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# Regularity of the $p$-Gauss Curvature Flow with Flat Sides*

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**Abstract** The authors study the regularity of the $p$-Gauss curvature flow with flat sides. In their previous paper [Huang, G. G., Wang, X.-J. and Zhou, Y., Long time regularity of the $p$-Gauss curvature flow with flat side, https://arxiv.org/abs/2403.12292], they obtained the regularity of the interface, namely the boundary of the flat part. In this paper, they study the regularity of the convex hypersurface near the interface.

**Keywords** Gauss curvature flow, Parabolic Monge-Ampère equation, Interface, Regularity
**2020 MR Subject Classification** 53E40, 35K96, 35R35, 35B65

## 1 Introduction

Let $\mathcal{M}_0$ be a closed convex hypersurface in $\mathbb{R}^{n+1}$, parametrized by $X_0(\omega)$, $\omega \in \mathbb{S}^n$. In this paper we study the Gauss curvature flow with power $p > 0$,

$$\begin{aligned} \frac{\partial X}{\partial t}(\omega, t) &= -K^p(\omega, t)\gamma(\omega, t), \\ X(\omega, 0) &= X_0(\omega), \end{aligned} \tag{1.1}$$

where $K$ is the Gauss curvature of $\mathcal{M}_t = X(\omega, t)$, $\gamma$ is the outer unit normal of $\mathcal{M}_t$ at $X(\omega, t)$.

The Gauss curvature flow has been extensively studied if the initial hypersurface $\mathcal{M}_0$ is strictly convex (see [1, 3–4, 6, 9]). Here we are concerned with the regularity in the case when the initial hypersurface $\mathcal{M}_0$ contains a flat side, a question first studied by Hamilton [12]. In this case, the solution will become strictly convex instantly when $t > 0$ if $p \le \frac{1}{n}$ (see [2, 6]), but the flat side will persist for a while before $\mathcal{M}_t$ becomes strictly convex if $p > \frac{1}{n}$ (see [5, 12]). In the latter case, the local $C^\infty$ regularity of the strictly convex part of $\mathcal{M}_t$ was proved in

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[6, 19], and the $C^{1,\alpha}$ regularity across the interface $\Gamma_t$ were obtained in [11], where $\Gamma_t$ denotes the boundary of the flat side $F_t \subset \mathcal{M}_t$.

When $p = 1$, the regularity of $\Gamma_t$ was obtained in [8] for small time $t > 0$, and the long time regularity of $\Gamma_t$ in the case $n = 2$ was obtained in [10], under certain non-degenerate conditions on the initial hypersurface $\mathcal{M}_0$. The results in [10] were extended to $p \in \left(\frac{1}{2}, 1\right]$ in [16] when $n = 2$. For general $n \geq 2$ and $p > \frac{1}{n}$, the long time regularity of $\Gamma_t$ was recently obtained by the authors (see [14]). We proved that the interface $\Gamma_t$ is smooth until it disappears.

In this paper, we study the regularity of the strictly convex part of $\mathcal{M}_t$ near the interface $\Gamma_t$. The regularity of $\mathcal{M}_t$ near $\Gamma_t$ does not follow directly from the regularity theory of parabolic equations, as the flow (1.1) is strongly degenerate near $\Gamma_t$, even though the regularity of $\Gamma_t$ has been obtained (see [14]). For simplicity we assume that $\mathcal{M}_0$ has only one flat part. Choosing the coordinates properly, we may assume that $\mathcal{M}_t \subset \{y_{n+1} \geq 0\}$ and the flat side lies on the plane $\{y_{n+1} = 0\}$. Then, locally $\mathcal{M}_t$ can be represented as the graph of a nonnegative function $v$,

$$y_{n+1} = v(y_1, \cdots, y_n, t)$$

over a bounded domain $\Omega_t$, such that $\Gamma_t$ is strictly contained in $\Omega_t$, $|Dv| \to \infty$ near $\partial\Omega_t$, and $v$ satisfies the equation

$$v_t(y,t) = \frac{(\det D^2 v(y,t))^p}{(1 + |Dv|^2)^{\frac{(n+2)p-1}{2}}}, \quad y \in \Omega_t,\ t > 0. \tag{1.2}$$

By the $C^{1,\alpha}$ regularity (see [11]), we have $|Dv(y,t)| \to 0$ as $y \to \Gamma_t$.

For the short time smoothness of the interface $\Gamma_t$, it is necessary to assume certain non-degeneracy conditions on the initial hypersurface $\mathcal{M}_0$ (see [7–8, 10]). Denote

$$g = \left(\frac{\sigma_p + 1}{\sigma_p} v\right)^{\frac{\sigma_p}{\sigma_p + 1}}, \quad \sigma_p = n - \frac{1}{p}. \tag{1.3}$$

The following non-degeneracy conditions were introduced in [7–8, 10].

(I1) The level set $\{v(y,0) = \varepsilon\}$ is uniformly convex for $\varepsilon \geq 0$ small, i.e., its principal curvatures have positive upper and lower bounds.

(I2) There exists a constant $\lambda_0 \in (0,1)$ such that

$$\lambda_0 \leq |Dg(y,0)| \leq \lambda_0^{-1} \quad \text{on } \Gamma_0.$$

Note that condition (I2) implies that

$$v(y,0) \approx \operatorname{dist}(y, \Gamma_0)^{\frac{\sigma_p+1}{\sigma_p}}.$$

We also assume the following (I3).

(I3) $\mathcal{M}_0$ is locally uniformly convex and smooth away from the flat region, and

$$g(y,0) \in C^{2+\alpha}_{\mu}(\overline{\{v > 0\}}),$$

where $C_\mu^{2+\alpha}$ will be introduce in (1.9) below.

We have the following regularity for the function $g$ near the interface $\Gamma_t$.

**Theorem 1.1** *Assume conditions* (I1)–(I3). *Then if* $p > \frac{1}{n}$, *we have* $g(\cdot,t) \in C_\mu^{2+\beta}(\overline{\{v>0\}})$ *on* $0 < t < T^*$ *for some* $\beta \in (0,1)$. *Moreover,*

(1) *if* $\frac{2}{\sigma_p} \in \mathbb{Z}^+$, $g$ *is* $C^\infty$*-smooth up to* $\Gamma_t$ *for* $0 < t < T^*$;

(2) *if* $\frac{2}{\sigma_p} \notin \mathbb{Z}^+$, $g \in C_\mu^{[\frac{2}{\sigma_p}],2+\beta_0}(\overline{\{g>0\}})$ *for* $0 < t < T^*$, *where* $\beta_0 = \min\{1, \frac{4}{\sigma_p} - 2[\frac{2}{\sigma_p}]\}$.

We remark that the regularity for $g$ in Theorem 1.1 is optimal, due to the term $g(y,t)^{\frac{2}{\sigma_p}} \approx \operatorname{dist}(y,\Gamma_t)^{\frac{2}{\sigma_p}}$ in the equation (3.3). As usual we use $[a]$ to denote the greatest integer less than $a$.

From Theorem 1.1, it follows the regularity of the height function $v$.

**Corollary 1.1** *Assume conditions* (I1)–(I3) *and assume* $p > \frac{1}{n}$.

(1) *If* $\frac{1}{\sigma_p} \in \mathbb{Z}^+$, $v$ *is* $C^\infty$*-smooth up to* $\Gamma_t$ *for* $0 < t < T^*$;

(2) *if* $\frac{1}{\sigma_p} \notin \mathbb{Z}^+$, $v \in C^{1+\left[\frac{1}{\sigma_p}\right],\frac{1}{\sigma_p}-\left[\frac{1}{\sigma_p}\right]}(\overline{\{v>0\}})$ *for* $0 < t < T^*$.

*Moreover, we have the following intermediate estimate*:

$$\sup_{t\in[\sigma,T]} \sup_{y,\widetilde{y}\in\overline{\{0<v(\cdot,t)<1\}}} d_{y,\widetilde{y}}(t)^{1+\frac{1}{\sigma_p}} \frac{|D_y^{k_0+2}v(y,t) - D_y^{k_0+2}v(\widetilde{y},t)|}{|y-\widetilde{y}|^{\frac{2}{\sigma_p}-k_0}} \le C \tag{1.4}$$

*for all* $0 < \sigma < T < T^*$, *where* $d_{y,\widetilde{y}}(t) := \min\{\operatorname{dist}(y,\Gamma_t), \operatorname{dist}(\widetilde{y},\Gamma_t)\}$, $k_0$ *is the greatest integer strictly less than* $\frac{2}{\sigma_p}$ *and* $C$ *is a positive constant depending only on* $\mathcal{M}_0, n, p, \sigma, T$.

For intermediate estimate to uniformly elliptic and parabolic equations, we refer the readers to [17, Chapter IV].

To prove Theorem 1.1, we introduce the Hodograph transformation $h$, given in (3.4), which satisfies the evolution equation

$$h_t = \frac{(\det \widetilde{H})^p}{(h_{n+1}^2 + y_{n+1}^{\frac{2}{\sigma_p}}(1 + h_1^2 + \cdots + h_{n-1}^2))^{\frac{(n+2)p-1}{2}}} \quad \text{in } \{y_{n+1} > 0\}, \tag{1.5}$$

where the matrix $\widetilde{H}$ is given in (3.8). Note that equation (1.5) is a degenerate fully nonlinear parabolic equation without the concavity condition, and the coefficient $y_{n+1}^{\frac{2}{\sigma_p}}$ is only Hölder continuous when $p > \frac{1}{n-2}$. Hence the regularity theories, such as [15, 17], do not apply. Here we make use of some estimates in [13].

The paper is organized as follows. In Section 2, we recall the short time regularity and some basic estimates. We then derive the equations for $g, h$ and prove the regularity (Theorem 1.1) in Section 3.

**Notation 1.1** Given two positive quantities $a$ and $b$, we denote $a \lesssim b$ if there is a constant $C > 0$, depending only on $\mathcal{M}_0, n, p, T$, such that $a \le Cb$, where $T \in (0, T^*)$ is any given constant. We also denote $a \approx b$ if $a \lesssim b$ and $b \lesssim a$.

Let $k \geq 0$ be an integer and $\alpha \in (0,1]$. Let $\Omega$ be a domain in $\mathbb{R}^n$. As usual, we define the norm $\|\cdot\|_{C^{k,\alpha}(\overline{\Omega})}$ by

$$\|U\|_{C^{k,\alpha}(\overline{\Omega})} = \sup_{|\gamma|\leq k} |D^\gamma U(x)| + \sup_{\substack{|\gamma|=k \\ x,y\in\Omega}} \frac{|D^\gamma U(x) - D^\gamma U(y)|}{|x-y|^\alpha}. \tag{1.6}$$

In the parabolic case, we denote

$$\begin{aligned} &\|U\|_{C_{x,t}^{k+\alpha,\frac{k+\alpha}{2}}(\overline{Q})} \\ &= \sup_{\substack{|\gamma|+2s\leq k \\ (x,t)\in Q}} |D_x^\gamma D_t^s U(x,t)| + \sup_{\substack{|\gamma|+2s=k \\ (x,t),(y,t')\in Q}} \frac{|D_x^\gamma D_t^s U(x,t) - D_x^\gamma D_t^s U(y,t')|}{(|x-y|^2 + |t-t'|)^{\frac{\alpha}{2}}}, \end{aligned} \tag{1.7}$$

where $Q$ is a domain in $\mathbb{R}^n \times \mathbb{R}^1$. If $\alpha \in (0,1)$, we will write $\|\cdot\|_{C_{x,t}^{k+\alpha,\frac{k+\alpha}{2}}(\overline{Q})}$ as $\|\cdot\|_{C^{k+\alpha}(\overline{Q})}$ for brevity.

To study the regularity of $g$, we introduce Hölder spaces with respect to the metric $\mu$ in $\mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}$ as in [8, 10],

$$\mu[(x,t),(y,s)] = |x'-y'| + |\sqrt{x_n} - \sqrt{y_n}| + \sqrt{|t-s|}.$$

Let $Q$ be a domain in $\mathbb{R}^{n,+} \times \mathbb{R}$, where $\mathbb{R}^{n,+} := \mathbb{R}^{n-1} \times \mathbb{R}^+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$. We denote

$$\|U\|_{C_\mu^{0,\alpha}(\overline{Q})} = \sup_{p\in Q} |U(p)| + \sup_{p_1,p_2\in Q} \frac{|U(p_1) - U(p_2)|}{\mu[p_1,p_2]^\alpha}, \tag{1.8}$$

$$\begin{aligned} \|U\|_{C_\mu^{2+\alpha}(\overline{Q})} &= \|x_n U_{nn}\|_{C_\mu^{0,\alpha}(\overline{Q})} + \sum_{i=1}^{n-1} \|\sqrt{x_n} U_{ni}\|_{C_\mu^{0,\alpha}(\overline{Q})} + \sum_{i,j=1}^{n-1} \|U_{ij}\|_{C_\mu^{0,\alpha}(\overline{Q})} \\ &\quad + \sum_{i=1}^{n} \|U_i\|_{C_\mu^{0,\alpha}(\overline{Q})} + \|U_t\|_{C_\mu^{0,\alpha}(\overline{Q})} + \|U\|_{C_\mu^{0,\alpha}(\overline{Q})} \end{aligned} \tag{1.9}$$

and

$$\|U\|_{C_\mu^{m,2+\alpha}(\overline{Q})} = \sum_{|\gamma|+2s\leq m} \|D_x^\gamma D_t^s U\|_{C_\mu^{2+\alpha}(\overline{Q})}. \tag{1.10}$$

## 2 Some Estimates

First we recall the short time existence and regularity in [8], where Daskalopoulos and Hamilton proved the following result.

**Proposition 2.1** (see [8, Theorem 9.1]) *Assume conditions* (I1)–(I3). *Then, there exists a time $T_0 > 0$ such that* (1.1) *admits a solution $\mathcal{M}_t$ for $0 < t \leq T_0$, and at any given time $t \in (0, T_0]$, $\mathcal{M}_t$ satisfies the conditions* (I1)–(I3).

**Remark 2.1** Proposition 2.1 is proved in [8] for $n = 2$. The proof also holds for high dimension case $n \geq 3$. Moreover, for $0 < t \leq T_0$, the proof also implies the following condition (see [8, Theorem 9.2]):

(I4) $g_{ij}\tau_i g_j \in L^\infty(\{v > 0\})$, where $\tau = (\tau_1, \cdots, \tau_n)$ is any tangent vector field of the level set of $g$, i.e., $\tau \cdot \nabla g = 0$.

Therefore, choosing a sufficiently small $t_0 > 0$ as the initial time, we may assume that (I4) holds at $t = 0$.

To prove Theorem 1.1, we then tap into some estimates obtained in previous work [14].

Let $u(\cdot, t)$ be the Legendre transformation of $v(\cdot, t)$, i.e.,

$$u(x,t) = \sup\{y \cdot x - v(y,t) \mid y \in \Omega_t\}, \quad x \in D_y v(\Omega_t) = \mathbb{R}^n. \tag{2.1}$$

Then $u(x,t)$ satisfies the equation

$$\det D^2 u = \frac{1}{(-u_t)^{\frac{1}{p}}(1+|x|^2)^{\frac{(n+2)p-1}{2p}}} + c_t\delta_0, \tag{2.2}$$

where $c_t$ is the volume of the flat part. Hence $c_t > 0$ for $t \in [0, T^*)$. Without loss of generality, we assume that the origin is an interior point of the convex set $\{v(\cdot,t) = 0\}$ for all $t \in [0, T^*)$. Then for any given $T \in (0, T^*)$, there is a positive constant $\rho_0$ such that

$$B_{\rho_0}(0) \subset\subset \{y \in \mathbb{R}^n \mid v(y,t) = 0\}, \quad \forall t \in [0, T]. \tag{2.3}$$

**Lemma 2.1** (see [14]) *Assume conditions* (I1)–(I4)*. Then*

$$\begin{aligned} -u_t(x,t) &\approx |x|, \\ u_{rr}(x,t) &\approx |x|^{n-1-\frac{1}{p}}, \\ u_{\xi\xi}(x,t) &\approx |x|^{-1} \end{aligned} \tag{2.4}$$

*for any* $x \in B_1(0)\backslash\{0\}$, $t \in [0,T]$ *and any unit vector* $\xi \perp \overrightarrow{ox}$, *where* $u_{rr}(x,t) := \frac{1}{|x|^2}x_i x_j u_{ij}(x,t)$.

Denote $r = |x|$. Let

$$\zeta(\theta, s, t) = \frac{u(\theta, r, t)}{r}, \quad s = r^{\frac{\sigma_p}{2}}, \tag{2.5}$$

where $(\theta, r)$ is the spherical coordinates for $x$. Then $\zeta$ satisfies the parabolic Monge-Ampère type equation (see [14]):

$$-\zeta_t \det \begin{pmatrix} \zeta_{ss} + \frac{2+\sigma_p}{\sigma_p}\frac{\zeta_s}{s} & \zeta_{s\theta_1} & \cdots & \zeta_{s\theta_{n-1}} \\ \zeta_{s\theta_1} & \zeta_{\theta_1\theta_1} + \zeta + \frac{\sigma_p}{2}s\zeta_s & \cdots & \zeta_{\theta_1\theta_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \zeta_{s\theta_{n-1}} & \zeta_{\theta_1\theta_{n-1}} & \cdots & \zeta_{\theta_{n-1}\theta_{n-1}} + \zeta + \frac{\sigma_p}{2}s\zeta_s \end{pmatrix}^p = \overline{F}(s), \tag{2.6}$$

in $\{s > 0\}$, where $\overline{F}(s) = 4^p\sigma_p^{-2p}\big(1 + s^{\frac{4}{\sigma_p}}\big)^{-\frac{(n+2)p-1}{2}}$.

**Lemma 2.2** (see [14, Theorem 6.3]) *Assume the conditions* (I1)–(I4). *We have*

$$\|\zeta\|_{C^{2+\alpha_0}(\mathbb{S}^{n-1}\times[0,1]\times[\sigma,T])} \leq C, \quad \forall\, 0<\sigma<T<T^*, \tag{2.7}$$

*where the constants* $\alpha_0 \in (0,1)$ *and* $C>0$ *depend only on* $\mathcal{M}_0, n, p, \sigma, T$.

Next we quote the $C^\alpha$ and $C^{2,\alpha}$ estimates for degenerate linear parabolic equations which are needed later. Given a point $p_0=(x_0,t_0)=(x_0', x_{0,n}, t_0)\in\mathbb{R}^{n,+}\times\mathbb{R}$ , denote

$$Q_\rho^*(p_0)=\{(x,t)\mid x_n>0, |x'-x_0'|<\rho, |x_n-x_{0,n}|<\rho^2, t_0-\rho^2<t\leq t_0\}, \tag{2.8}$$

which is a cylinder in $\mathbb{R}^{n,+}\times\mathbb{R}$. When $p_0=(0,0)$, we simply write $Q_\rho^*=Q_\rho^*(p_0)$.

Consider the following linear degenerate operator

$$L_+U := -U_t + a_{nn}x_n\partial_{nn}U + \sum_{i=1}^{n-1}2a_{in}\sqrt{x_n}\partial_{in}U + \sum_{i,j=1}^{n-1}a_{ij}\partial_{ij}U + \sum_{i=1}^{n}b_i\partial_iU \tag{2.9}$$

with variable coefficients $a_{ij}, b_i$ defined in the cylinder $Q_\rho^*$.

**Lemma 2.3** *Assume that the coefficients* $a_{ij}, b_i$ *are measurable and satisfy*

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall\xi\in\mathbb{R}^n,$$
$$|a_{ij}|, |b_i| \leq \lambda^{-1}$$

*and*

$$\frac{2b_n}{a_{nn}} \geq \nu$$

*for some constants* $\lambda, \nu \in (0,1)$. *Let* $U\in C^2(\overline{Q_\rho})$ *be the solution to* $L_+U=f$. *Then there exists* $\alpha\in(0,1)$ *such that for any* $\rho'\in(0,\rho)$, *it holds*

$$\|U\|_{C_\mu^\alpha(Q_{\rho'}^*)} \leq C\Big(\sup_{Q_\rho^*}|U| + \Big(\int_{Q_\rho^*}|f|^{n+1}x_n^{\frac{\nu}{2}-1}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{n+1}}\Big), \tag{2.10}$$

*where the positive constant* $C$ *depends only on* $n, \rho, \rho', \lambda$ *and* $\nu$.

For the proof of Lemma 2.3, we refer the readers to [9, Theorem 3.1] or [18, Theorem 3.3].

**Lemma 2.4** (Schauder estimate) (see [8]) *Assume that the coefficients* $a_{ij}, b_i \in C_\mu^\alpha(\overline{Q_\rho^*})$ *for some* $\alpha\in(0,1)$ *and satisfy*

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall\xi\in\mathbb{R}^n,$$
$$\|a_{ij}\|_{C_\mu^\alpha(\overline{Q_\rho^*})}, \quad \|b_i\|_{C_\mu^\alpha(\overline{Q_\rho^*})} \leq \lambda^{-1}$$

*and*

$$b_n \geq \lambda \quad \textit{at } \{x_n=0\}$$

*for some constant $\lambda \in (0,1)$. Let $U \in C_\mu^{2+\alpha}(\overline{Q_\rho^*})$ be the solution to $L_+U = f$. Then for any given $\rho' \in (0,\rho)$, it holds*

$$\|U\|_{C_\mu^{2+\alpha}(\overline{Q_{\rho'}^*})} \le C(\|U\|_{L^\infty(\overline{Q_\rho^*})} + \|f\|_{C_\mu^\alpha(\overline{Q_\rho^*})}), \tag{2.11}$$

*where the positive constant $C$ depends only on $n, \alpha, \rho, \rho'$ and $\lambda$.*

## 3 The Regularity for the Graph

In this section, we will first derive the evolution equations of $g$ and $h$. Then, we utilize the a priori estimates of $u$ and $\zeta$ to obtain the $C_\mu^{2+\beta}$ regularity of $g$ and $h$, which allows us to prove Theorem 1.1 and Corollary 1.1.

### 3.1 Derivation of equations

Recall that the function $v$ satisfies equation (1.2) and $g$ is defined in (1.3). A direct computation yields that, for $1 \le i, j \le n$,

$$v_i = g^{\frac{1}{\sigma_p}} g_i, \quad v_t = g^{\frac{1}{\sigma_p}} g_t \tag{3.1}$$

and

$$v_{ij} = g^{\frac{1}{\sigma_p}} g_{ij} + \frac{1}{\sigma_p} g^{\frac{1}{\sigma_p}-1} g_i g_j. \tag{3.2}$$

Then, $g$ satisfies

$$g_t = \frac{\Big(g \det\Big(D^2 g + \frac{1}{\sigma_p} g^{-1} Dg \otimes Dg\Big)\Big)^p}{(1 + g^{\frac{2}{\sigma_p}} |Dg|^2)^{\frac{(n+2)p-1}{2}}}, \tag{3.3}$$

where $Dg \otimes Dg$ is a matrix with $(i,j)$-entries $g_i g_j$.

Then, we perform the following Hodograph transformation mentioned in Section 1. Let $\overline{p}_0 = (\overline{y}_0, \overline{t}_0)$ be a point on the interface $\Gamma_{\overline{t}_0}$ with $0 < \overline{t}_0 < T^*$. By a rotation of the coordinates, we may assume that $e_n = (0, \cdots, 0, 1)$ is the unit outer normal of the flat part $\{v(y, \overline{t}_0) = 0\}$ at $\overline{p}_0$, so that at the point, we have

$$g_i(\overline{p}_0) = 0, \quad i = 1, \cdots, n-1, \quad g_n(\overline{p}_0) > 0.$$

The above condition can be guaranteed by the initial conditions on $g(y,0)$ and later by the a priori estimates on $|Dg|$ (see (3.12) below). Hence, we can solve the equation $y_{n+1} = g(y_1, \cdots, y_n, t)$ with respect to $y_n$ around the point $\overline{p}_0$ and yield a map

$$y_n = -h(y_1, \cdots, y_{n-1}, y_{n+1}, t), \tag{3.4}$$

defined for all $(y_1, \cdots, y_{n-1}, y_{n+1}, t)$ sufficiently close to $\overline{q}_0 = (\overline{y}_{0,1}, \cdots, \overline{y}_{0,n-1}, 0, \overline{t}_0)$. Then, direct computations give that

$$Dg = -\frac{1}{h_{n+1}}(h_1, \cdots, h_{n-1}, 1), \quad g_t = -\frac{h_t}{h_{n+1}} \tag{3.5}$$

and for $1 \le i, j \le n-1$,

$$g_{ij} = -\frac{1}{h_{n+1}}\Big(h_{ij} - h_{i,n+1}\frac{h_j}{h_{n+1}} - h_{j,n+1}\frac{h_i}{h_{n+1}} + h_{n+1,n+1}\frac{h_i h_j}{h_{n+1}^2}\Big),$$
$$g_{in} = \frac{1}{h_{n+1}^2}\Big(h_{i,n+1} - h_{n+1,n+1}\frac{h_i}{h_{n+1}}\Big)$$

and

$$g_{nn} = -\frac{1}{h_{n+1}^3}h_{n+1,n+1}.$$

According to (3.3), $h$ satisfies

$$h_t = \frac{(y_{n+1}\det H)^p}{(h_{n+1}^2 + y_{n+1}^{\frac{2}{\sigma_p}}(1 + h_1^2 + \cdots + h_{n-1}^2))^{\frac{(n+2)p-1}{2}}}, \tag{3.6}$$

where $H$ is an $n \times n$ matrix with entries

$$H_{ij} = h_{ij} - h_{i,n+1}\frac{h_j}{h_{n+1}} - h_{j,n+1}\frac{h_i}{h_{n+1}} + h_{n+1,n+1}\frac{h_i h_j}{h_{n+1}^2} - \frac{\sigma_p^{-1}h_i h_j}{y_{n+1}h_{n+1}},$$
$$H_{i,n+1} = h_{i,n+1} - h_{n+1,n+1}\frac{h_i}{h_{n+1}} + \frac{\sigma_p^{-1}h_i}{y_{n+1}}$$

and

$$H_{n+1,n+1} = h_{n+1,n+1} - \frac{\sigma_p^{-1}h_{n+1}}{y_{n+1}}.$$

By the elementary properties of determinants, equation (3.6) can be transformed into

$$h_t = \frac{(\det \widetilde{H})^p}{(h_{n+1}^2 + y_{n+1}^{\frac{2}{\sigma_p}}(1 + |D_{y'}h|^2))^{\frac{(n+2)p-1}{2}}}, \tag{3.7}$$

where

$$\widetilde{H} = \begin{pmatrix} h_{11} & \cdots & h_{1,n-1} & \sqrt{y_{n+1}}h_{1,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ h_{1,n-1} & \cdots & h_{n-1,n-1} & \sqrt{y_{n+1}}h_{n-1,n+1} \\ \sqrt{y_{n+1}}h_{1,n+1} & \cdots & \sqrt{y_{n+1}}h_{n-1,n+1} & y_{n+1}h_{n+1,n+1} - \sigma_p^{-1}h_{n+1} \end{pmatrix}. \tag{3.8}$$

One can calculate that the linearized operator of (3.7),

$$\begin{aligned}\mathcal{L} :&= -\frac{1}{h_t}\partial_t + \sum_{i,j=1}^{n-1} p\widetilde{H}^{ij}\partial_{y_i y_j} + 2\sum_{i=1}^{n-1} p\widetilde{H}^{i,n+1}\sqrt{y_{n+1}}\partial_{y_i y_{n+1}} + p y_{n+1}\widetilde{H}^{n+1,n+1}\partial_{y_{n+1}y_{n+1}} \\ &\quad + \widehat{b}\partial_{y_{n+1}} - \sum_{i=1}^{n-1}\frac{[(n+2)p-1]y_{n+1}^{\frac{2}{\sigma_p}}}{h_{n+1}^2 + y_{n+1}^{\frac{2}{\sigma_p}}(1 + |D_{y'}h|^2)}\, h_i\partial_{y_i},\end{aligned} \tag{3.9}$$

where $\widetilde{H}^{ij}, \widetilde{H}^{i,n+1}, \widetilde{H}^{n+1,n+1}$, $i, j = 1, \cdots, n-1$ are the elements of the inverse matrix of $\widetilde{H}$ and

$$\widehat{b}(y', y_{n+1}, t) := -\Big(\frac{[(n+2)p-1]h_{n+1}}{h_{n+1}^2 + y_{n+1}^{\frac{2}{\sigma_p}}(1 + |D_{y'}h|^2)} + \frac{p}{\sigma_p}\widetilde{H}^{n+1,n+1}\Big). \tag{3.10}$$

### 3.2 Regularity for $g$ and $h$

**Lemma 3.1** *Assume the conditions* (I1)–(I4). *Then for some* $\beta \in (0,1)$*, it holds that*

$$\max_{t\in[\sigma,T]} \|g(\cdot,t)\|_{C^{2+\beta}_{\mu}(\overline{\{0<g(\cdot,t)<1\}})} \le C(\mathcal{M}_0, n, p, \sigma, T) \tag{3.11}$$

*for all* $0 < \sigma < T < T^*$.

**Proof Step 1** First order derivative estimates.

By (2.1) and (2.4), we have

$$\begin{aligned} v &= x \cdot Du - u = ru_r - u \\ &= \int_0^r (\rho u_\rho - u)_\rho \mathrm{d}\rho = \int_0^r \rho u_{\rho\rho} \mathrm{d}\rho \approx r^{\sigma_p+1}. \end{aligned}$$

Hence $g \approx r^{\sigma_p}$. It follows from (3.1) and Lemma 2.1 that

$$g_t \approx 1, \quad |Dg| \approx 1, \tag{3.12}$$

uniformly near the interface $\Gamma_t$ for $t \in (0,T]$. Note that estimate (3.12) allows us to perform the local coordinate transform (3.4).

Fix a point $\overline{p}_0 = (\overline{y}_0, \overline{t}_0)$ on the interface $\Gamma_{\overline{t}_0}$, where $\overline{t}_0 \in (0,T]$. By a rotation of the coordinates, we may assume that the unit outer normal of the flat part $\{v(y,\overline{t}_0) = 0\}$ at $\overline{p}_0$ is $e_n = (0, \cdots, 0, 1)$, i.e., $\frac{Dg(\overline{p}_0)}{|Dg(\overline{p}_0)|} = e_n$. By [10, Lemmas 4.7–4.8], there exist positive constants $\eta > 0$ and $\gamma_0 > 0$, depending only on the initial data and $\rho_0$, such that

$$e_n \cdot \frac{Dg(p)}{|Dg(p)|} \ge \gamma_0, \quad \forall p = (y,t) \text{ with } g(p) > 0 \text{ and } |p - \overline{p}_0| < \eta, \quad t \in (0, \overline{t}_0]. \tag{3.13}$$

From (3.5) and (3.12)–(3.13), one knows, for a small constant $\eta > 0$,

$$h_t \approx 1, \quad -h_{n+1} \approx \sqrt{1 + |D_{y'}h|^2} \approx 1, \quad \forall\ (y', y_{n+1}, t) \in Q^*_\eta(\overline{q}_0), \tag{3.14}$$

where the cylinder $Q^*_\eta$ is defined in (2.8).

**Step 2** Second order derivative estimates.

Now, we fix a point $(y_0, t_0) \in \{(y,t) \mid g(y,t) > 0, |(y,t) - (\overline{y}_0, \overline{t}_0)| < \eta, \overline{t}_0 - \eta^2 < t \le \overline{t}_0\}$ for small constant $\eta > 0$. Let $\xi^{(1)}, \cdots, \xi^{(n-1)}, \xi^{(n)} = \frac{Dg}{|Dg|}$ be $n$ vectors at the point $(y_0, t_0)$ with $\xi^{(i)} = \frac{g_n e_i - g_i e_n}{|g_n e_i - g_i e_n|}$, $i = 1, \cdots, n-1$. Note that $\xi^{(i)} \perp \xi^{(n)}$ for $i = 1, \cdots, n-1$. From (3.13), one gets

$$\det(\xi^{(1)}, \cdots, \xi^{(n-1)}, \xi^{(n)}) \ge \Big(\frac{g_n}{|Dg|}\Big)^{n-2} \approx 1. \tag{3.15}$$

At point $(y_0, t_0)$, consider the following matrix

$$G = \begin{pmatrix} g_{\xi^{(1)}\xi^{(1)}} & \cdots & g_{\xi^{(1)}\xi^{(n-1)}} & \sqrt{g} g_{\xi^{(1)}\xi^{(n)}} \\ g_{\xi^{(2)}\xi^{(1)}} & \cdots & g_{\xi^{(2)}\xi^{(n-1)}} & \sqrt{g} g_{\xi^{(2)}\xi^{(n)}} \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{g} g_{\xi^{(n)}\xi^{(1)}} & \cdots & \sqrt{g} g_{\xi^{(n)}\xi^{(n-1)}} & g g_{\xi^{(n)}\xi^{(n)}} + \frac{1}{\sigma_p}|Dg|^2 \end{pmatrix}. \tag{3.16}$$

Then, (3.15) gives that

$$\det G = g\det\Big(D^2 g + \frac{1}{\sigma_p}g^{-1}Dg\otimes Dg\Big)(\det(\xi^{(1)},\cdots,\xi^{(n-1)},\xi^{(n)}))^2$$
$$\approx g\det\Big(D^2 g + \frac{1}{\sigma_p}g^{-1}Dg\otimes Dg\Big). \tag{3.17}$$

The first aim in this part is to show that

$$G \approx I_{n\times n}, \tag{3.18}$$

where $I_{n\times n}$ is the identity matrix. For this, let $\nu$ be the unit eigenvector which corresponds to the smallest eigenvalue of $D^2u$. Recall that $x_0 = D_y v(y_0,t_0)$. Then $r=|x_0|$ can be arbitrary small if we take $\eta>0$ small.

$$\text{Claim:}\quad |\xi^{(n)}\cdot\nu-1|\lesssim r^{\sigma_p},\quad |\xi^{(i)}\cdot\nu|\lesssim r^{\frac{\sigma_p}{2}},\quad i=1,\cdots,n-1.$$

By estimate (2.4) and

$$\xi^{(n)} = \frac{Dg(y_0,t_0)}{|Dg(y_0,t_0)|} = \frac{Dv(y_0,t_0)}{|Dv(y_0,t_0)|} = \frac{x_0}{|x_0|} = \frac{x_0}{r},$$

we have $u_{\xi^{(n)}\xi^{(n)}} = u_{rr}\approx r^{\sigma_p-1}$ at the point $(x_0,t_0)$ (see [14]). Suppose the first part of the Claim fails, i.e., $|\xi^{(n)}\cdot\nu-1| >> r^{\sigma_p}$. Denote

$$\xi^{(n)} = \tau_1\nu+\tau_2\overline{\xi}\quad\text{for some unit vector } \overline{\xi}\perp\nu,$$

where $\tau_1=\xi^{(n)}\cdot\nu$ and $\tau_2=\xi^{(n)}\cdot\bar{\xi}$. Then, it holds

$$||\tau_1|-1|\gg r^{\sigma_p},\quad \tau_2\gg r^{\frac{\sigma_p}{2}},$$

which yields

$$\begin{aligned} r^{\sigma_p-1}\approx u_{\xi^{(n)}\xi^{(n)}} &= \tau_1^2 u_{\nu\nu}+2\tau_1\tau_2 u_{\nu\overline{\xi}}+\tau_2^2 u_{\overline{\xi}\overline{\xi}}\\ &\ge (\tau_2\sqrt{u_{\overline{\xi}\overline{\xi}}}-|\tau_1|\sqrt{u_{\nu\nu}})^2\\ &\gg r^{\sigma_p-1}. \end{aligned}$$

This contradiction proves the first part of the Claim.

It then follows that

$$|\xi^{(n)}-\nu|^2 = 2|\xi^{(n)}\cdot\nu-1|\lesssim r^{\sigma_p}.$$

Hence

$$|\xi^{(i)}\cdot\nu| = |\xi^{(i)}\cdot(\xi^{(n)}-\nu)|\lesssim r^{\frac{\sigma_p}{2}},\quad i=1,\cdots,n-1,$$

which proves the second part of the Claim.

By the above Claim and Lemma 2.1, for $\overline{\xi} \perp \nu$, we get

$$
\begin{aligned}
g_{\xi^{(i)}\xi^{(i)}} &= \Big(\frac{1+\sigma_p}{\sigma_p}v\Big)^{-\frac{1}{1+\sigma_p}} v_{\xi^{(i)}\xi^{(i)}} \approx r^{-1}u^{\xi^{(i)}\xi^{(i)}} \\
&= r^{-1}(u^{\overline{\xi}\overline{\xi}}(\overline{\xi}\cdot\xi^{(i)})^2 + 2u^{\overline{\xi}\nu}(\overline{\xi}\cdot\xi^{(i)})(\nu\cdot\xi^{(i)}) + u^{\nu\nu}(\nu\cdot\xi^{(i)})^2) \\
&\lesssim r^{-1}(r + r^{1-\frac{\sigma_p}{2}}\cdot r^{\frac{\sigma_p}{2}} + r^{1-\sigma_p}r^{\sigma_p}) \lesssim 1
\end{aligned}
$$

for $i = 1, \cdots, n-1$, and

$$
\begin{aligned}
gg_{\xi^{(n)}\xi^{(n)}} + \frac{1}{\sigma_p}|Dg|^2 &= g\Big(\frac{1+\sigma_p}{\sigma_p}v\Big)^{-\frac{1}{1+\sigma_p}} v_{\xi^{(n)}\xi^{(n)}} \\
&\approx r^{\sigma_p-1}u^{\xi^{(n)}\xi^{(n)}} \\
&\lesssim r^{\sigma_p-1}r^{1-\sigma_p} \approx 1.
\end{aligned}
$$

Then from equation (3.3) and estimates (3.12), (3.17), one knows $G \approx I_{n\times n}$.

We next claim that the matrix $\widetilde{H}$, defined in (3.8), satisfies

$$
\widetilde{H} \approx I_{n\times n} \quad \text{in } Q_\eta^*(\overline{q}_0), \tag{3.19}
$$

where $\overline{q}_0 = (\overline{y}_0', 0, \overline{t}_0)$. Indeed, by the definition of $h$, one has

$$
\begin{aligned}
\xi^{(i)} &= \frac{e_i - h_i e_n}{\sqrt{1+h_i^2}}, \qquad i = 1, \cdots, n-1, \\
\xi^{(n)} &= \frac{(D_{y'}h, 1)}{\sqrt{1+|D_{y'}h|^2}}
\end{aligned}
$$

at the point $(y_0', g(y_0, t_0), t_0)$. Then a direct computation implies

$$
\begin{aligned}
g_{\xi^{(i)}\xi^{(j)}} &= -\frac{h_{ij}}{h_{n+1}\sqrt{(1+h_i^2)(1+h_j^2)}}, \qquad 1 \le i, j \le n-1, \\
g_{\xi^{(i)}\xi^{(n)}} &= \frac{1}{\sqrt{(1+h_i^2)(1+|D_{y'}h|^2)}}\Big(-\frac{h_k h_{ik}}{h_{n+1}} + \frac{1+|D_{y'}h|^2}{h_{n+1}^2}h_{i,n+1}\Big), \quad 1 \le i \le n-1, \\
g_{\xi^{(n)}\xi^{(n)}} &= -\frac{h_k h_l h_{kl}}{h_{n+1}(1+|D_{y'}h|^2)} + \frac{2h_l h_{l,n+1}}{h_{n+1}^2} - \frac{(1+|D_{y'}h|^2)h_{n+1,n+1}}{h_{n+1}^3}.
\end{aligned} \tag{3.20}
$$

Here the subscripts $k, l$ obey the Einstein summation convention from 1 to $n-1$. From (3.14), (3.18) and (3.20), it follows that

$$
\sum_{i,j=1}^{n-1}(|h_{ij}| + |\sqrt{y_{n+1}}h_{i,n+1}|) + |y_{n+1}h_{n+1,n+1}| \lesssim 1, \tag{3.21}
$$

which gives (3.19) by (3.7) and the arbitrariness of $(y_0, t_0)$.

**Step 3** $C_\mu^{2+\beta}$-estimate.

Now we refine the estimates of $gg_{\xi^{(n)}\xi^{(n)}}$ and $\sqrt{g}g_{\xi^{(i)}\xi^{(n)}}$ according to the regularity of $\zeta$. By Lemma 2.2, $\zeta(\theta, s, t) \in C^{2+\alpha_0}(\mathbb{S}^{n-1}\times[0,1]\times(0,T])$. At the point $(x_0, t_0) = (D_y v(y_0, t_0), t_0)$, where $(y_0, t_0)$ is a fixed point in $\{(y,t) \mid g(y,t) > 0, |(y,t) - (\overline{y}_0, \overline{t}_0)| < \eta, \overline{t}_0 - \eta^2 < t \leq \overline{t}_0\}$ for small constant $\eta > 0$, one knows that

$$\begin{aligned} &u_{\xi^{(n)}\xi^{(n)}} = r^{\sigma_p - 1}(a_0 + O(r^{\frac{\alpha_0\sigma_p}{2}})), \\ &|u_{\bar{\xi}^{(i)}\xi^{(n)}}| \lesssim r^{\sigma_p - 1}, \quad i = 1, \cdots, n-1, \end{aligned} \tag{3.22}$$

where $\bar{\xi}^{(n)} := \xi^{(n)}$, $\{\bar{\xi}^{(i)}\}_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$ and

$$a_0 := \lim_{\rho \to 0^+} \frac{u_{\rho\rho}\Big(\dfrac{\rho x_0}{|x_0|}, t_0\Big)}{\rho^{\sigma_p - 1}}.$$

Then, by Lemma 2.1 and (3.22), we get

$$\begin{aligned} u^{\xi^{(n)}\xi^{(n)}} = \frac{U^{\xi^{(n)}\xi^{(n)}}}{\det D^2 u} &= \frac{U^{\xi^{(n)}\xi^{(n)}}}{u_{\xi^{(n)}\xi^{(n)}} U^{\xi^{(n)}\xi^{(n)}} + O(u_{\bar{\xi}^{(i)}\xi^{(n)}} u_{\bar{\xi}^{(j)}\xi^{(n)}} r^{-(n-2)})} \\ &= \frac{1}{u_{\xi^{(n)}\xi^{(n)}}} \Big[1 + O\Big(\frac{u_{\bar{\xi}^{(i)}\xi^{(n)}} u_{\bar{\xi}^{(j)}\xi^{(n)}} r^{-(n-2)}}{u_{\xi^{(n)}\xi^{(n)}} U^{\xi^{(n)}\xi^{(n)}}}\Big)\Big]^{-1} \\ &= \frac{1}{u_{\xi^{(n)}\xi^{(n)}}} \Big[1 + O\Big(\frac{r^{2\sigma_p - n}}{u_{\xi^{(n)}\xi^{(n)}} u^{\xi^{(n)}\xi^{(n)}} \det D^2 u}\Big)\Big]^{-1}. \end{aligned} \tag{3.23}$$

Here we denote by $U^{\bar{\xi}^{(i)}\bar{\xi}^{(j)}}$ the elements of the adjoint matrix of $\{u_{\bar{\xi}^{(i)}\bar{\xi}^{(j)}}\}_{i,j=1}^n$.

Since

$$\begin{aligned} &\det D^2 u \approx (-u_t)^{-\frac{1}{p}} \approx r^{-\frac{1}{p}}, \\ &u_{\xi^{(n)}\xi^{(n)}} \geq \frac{1}{u^{\xi^{(n)}\xi^{(n)}}}, \end{aligned}$$

(3.23) implies

$$u^{\xi^{(n)}\xi^{(n)}} = \frac{1}{u_{\xi^{(n)}\xi^{(n)}}}(1 + O(r^{\sigma_p})). \tag{3.24}$$

As a result, by (3.1)–(3.2), (3.22) and (3.24), we have

$$\begin{aligned} g g_{\xi^{(n)}\xi^{(n)}} &= g\Big(\frac{1+\sigma_p}{\sigma_p} v\Big)^{-\frac{1}{1+\sigma_p}} v_{\xi^{(n)}\xi^{(n)}} - \frac{1}{\sigma_p} g^2_{\xi^{(n)}} \\ &= \frac{1}{\sigma_p}\Big(\frac{1+\sigma_p}{\sigma_p} v\Big)^{-\frac{2}{1+\sigma_p}} ((\sigma_p + 1)(r u_r - u) u^{\xi^{(n)}\xi^{(n)}} - r^2) \\ &= \frac{r^2}{\sigma_p}\Big(\frac{1+\sigma_p}{\sigma_p} v\Big)^{-\frac{2}{1+\sigma_p}} \Big((\sigma_p + 1)\frac{(r u_r - u)}{r^2 u_{\xi^{(n)}\xi^{(n)}}}(1 + O(r^{\sigma_p})) - 1\Big) \\ &= \frac{r^2}{\sigma_p}\Big(\frac{1+\sigma_p}{\sigma_p} v\Big)^{-\frac{2}{1+\sigma_p}} \Big(\frac{(\sigma_p + 1)\displaystyle\int_0^r \rho u_{\rho\rho} \mathrm{d}\rho}{r^{\sigma_p+1}(a_0 + O(r^{\frac{\alpha_0\sigma_p}{2}}))}(1 + O(r^{\sigma_p})) - 1\Big) \\ &\approx \Big(\frac{(\sigma_p + 1)\displaystyle\int_0^r \rho u_{\rho\rho} \mathrm{d}\rho}{a_0 r^{\sigma_p+1}}(1 + O(r^{\frac{\alpha_0\sigma_p}{2}})) - 1\Big) \lesssim r^{\frac{\alpha_0\sigma_p}{2}} \end{aligned} \tag{3.25}$$

and for $j = 1, \cdots, n-1$,

$$
\begin{aligned}
|\sqrt{g} g_{\bar{\xi}^{(j)}\xi^{(n)}}| &= \sqrt{g}\Big(\frac{1+\sigma_p}{\sigma_p} v\Big)^{-\frac{1}{1+\sigma_p}} |v_{\bar{\xi}^{(j)}\xi^{(n)}}| \lesssim r^{\frac{\sigma_p}{2}-1} |u^{\bar{\xi}^{(j)}\xi^{(n)}}| \\
&\lesssim r^{\frac{\sigma_p}{2}-1} \frac{|U^{\bar{\xi}^{(j)}\xi^{(n)}}|}{\det D^2 u} \lesssim \sum_{i=1}^{n-1} r^{\frac{\sigma_p}{2}-1} \frac{|u_{\bar{\xi}^{(i)}\xi^{(n)}}| r^{-(n-2)}}{r^{-1/p}} \\
&\lesssim r^{\frac{\sigma_p}{2}}.
\end{aligned} \tag{3.26}
$$

According to estimates (3.14) and (3.19), by scaling, we can apply interior estimates for uniformly parabolic equations (see [17]) to get

$$
|h_{t,n+1}(y', y_{n+1}, t)| \lesssim y_{n+1}^{-1}, \quad \forall (y', y_{n+1}, t) \in Q_\eta^*(\overline{q}_0), \ \overline{q}_0 = (\overline{y}_0', 0, \overline{t}_0). \tag{3.27}
$$

By the relationship between $D^2 h$ and $D^2 g$ in (3.20), the refined estimates (3.25) and (3.26) give that

$$
y_{n+1} h_{n+1,n+1} \lesssim y_{n+1}^{\frac{\alpha_0}{2}}, \quad |h_{i,n+1}| \lesssim 1, \quad i = 1, \cdots, n-1 \quad \text{in } Q_\eta^*(\overline{q}_0). \tag{3.28}
$$

We claim that (3.27) and (3.28) imply $h_{n+1} \in C_\mu^\beta(\overline{Q_\eta^*(\overline{q}_0)})$ for some $\beta \in \big(0, \frac{\alpha_0}{8}\big)$. In fact, for all $(y', y_{n+1}, t), (\widetilde{y}', \widetilde{y}_{n+1}, \widetilde{t}) \in Q_\eta^*(\overline{q}_0)$, one has

$$
\begin{aligned}
|h_{n+1}(y', y_{n+1}, t) - h_{n+1}(y', \widetilde{y}_{n+1}, t)| &\le \Big| \int_{\widetilde{y}_{n+1}}^{y_{n+1}} h_{n+1,n+1}(y', \lambda, t) \mathrm{d}\lambda \Big| \\
&\lesssim |\widetilde{y}_{n+1} - y_{n+1}|^{\frac{\alpha_0}{2}} \lesssim |\sqrt{\widetilde{y}_{n+1}} - \sqrt{y_{n+1}}|^{\frac{\alpha_0}{2}}
\end{aligned}
$$

and

$$
|h_{n+1}(y', y_{n+1}, t) - h_{n+1}(\widetilde{y}', y_{n+1}, t)| \lesssim |\widetilde{y}' - y'|.
$$

Also for $|t - \widetilde{t}| \le y_{n+1}^2$,

$$
|h_{n+1}(y', y_{n+1}, t) - h_{n+1}(y', y_{n+1}, \widetilde{t})| \lesssim |t - \widetilde{t}| y_{n+1}^{-1} \le |t - \widetilde{t}|^{\frac{1}{2}},
$$

for $|t - \widetilde{t}| \ge y_{n+1}^2$,

$$
\begin{aligned}
&|h_{n+1}(y', y_{n+1}, t) - h_{n+1}(y', y_{n+1}, \widetilde{t})| \\
\le\ & |h_{n+1}(y', |t - \widetilde{t}|^{\frac{1}{4}}, t) - h_{n+1}(y', y_{n+1}, t)| \\
&+ |h_{n+1}(y', |t - \widetilde{t}|^{\frac{1}{4}}, t) - h_{n+1}(y', |t - \widetilde{t}|^{\frac{1}{4}}, \widetilde{t})| \\
&+ |h_{n+1}(y', |t - \widetilde{t}|^{\frac{1}{4}}, \widetilde{t}) - h_{n+1}(y', y_{n+1}, \widetilde{t})| \\
\lesssim\ & ||t - \widetilde{t}|^{\frac{1}{4}} - y_{n+1}|^{\frac{\alpha_0}{2}} + |t - \widetilde{t}|^{\frac{3}{4}} \\
\lesssim\ & |t - \widetilde{t}|^{\frac{\alpha_0}{8}}.
\end{aligned}
$$

The above estimates conclude that $h_{n+1} \in C_\mu^\beta(\overline{Q_\eta^*(\overline{q}_0)})$.

Moreover, the estimate (3.28) also gives that

$$\begin{aligned}\widehat{b}(y',0,t) &= -\Big(\frac{[(n+2)p-1]h_{n+1}}{h_{n+1}^2+y_{n+1}^{\frac{2}{\sigma_p}}(1+|D_{y'}h|^2)}+\frac{p}{\sigma_p}\widetilde{H}^{n+1,n+1}\Big)\Big|_{y_{n+1}=0}\\ &= -\frac{(n+1)p-1}{h_{n+1}}\gtrsim 1,\end{aligned}$$

which yields that the coefficient of $\partial_{y_{n+1}}$,

$$\widehat{b}(y',y_{n+1},t)\gtrsim 1,\quad \forall (y',y_{n+1},t)\in Q^*_\eta(\overline{q}_0). \tag{3.29}$$

By (3.14), (3.19) and (3.28), there holds

$$\widehat{b}/(p\widetilde{H}^{n+1,n+1})\gtrsim 1 \quad \text{in } Q^*_\eta(\overline{q}_0).$$

Hence, by Lemma 2.3, we obtain $h_1,\cdots,h_{n-1},h_t\in C^\beta_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$ for some $\beta\in(0,1)$.

Consequently, by the proof in [14, Lemma 4.4] or the argument in [10, Section 6], we obtain $h\in C^{2+\beta}_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$. Therefore the coefficients of the operator $\mathcal{L}$ belong to $C^\beta_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$, for a small positive constant $\eta$ depending only on $\mathcal{M}_0,n,p,T$. By (3.12),

$$g(y,t)\approx \operatorname{dist}(y,\partial\Gamma_t) \tag{3.30}$$

near the interface $\Gamma_t$ for $t\in(0,T]$. Hence $g$ is $C^{2+\beta}_\mu$-smooth up to the interface $\Gamma_{\overline{t}_0}$, and the desired a priori estimate (3.11) follows.

**Proof of Theorem 1.1** We still consider equation (3.7) in $Q^*_\eta(\overline{q}_0)$ with $\overline{q}_0=(\overline{y}'_0,0,\overline{t}_0)$, where $\overline{y}_0=(\overline{y}'_0,\overline{y}_{0,n})\in\Gamma_{\overline{t}_0}$, $t_0\in(0,T^*)$. Differentiating the equation with respect to $t$ gives

$$\mathcal{L}(h_t)=0,$$

where $\mathcal{L}$ is the linearized operator in (3.9). Since the coefficients of the operator $\mathcal{L}$ all belong to $C^\beta_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$ and $\widehat{b}\gtrsim 1$ in $Q^*_{\frac{\eta}{2}}(\overline{q}_0)$, by Lemma 2.4, one gets $h_t\in C^{2+\beta}_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$. Similarly, differentiating equation (3.7) in $y_i$, $i=1,\cdots,n-1$, we have $h_{y_i}\in C^{2+\beta}_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$. It follows by the Schauder estimate that $D^k_{t,y'}h\in C^{2+\beta}_\mu(\overline{Q^*_{\frac{\eta}{2}}(\overline{q}_0)})$ for each $k\in\mathbb{N}^+$, after differentiating equation (3.7) with respect to $t,y_i$ up to $k$ times.

As for the regularity of $h$ in $y_{n+1}$, we need to take care of the term $y_{n+1}^{\frac{2}{\sigma_p}}$ in equation (3.7), which is not smooth if $\frac{2}{\sigma_p}$ is not an integer.

**Case 1** $\frac{2}{\sigma_p}\in\mathbb{N}^+$. Then $y_{n+1}^{\frac{2}{\sigma_p}}$ is smooth. In this case, one can differentiate equation (3.7) in $y_{n+1}$ to obtain higher regularity as above.

**Case 2a** $\frac{2}{\sigma_p}\notin\mathbb{Z}^+$ and $\frac{2}{\sigma_p}<1$. Let

$$z_1=y_1,\ \cdots,\ z_{n-1}=y_{n-1},\ z_n=2\sqrt{y_{n+1}},$$

then $h(z,t),h_{z_i}(z,t),h_t(z,t)\in C^{2+\beta}(\overline{Q_\eta(\overline{q}_0)})$, $i=1,\cdots,n-1$. Here $Q_\eta(\overline{q}_0)$, $\overline{q}_0=(\overline{y}'_0,0,\overline{t}_0)$, is the cylinder given by

$$Q_\eta(\overline{q}_0)=\{(z,t)\in\mathbb{R}^{n,+}\times\mathbb{R}\mid z_n>0,|z-(\overline{y}'_0,0)|<\eta,\overline{t}_0-\eta^2<t\le\overline{t}_0\}. \tag{3.31}$$

Hence, one can rewrite equation (3.7) in coordinates $(z,t)$ as

$$h_{z_n z_n} - \frac{\sigma_p + 2}{\sigma_p}\frac{h_{z_n}}{z_n} = \widetilde{f}\Big(z_n^{\frac{4}{\sigma_p}}, h_t, D_{z'}h, \frac{h_{z_n}}{z_n}, D_{z'}D_z h\Big), \tag{3.32}$$

where $\widetilde{f}$ is $C^{1+\beta}$ smooth in its all arguments. Moreover, by (3.29),

$$-\frac{\partial \widetilde{f}}{\partial\Big(\dfrac{h_{z_n}}{z_n}\Big)} - \frac{\sigma_p+2}{\sigma_p} \gtrsim 1 \quad \text{in} \quad \overline{Q_\eta(\overline{q}_0)}.$$

To prove the regularity of $h_{z_n z_n}$ in $z_n$, we take a fixed point $(z_0,t_0) = (z_0', 0, t_0) \in \overline{Q_\eta(\overline{q}_0)}$, and denote

$$\varrho(z_n) = (\varrho_1(z_n), \varrho_2(z_n)) = \Big(z_n^{\frac{4}{\sigma_p}}, \ \frac{h_{z_n}}{z_n}\Big|_{(z_0', z_n, t_0)}\Big) \quad \text{with} \quad (z_0', z_n, t_0) \in \overline{Q_\eta(\overline{q}_0)}$$

and

$$\overline{f}(\varrho(z_n)) = \overline{f}(\varrho_1(z_n), \varrho_2(z_n)) = \widetilde{f}\Big(z_n^{\frac{4}{\sigma_p}}, h_t, D_{z'}h, \frac{h_{z_n}}{z_n}, D_{z'}D_z h\Big)\Big|_{(z_0', z_n, t_0)}.$$

We also define two constants:

$$\kappa_0 = \frac{h_{z_n}}{z_n}\Big|_{(z_0,t_0)}, \quad b_0 = -\frac{\partial \overline{f}}{\partial \varrho_2}\Big|_{\varrho=(0,\kappa_0)} - \frac{\sigma_p+2}{\sigma_p} > 0,$$

which are well-defined as $h(z,t) \in C^{2+\beta}(\overline{Q_\eta(\overline{q}_0)})$. Then, equation (3.32) can be regarded as an ODE of the variable $z_n$ and rewritten as

$$h_{z_n z_n}\Big|_{(z_0', z_n, t_0)} + b_0 \frac{h_{z_n}}{z_n}\Big|_{(z_0', z_n, t_0)} = \overline{f}(\varrho(z_n)) - \overline{f}_{\varrho_2}(0,\kappa_0)\varrho_2(z_n) =: \check{f}(\varrho(z_n)), \tag{3.33}$$

which yields that

$$h_{z_n}\Big|_{(z_0', z_n, t_0)} = z_n^{-b_0}\int_0^{z_n} \rho^{b_0}\check{f}(\varrho(\rho))\mathrm{d}\rho$$

and

$$\frac{h_{z_n}}{z_n}\Big|_{(z_0', \lambda, t_0)} = \lambda^{-b_0-1}\int_0^{\lambda} \rho^{b_0}\check{f}(\varrho(\rho))\mathrm{d}\rho = \int_0^1 \rho^{b_0}\check{f}(\varrho(\lambda\rho))\mathrm{d}\rho.$$

Note that $\partial \varrho_2 \check{f}(\varrho(0)) = 0$. Then, for $(z_0', \lambda, t_0), (z_0', \widetilde{\lambda}, t_0) \in \overline{Q_\eta(\overline{q}_0)}$, we get

$$\begin{aligned}
&\Big|\frac{h_{z_n}}{z_n}\Big|_{(z_0', \lambda, t_0)} - \frac{h_{z_n}}{z_n}\Big|_{(z_0', \widetilde{\lambda}, t_0)}\Big| \le \int_0^1 \rho^{b_0}|\check{f}(\varrho(\lambda\rho)) - \check{f}(\varrho(\widetilde{\lambda}\rho))|\mathrm{d}\rho \\
&\le \int_0^1 |\check{f}(\varrho_1(\lambda\rho), \varrho_2(\lambda\rho)) - \check{f}(\varrho_1(\widetilde{\lambda}\rho), \varrho_2(\lambda\rho))|\mathrm{d}\rho \\
&\quad + \int_0^1 \rho^{b_0}|\check{f}(\varrho_1(\widetilde{\lambda}\rho), \varrho_2(\lambda\rho)) - \check{f}(\varrho_1(\widetilde{\lambda}\rho), \varrho_2(\widetilde{\lambda}\rho))|\mathrm{d}\rho \\
&\le \|D\widetilde{f}\|_{L^\infty} \cdot |\lambda^{\frac{4}{\sigma_p}} - \widetilde{\lambda}^{\frac{4}{\sigma_p}}| + o_\eta(1)\|\widetilde{f}\|_{C^{1,\beta}} \cdot \sup_{\rho\in[0,1]}\Big|\frac{h_{z_n}}{z_n}\Big|_{(z_0', \lambda\rho, t_0)} - \frac{h_{z_n}}{z_n}\Big|_{(z_0', \widetilde{\lambda}\rho, t_0)}\Big|,
\end{aligned} \tag{3.34}$$

where $o_\eta(1) \le O(\eta^{\beta\min\{1,\frac{4}{\sigma_p}\}}) \to 0$ as $\eta \to 0$. Therefore, for $\eta > 0$ small, one gets

$$\left|\frac{h_{z_n}}{z_n}\Big|_{(z_0',\lambda,t_0)} - \frac{h_{z_n}}{z_n}\Big|_{(z_0',\widetilde{\lambda},t_0)}\right| \lesssim |\lambda - \widetilde{\lambda}|^{\min\{1,\frac{4}{\sigma_p}\}}. \tag{3.35}$$

This implies $\frac{h_{z_n}}{z_n}$ is $C^{0,\min\{1,\frac{4}{\sigma_p}\}}$-smooth with respect to $z_n$, so is $h_{z_nz_n}$ from (3.32). Recall that $y_{n+1} = \frac{1}{4}z_n^2$, we obtain $h(y', y_{n+1}, t) \in C_\mu^{2+\min\{1,\frac{4}{\sigma_p}\}}(\overline{Q^*_{\frac{\eta}{4}}(\overline{q}_0)})$.

**Case 2b** $\frac{2}{\sigma_p} \notin \mathbb{Z}^+$ and $\frac{2}{\sigma_p} > 1$. Differentiating equation (3.7) in $y_{n+1}$ up to $k_0 = \left[\frac{2}{\sigma_p}\right]$ times, one gets

$$V := D^{k_0}_{y_{n+1}} h(y', y_{n+1}, t) \in C_\mu^{2+\beta}\overline{(Q^*_{\frac{\eta}{2}}(\overline{q}_0))}$$

by Lemma 2.4. Similarly, $V_t, V_{y_i} \in C_\mu^{2+\beta}\overline{(Q^*_{\frac{\eta}{2}}(\overline{q}_0))}$, $i = 1, \cdots, n-1$.

Let

$$z_1 = y_1,\ \cdots,\ z_{n-1} = y_{n-1},\ z_n = 2\sqrt{y_{n+1}},$$

then $V, V_{z_i}, V_t \in C^{2+\beta}\overline{(Q_\eta(\overline{q}_0))}$ as a function in $(z, t)$ for $i = 1, \cdots, n-1$. Consider the equation for $V$ in coordinates $(z, t)$ as

$$V_{z_nz_n} - \frac{\sigma_p + 2}{\sigma_p}\frac{V_{z_n}}{z_n} = \widehat{f}\Big(z_n^{\frac{4}{\sigma_p}-2k_0}, V_t, D_{z'}V, \frac{V_{z_n}}{z_n}, D_{z'}D_zV\Big), \tag{3.36}$$

where $\widehat{f}$ is a $C^{1+\beta}$ smooth function of all its arguments. Hence, one obtains $V(y', y_{n+1}, t) \in C_\mu^{2+\min\{1,\frac{4}{\sigma_p}-2k_0\}}(\overline{Q^*_{\frac{\eta}{4}}(\overline{q}_0)})$ by the same argument as in Case 2a.

From the arbitrariness of $\overline{q}_0$, we obtain Theorem 1.1.

**Proof of Corollary 1.1** Fix a point $\overline{p}_0 = (\overline{y}_0, \overline{t}_0)$ on the interface $\Gamma_{\overline{t}_0}$, $\overline{t}_0 \in (0, T]$. By a rotation of the coordinates, we may assume that the unit outer normal of the flat part $\{v(y, \overline{t}_0) = 0\}$ at $\overline{p}_0$ is $e_n = (0, \cdots, 0, 1)$.

If $\frac{1}{\sigma_p} \in \mathbb{Z}^+$, by Theorem 1.1, $g$ is $C^\infty$-smooth up to the interface $\Gamma_t$ for $0 < t < T^*$. Hence $v = \frac{\sigma_p}{\sigma_p+1} g^{1+\frac{1}{\sigma_p}}$ is also $C^\infty$ smooth.

Next we consider the case $\frac{1}{\sigma_p} \notin \mathbb{Z}^+$.

If $\frac{1}{\sigma_p} \in \big(0, \frac{1}{2}\big]$, then $\frac{2}{\sigma_p} \in (0, 1]$ and $k_0 = 0$, where $k_0$ is the greatest integer strictly less than $\frac{2}{\sigma_p}$. By Theorem 1.1, we have $g \in C_\mu^{2+\beta_0}(\overline{\{v > 0\}})$, where $\beta_0 := \min\big\{1, \frac{4}{\sigma_p} - 2k_0\big\}$. Hence $g \in C^{0,1}(\overline{\{v > 0\}})$ and $D_y g \in C^{0,\frac{1}{\sigma_p}}(\overline{\{v > 0\}})$ as $\frac{1}{\sigma_p} \le \frac{1}{2}\beta_0$. Hence,

$$D_y v = g^{\frac{1}{\sigma_p}} D_y g \in C^{0,\frac{1}{\sigma_p}}(\overline{\{v > 0\}}), \tag{3.37}$$

which yields $v \in C^{1,\frac{1}{\sigma_p}}(\overline{\{v > 0\}})$.

If $\frac{1}{\sigma_p} \in \big(\frac{1}{2}, 1\big)$, then $\frac{2}{\sigma_p} \in (1, 2)$ and $k_0 = 1$. By Theorem 1.1, $g \in C_\mu^{1,2+\beta_0}(\overline{\{v > 0\}})$, and so $g, D_y g \in C^{0,1}(\overline{\{v > 0\}})$. Hence, (3.37) gives $v \in C^{1,\frac{1}{\sigma_p}}(\overline{\{v > 0\}})$.

If $\frac{1}{\sigma_p} > 1$. Denote $l_0 := \left[\frac{1}{\sigma_p}\right]$, then $\frac{2}{\sigma_p} > 2l_0 \ge 2$. In this case, by Theorem 1.1, $g$ is at least of class $C_\mu^{2l_0,2+\varepsilon}(\overline{\{v > 0\}})$ for some small $\varepsilon > 0$. Hence $g, D_y g, D_y^2 g, \cdots, D_y^{l_0+1} g \in C^{0,1}(\overline{\{v > 0\}})$ as $l_0 + 1 \le 2l_0$. Differentiating $D_y v = g^{\frac{1}{\sigma_p}} D_y g$ $l_0$ times in space variables $y$, we get $D_y^{l_0+1} v \in C^{0,\frac{1}{\sigma_p}-l_0}(\overline{\{v > 0\}})$. It follows that $v \in C^{1+l_0,\frac{1}{\sigma_p}-l_0}(\overline{\{v > 0\}})$.

As for the estimate (1.4), we fix $\overline{t}_0 \in [\sigma, T]$ and take $(y, \overline{t}_0), (\widetilde{y}, \overline{t}_0) \in \overline{\{0 < v(\cdot, \overline{t}_0) < 1\}}$ with $d_{y,\widetilde{y}}(\overline{t}_0) := \min\{\mathrm{dist}(y, \Gamma_{\overline{t}_0}), \mathrm{dist}(\widetilde{y}, \Gamma_{\overline{t}_0})\} = \mathrm{dist}(\widetilde{y}, \Gamma_{\overline{t}_0})$.

If $\frac{2}{\sigma_p} \le 1$, then $k_0 = 0$. By Theorem 1.1 and (3.2), it holds

$$g^{1-\frac{1}{\sigma_p}} v_{ij} = g g_{ij} + \frac{1}{\sigma_p} g_i g_j \in C_\mu^{0, \frac{2}{\sigma_p}}(\overline{\{v > 0\}}), \quad 1 \le i, j \le n. \tag{3.38}$$

In the case $|y - \widetilde{y}| \ge \mathrm{dist}(\widetilde{y}, \Gamma_{\overline{t}_0})$, (3.30) and (3.38) give that

$$d_{y,\widetilde{y}}(\overline{t}_0)^{1+\frac{1}{\sigma_p}} \frac{|v_{ij}(y, \overline{t}_0) - v_{ij}(\widetilde{y}, \overline{t}_0)|}{|y - \widetilde{y}|^{\frac{2}{\sigma_p}}} \le C \mathrm{dist}(\widetilde{y}, \Gamma_{\overline{t}_0})^{1-\frac{1}{\sigma_p}} |v_{ij}(y, \overline{t}_0) - v_{ij}(\widetilde{y}, \overline{t}_0)| \le C. \tag{3.39}$$

In the case $|y - \widetilde{y}| \le \mathrm{dist}(\widetilde{y}, \Gamma_{\overline{t}_0})$, by (3.12), (3.30) and (3.38), we have

$$\begin{aligned}
& d_{y,\widetilde{y}}(\overline{t}_0)^{1+\frac{1}{\sigma_p}} \frac{|v_{ij}(y, \overline{t}_0) - v_{ij}(\widetilde{y}, \overline{t}_0)|}{|y - \widetilde{y}|^{\frac{2}{\sigma_p}}} \\
\le\, & C d_{y,\widetilde{y}}(\overline{t}_0)^{\frac{2}{\sigma_p}} \frac{|g(y, \overline{t}_0)^{1-\frac{1}{\sigma_p}} v_{ij}(y, \overline{t}_0) - g(y, \overline{t}_0)^{1-\frac{1}{\sigma_p}} v_{ij}(\widetilde{y}, \overline{t}_0)|}{|y - \widetilde{y}|^{\frac{2}{\sigma_p}}} \\
\le\, & C d_{y,\widetilde{y}}(\overline{t}_0)^{\frac{2}{\sigma_p}} \frac{|g(y, \overline{t}_0)^{1-\frac{1}{\sigma_p}} v_{ij}(y, \overline{t}_0) - g(\widetilde{y}, \overline{t}_0)^{1-\frac{1}{\sigma_p}} v_{ij}(\widetilde{y}, \overline{t}_0)|}{\mu[(y, \overline{t}_0), (\widetilde{y}, \overline{t}_0)]^{\frac{2}{\sigma_p}} \cdot |\sqrt{y_n} + \sqrt{\widetilde{y}_n}|^{\frac{2}{\sigma_p}}} \\
& + C d_{y,\widetilde{y}}(\overline{t}_0)^{\frac{2}{\sigma_p}} \frac{v_{ij}(\widetilde{y}, \overline{t}_0) |g(y, \overline{t}_0)^{1-\frac{1}{\sigma_p}} - g(\widetilde{y}, \overline{t}_0)^{1-\frac{1}{\sigma_p}}|}{|y - \widetilde{y}|^{\frac{2}{\sigma_p}}} \\
\le\, & C + C d_{y,\widetilde{y}}(\overline{t}_0)^{\frac{2}{\sigma_p}} v_{ij}(\widetilde{y}, \overline{t}_0) \cdot (g_n g^{-\frac{1}{\sigma_p}})\big|_{(\lambda y + (1-\lambda)\widetilde{y}, \overline{t}_0)} \cdot |y - \widetilde{y}|^{1-\frac{2}{\sigma_p}} \quad (\lambda \in [0, 1]) \\
\le\, & C + C \mathrm{dist}(\widetilde{y}, \Gamma_{\overline{t}_0})^{1-\frac{1}{\sigma_p}} v_{ij}(\widetilde{y}, \overline{t}_0) \le C.
\end{aligned} \tag{3.40}$$

Hence, (1.4) holds when $\frac{2}{\sigma_p} \le 1$.

If $\frac{2}{\sigma_p} > 1$, then $k_0 \ge 1$. By differentiating equation $D_y v = g^{\frac{1}{\sigma_p}} D_y g$ with respect to $y$ $k_0 + 1$ times and by Theorem 1.1, we have

$$g^{k_0+1-\frac{1}{\sigma_p}} D_y^{k_0+2} v \in C_\mu^{0, \frac{2}{\sigma_p} - k_0}(\overline{\{v > 0\}}).$$

Therefore, by similar computations as in (3.39)–(3.40), it follows that

$$d_{y,\widetilde{y}}(\overline{t}_0)^{1+\frac{1}{\sigma_p}} \frac{|D_y^{k_0+2} v(y, \overline{t}_0) - D_y^{k_0+2} v(\widetilde{y}, \overline{t}_0)|}{|y - \widetilde{y}|^{\frac{2}{\sigma_p} - k_0}} \le C.$$

As a result, (1.4) follows.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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