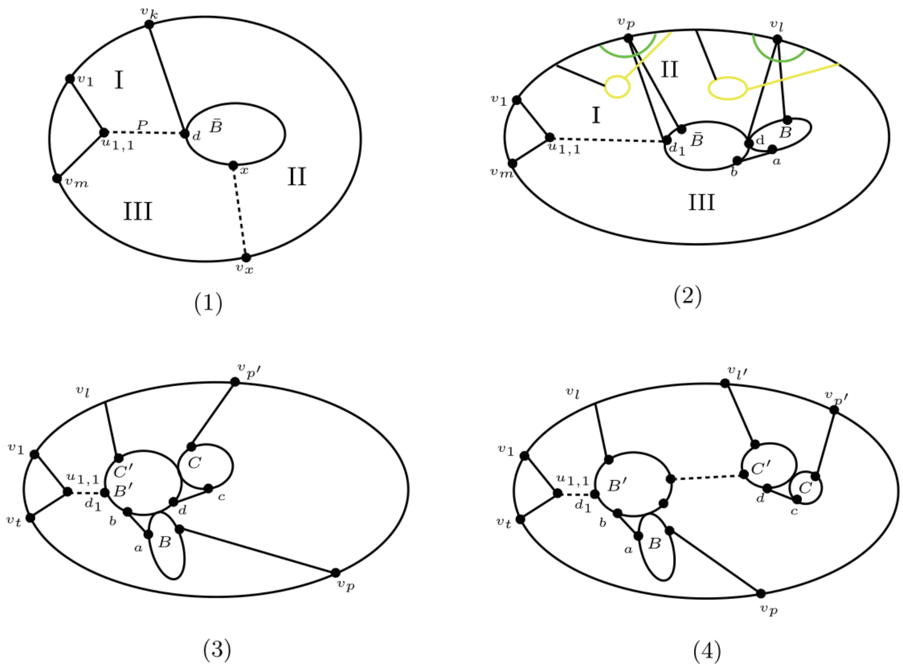

Figure 5 Illustrations for the proof of Lemma 3.4.

Next, we show that no binding edges in X are in conflict with any other binding edges or level i edges in X on page c_1. Consider a binding edge (v_x, x) assigned to page c_1, where v_x is an outer vertex and x is an inner vertex assigned to the block $\overline{B}$ whose dominator and leader are v_k and d, respectively (see Figure 5(1)). Also consider a path P from the first inner vertex $u_{1,1}$ to d in the planar skeleton of $H(C_i)$. According to the block $\overline{B}$, the two edges (x, v_x) and (d, v_k), along with the path P and the two edges $(u_{1,1}, v_1)$, $(u_{1,1}, v_m)$, we partition the interior of C_i into three parts as follows: (i) The interior of $\overline{B}$, (ii) the interior of the triangle $(u_{1,1}, v_1, v_m)$, and (iii) the three regions marked by I, II and III in Figure 5(1). Since the path P and the boundary of $\overline{B}$ belong to the planar skeleton of $H(C_i)$ and since the edge (v_x, x) is a crossing edge, each edge assigned to page c_1 is embedded in the interior of one of the three regions I, II or III.

All the level i vertices in region I are placed on or before v_k in L. Since v_k is the dominator of $\overline{B}$, it is placed before any vertex assigned to $\overline{B}$, including x. Thus for any level i edge in X placed in region I, both its end-vertices are placed before both x and v_x. Hence there is no crossing between these level i edges and (v_x, x). Note that all level $i+1$ vertices u in region I including the ones on P are also placed before x, that is, $L(u) < L(x)$. If u is assigned to the block $\overline{B}'$, then $\overline{B}'$ is dominated by either an outer vertex placed before v_k or the outer vertex v_k. However the vertices on $\overline{B}'$ are placed before those of $\overline{B}$ for both case, following the consecutive method of placement. Then both end-vertices of any binding edge in X lying in region I are placed before x and v_k. Hence binding edges in X lying in region I do not create any conflict with (v_x, x).

All the level i vertices in region II, except for v_k, are placed after x and before v_x. Similarly, all the level $i+1$ vertices in region II, except for d, are placed on or after x and before v_x. Hence, there is no binding edge or level i edge, except for a level i edge, that conflicts with a trivial block of D lying in region II incident to v_k. The level i edges, that conflict with a trivial block of D are not assigned to X according to Case 5 of the placement of edges in X, that is, these level i edges are not assigned to page c_1. Furthermore, no binding edges incident to d are assigned to page c_1. Thus all the binding edges and level i edges assigned to page c_1 have end-vertices placed between x and v_x; hence they create no conflict with (v_x, x).

All the level i vertices in region III are placed on or after v_x. Then all level i edges in X lying in region III have both their end vertices placed after both x and v_x, and hence they create no conflict with (v_x, x). On the other hand, the level $i+1$ vertices on P or on the boundary of $\overline{B}$ lying region III are placed before x and the binding edge incident to them does not create conflict with (v_x, x). Moreover, all the level $i+1$ blocks strictly in region III are dominated by the vertices placed on or after v_x. Indeed, the only possible planar edge crossing the region boundary would have been incident to the level i vertex v_x just before v_x, and it would have crossed the edge (v_x, x). However in that case, the other end vertex of such an edge would have been on a block dominated by v_x and x would have been its leader, which is a contradiction since the edge (v_x, x) is assigned to page c_1. Thus all the binding edges in region III incident to some level $i+1$ vertex neither on P nor $\overline{B}$, have both the end-vertices placed after x and v_x. Hence they do not create conflict with (v_x, x).

Finally, we show that the first crossing edges of blocks are not in conflict with any other edge on page c_1. Let (a, b) be the first crossing edge of some block B assigned to page c_1, where a is assigned to block B and b is assigned to block B'. Then B' is the parent block of B, that is, $L(b) < L(a)$. Assume that B and B' are dominated by the vertex v_p and the vertex v_l, respectively. Then $L(v_p) \geq L(v_l)$, $L(v_p) < L(a) < L(v_{p+1})$ and $L(v_l) < L(b) < L(v_{l+1})$. Note that G is a normal maximal 1-planar graph. We have that its leader and its dominator are adjacent for each block. Then that there are no level i edges so that whose one end is v_x $(p+1 \leq x \leq l)$ and the other end is v_y $(1 \leq y \leq p$ or $l+1 \leq y \leq t)$ in the layout L. Moreover, there are no binding edges so that whose one end is v_r $(p+1 \leq r \leq l-1)$ and the other end is assigned to some block dominated by the vertex v_z $(1 \leq z \leq p-1)$, or whose one end is v_r $(l+1 \leq r \leq t)$ and the other end is assigned to some block dominated by the vertex v_z $(p+1 \leq z \leq l-1)$ (see Figure 5(2)). Therefore there is no crossing between (a, b) and the binding edges and the level i edges assigned to page c_1.

Let (c, d) be the first crossing edge of some block C assigned to page c_1, where c is assigned to block C and d is assigned to block C'. Then C' is the parent block of C, that is, $L(d) < L(c)$. Assume that C and C' are dominated by vertex $v_{p'}$ and vertex $v_{l'}$, respectively. Then $L(v_{p'}) \geq L(v_{l'})$, $L(v_{p'}) < L(c) < L(v_{(p'+1)})$ and $L(v_{l'}) < L(d) < L(v_{(l'+1)})$. If $B' = C'$, then $L(d) < L(a)$ and $L(b) < L(c)$. However, $L(b) < L(d)$ or $L(b) < L(d)$ is possible. We can also assume without loss of generality that $L(b) < L(d)$. Since G is a normal maximal 1-planar graph, $L(a) > L(c)$, that is, $L(b) < L(d) < L(c) < L(a)$ (see Figure 5(3)). Therefore there is no crossing between (a, b) and (c, d). If $B' \neq C'$ (we can assume without loss of generality that $L(v_l) < L(v_{l'})$, i.e., $L(b) < L(d)$), then $L(v_{p'}) < L(v_p)$, that is, $L(b) < L(a) < L(d) < L(c)$ (see Figure 5(4)). Therefore there is no crossing between (a, b) and (c, d), that is, there is no crossing between the first crossing edges assigned to page c_1.

To sum up, there is no crossing between edges assigned to page c_1.

Next, we prove the case for pages c_2 and c_3.

Lemma 3.5 *There is no crossing between edges assigned to pages c_2 and c_3.*

Proof The edges assigned to page c_2 consist of the forbidden edge and the last crossing edge of some blocks.

We firstly consider the forbidden edge (a, b) of some block B_j dominated by v_k, where a is assigned to the block B_j. Since the forbidden edge (a, b) of the block B_j joints its leader to v_{k+1}, $b = v_{k+1}$ and $L(a) < L(b)$. Note that there is at most a forbidden edge or a last crossing edge for a block B. Note that G is a maximal 1-planar graph and v_k is not the dominator of any child block of B_j in $\mathcal{T}$. Then there are no binding edges or $i+1$ level edges (a', b') satisfying $L(a) < L(a') < L(b) < L(b')$ or $L(a') < L(a) < L(b') < L(b)$. Therefore, there are no edges crossing to (a, b) on page c_2, that is, no forbidden edge of any block is in conflict with any other edge on page c_2.

Next, we show that there is no crossing between any two the last crossing edges assigned to page c_2. Consider the last crossing edge (c, d) of some block B, where c and d are assigned to the block B and B', respectively. Assume we know that if B' is the parent of B in $\mathcal{T}$, then $L(d) < L(c)$. Otherwise, $L(c) < L(d)$. Assume that $L(c) < L(d)$. Since G is a maximal 1-planar graph, there is no last crossing edge (c', d') of any block satisfying $L(c') < L(c) < L(d') < L(d)$ or $L(c) < L(c') < L(d) < L(d')$ except for the edges which are either the last crossing edge $e_1 = fg$ of B'' coming directly before B, or the last crossing edge e_2 of B', or the trivial block e_3 incident to some vertex h of B so that $L(a) < L(h) < L(f)$. We have that e_1 and e_2 are assigned to page c_3 according to the assignment of edges in X. Therefore, (c, d) and other last crossing edges on page c_2 are not crossing. Assume that $L(d) < L(c)$. Then we can prove that there is no crossing between (c, d) and other last crossing edges on page c_2 similar to the case of $L(c) < L(d)$.

We can prove that there is no crossing between edges assigned to page c_3 similar to c_2.

Lemmas 3.4–3.5 imply the following lemma.

Lemma 3.6 *The edges of $H(C_i)$ in X can be assigned to three pages c_1, c_2 and c_3 so that there is no crossing between edges assigned to the same page.*

We next describe a lemma as following for the block B with an arbitrary cycle C_{i+1} (C_{i+1} : $w_1 \to \cdots \to w_t \to w_1$) as its outer boundary in $H(C_i)$. Without loss of generality, we assume that if there is a forbidden edge or a last crossing edge for the block B, then it is assigned to page c_3.

Lemma 3.7 *On page c_3, there are no edges incident to vertices of B except for w_1, w_2 and w_t.*

Proof Note that the edges assigned to page c_3 consist of the forbidden edges and the last crossing edges for the blocks. We distinguish two cases to discuss as follows.

Case 1 Assume that there exists no forbidden edge and last crossing edge for the block B. Clearly, no forbidden edges are incident to vertices of B according to the definition of the forbidden edge. Then we also distinguish two subcases to discuss.

Subcase 1.1 There are no last crossing edges incident to vertices of B. It is obvious that there are no edges incident to vertices of B on page c_3.

Subcase 1.2 There are last crossing edges incident to vertices of B. Let one of these last crossing edges be the last crossing edge the block B'. Then B' is either a child of B, or B' and B have common leader d, moreover, B comes directly before B' in the cw-order around d from their parent block according to Lemma 3.3. If B' is a child of B, then the last crossing edge of B' is assigned to page c_2 according to the assignment of edges in X. If B' and B have common leader, then the last crossing edge of B' is also assigned to page c_2 according to the assignment of edges in X. Hence, there are no edges incident to last crossing edges of B on page c_3.

Case 2 Assume that there exists a forbidden edge or a last crossing edge for the block B. Then it is assigned to page c_3.

Subcase 2.1 Suppose that there is a forbidden edge for the block B. Then the forbidden edge is between its leader w_1 and v_{k+1}, where v_k is the dominator of B. Note that w_1 is placed before v_{k+1} and no edges on page c_3 cross each other. Therefore, there are no edges incident to vertices of B except for w_1 on page c_3.

Subcase 2.2 There is a last crossing edge for the block B. According to Lemma 3.3, one end of the last crossing edge is w_t, and the other end is some vertex of the block B', where B' is either the parent of B, or B' and B have common leader d. Moreover, B' comes directly before B in the cw-order around d from their parent block. Therefore, the vertices of B' are all placed before the vertices of B except for its leader w_1. Note that no edges on page c_3 cross each other and there are no last crossing edges incident to last crossing edges of B except for w_t on page c_3 according to Subcase 1.2. We obtain that there are no edges incident to vertices of B except for w_t on page c_3. Therefore, the result holds.

Theorem 3.1 *3-connected 1-planar graphs can be embedded in 10-page books.*

Proof Let G be a 3-connected 1-planar graph. Then G is a spanning subgraph of some normal planar maximal 1-planar graph G'. The vertices of G' are assigned level-by-level using the Algorithm Order-Vertices. In fact, this order of vertices is the same as the order of vertices in algorithm by Yannakakis [30]. Let $p_1, \cdots, p_5, c_1, \cdots, c_5$ denote 10 pages. We use the pages p_1, p_2, p_3, p_4, p_5 to embed edges of $G \setminus X$ by the algorithm of Yannakakis [30]. Assume that edges in X inside $H(C_i)$ have been embedded in pages c_1, c_2 and c_3 using Lemma 3.6. Lemma 3.7 implies that for each block B inside $H(C_i)$, there is at least one page of c_1, c_2, c_3, so that no edges are incident to its vertices except for its leader, first vertex and last vertex. We assume without loss of generality that it is c_3. Since the vertices of $H(C_{i+1})$ are assigned to the first vertex and the last vertex of C_{i+1}, we place edges in X inside $H(C_{i+1})$ to page c_3, c_4 and c_5, where C_{i+1} is the outer boundary of B. Lemma 3.2 implies that with this vertex order, no level $i+1$ edge in G' conflicts with any level j edge with $j < i$. Therefore, we can iteratively use the algorithm to obtain a 9-page book embedding of G'. The claim holds.

4 Book Embedding of General 1-Planar Graphs

If a graph can be embedded in a given number of pages, then the same is also true for its subgraphs. Then for the general 1-planar graphs we may assume that the input graph is a maximal 1-planar graph G, that is, it is 2-connected.

We first normalize the maximal 1-planar graph similarly to the case of a 3-connected 1-planar graph. Lemma 2.2 (see [1]) implies that a pair of vertices $\{u, v\}$ of G shares more than two crossing edge pairs if and only if $\{u, v\}$ forms a separation pair in G. During the normalization, for any separation pair $\{u, v\}$, we route the edge (u, v) such that all the crossing edge pairs with u, v as end-vertices fall on the same side of (u, v) (see Figure 5).

Assume that there is a separation pair $\{u, v\}$ with a decomposition $G - \{u, v\} = H_0, \cdots, H_k$ for some $k \geq 1$. For a component H_j $(1 \leq j \leq k)$, let H_j^* be the subgraph of G induced by the vertices of H_j and $\{u, v\}$, that is, $H_j^* = G[V(H_j) \cup (u, v)]$. We have that there is at most one component H_j such that u and v are not on the outer face of H_j^*. Then assume without loss of generality that $H_1^*, \cdots, H_k^*$ all have u, v on the outer face. Then H_0 and H_0^* are called the main component and the extended main component for $\{u, v\}$, respectively. $H_1, \cdots, H_k$ and $H_1^*, \cdots, H_k^*$ are called the inner components and the extended inner components for $\{u, v\}$, respectively. The edge (u, v) is called separating edge. Note that the inner components can be arbitrary permuted and flipped at $\{u, v\}$. Moreover, in a normalized planar maximal embedding $\mathcal{E}(G)$ of G, the inner components $H_1, \cdots, H_k$ are attached to (u, v) and are embedded on one side of (u, v), say in this ccw-order at u. The components are separated by one or two pairs of crossing edges (see Figure 6). And they may also be separated by copies of the separation edge (see [7–8]). We know that the boundaries of the inner components are triangles and quadrangles according to the definition of B- or W-configurations in [27].

We now extend our 10-page book embedding of 3-connected 1-planar graphs to general 1-planar graphs.

Theorem 4.1 *1-planar graphs can be embedded in 10-page books.*

Proof We proceed as in the case of 3-connected graphs. However we extend the peeling technique here to deal with the inner components for the separation pairs. Let the main graph G_0 be obtained from G by deleting all the inner components for all the separation pairs. Clearly, G_0 is 3-connected. For each separation pair $\{u, v\}$, the edge (u, v) is a planar edge and if (u, v) is an edge of the main graph, then by the peeling technique, u, v are on the same level or on consecutive levels. Let $H_1, \cdots, H_k$ be the inner components for u, v. We then assign the vertices on the outer boundary O_j for each inner component H_j on the higher (i.e., deeper) of the two levels for u and v. For the remaining vertices of H_j we proceed with the peeling technique recursively and assign them to subsequent levels. Let u, v belong to some 2-level subgraph $H(C_i)$ of the main graph, where C_i is a level i cycle. Then the vertices on the outer boundary for each inner component and the edges between these outer vertices and u, v are

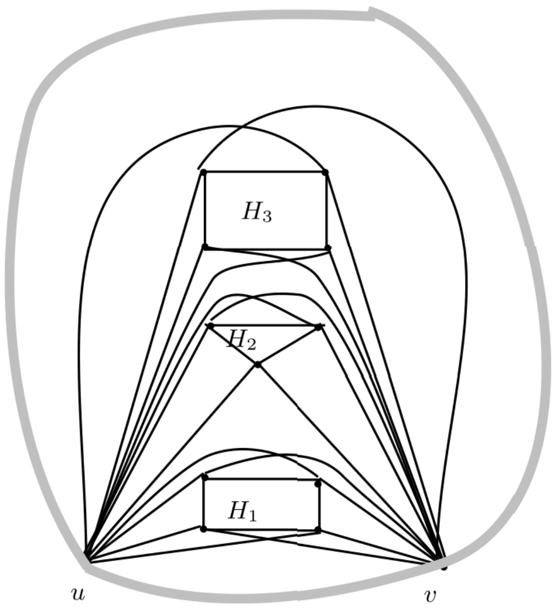

Figure 6 A separation pair and the corresponding components.

on the 2-level subgraph for G. We now show how to place these vertices and assign the edges to augment the book embedding of G_0. Let $p_1, \cdots, p_5, c_1, \cdots, c_5$ denote 10 pages. Without loss of generality, we can assume that the edges of the 2-level subgraph $H(C_i)$ are embedded into pages p_1, p_2, p_3, c_1, c_2, c_3, and uv is embedded in page p_1. And the edges of the 2-level subgraph $H(O_j)$ are embedded into pages p_3, p_4, p_5, c_3, c_4, c_5, where O_j is the outer boundary O_j for some inner component H_j.

For each separating edge (u, v) on the main graph G_0, with $L(v) < L(u)$, insert the vertices on the outer boundary of each inner component for u, v consecutively, to the immediate left of v (in cw-order and ccw-order if v is on odd and even level, respectively). If there is more than one inner component for separating edge (u, v), then the order of their placement is arbitrary. If there is more than one separating edge incident to v, with the other end-vertex, say $w_1, \cdots, w_q$, all placed before v and $L(w_1) < L(w_2) < \cdots < L(w_q)$, insert the vertices of the corresponding inner components in reverse order (i.e., the inner components for $w_q, \cdots$, the inner components for w_1). Since they form simple cycles of length 3 or 4 and the vertices are consecutive, they do not create conflicts.

For each inner component H_j for separation pair $\{u, v\}$, the edges from u to the vertices on O_j are assigned to page p_1, and the edges from v to the vertices of O_j are assigned to one of pages p_2, c_1, c_2.

Since the edges are all incident with v, the edges do not create conflicts with each other. Since they are all placed immediately before v and the edges of $H(O_j)$ are embedded into pages p_3, p_4, p_5, c_3, c_4, c_5, they do not create conflicts with other edges on this page. Similarly, the edges incident with u do not cross each other on q_1.

We recursively place the vertices inside each inner component during the computation for 2-level subgraphs on subsequent levels. Thus, 1-planar graphs can be embedded in 10-page books.

The proof of above Theorem 4.1 is similar to the proof of [2, Theorem 3]. Since the article [2] is unpublished, the proof is written again. This paper concludes with proof of Theorem 1.1.

Proof of Theorem 1.1 Given a 1-planar graph with n vertices and its 1-planar embedding, the computation of the normal planar maximal augmentation, removing the crossing edge, algorithm of Yannakakis for planar graphs, the assignment of removed edges can be obtained by linear time in the size of the graph. Since there are at most $4n-8$ edges for 1-planar graphs with n vertices, the algorithm takes linear time for the order of the graph.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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