Chin. Ann. Math. Ser. B
46(2), 2025, 303–320
DOI: 10.1007/s11401-025-0017-0

Chinese Annals of
Mathematics, Series B


# Waring-Goldbach Problem for Unlike Powers*

Zhenzhen FENG[1] Jing MA[2]

**Abstract** In this paper, the authors investigate exceptional sets in the Waring-Goldbach problem for unlike powers. For example, estimates are obtained for sufficiently large integers below a parameter subject to the necessary local conditions that do not have a representation as the sum of a square of prime, a cube of prime and a sixth power of prime and a $k$-th power of prime. These results improve the recent result due to Brüdern in the order of magnitude. Furthermore, the method can be also applied to the similar estimates for the exceptional sets for Waring-Goldbach problem for unlike powers.

**Keywords** Exceptional sets, Waring-Goldbach problem, Circle method
**2020 MR Subject Classification** 11P32, 11P55

## 1 Introduction

Let $N$, $k_1$, $k_2$, $\cdots$, $k_r$ be natural numbers such that $2 \le k_1 \le k_2 \le \cdots \le k_r$. The Waring-Goldbach problem for unlike powers concerns the representation of $N$ as the form

$$N = p_1^{k_1} + p_2^{k_2} + \cdots + p_r^{k_r}.$$

Not very much is known about results of this kind if $\frac{1}{k_1} + \cdots + \frac{1}{k_r} < 2$. However, these topics have attracted mathematicians' attentions.

Schwarz [14] considered the exceptional set of expressing a positive even number as the sum of a square of prime, a cube of prime, a sixth power of prime and a $k$-th power of prime, i.e.,

$$n = p_1^2 + p_2^3 + p_3^6 + p_4^k, \tag{1.1}$$

where $p_1$, $p_2$, $p_3$, $p_4$ are primes. Let $E_1(k,N)$ be the number of positive even integers $n$ up to $N$ which cannot be written in the form (1.1). Exactly, Schwarz [14] showed that $E_1(k,N) \ll N(\log N)^{-A}$ for any fixed $A > 0$. Recently, Brüdern [4] improved this result and established that $E_1(k,N) \ll N^{1-\frac{1}{8k^2}+\varepsilon}$. In this paper, we further improve the result of Brüdern by giving the following theorem.

**Theorem 1.1** *Let $E_1(k,N)$ be defined as above. We have*

$$E_1(k,N) \ll N^{1-\theta_1(k)+\varepsilon},$$

Manuscript received September 12, 2019. Revised March 9, 2023.
[1]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, Fujian, China.
E-mail: zhenzhenfeng@mnnu.edu.cn
[2]Corresponding author. School of Mathematics, Jilin University, Changchun 130012, China.
E-mail: jma@jlu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11771252) and the Natural Science Foundation of Fujian Province (Nos. 2024J01805, 2022J02046).

*among which*

$$\theta_1(k) = \begin{cases} \dfrac{1}{54}, & k = 6, \\ \dfrac{1}{81}, & k = 7, \\ \dfrac{1}{54x}, & k \geq 8, \end{cases}$$

*where*

$$x = \begin{cases} \Big\lceil \Big(\dfrac{k}{6} + 1 - \Big[\dfrac{k}{6}\Big]\Big) 2^{[\frac{k}{6}]-1} \Big\rceil, & 8 \leq k \leq 23, \\ \Big\lceil \dfrac{7k}{6} - 20 \Big\rceil, & 24 \leq k \leq 29, \\ \Big\lceil \Big(\dfrac{k}{6} - \dfrac{1}{2}\Big[\dfrac{k}{6}\Big]\Big)\Big(\Big[\dfrac{k}{6}\Big] + 1\Big) \Big\rceil, & k \geq 30. \end{cases} \tag{1.2}$$

*Here $\lceil a \rceil$ means the smallest integer no smaller than $a$ and $[a]$ means the biggest integer no bigger than $a$.*

**Remark 1.1** We can compare the results of Theorem 1.1 with those of Brüdern [4]. For example, we obtain $E_1(6, N) \ll N^{1-\frac{1}{54}+\varepsilon}$ and $E_1(7, N) \ll N^{1-\frac{1}{81}+\varepsilon}$. Meanwhile, Brüdern's results indicated that $E_1(6, N) \ll N^{1-\frac{1}{288}+\varepsilon}$ and $E_1(7, N) \ll N^{1-\frac{1}{392}+\varepsilon}$. In addition, for large value $k$, Theorem 1.1 gives that $E_1(k, N) \ll N^{1-\frac{1}{\frac{3}{4}k^2+O(k)}+\varepsilon}$, whereas Brüdern's result (see [4]) showed that $E_1(k, N) \ll N^{1-\frac{1}{8k^2}+\varepsilon}$.

In the same paper [14], Schwarz also considered the problem of representing a large even integer $n$ in the form

$$n = p_1^2 + p_2^4 + p_3^4 + p_4^k, \tag{1.3}$$

where $p_1, p_2, p_3, p_4$ are primes. Let $E_2(k, N)$ denote the number of positive even integers $n$ up to $N$ which cannot be written in the form (1.3). In fact, Schwarz [14] proved that $E_2(k, N) \ll N(\log N)^{-A}$ for any fixed $A > 0$. Using the similar method to treat Theorem 1.1, we obtain the following result.

**Theorem 1.2** *Let $E_2(k, N)$ be defined as above. We have*

$$E_2(k, N) \ll N^{1-\theta_2(k)+\varepsilon},$$

*here*

$$\theta_2(k) = \begin{cases} \dfrac{1}{32}, & k = 4, \\ \dfrac{1}{48}, & k = 5, \\ \dfrac{1}{64}, & 6 \leq k \leq 8, \\ \dfrac{1}{48x}, & k \geq 9, \end{cases}$$

*where*

$$x=\begin{cases}\left\lceil\left(\frac{k}{4}+1-\left[\frac{k}{4}\right]\right)2^{[\frac{k}{4}]-1}\right\rceil, & 9\le k\le 19,\\ \left\lceil\left(\frac{k}{4}-\frac{1}{2}\left[\frac{k}{4}\right]\right)\left(\left[\frac{k}{4}\right]+1\right)\right\rceil, & k\ge 20.\end{cases}$$

**Remark 1.2** For example, we obtain that $E_2(4,N)\ll N^{1-\frac{1}{32}+\varepsilon}$ and $E_2(6,N)\ll N^{1-\frac{1}{64}+\varepsilon}$. Meanwhile, Brüdern's method in [4] indicated that $E_2(4,N)\ll N^{1-\frac{1}{128}+\varepsilon}$ and $E_2(6,N)\ll N^{1-\frac{1}{288}+\varepsilon}$. In addition, for large value $k$, Theorem 1.2 gives that $E_2(k,N)\ll N^{1-\frac{1}{\frac{3}{2}k^2+O(k)}+\varepsilon}$, whereas Brüdern's method in [4] showed that $E_2(k,N)\ll N^{1-\frac{1}{8k^2}+\varepsilon}$.

Another related problem is to study for the diophantine equation

$$n=p_1^2+p_2^3+p_3^5+p_4^k, \tag{1.4}$$

where $p_1$, $p_2$, $p_3$, $p_4$ are primes. Let $E_3(k,N)$ be the number of even integers $n\le N$ that cannot be represented in the form (1.4). In 1953, Prachar [11] proved that $E_3(4,N)\ll N(\log N)^{-\frac{30}{47}+\varepsilon}$. This has been improved by a number of authors (see [1–2, 12–13]). The latest result is

$$E_3(4,N)\ll N^{1-\frac{1}{16}+\varepsilon}$$

given by Zhao [16]. For general $k\ge 5$, Lu and Shan [10] proved that $E_3(k,N)\ll N(\log N)^{-c}$ for some $c>0$. Lately, it was improved to $E_3(k,N)\ll N^{1-\frac{1}{3k\times 2^{k-2}}+\varepsilon}$ by Liu [9]. The current best result was given by Hoffman and Yu [5] which is

$$E_3(k,N)\ll N^{1-\frac{47}{420\cdot 2^s}+\varepsilon} \tag{1.5}$$

where $s=\left[\frac{k+1}{2}\right]$. In this paper, we established the following result which improves (1.5).

**Theorem 1.3** *Let $E_3(k,N)$ be defined as above. We have*

$$E_3(k,N)\ll N^{1-\theta_3(k)+\varepsilon},$$

*here*

$$\theta_3(k)=\begin{cases}\frac{1}{24}, & k=5,\\ \frac{2}{81}, & k=6,\\ \frac{1}{36x}, & k\ge 7,\end{cases}$$

*where*

$$x=\begin{cases}\left\lceil\left(\frac{k}{6}+1-\left[\frac{k}{6}\right]\right)2^{[\frac{k}{6}]-1}\right\rceil, & 7\le k\le 23,\\ \left\lceil\frac{7k}{6}-20\right\rceil, & 24\le k\le 29,\\ \left\lceil\left(\frac{k}{6}-\frac{1}{2}\left[\frac{k}{6}\right]\right)\left(\left[\frac{k}{6}\right]+1\right)\right\rceil, & k\ge 30.\end{cases}$$

**Remark 1.3** Our results indeed improve the result of Hoffman and Yu [5]. For example, we obtain that $E_3(5,N) \ll N^{1-\frac{1}{24}+\varepsilon}$ and $E_3(7,N) \ll N^{1-\frac{1}{72}+\varepsilon}$. Meanwhile, Hoffman and Yu's results indicated that $E_3(5,N) \ll N^{1-\frac{47}{3360}+\varepsilon}$ and $E_3(7,N) \ll N^{1-\frac{47}{6720}+\varepsilon}$. In addition, for large value $k$, $E_3(k,N) \ll N^{1-\frac{1}{\theta(k)}+\varepsilon}$, Hoffman and Yu [5] showed that $\theta(k)$ grows exponentially, whereas, Theorem 1.3 implicates that $\theta(k) = \frac{k^2}{2} + O(k)$ with polynomial growth.

Finally, we consider the problem of representing a large odd integer $n$ in the form

$$n = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^k, \tag{1.6}$$

where $p_1$, $p_2$, $p_3$, $p_4$ and $p_5$ are primes. Let $E_4(k,N)$ denote the number of positive odd integers $n$ up to $N$ which cannot be written in the form (1.6). In the following result, we will give an up bound for $E_4(k,N)$ for $k \geq 4$.

**Theorem 1.4** *Let $E_4(k,N)$ be defined as above. We have*

$$E_4(k,N) \ll N^{1-\theta_4(k)+\varepsilon},$$

*here*

$$\theta_4(k) = \begin{cases} \dfrac{1}{24}, & k = 4, \\ \dfrac{1}{54}, & k = 5, \\ \dfrac{1}{9x}, & k \geq 6, \end{cases}$$

*where*

$$x = \begin{cases} \left\lceil \dfrac{14k}{3} - 20 \right\rceil, & k = 6, 7, \\ \left\lceil \Big(\dfrac{2k}{3} - \dfrac{1}{2}\Big[\dfrac{2k}{3}\Big]\Big)\Big(\Big[\dfrac{2k}{3}\Big] + 1\Big) \right\rceil, & k \geq 8. \end{cases}$$

As usual, we abbreviate $\mathrm{e}^{2\pi\mathrm{i}\alpha}$ to $e(\alpha)$. The letter $p$, with or without indices, is prime number. The letter $\varepsilon$ denotes a sufficiently small positive real number, and the value of $\varepsilon$ may change from statement to statement. Let $N$ be a real number sufficiently large in terms of $\varepsilon$ and $k$. We use $\ll$ and $\gg$ to denote Vinogradov's well-know notation, while implied constant may depend on $\varepsilon$ and $k$.

## 2 Preliminaries and Lemmas

We will prove Theorems 1.1–1.4 by using the circle method. Now the treatment for major arcs of Hardy-Littlewood method are standard nowadays, for example Liu and Zhan [8]. We need the following lemmas to control the minor arcs of circle method.

**Lemma 2.1** *Let*

$$S_k(\alpha) = \sum_{\frac{N}{4} < p^k \leq N} (\log p) e(\alpha p^k).$$

*Then for $1 \leq j \leq k$, we have*

$$\int_0^1 |S_k^{2^j}(\alpha)| \mathrm{d}\alpha \ll N^{\frac{1}{k}(2^j - j)+\varepsilon}$$

*and*

$$\int_0^1 |S_k^{j(j+1)}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{j^2}{k}+\varepsilon}.$$

In fact, Lemma 2.1 is the classical result of Hua [6] and the recent work of Bourgain [3]. The next lemma is a generalization of Lemma 2.1.

**Lemma 2.2** *Let* $S_k(\alpha)$ *be defined as Lemma* 2.1. *For* $0 < \delta \le 1$,

$$\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{2x}{k}-\delta+\varepsilon},$$

*where*

$$x = \begin{cases} \lceil (k\delta + 1 - [k\delta])2^{[k\delta]-1} \rceil, & [k\delta] \le 3, \\ \lceil 7k\delta - 20 \rceil, & [k\delta] = 4, \\ \left\lceil \left(k\delta - \frac{1}{2}[k\delta]\right)([k\delta]+1) \right\rceil, & [k\delta] \ge 5. \end{cases} \tag{2.1}$$

**Proof** For $\delta = 1$, this is Lemma 2.1. Next, we consider the case $0 < \delta < 1$.

For $[k\delta] \le 3$, clearly by (2.1) we have

$$2^{[k\delta]} \le 2x \le 2^{[k\delta]+1}.$$

Applying Hölder's inequality and Hua's lemma, one has

$$\begin{aligned} \int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll& \Big( \int_0^1 |S_k^{2^{[k\delta]}}(\alpha)|\mathrm{d}\alpha \Big)^a \Big( \int_0^1 |S_k^{2^{[k\delta]+1}}(\alpha)|\mathrm{d}\alpha \Big)^b \\ \ll& N^{\frac{2x}{k}-c+\varepsilon}, \end{aligned}$$

where

$$a = 2 - \frac{x}{2^{[k\delta]-1}}, \quad b = \frac{x}{2^{[k\delta]-1}} - 1, \quad c = \frac{[k\delta] + \frac{2x}{2^{[k\delta]}} - 1}{k}.$$

Recall that $x \ge (k\delta + 1 - [k\delta])2^{[k\delta]-1}$, so we have $c \ge \delta$. Thus this lemma holds for $[k\delta] \le 3$.

For $[k\delta] = 4$, obviously by (2.1) we have

$$16 < 2x \le 30.$$

Applying Hölder's inequality and Lemma 2.1, one has

$$\begin{aligned} \int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll& \Big( \int_0^1 |S_k^{16}(\alpha)|\mathrm{d}\alpha \Big)^{\frac{15}{7}-\frac{x}{7}} \Big( \int_0^1 |S_k^{30}(\alpha)|\mathrm{d}\alpha \Big)^{\frac{x}{7}-\frac{8}{7}} \\ \ll& N^{\frac{2x}{k}-\frac{x+20}{7k}+\varepsilon}. \end{aligned}$$

This combining with $x \ge 7k\delta - 20$ gives $\frac{x+20}{7k} \ge \delta$. Thus this lemma holds for $[k\delta] = 4$.

For $[k\delta] \ge 5$, by (2.1) we have

$$[k\delta]([k\delta]+1) \le 2x \le ([k\delta]+1)([k\delta]+2).$$

Applying Hölder's inequality and Lemma 2.1, one has

$$\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll \Big( \int_0^1 |S_k^{[k\delta]([k\delta]+1)}(\alpha)|\mathrm{d}\alpha \Big)^a \Big( \int_0^1 |S_k^{([k\delta]+1)([k\delta]+2)}(\alpha)|\mathrm{d}\alpha \Big)^b$$

$$\ll N^{\frac{2x}{k}-c+\varepsilon},$$

where

$$a = 1 - \frac{x}{[k\delta]+1} + \frac{[k\delta]}{2}, \quad b = \frac{x}{[k\delta]+1} - \frac{[k\delta]}{2}, \quad c = \frac{\frac{x}{[k\delta]+1} + \frac{[k\delta]}{2}}{k}.$$

Then we have $c \geq \delta$ because of $x \geq \left(k\delta - \frac{[k\delta]}{2}\right)([k\delta]+1)$. This lemma holds for $[k\delta] \geq 5$. Hence, this lemma holds for $0 < \delta \leq 1$.

**Lemma 2.3** *For $k \geq 3$, we have*

$$\int_0^1 |S_2^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{2x}{k}+\varepsilon},$$

*where*

$$x = \begin{cases} \left\lceil \left(\frac{k}{2} + 1 - \left[\frac{k}{2}\right]\right) 2^{[\frac{k}{2}]-1} \right\rceil, & 3 \leq k \leq 9, \\ \left\lceil \left(\frac{k}{2} - \frac{1}{2}\left[\frac{k}{2}\right]\right)\left(\left[\frac{k}{2}\right]+1\right) \right\rceil, & k \geq 10. \end{cases}$$

**Proof** $\int_0^1 |S_2^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha$ is no more than $N^{\varepsilon}$ times the number of solutions of the equation

$$t_1^2 - t_2^2 = y_1^k + y_2^k + \cdots + y_x^k - y_{x+1}^k - \cdots - y_{2x}^k$$

with $N^{\frac{1}{2}} < t_1, t_2 \leq 2N^{\frac{1}{2}}$ and $N^{\frac{1}{k}} < y_1, y_2, \cdots, y_{2x} \leq 2N^{\frac{1}{k}}$. If $t_1 \neq t_2$, the contribution is bounded by $N^{\frac{2x}{k}+\varepsilon}$. If $t_1 = t_2$, the contribution is bounded by $N^{\frac{1}{2}+\varepsilon}\int_0^1 |S_k^{2x}|\mathrm{d}\alpha$. Thus

$$\int_0^1 |S_2^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{2x}{k}+\varepsilon} + N^{\frac{1}{2}+\varepsilon}\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha.$$

What we need is

$$\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{2x}{k}-\frac{1}{2}+\varepsilon}.$$

Hence this lemma holds by Lemma 2.2 with $\delta = \frac{1}{2}$.

**Lemma 2.4** *For $k \geq 4$, we have*

$$\int_0^1 |S_2^2(\alpha)S_4^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{2x}{k}+\frac{1}{2}+\varepsilon},$$

*where*

$$x = \begin{cases} \left\lceil \left(\frac{k}{4} + 1 - \left[\frac{k}{4}\right]\right) 2^{[\frac{k}{4}]-1} \right\rceil, & 4 \leq k \leq 19, \\ \left\lceil \left(\frac{k}{4} - \frac{1}{2}\left[\frac{k}{4}\right]\right)\left(\left[\frac{k}{4}\right]+1\right) \right\rceil, & k \geq 20. \end{cases}$$

**Proof** $\int_0^1 |S_2^2(\alpha)S_4^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha$ is no more than $N^{\varepsilon}$ times the number of solutions for the equation

$$t_1^2 - t_2^2 = y_1^4 - y_2^4 + z_1^k + z_2^k + \cdots + z_x^k - z_{x+1}^k - \cdots - z_{2x}^k$$

with $P_2 < t_1, t_2 \le 2P_2$, $P_4 < y_1, y_2 \le 2P_4$ and $P_k < z_1, z_2, \cdots, z_{2x} \le 2P_k$, where $\frac{N}{4} < P_2^2, P_4^4, P_k^k \le N$. If $t_1 \ne t_2$, the contribution is bounded by $P_4^{2+\varepsilon} P_k^{2x}$. If $t_1 = t_2, y_1 \ne y_2$, the contribution is bounded by $P_2 P_k^{2x+\varepsilon}$. If $t_1 = t_2, y_1 = y_2$, the contribution is bounded by $P_2^{1+\varepsilon} P_4 \int_0^1 |S_k(\alpha)|^{2x} \mathrm{d}\alpha$. Thus

$$\int_0^1 |S_2^2(\alpha) S_4^2(\alpha) S_k^{2x}(\alpha)| \mathrm{d}\alpha \ll N^\varepsilon P_4^{2+\varepsilon} P_k^{2x} + N^\varepsilon P_2 P_4 \int_0^1 |S_k(\alpha)|^{2x} \mathrm{d}\alpha.$$

What we need is

$$\int_0^1 |S_k^{2x}(\alpha)| \mathrm{d}\alpha \ll N^{\frac{2x}{k} - \frac{1}{4} + \varepsilon}.$$

Hence this lemma holds by Lemma 2.2 with $\delta = \frac{1}{4}$.

**Lemma 2.5** *For $k \ge 3$, we have*

$$\int_0^1 |S_2^2(\alpha) S_3^2(\alpha) S_k^{2x}(\alpha)| \mathrm{d}\alpha \ll N^{\frac{2x}{k} + \frac{2}{3} + \varepsilon},$$

*where*

$$x = \begin{cases} \left\lceil \left(\frac{k}{6} + 1 - \left[\frac{k}{6}\right]\right) 2^{[\frac{k}{6}] - 1} \right\rceil, & 3 \le k \le 23, \\ \left\lceil \frac{7k}{6} - 20 \right\rceil, & 24 \le k \le 29, \\ \left\lceil \left(\frac{k}{6} - \frac{1}{2}\left[\frac{k}{6}\right]\right)\left(\left[\frac{k}{6}\right] + 1\right) \right\rceil, & k \ge 30. \end{cases}$$

**Proof** The proof is similar as the proof of Lemma 2.4 with $\delta = \frac{1}{6}$.

**Lemma 2.6** *For $k \ge 3$, we have*

$$\int_0^1 |S_3^4(\alpha) S_k^{2x}(\alpha)| \mathrm{d}\alpha \ll N^{\frac{2x}{k} + \frac{1}{3} + \varepsilon},$$

*where*

$$x = \begin{cases} \left\lceil \left(\frac{2k}{3} + 1 - \left[\frac{2k}{3}\right]\right) 2^{[\frac{2k}{3}] - 1} \right\rceil, & 3 \le k \le 5, \\ \left\lceil \frac{14k}{3} - 20 \right\rceil, & k = 6, 7, \\ \left\lceil \left(\frac{2k}{3} - \frac{1}{2}\left[\frac{2k}{3}\right]\right)\left(\left[\frac{2k}{3}\right] + 1\right) \right\rceil, & k \ge 8. \end{cases}$$

**Proof** We have

$$\int_0^1 |S_3^4(\alpha) S_k^{2x}(\alpha)| \mathrm{d}\alpha \ll N^\varepsilon \int_0^1 |f_3^4(\alpha) S_k^{2x}(\alpha)| \mathrm{d}\alpha, \tag{2.2}$$

where

$$f_3(\alpha) = \sum_{t \sim P_3} e(\alpha t^3).$$

By [15, Lemma 2.3], one has

$$|f_3(\alpha)|^4 \ll P_3 \sum_{|h_1|<P_3} \sum_{|h_2|<P_3} \sum_{x\in\mathcal{J}} e(\alpha\Delta(t^3;\mathbf{h})),$$

where $\mathcal{J} = \mathcal{J}(\mathbf{h})$ is a subinterval of $[P_3, 2P_3)$ and $\Delta(t^3;\mathbf{h})$ is the second-order forward difference of the function $t \to t^3$ with steps $h_1, h_2$, that is,

$$\Delta(t^3;\mathbf{h}) = 3h_1h_2(2t + h_1 + h_2).$$

Thus, we deduce from (2.2) that

$$\int_0^1 |S_3^4(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll P_3J(P_3),$$

where $J(P_3)$ is the number of solutions of the diophantine equation

$$\Delta(t^3;\mathbf{h}) = 3h_1h_2(2t + h_1 + h_2) = p_1^k + p_2^k + \cdots + p_x^k - q_1^k - q_2^k - \cdots - q_x^k \tag{2.3}$$

subject to

$$P_3 \le t \le 2P_3, \quad |h_i| < P_3, \quad P_k < p_1, \cdots, p_x, q_1, \cdots, q_x \le 2P_k, \quad \frac{N}{4} < P_3^3, P_k^k \le N. \tag{2.4}$$

The number of solutions of (2.3)–(2.4) with $\Delta(t^3;\mathbf{h}) = 0$ is bounded by $P_3^{2+\varepsilon}\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha$. The number of solutions of (2.3)–(2.4) with $\Delta(t^3;\mathbf{h}) \ne 0$ is bounded by $N^{\frac{2x}{k}+\varepsilon}$. Then

$$\int_0^1 |S_3^4(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{1+\varepsilon}\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha + N^{\frac{1}{3}+\frac{2x}{k}+\varepsilon}.$$

Thus we just need

$$\int_0^1 |S_k^{2x}(\alpha)|\mathrm{d}\alpha \ll N^{\frac{2x}{k}-\frac{2}{3}+\varepsilon}.$$

Hence it establishes this lemma by Lemma 2.2 with $\delta = \frac{2}{3}$.

## 3 Proof of Theorem 1.1

The purpose of this section is to concentrate on proving Theorem 1.1. We establish Theorem 1.1 by means of the Hardy-Littlewood method. We will give the proof of Theorem 1.2 in Section 4 and will describe the straight forward modifications needed for Theorem 1.3 in Section 5. In Section 6, we will give the outline of the proof of Theorem 1.4.

Let $S_k(\alpha)$ be defined as in Lemma 2.1. We denote

$$r(n) = \sum_{\substack{p_1^2+p_2^3+p_3^6+p_4^k=n \\ \frac{N}{4}<p_1^2,p_2^3,p_3^6,p_4^k\le N}} (\log p_1)(\log p_2)(\log p_3)(\log p_4), \tag{3.1}$$

where $p_1$, $p_2$, $p_3$, $p_4$ are primes. Let $Q = N^{\frac{2}{5k}}$, and write $\mathfrak{M}(Q)$ for the union of the intervals

$$\{\alpha \in [0,1] : |q\alpha - a| \le QN^{-1}\}$$

with $1 \le a \le q, (a,q) = 1$ and $1 \le q \le Q$. We define $\mathfrak{M} = \mathfrak{M}(Q)$, $\mathfrak{m} = [0,1]\backslash\mathfrak{M}$. Thus the formula (3.1) becomes

$$r(n) = \Big\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \Big\} S_2(\alpha)S_3(\alpha)S_6(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Whenever $\mathfrak{B} \subset [0,1]$ is measurable, we put

$$r_{\mathfrak{B}}(n,N) = \int_{\mathfrak{B}} S_2(\alpha)S_3(\alpha)S_6(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Then we have

$$r(n) = r_{[0,1]}(n,N) = \int_0^1 S_2(\alpha)S_3(\alpha)S_6(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Next we will deal with the integral of major arcs and minor arcs, respectively. Applying the now standard methods of enlarging major arcs (see [8]), we can get the following result.

**Lemma 3.1** *For all even integer $n$ with $N < n \le 2N$, one has $r_{\mathfrak{M}}(n,N) \gg N^{\frac{1}{k}-\varepsilon}$.*

To estimate the integral of the minor arcs, we split the minor arcs in two part. Let $1 \le Y \le N^{\frac{1}{8}}$, and denote $\mathfrak{N}$ the union of the pairwise disjoint intervals

$$\mathfrak{N}_{q,a}(Y) = \Big\{\alpha \in [0,1] : |q\alpha - a| \le \frac{Y}{N}\Big\}$$

with $1 \le a \le q, (a,q) = 1$ and $1 \le q \le Y$. We write $\mathfrak{N} = \mathfrak{N}(N^{\frac{1}{8}})$ and $\mathfrak{n} = \mathfrak{m}\backslash\mathfrak{N}$.

**Lemma 3.2** *For $\alpha \in \mathfrak{n}$, we have*

$$S_2(\alpha) \ll N^{\frac{1}{2}-\frac{1}{16}+\varepsilon}, \quad S_3(\alpha) \ll N^{\frac{1}{3}-\frac{1}{36}+\varepsilon}, \quad S_4(\alpha) \ll N^{\frac{1}{4}-\frac{1}{96}+\varepsilon}.$$

**Proof** For any given $\alpha \in \mathbb{N}$, by Dirichlet's approximation theorem, there exists $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q) = 1, \quad 1 \le q \le N^{\frac{5}{12}} \quad \text{and} \quad |q\alpha - a| \le N^{-\frac{5}{12}}.$$

Then by [7, Theorem 1], one has

$$S_2(\alpha) \ll N^{\frac{1}{2}-\frac{1}{16}+\varepsilon} + \frac{N^{\frac{1}{2}+\varepsilon}}{(q+N|q\alpha-a|)^{\frac{1}{2}}}$$

and

$$S_4(\alpha) \ll N^{\frac{1}{4}-\frac{1}{96}+\varepsilon} + \frac{N^{\frac{1}{4}+\varepsilon}}{(q+N|q\alpha-a|)^{\frac{1}{2}}},$$

and by [17, Lemma 2.3], one has

$$S_3(\alpha) \ll N^{\frac{1}{3}-\frac{1}{36}+\varepsilon} + \frac{N^{\frac{1}{3}+\varepsilon}}{(q+N|q\alpha-a|)^{\frac{1}{2}}}.$$

If $\alpha \in \mathfrak{n}$, then

$$q > N^{\frac{1}{8}} \quad \text{or} \quad q \le N^{\frac{1}{8}}, \quad N^{-\frac{7}{8}} \le |q\alpha - a| < N^{-\frac{5}{12}}.$$

In any case, we have

$$q + |q\alpha - a| \gg N^{\frac{1}{8}},$$

then this lemma clearly holds.

**Proof of Theorem 1.1** By Bessel's inequality, we have

$$\sum_{N<n\le 2N}\Big|\int_{\mathfrak{m}} S_2(\alpha)S_3(\alpha)S_6(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha\Big|^2 \le \int_{\mathfrak{m}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.$$

To prove Theorem 1.1, it suffices to show that

$$\int_{\mathfrak{m}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\theta_1(k)+\varepsilon}, \tag{3.2}$$

where $\theta_1(k)$ is defined in Theorem 1.1.

Obviously, we know that

$$\begin{aligned}\int_{\mathfrak{m}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll &\int_{\mathfrak{N}\backslash\mathfrak{M}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\\ &+\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.\end{aligned}$$

By the estimate on [4, p. 80], one has

$$\int_{\mathfrak{N}\backslash\mathfrak{M}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\frac{1}{4k}+\varepsilon} \ll N^{1+\frac{2}{k}-\theta_1(k)+\varepsilon} \tag{3.3}$$

since $\frac{1}{4k} > \theta_1(k)$ for all $k \ge 6$.

Next we estimate $\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha$.

For $k = 6$, by Lemmas 2.3 and 3.2, one has

$$\begin{aligned}&\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_6^4(\alpha)|\mathrm{d}\alpha\\ &\ll \sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^{\frac{2}{3}}\Big(\int_0^1 |S_2^2(\alpha)S_6^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{3}}\Big(\int_0^1 |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)|\mathrm{d}\alpha\Big)^{\frac{2}{3}}\\ &\ll N^{1+\frac{1}{3}-\frac{1}{54}+\varepsilon}.\end{aligned} \tag{3.4}$$

For $k = 7$, by Lemmas 2.3 and 3.2, we have

$$\begin{aligned}&\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_7^2(\alpha)|\mathrm{d}\alpha\\ &\ll \sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^{\frac{4}{9}}\Big(\int_0^1 |S_2^2(\alpha)S_6^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{18}}\Big(\int_0^1 |S_2^2(\alpha)S_7^{12}(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{6}}\\ &\quad\times\Big(\int_0^1 |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)|\mathrm{d}\alpha\Big)^{\frac{7}{9}}\\ &\ll N^{1+\frac{2}{7}-\frac{1}{81}+\varepsilon}.\end{aligned} \tag{3.5}$$

For $k \ge 8$ and $x$ in the form (1.2), by Lemmas 2.3 and 3.2, we have

$$\begin{aligned}&\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\\ &\ll \sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^{\frac{2}{3x}}\Big(\int_0^1 |S_2^2(\alpha)S_6^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{3x}}\Big(\int_0^1 |S_2^2(\alpha)S_3^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{x}}\end{aligned}$$

$$\times \Big( \int_0^1 |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)|\mathrm{d}\alpha \Big)^{1-\frac{4}{3x}}$$
$$\ll N^{1+\frac{2}{k}-\frac{1}{54x}+\varepsilon}. \tag{3.6}$$

By (3.4)–(3.6), we have

$$\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\theta_1(k)+\varepsilon}. \tag{3.7}$$

Thus, it establishes (3.2) by (3.3) and (3.7). Hence, Theorem 1.1 holds.

## 4 Proof of Theorem 1.2

Suppose that $N$ is a large positive integer. Let $S_k(\alpha)$ be defined as in Lemma 2.1. Let

$$r(n) = \sum_{\substack{p_1^2+p_2^4+p_3^4+p_4^k=n \\ \frac{N}{4}<p_1^2,p_2^4,p_3^4,p_4^k\le N}} (\log p_1)(\log p_2)(\log p_3)(\log p_4),$$

where $p_1$, $p_2$, $p_3$, $p_4$ are primes and the major arcs $\mathfrak{M}$, minor arcs $\mathfrak{m}$, $\mathfrak{N}$ and $\mathfrak{n}$ be defined as in Section 3. Then the weighted number of representations of $n$ in the form of (1.3) equals

$$r(n) = \int_0^1 S_2(\alpha)S_4^2(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

Whenever $\mathfrak{B} \subset [0,1]$ is measurable, we put

$$r_{\mathfrak{B}}(n,N) = \int_{\mathfrak{B}} S_2(\alpha)S_4^2(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Then we have

$$r(n) = r_{[0,1]}(n,N) = \int_0^1 S_2(\alpha)S_4^2(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Next we will deal with the integral of major arcs and minor arcs, respectively. Applying the now standard methods of enlarging major arcs (see [8]), we can get the following result.

**Lemma 4.1** *For all even integer $n$ with $N < n \le 2N$, one has $r_{\mathfrak{M}}(n,N) \gg N^{\frac{1}{k}-\varepsilon}$.*

**Lemma 4.2** *For $\alpha \in \mathfrak{M}(2K)\backslash\mathfrak{M}(K)$, $N^{\frac{2}{5k}} \ll K \ll N^{\frac{1}{8}}$, one has*

$$\begin{aligned} S_2(\alpha) &\ll N^{\frac{1}{2}+\varepsilon}K^{-\frac{1}{2}}, \\ S_3(\alpha) &\ll N^{\frac{1}{3}+\varepsilon}K^{-\frac{1}{2}}, \\ S_4(\alpha) &\ll N^{\frac{11}{80}+\varepsilon}K^{\frac{1}{2}} + N^{\frac{1}{4}+\varepsilon}K^{-\frac{1}{2}}, \\ S_5(\alpha) &\ll N^{\frac{11}{100}+\varepsilon}K^{\frac{1}{2}} + N^{\frac{1}{5}+\varepsilon}K^{-\frac{1}{2}}. \end{aligned}$$

**Proof** The [7, Theorem 2] implies that, if $1 \le q \le H$, $(a,q)=1$, $|q\alpha - a| < HN^{-1}$ with $H \ll N^{\frac{1}{k}}$, then

$$\sum_{p\sim N^{\frac{1}{k}}} e(\alpha p^k) \ll H^{\frac{1}{2}}N^{\frac{11}{20k}+\varepsilon} + \frac{N^{\frac{1}{k}+\varepsilon}}{(q+N|q\alpha-a|)^{\frac{1}{2}}}. \tag{4.1}$$

If $\alpha \in \mathfrak{M}(2K)\backslash\mathfrak{M}(K)$, $N^{\frac{2}{5k}} \ll K \ll N^{\frac{1}{8}}$, then this lemma clearly follows by (4.1).

**Proof of Theorem 1.2** By Bessel's inequality, we have

$$\sum_{N<n\le 2N}\Big|\int_{\mathfrak{m}} S_2(\alpha)S_4^2(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha\Big|^2 \le \int_{\mathfrak{m}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.$$

Thus, to prove Theorem 1.2, it suffices to show that

$$\int_{\mathfrak{m}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\theta_2(k)+\varepsilon}, \tag{4.2}$$

where $\theta_2(k)$ is defined in Theorem 1.2.

Obviously, we know that

$$\int_{\mathfrak{m}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll \int_{\mathfrak{N}\backslash\mathfrak{M}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha + \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.$$

First, we show that

$$\int_{\mathfrak{N}\backslash\mathfrak{M}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\theta_2(k)+\varepsilon}. \tag{4.3}$$

It suffices to prove that

$$\int_{\mathfrak{M}(2K)\backslash\mathfrak{M}(K)} \big|S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)\big|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\theta_2(k)+\varepsilon}$$

for $N^{\frac{2}{5k}} \ll K \ll N^{\frac{1}{8}}$. By [5, Lemmas 4.2 and 5.2 ], we have

$$\begin{aligned}
&\int_{\mathfrak{M}(2K)\backslash\mathfrak{M}(K)} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\\
\ll &\sup_{\alpha\in\mathfrak{M}(2K)\backslash\mathfrak{M}(K)} |S_2^2(\alpha)S_4^4(\alpha)| \int_{\mathfrak{M}(2K)} |S_k^2(\alpha)|\mathrm{d}\alpha\\
\ll &\frac{N^{1+\varepsilon}}{K}(K^2N^{\frac{11}{20}+\varepsilon} + N^{1+\varepsilon}K^{-2})(N^{-1}K(N^{\frac{1}{k}}K + N^{\frac{2}{k}}))\\
\ll &N^{\frac{11}{20}+\frac{1}{k}+\varepsilon}K^3 + N^{\frac{11}{20}+\frac{2}{k}+\varepsilon}K^2 + N^{1+\frac{1}{k}+\varepsilon}K^{-1} + N^{1+\frac{2}{k}+\varepsilon}K^{-2}\\
\ll &N^{1+\frac{2}{k}-\frac{4}{5k}+\varepsilon} \ll N^{1+\frac{2}{k}-\theta_2(k)+\varepsilon},
\end{aligned}$$

since $\frac{4}{5k} > \theta_2(k)$ for all $k \ge 4$.

Next we show that

$$\int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{k}-\theta_2(k)+\varepsilon}. \tag{4.4}$$

For $k \ge 9$ and $x$ in the form in Theorem 1.2, by Lemmas 2.3–2.4 , one has

$$\begin{aligned}
&\int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\\
\ll &\Big(\int_0^1 |S_2^2(\alpha)S_4^4(\alpha)|\mathrm{d}\alpha\Big)^{1-\frac{1}{x}}\Big(\int_0^1 |S_2^2(\alpha)S_4^2(\alpha)S_k^{2x}|\mathrm{d}\alpha\Big)^{\frac{1}{x}} \sup_{\alpha\in\mathfrak{m}} |S_4(\alpha)|^{\frac{2}{x}}
\end{aligned}$$

$$\ll N^{1+\frac{2}{k}-\frac{1}{48x}+\varepsilon}.$$

We use Zhao [18, Lemma 3.1] to prove (4.4) for $4 \le k \le 8$, since the methods are same, we only give the proof for $k = 5$ for simplicity.

For $k = 5$, by [18, Lemma 3.1] with $g(\alpha) = S_4(\alpha)$ and $h(\alpha) = S_5(\alpha)$, one has

$$\begin{aligned}
&\int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \\
\ll &N^{\frac{1}{4}} J_0^{\frac{1}{4}} \Big( \int_{\mathfrak{n}} |S_2^4(\alpha)S_4^6(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \Big)^{\frac{1}{4}} \Big( \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^3(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \Big)^{\frac{1}{2}} \\
&+ N^{\frac{1}{4}(1-2^{-4})+\varepsilon} \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^3(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha,
\end{aligned} \tag{4.5}$$

where

$$J_0 \ll N^{-\frac{3}{5}+\varepsilon}$$

by [18, Lemma 2.2].

By Hölder's inequality, Lemmas 2.3 and 2.5, one has

$$\begin{aligned}
&\int_{\mathfrak{n}} |S_2^2(\alpha)S_4^3(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \\
\ll &\Big( \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \Big)^{\frac{1}{4}} \Big( \int_0^1 |S_2^2(\alpha)S_4^4(\alpha)|\mathrm{d}\alpha \Big)^{\frac{1}{2}} \Big( \int_0^1 |S_2^2(\alpha)S_5^6(\alpha)|\mathrm{d}\alpha \Big)^{\frac{1}{4}} \\
\ll &N^{\frac{4}{5}+\varepsilon} \Big( \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \Big)^{\frac{1}{4}}.
\end{aligned} \tag{4.6}$$

By Lemma 3.2,

$$\begin{aligned}
\int_{\mathfrak{n}} |S_2^4(\alpha)S_4^6(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha &\ll \sup_{\alpha\in\mathfrak{n}} |S_2(\alpha)S_4(\alpha)|^2 \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \\
&\ll N^{\frac{65}{48}+\varepsilon} \int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha.
\end{aligned} \tag{4.7}$$

By (4.5)–(4.7), we have

$$\int_{\mathfrak{n}} |S_2^2(\alpha)S_4^4(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha \ll N^{1+\frac{2}{5}-\frac{1}{48}+\varepsilon}.$$

Thus, it establishes (4.2) by (4.3)–(4.4).

## 5 Proof of Theorem 1.3

Suppose that $N$ is a large positive integer. Let $S_k(\alpha)$ be defined as Lemma 2.1. Denote

$$r(n) = \sum_{\substack{p_1^2+p_2^3+p_3^5+p_4^k=n \\ \frac{N}{4}<p_1^2,p_2^3,p_3^5,p_4^k\le N}} (\log p_1)(\log p_2)(\log p_3)(\log p_4)$$

and let the major arcs $\mathfrak{M}$, minor arcs $\mathfrak{m}$, $\mathfrak{N}$ and $\mathfrak{n}$ be defined as in Section 3. Then the weighted number of representations of $n$ in the form of (1.4) equals

$$r(n) = \int_0^1 S_2(\alpha)S_3(\alpha)S_5(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

Whenever $\mathfrak{B} \subset [0,1]$ is measurable, we put

$$r_{\mathfrak{B}}(n,N) = \int_{\mathfrak{B}} S_2(\alpha)S_3(\alpha)S_5(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Then we have

$$r(n) = r_{[0,1]}(n,N) = \int_0^1 S_2(\alpha)S_3(\alpha)S_5(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Next we will deal with the integral of major arcs and minor arcs, respectively. Applying the now standard methods of enlarging major arcs (see [8]), we can get the following result.

**Lemma 5.1** *For all even integer $n$ with $N < n \le 2N$, one has $r_{\mathfrak{M}}(n,N) \gg N^{\frac{1}{30}+\frac{1}{k}-\varepsilon}$.*

**Proof of Theorem 1.3** By Bessel's inequality, we have

$$\sum_{N<n\le 2N}\Big|\int_{\mathfrak{m}} S_2(\alpha)S_3(\alpha)S_5(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha\Big|^2 \le \int_{\mathfrak{m}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.$$

Thus, to prove Theorem 1.3, it suffices to show that

$$\int_{\mathfrak{m}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{\frac{16}{15}+\frac{2}{k}-\theta_3(k)+\varepsilon}, \tag{5.1}$$

where $\theta_3(k)$ is defined in Theorem 1.3. Obviously, one has

$$\begin{aligned}\int_{\mathfrak{m}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll& \int_{\mathfrak{N}\setminus\mathfrak{M}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \\ &+ \int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.\end{aligned}$$

First, we show that

$$\int_{\mathfrak{N}\setminus\mathfrak{M}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{\frac{16}{15}+\frac{2}{k}-\theta_3(k)+\varepsilon}. \tag{5.2}$$

It suffices to prove that

$$\int_{\mathfrak{M}(2K)\setminus\mathfrak{M}(K)} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{\frac{16}{15}+\frac{2}{k}-\theta_3(k)+\varepsilon}$$

with $N^{\frac{1}{5k}} \ll K \ll N^{\frac{1}{8}}$. By [5, Lemmas 4.2 and 5.2], one has

$$\begin{aligned}&\int_{\mathfrak{M}(2K)\setminus\mathfrak{M}(K)} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \\ \ll& \frac{N^{1+\varepsilon}}{K}\frac{N^{\frac{2}{3}+\varepsilon}}{K}(N^{\frac{11}{50}+\varepsilon}K + N^{\frac{2}{5}+\varepsilon}K^{-1})(N^{-1}K(N^{\frac{1}{k}}K + N^{\frac{2}{k}})) \\ \ll& N^{\frac{16}{15}+\frac{2}{k}-\frac{4}{5k}+\varepsilon} \ll N^{\frac{16}{15}+\frac{2}{k}-\theta_3(k)+\varepsilon},\end{aligned}$$

since $\frac{4}{5k} > \theta_3(k)$ for all $k \ge 5$. This establishes (5.2).

Next, we show that

$$\int_{\mathfrak{n}} |S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{\frac{16}{15}+\frac{2}{k}-\theta_3(k)+\varepsilon}. \tag{5.3}$$

For $k=5$, by [18, Lemma 3.1] with $g(\alpha)=S_3(\alpha)$ and $h(\alpha)=S_5(\alpha)$, one has

$$\begin{aligned}&\int_{\mathfrak{n}}|S_2^2(\alpha)S_3^2(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha\\ \ll&N^{\frac{1}{3}}J_1^{\frac{1}{4}}\Big(\int_0^1|S_2^4(\alpha)S_3^2(\alpha)S_5^6(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{4}}\Big(\int_0^1|S_2^2(\alpha)S_3(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2}}\\ &+N^{\frac{1}{3}(1-2^{-3})}\int_0^1|S_2^2(\alpha)S_3(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha, \end{aligned}\tag{5.4}$$

where

$$J_1\ll N^{-\frac{3}{5}+\varepsilon}$$

by [18, Lemma 2.2]. By Hölder's inequality, Lemmas 2.3 and 2.5, one has

$$\begin{aligned}\int_{\mathfrak{n}}|S_2^2(\alpha)S_3(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha\ll&\Big(\int_0^1|S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1|S_2^2(\alpha)S_5^6(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2}}\\ \ll&N^{\frac{17}{15}+\varepsilon}.\end{aligned}\tag{5.5}$$

Also,

$$\begin{aligned}\int_{\mathfrak{n}}|S_2^4(\alpha)S_3^2(\alpha)S_5^6(\alpha)|\mathrm{d}\alpha\ll&\sup_{\alpha\in\mathfrak{n}}|S_2^2(\alpha)S_5^2(\alpha)|\int_0^1|S_2^2(\alpha)S_3^2(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha\\ \ll&N^{\frac{19}{15}+\varepsilon}\int_0^1|S_2^2(\alpha)S_3^2(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha.\end{aligned}\tag{5.6}$$

Thus, by (5.4)–(5.6), we obtain that

$$\int_{\mathfrak{n}}|S_2^2(\alpha)S_3^2(\alpha)S_5^4(\alpha)|\mathrm{d}\alpha\ll N^{1+\frac{7}{15}-\frac{1}{24}+\varepsilon}.$$

It establishes (5.3) for $k=5$.

For $k=6$, applying Hölder's inequality, Lemmas 2.3 and 2.5, one has

$$\begin{aligned}&\int_{\mathfrak{n}}|S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_6^2(\alpha)|\mathrm{d}\alpha\\ \ll&\sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^{\frac{8}{9}}\Big(\int_0^1|S_2^2(\alpha)S_5^6(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{3}}\Big(\int_0^1|S_2^2(\alpha)S_6^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{9}}\\ &\times\Big(\int_0^1|S_2^2(\alpha)S_3^2(\alpha)S_6^2(\alpha)|\mathrm{d}\alpha\Big)^{\frac{5}{9}}\\ \ll&N^{1+\frac{2}{5}-\frac{2}{81}+\varepsilon}.\end{aligned}$$

It establishes (5.3) for $k=6$.

For $k\geq 7$ and $x$ in the form in Theorem 1.3, by Lemmas 2.3, 2.5 and 3.2, one has

$$\begin{aligned}&\int_{\mathfrak{n}}|S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\\ \ll&\sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^{\frac{1}{x}}\Big(\int_0^1|S_2^2(\alpha)S_5^6(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2x}}\Big(\int_0^1|S_2^2(\alpha)S_3^2(\alpha)S_5^2(\alpha)|\mathrm{d}\alpha\Big)^{1-\frac{3}{2x}}\\ &\times\Big(\int_0^1|S_2^2(\alpha)S_3^2(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{x}}\\ \ll&N^{\frac{16}{15}+\frac{2}{k}-\frac{1}{36x}+\varepsilon}.\end{aligned}$$

It establishes (5.3) for $k\geq 7$. Hence, (5.1) holds by (5.2)–(5.3), and it establishes Theorem 1.3.

## 6 Proof of Theorem 1.4

Suppose that $N$ is a large positive integer. Let $S_k(\alpha)$ be defined as in Lemma 2.1. Let

$$r(n)=\sum_{\substack{p_1^3+p_2^3+p_3^3+p_4^3+p_5^k=n\\ \frac{N}{5}<p_1^3,p_2^3,p_3^3,p_4^3,p_5^k\le N}}(\log p_1)(\log p_2)(\log p_3)(\log p_4)(\log p_5)$$

and the major arcs $\mathfrak{M}=\mathfrak{M}(Q)$, minor arcs $\mathfrak{m}$, $\mathfrak{N}$ and $\mathfrak{n}$ be defined as in Section 3 with $Q=N^{\frac{1}{2k}}$. Then the weighted number of representations of $n$ in the form of (1.6) equals

$$r(n)=\int_0^1 S_3^4(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha=\int_{\mathfrak{M}}+\int_{\mathfrak{m}}.$$

Whenever $\mathfrak{B}\subset[0,1]$ is measurable, we put

$$r_{\mathfrak{B}}(n,N)=\int_{\mathfrak{B}} S_3^4(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Then we have

$$r(n)=r_{[0,1]}(n,N)=\int_0^1 S_3^4(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

Next we will deal with the integral of major arcs and minor arcs, respectively. Applying the now standard methods of enlarging major arcs (see [8]), we can get the following result.

**Lemma 6.1** *For all odd integer $n$ with $N<n\le 2N$, one has $r_{\mathfrak{M}}(n,N)\gg N^{\frac{1}{k}+\frac{1}{3}-\varepsilon}$.*

**Proof of Theorem 1.4** By Bessel's inequality, we have

$$\sum_{N<n\le 2N}\Big|\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)e(-n\alpha)\mathrm{d}\alpha\Big|^2\le\int_{\mathfrak{m}}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.$$

Thus, to prove Theorem 1.4, it suffices to show that

$$\int_{\mathfrak{m}}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\ll N^{\frac{5}{3}+\frac{2}{k}-\theta_4(k)+\varepsilon}, \tag{6.1}$$

where $\theta_4(k)$ is defined in Theorem 1.4. Obviously, one has

$$\int_{\mathfrak{m}}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\ll\int_{\mathfrak{N}\backslash\mathfrak{M}}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha+\int_{\mathfrak{n}}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha.$$

First, we show that

$$\int_{\mathfrak{N}\backslash\mathfrak{M}}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\ll N^{\frac{5}{3}+\frac{2}{k}-\theta_4(k)+\varepsilon}. \tag{6.2}$$

It suffices to prove that

$$\int_{\mathfrak{M}(2K)\backslash\mathfrak{M}(K)}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\ll N^{\frac{5}{3}+\frac{2}{k}-\theta_4(k)+\varepsilon}$$

with $N^{\frac{1}{2k}}\ll K\ll N^{\frac{1}{8}}$. By [5, Lemmas 4.2 and 5.2], one has

$$\int_{\mathfrak{M}(2K)\backslash\mathfrak{M}(K)}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\ll N^{\frac{8}{3}+\varepsilon}K^{-4}(N^{-1}K(N^{\frac{1}{k}}K+N^{\frac{2}{k}}))$$

$$\ll N^{\frac{5}{3}+\frac{2}{k}-\frac{3}{2k}+\varepsilon}$$
$$\ll N^{\frac{5}{3}+\frac{2}{k}-\theta_4(k)+\varepsilon},$$

since $\frac{3}{2k} > \theta_4(k)$ for all $k \geq 4$. This establishes (6.2).

Next, we show that

$$\int_{\mathfrak{n}} |S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll N^{\frac{5}{3}+\frac{2}{k}-\theta_4(k)+\varepsilon}. \tag{6.3}$$

For $k = 4$, by [18, Lemma 3.1] with $g(\alpha) = h(\alpha) = S_3(\alpha)$, one has

$$\begin{aligned}\int_{\mathfrak{n}} |S_3^8(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha \ll & N^{\frac{1}{3}}J^{\frac{1}{4}}\Big(\int_{\mathfrak{n}} |S_3^{12}(\alpha)S_4^4(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{4}}\Big(\int_{\mathfrak{n}} |S_3^7(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2}}\\ &+ N^{\frac{1}{3}(1-2^{-3})}\int_{\mathfrak{n}} |S_3^7(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha,\end{aligned} \tag{6.4}$$

where

$$J \ll N^{-\frac{1}{3}+\varepsilon}$$

by [18, Lemma 2.2]. By Hölder's inequality and Lemma 2.6, one has

$$\begin{aligned}\int_{\mathfrak{n}} |S_3^7(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha &\ll \Big(\int_0^1 |S_3^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{3}{4}}\Big(\int_0^1 |S_3^4(\alpha)S_4^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{4}}\\ &\ll N^{\frac{11}{6}+\varepsilon}.\end{aligned} \tag{6.5}$$

Also,

$$\begin{aligned}\int_{\mathfrak{n}} |S_3^{12}(\alpha)S_4^4(\alpha)|\mathrm{d}\alpha &\ll \sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^6\Big(\int_0^1 |S_3^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1 |S_3^4(\alpha)S_4^8(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{2}}\\ &\ll N^{\frac{23}{6}+\varepsilon}.\end{aligned} \tag{6.6}$$

Thus, by (6.4)–(6.6), we obtain that

$$\int_{\mathfrak{n}} |S_3^8(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha \ll N^{\frac{5}{3}+\frac{1}{2}-\frac{1}{24}+\varepsilon}.$$

It establishes (6.3) for $k = 4$.

For $k \geq 5$ and $x$ in the form in Theorem 1.4, by Hölder's inequality, Lemmas 2.6 and 3.2, one has

$$\begin{aligned}\int_{\mathfrak{n}} |S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha \ll & \sup_{\alpha\in\mathfrak{n}}|S_3(\alpha)|^{\frac{4}{x}}\Big(\int_0^1 |S_3^8(\alpha)|\mathrm{d}\alpha\Big)^{1-\frac{1}{x}}\Big(\int_0^1 |S_3^4(\alpha)S_k^{2x}(\alpha)|\mathrm{d}\alpha\Big)^{\frac{1}{x}}\\ \ll & N^{\frac{5}{3}+\frac{2}{k}-\theta_4(k)+\varepsilon}.\end{aligned}$$

This establishes (6.3) for $k \geq 5$. Hence, (6.1) holds by (6.2)–(6.3). Thus it establishes Theorem 1.4.

**Acknowledgements** We express our sincere thanks to reviewer for valuable suggestions and comments. The authors also would like to express the most sincere gratitude to Prof. Zhixin Liu for his valuable advice and constant assistance.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

## References


[1] Bauer, C., An improvement on a theorem of the Goldbach-Waring type, *Rocky Mount. J. Math.*, **31**, 2001, 1151–1170.

[2] Bauer, C., A remark on a theorem of the Goldbach Waring type, *Studia Sci. Math. Hungar.*, **41**, 2004, 309–324.

[3] Bourgain, J., On the Vinogradov mean value, *Proc. Steklov Inst. Math.*, **296**, 2017, 30–40.

[4] Brüdern, J., A ternary problem in additive prime number theory, Sander J., Steuding J., Steuding R. (eds), From Arithmetic to Zeta-Functions, Springer-Verlag, Cham, 2016, 57–81.

[5] Hoffman, J. W. and Yu, G., A ternary addtive problem, *Monatsh Math.*, **172**, 2013, 293–321.

[6] Hua, L. K., Additive Theory of Prime Numbers, Science Press, American Mathematical Society, Providence, RI, 1965.

[7] Kumchev, A., On weyl sums over primes and almost primes, *Michigan Math. J.*, **54**, 2006, 243–268.

[8] Liu, J. Y. and Zhan, T., Sums of five almost equal prime squares (II), *Sci. in China.*, **41**, 1998, 710–722.

[9] Liu, Z. X., An improvement on Waring-Goldbach problem for unlike powers, *Acta Math. Hungar.*, **130**, 2011, 118–139.

[10] Lu, M. G. and Shan, Z., A problem of Waring-Goldbach type, *J. China Univ. Sci. Tech. Suppl. I*, 1982, 1–8 (in Chinese).

[11] Prachar, K., Über ein Problem vom Waring-Goldbach'schen Typ, *Monatsh. Math.*, **57**, 1953, 66–74.

[12] Ren, X. M. and Tsang, K. M., Goldbach-Waring problem for unlike powers, *Acta Math. Sin., Engl. Ser.*, **23**, 2007, 265–280.

[13] Ren, X. M. and Tsang, K. M., Goldbach-Waring problem for unlike powers (II), *Acta Math. Sin., Chin. Ser.*, **50**, 2007, 175–182.

[14] Schwarz, W., Zur Darstellung von Zahlen durch Summen von Primzahlpotenzen, *J. Reine Angew. Math.*, **206**, 1961, 78–112.

[15] Vaughan, R. C., The Hardy-Littlewood Method, Cambridge University Press, Cambridge, 1997.

[16] Zhao, L. L., The exceptional set for sums of unlike powers of primes, *Acta Math. Sin., Engl. Ser.*, **30**, 2014, 1897–1904.

[17] Zhao, L. L., The additive problem with one cube and three cubes of primes, *Michigan Math. J.*, **63**, 2014, 763–779.

[18] Zhao, L. L., On the Waring-Goldbach problem for fourth and sixth powers, *Proc. London Math. Soc.*, **108**(6), 2014, 1593–1622.