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Lagrange Stability and KAM Tori for Duffing Equations with Quasi-periodic Coefficients*

Huining XUE[1]   Xiaoping YUAN[1]

Abstract It is proved that there are many (positive Lebesgue measure) Kolmogorov-Arnold-Moser (KAM for short) tori at infinity and thus all solutions are bounded for the Duffing equations $\ddot{x}+x^{2n+1}+\sum\limits_{j=0}^{2n}p_i(t)x^j=0$ with $p_j(t)$'s being time-quasi-periodic smooth functions.

Keywords KAM tori, Lagrangian stability, Duffing equation, Quasi-periodic function
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1 Introduction

In mechanics one frequently encounters the Duffing equations

$$\ddot{x}+ax+bx^3=p\left(t\right),$$

where $p(t)=p(t+2\pi)$ is a periodic forcing function. If $p=0$ and $b>0$, it is well known that all solutions are periodic, with a period depending on the amplitudes. However, even if the exterior force $p(t)\not\equiv 0$ is small, it is a complicated problem to decide the boundedness of all solutions (that is, Lagrangian stability). Moser [1] proposed to investigate this problem using Kolmogorov-Arnold-Moser theory (KAM for short) (see [2–4]).

The first result is due to Morris [5] who proved that all solutions of $\ddot{x}+2x^3=p(t)$ are bounded, that is, there exists a constant C (depending on the initial data) such that

$$|x(t)|+|\dot{x}(t)|<C,\quad t\in\mathbb{R}.$$

The Morris's result was generalised by Diekerhoff-Zehhder [6] to the equation of more general form

$$\ddot{x}+x^{2n+1}+\sum_{j=0}^{2n}p_j(t)x^j=0,\tag{1.1}$$

Manuscript received May 7, 2024. Revised September 24, 2024.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: 21110180025@m.fudan.edu.cn   xpyuan@fudan.edu.cn
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where $p_j(t) = p_j(t + 2\pi)$ $(j = 0, 1, \cdots, 2n)$ are C^ν-smooth functions with $\nu \geq 1 + \frac{4}{n} + \log_2^n$. See [7–11] for more backgrounds.

A natural question is what happens to (1.1) when the coefficients $p_j(t)$ are quasi-periodic in time t. We say that a function $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is quasi-periodic in time with frequency $\omega \in \mathbb{R}^d$, if there exists a function $F : \mathbb{T}^d \to \mathbb{R}$ (or $\mathbb{C}$) such that $f(t) = F(\omega t)$, where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. We say that F is the hull of f. (see [12] for the notations of quasi-periodic functions and their hulls). In the following arguments, denote by P_j the hulls of the coefficients $p_j(t)$ in (1.1).

In the present paper, we will prove the following theorem.

Theorem 1.1 *Assume that the hulls P_j of the coefficients p_j $(j = 1, 2, \cdots, 2n)$ are real analytic in $\mathbb{T}^d$, and assume that the frequency $\omega \in \mathbb{R}^d$ of the coefficients P_j's obeys Diophantine condition (DC_{γ_0}),*

$$|\langle k, \omega\rangle| \geq \gamma_0/|k|^{d+2}, \quad \forall k \in \mathbb{Z}^d \setminus 0,$$

where $0 < \gamma_0 \ll 1$ is a constant. Then (1.1) has many (positive Lebesgue measure) $(d + 1)$-dimensional KAM tori clustering at infinity in the $(d + 2)$-dimensional extended phase space[1] *$\mathbb{T}^{d+1} \times \mathbb{R}^1$, with frequency $(\omega, 1) \in \mathbb{R}^{d+1}$. Therefore, all solutions of (1.1) are bounded, that is, $|x(t)| + |\dot{x}(t)| \leq C$ for $t \in \mathbb{R}$, where the constant C depends on the initial values $(x(0), \dot{x}(0))$.*

Remark 1.1 When $p_j(t)$'s are periodic in time t, in [6], by a series of symplectic coordinates which are close to identity, the Hamiltonian H corresponding to (1.1) can be reduced to

$$H = I^a + h_1(I, t) + h_2(I, \theta, t), \quad a > 0, \tag{1.2}$$

where (I, θ) are the action angle variables and the size of h_2 is small enough. Note that the system defined by H is periodic in time t. It follows that the Poincar'e mapping obeys the Moser's twist theorem (see [4]). Thus the boundedness of all solutions follows. As for our case where $p_j(t)$'s are quasi-periodic in time t, the Poincaré mapping could not be defined directly. In an early work [13], the existence of many KAM tori was obtained for $\ddot{x} + x^{2n+1} + cx = P(\omega_1 t, \cdots, \omega_d t)$, a special form of (1.1), but there were no results of the boundedness of all solutions.

Remark 1.2 In the present paper, we decompose $p_j(t)$ into $p_{j\leq}(t)$ of lower Fourier frequencies (refer to (3.3)) and $p_{j>}(t)$ of higher Fourier frequencies (refer to (3.4)). We can choose sufficiently high Fourier frequencies such that $p_{j>}(t)$ is small enough. In order to apply KAM theorem, it suffices to eliminate all terms (3.3) involving $p_{j\leq}(t)$. Fortunately, while removing (3.3), the divisors are large enough instead of being small in the homological equation (3.16). This is key point in our proof. In addition, as in [6], we also derive a reduced Hamiltonian of the same form as (1.2) (refer to (3.41)). In our case, $h_1(I, t) = \widehat{R}_{\leq}^{(M)}(I, 0, \varphi)$ with $\varphi = \omega t$, which is quasi-periodic with frequency $\omega \in \mathbb{R}^d$ $(d > 1)$ in time t. So the Pioncaré mapping could not defined directly. We will find a symplectic coordinate change which is not close to identity to removing the dependence on time t of $h_1(I, t)$ (see (4.7)).

[1] See Section 2 for the extended phase space.

Remark 1.3 We also relax the analyticity of the coefficients $p_j(t)$'s to C^ν with $\nu \gg 1$. We do not pursue this end.

Proof outline In Section 2, using the periodic solution of the autonomous system, we introduce the action and angle variables. Then we introduce an angle variable $\phi \in \mathbb{T}^d$ and an artificial action variable $J \in \mathbb{R}^d$ such that the considered Hamiltonian system is transformed into an autonomous Hamiltonian $H = \omega \cdot J + H_0(I) + R(I, \theta, \phi)$ with the extended phase space $\mathbb{T}^{d+1} \times \mathbb{R}^{d+1}$ (see (2.10)–(2.11)). In Section 3, performing a series of symplectic transforms, we change the perturbation R to a small one. When the system (1.1) is periodic in time t, the perturbation is independent of ϕ, and thus we do not encounter any small divisor problem. In the present paper, the perturbation R is indeed dependent on ϕ. Write $R = R_{\leq} + R_{>}$ where $R_{\leq}$ and $R_{>}$ are the part of Fourier series of R in ϕ with lower frequencies and one with higher frequencies, respectively. We observe that there is no small divisor problem arising when eliminating the $R_{\leq}$ of lower frequencies, when $H_0(I)$ is large. Using this crucial observation, we change the perburbation R into a small $R^{(M)}$ by a series symplectic transformations without small divisor conditions (see (3.43)). In Section 4, we further change $R^{(M)}$ into $R^{(M+1)}$ such that the changed Hamiltonian system obeys the conditions of the Kolmogorov Theorem (KAM theorem), by which the proof is finished.

2 Action-Angle Variable

Replacing x by Ax in (1.1) with a large constant $A > 0$, we get

$$A\ddot{x} + A^{2n+1}x^{2n+1} + \sum_{j=0}^{2n} p_j\,(t)\,x^j A^j = 0. \tag{2.1}$$

Let

$$y = A^{-n}\dot{x} \quad \text{or} \quad \dot{x} = A^n y.$$

Then

$$\dot{y} = -A^n x^{2n+1} - \sum_{j=0}^{2n} p_j(t) x^j A^{j-n-1}.$$

Thus

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \tag{2.2}$$

where

$$H = A^n\Big(\frac{1}{2}y^2 + \frac{1}{2(n+1)}x^{2(n+1)}\Big) + \sum_{j=0}^{2n} \frac{p_j(t)}{j+1} x^{j+1} A^{j-n-1}. \tag{2.3}$$

Consider an auxiliary Hamiltonian system

$$\dot{x} = \frac{\partial H_0}{\partial y}, \quad \dot{y} = -\frac{\partial H_0}{\partial x}, \quad H_0 = \frac{1}{2}y^2 + \frac{1}{2(n+1)}x^{2(n+1)}, \tag{2.4}$$

let $(x_0(t), y_0(t))$ be the solution to (2.4) with initial $(x_0(t), y_0(t)) = (1, 0)$. Then this solution is clearly periodic. Let T_0 be its minimal positive period. By energy conservation, we have

$$(n+1)y_0^2(t) + x_0^{2n+2}(t) \equiv 1, \quad t \in \mathbb{R}. \tag{2.5}$$

We construct the symplectic transformation

$$\Psi_0 : \begin{cases} x = c^{\alpha} I^{\alpha} x_0(\theta T_0), \\ y = c^{\beta} I^{\beta} y_0(\theta T_0), \end{cases}$$

where $\alpha = \frac{1}{n+2}, \beta = 1 - \alpha = \frac{n+1}{n+2}, c = \frac{1}{\alpha T_0}$, and where $(I, \theta) \in \mathbb{R}^+ \times \mathbb{T}^1$ is action-angle variables. By (2.5), we have $\det \frac{\partial(x,y)}{\partial(I,\theta)} = 1$. Thus the transformation is indeed symplectic. Clearly $\Psi_0(I, \theta)$ is analytic in $(I, \theta) \in \mathbb{R}^+ \times \mathbb{T}^1$.

Under Ψ_0, equation (2.2) with Hamiltonian (2.3) is changed to

$$\dot{\theta} = \frac{\partial H}{\partial I}, \quad \dot{I} = -\frac{\partial H}{\partial \theta}, \tag{2.6}$$

where $H = H_0(I) + R(I, \theta, t)$ with

$$H_0(I) = \widetilde{d} \cdot A^n \cdot I^{2\beta} = \widetilde{d} \cdot A^n \cdot I^{\frac{2(n+1)}{n+2}}, \quad \widetilde{d} = \frac{c^{2\beta}}{2(n+1)} \tag{2.7}$$

and

$$R(I, \theta, t) = \sum_{j=0}^{2n} \frac{p_j(t)}{j+1} (c^{\frac{1}{n+1}} x_0(\theta T_0))^{j+1} A^{j-n-1} I^{\frac{j+1}{n+2}}. \tag{2.8}$$

Clearly, $R(I, \theta, t) = O(A^{n-1})$ for $A \to \infty$. Restrict I to some compact interval, say, $I \in [1, 2]$. Let $\varphi = \omega t$. Then (2.8) can be rewritten as

$$R(I, \theta, \varphi) = \sum_{j=0}^{2n} \frac{p_j(\varphi)}{j+1} (c^{\frac{1}{n+1}} x_0(\theta T_0))^{j+1} A^{j-n-1} I^{\frac{j+1}{n+2}}. \tag{2.9}$$

Introduce an artificial action $J \in \mathbb{R}^d$. Then we can lift the Hamiltonian system (2.6) to an autonomous system

$$\dot{\theta} = \frac{\partial H}{\partial I}, \quad \dot{I} = -\frac{\partial H}{\partial \theta}, \quad \dot{\varphi} = \omega, \quad \dot{J} = -\frac{\partial H}{\partial \varphi}, \tag{2.10}$$

where

$$H(I, \theta, \varphi) = \omega \cdot J + H_0(I) + R(I, \theta, \varphi). \tag{2.11}$$

For (2.10), our phase space is $(\theta, \phi, I, J) \in \mathbb{T}^{d+1} \times \mathbb{R}^{d+1}$. Since J is artificial, it can fixed. The phase space can be taken as $(\theta, \phi, I) \in \mathbb{T}^{d+1} \times \mathbb{R}^1$ which is called the extended phase space for the quasi-periodic system (1.1). Clearly, (2.6) is a sub-system of (2.11). It suffices to investigate the existence of KAM tori of (2.11). It is easy to see that $H(I, \theta, \varphi)$ is real analytic in $(I, \theta, \varphi) \in [1, 2] \times \mathbb{T} \times \mathbb{T}^d$. Write $\mathbb{T}^{1+d} =: \mathbb{T} \times \mathbb{T}^d$. By the compactness $[1, 2] \times \mathbb{T}^{1+d}$, we can assume that $H(I, \theta, \varphi)$ is real analytic in the complex domain $[1, 2] \times \mathbb{T}^{1+d}_{s_0}$ with some $s_0 > 0$ For a function of complex variables, we call it real analytic if it is analytic, and it is real for real arguments.

3 To Change Large Perturbation into Small One

For an analytic function $f : [1, 2] \times \mathbb{T}_s \times \mathbb{T}_s^d \to \mathbb{C}$, satisfying

$$\sup_{[1,2]\times\mathbb{T}_s^{1+d}} |f(I, \theta, \varphi)| \le CA^\alpha, \quad A \to +\infty,$$

for some constant C which might depends on the dimensional number d, n and s_0, we write $f = O_s(A^\alpha)$. In the following arguments, we will denote by C a universal constant which may be different in different places and which may depend on d, n, s_0, when we do not care about its size. It follows from (2.9) that

$$R(I, \theta, \varphi) = O_{s_0}(A^{n-1}). \tag{3.1}$$

Let $K = c_0 \log A$, $\mathbb{Z} \times \mathbb{Z}^d = \mathbb{Z}^{1+d}$ and $\mathbb{T} \times \mathbb{T}^d = \mathbb{T}^{1+d}$, where $c_0 = c_0(d)$ is a constant depending on only d. We will specify the constant $c_0 \gg 1$ in Section 4 (see (4.21)). Write

$$R = R_{\le}(I, \theta, \varphi) + R_{>}(I, \theta, \varphi), \tag{3.2}$$

where

$$R_{\le}(I, \theta, \varphi) = \sum_{\substack{|k|+|l|\le K \\ (k,l)\in\mathbb{Z}^{1+d}}} \widehat{R}(I, k, l)\mathrm{e}^{\mathrm{i}(k\theta+l\cdot\varphi)}, \tag{3.3}$$

$$R_{>}(I, \theta, \varphi) = \sum_{\substack{|k|+|l|> K \\ (k,l)\in\mathbb{Z}^{1+d}}} \widehat{R}(I, k, l)\mathrm{e}^{\mathrm{i}(k\theta+l\cdot\varphi)}, \tag{3.4}$$

$$\widehat{R}(I, k, l) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{T}^{1+d}} R(I, \theta, \varphi)\mathrm{e}^{-\mathrm{i}(k\theta+l\cdot\varphi)}\mathrm{d}\theta\mathrm{d}\varphi. \tag{3.5}$$

Then

$$H = \omega \cdot J + H_0(I) + R_{\le} + R_{>}. \tag{3.6}$$

Noting that $R(I, \theta, \varphi)$ is analytic in $I \times \mathbb{T}_{s_0}^{1+d}$, and in view of (3.1), we have

$$|\widehat{R}(I, k, l)| \le CA^{n-1} \exp(-s_0(|k| + |l|)), \quad \forall (k, l) \in \mathbb{Z}^{1+d}. \tag{3.7}$$

It follows

$$R_{\ge} = O_{\frac{s_0}{2}}(A^{-c_0 n}). \tag{3.8}$$

Our aim is now to find a series of symplectic coordinate changes to eliminate $R_{\le}$. To this end, let

$$F(I, \theta, \varphi) = \sum_{\substack{|k|+|l|\le K \\ (k,l)\in\mathbb{Z}^{1+d} \\ k\neq 0}} \widehat{F}(I, k, l)\mathrm{e}^{\mathrm{i}(k\cdot\theta+l\cdot\varphi)},$$

where $\widehat{F}(I,k,l)$ is to be specified later on. Let X_F^t be the flow of the Hamiltonian system

$$\dot{\theta}=\frac{\partial F}{\partial I},\quad \dot{I}=-\frac{\partial F}{\partial \theta},\quad \dot{\varphi}=\omega,\quad \dot{J}=-\frac{\partial F}{\partial \varphi}. \tag{3.9}$$

Then $X_F^1=X_F^t|_{t=1}$ is a symplectic coordinate change, and

$$\begin{aligned} H^{(1)}(I,\theta,\varphi)=H\circ X_F^1&=\omega\cdot J+H_0(I)+R_{\le}+R_{>}+\{\omega\cdot J+H_0(I),F\}\\ &\quad+\{R_{\le},F\}+\frac{1}{2}\{\{H,F\},F\}\circ X_F^1, \end{aligned}\tag{3.10}$$

where Poisson bracket is defined by

$$\{X,Y\}=\omega\cdot\partial_\varphi X-\partial_J X\cdot\partial_\varphi Y+\partial_\theta X\cdot\partial_I Y-\partial_I X\cdot\partial_\theta Y. \tag{3.11}$$

Let

$$\{\omega\cdot J+H_0(I),F\}+R_{\le}=\widehat{R}_{\le}(I,0,\varphi), \tag{3.12}$$

where

$$\widehat{R}_{\le}(I,0,\varphi)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}R_{\le}(I,\theta,\varphi)\mathrm{e}^{-\mathrm{i}k\cdot\theta}\mathrm{d}\theta. \tag{3.13}$$

Then (3.10) reads

$$H^{(1)}(I,\theta,\varphi)=\omega\cdot J+H_0(I)+\widehat{R}_{\le}(I,0,\varphi)+R_{>}+\{R_{\le},F\}+\frac{1}{2}\{\{H,F\},F\}\circ X_F^1. \tag{3.14}$$

By (3.11), we can rewrite (3.12) as

$$-\omega\cdot\partial_\varphi F-H_0'(I)\partial_\theta F=\widehat{R}_{\le}(I,0,\varphi)-R_{\le}. \tag{3.15}$$

Passing to Fourier coefficients, we have

$$\widehat{F}(I,k,l)=\frac{\widehat{R}(I,k,l)}{\mathrm{i}(\langle l,\omega\rangle+kH_0'(I))},\quad (k,l)\in\mathbb{Z}^{1+d},\ k\ne 0,\ |k|+|l|\le K. \tag{3.16}$$

We are now in position to investigate the denominator in (3.16). Fix $\omega\in DC_{\gamma_0}$. Let

$$©_{k,l}(w)=\Big\{I\in[1,2]\ \big|\ |\langle l,\omega\rangle+H_0'(I)k|<\frac{A^n\gamma}{(1+|l|)^\tau}\Big\}$$

for $(k,l)\in\mathbb{Z}^{1+d}$, and $k\in\mathbb{Z}\backslash\{0\}$, where γ is to be specified later on. Note,

$$H_0''(I)=\frac{2(n+1)nd}{(n+2)^2}I^{-\frac{2}{n+2}}A^n\ge C_0A^n,\quad C_0=\frac{2(n+1)n}{(n+2)^2}2^{-\frac{2}{n+2}}. \tag{3.17}$$

Thus

$$\Big|\frac{\mathrm{d}}{\mathrm{d}I}(\langle l,\omega\rangle+H_0'(I)k)\Big|\ge C_0A^n,\quad k\ne 0. \tag{3.18}$$

It follows

$$\mathrm{Leb}\bigcup_{\substack{|k|+|l|\le K\\ l\ne 0}}©_{k,l}(\omega)\le C^*K^{d+1}\gamma,\quad C^*=\frac{1}{C_0}\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|l|)^{d+2}}. \tag{3.19}$$

Take

$$\gamma = (C^* K^{d+1})^{-1} K^{-40d} = C^{**}(\log A)^{-41d+1}, \quad C^{**} = (C^* C_*^{41d+1})^{-1}. \tag{3.20}$$

Thus

$$\mathrm{Leb} \bigcup_{\substack{|k|+|l|\le K \\ l\neq 0}} ©_{k,l}(\omega) < C(\log A)^{-40d}. \tag{3.21}$$

Again by (3.18), we have that the set $©_{k,l}$ (with $k \neq 0$) consists of, at most, 2 connected components. So the set $[1,2] \setminus \bigcup_{|k|+|l|\le K} ©_{k,l}(\omega)$ consists of, at most, $(4K)^{d+1}$ many connected components. Furthermore, there is a subinterval, $\Gamma \subset [1,2] \setminus \bigcup_{|k|+|l|\le K} ©_{k,l}$ such that

$$\mathrm{Leb}(\Gamma) > C(\log A)^{-50d}. \tag{3.22}$$

Write $\Gamma = [\Gamma_-, \Gamma_+]$. Then $\Gamma_+ - \Gamma_- > C(\log A)^{-50d})$. By the definition of $©_{k,l}$, we have for $\forall I \in \Gamma$,

$$|\langle l, \omega\rangle + k\, H_0'(I)| \ge \frac{\gamma A^n}{(1+|l|)^\tau}, \quad (k,l) \in \mathbb{Z}^{1+d},\ k \neq 0,\ |k|+|l| \le K. \tag{3.23}$$

It follows that

$$|\widehat{F}(I,k,l)| = \frac{|\widehat{R}(I,k,l)|}{|\langle l,\omega\rangle + H_0'(I)k|} \le \frac{(1+|l|)^{d+2}}{\gamma} A^{-1} C \exp(-s_0(|k|+|l|)).$$

Moreover,

$$\begin{aligned}\sup_{\Gamma\times\mathbb{T}^{1+d}_{\frac{s_0}{2}}} |F(I,\theta,\varphi)| &\le \sum_{|k|+|l|\le K} C\gamma^{-1}(1+|l|)^{-(d+2)} A^{-1} \exp\Big(-\frac{s_0}{2}(|k|+|l|)\Big)\\ &\le C\gamma K A^{-1}\\ &\le (\log A)^C A^{-1}.\end{aligned} \tag{3.24}$$

By (2.10), we have

$$\partial_I^\alpha R(I,\theta,\varphi) = O_{s_0}(A^{n-1}), \quad |\alpha| \le C. \tag{3.25}$$

By (3.16) and(3.23), we have

$$\partial_I^\alpha F(I,\theta,\varphi) = O_{\frac{s_0}{2}}((\log A)^c A^{-1}), \quad |\alpha| \le C. \tag{3.26}$$

By Cauchy estimate, we have, furthermore,

$$\partial_I^\alpha \partial_\theta^{\beta_1} \partial_\varphi^{\beta_2} F(I,\theta,\varphi) = O_{\frac{s_0}{3}}((\log A)^c A^{-1}), \quad \alpha+\beta_1+\beta_2 \le C. \tag{3.27}$$

Note that the solution $X_F^t(I,\theta,\varphi)$ depends analytically on the initial values $(I(0),\theta(0),\varphi(0)) = (I,\theta,\varphi)$. It follows from contraction mapping principle that flow X_F^t does exist for $t \in [0,1]$, in particular,

$$\sup_{\Gamma_1\times\mathbb{T}^{1+d}_{\frac{s_0}{3}}} \|X_F^1(I,\theta,\varphi) - (I,\theta,\varphi)\| \le C(\log A)^c A^{-1}, \tag{3.28}$$

where $\Gamma^1 = [\Gamma_-(\log A)^{-C}, \Gamma_+(\log A)^{-C}]$, $\Gamma := [\Gamma_-, \Gamma_+]$. Thus $X_F^1(\Gamma_1 \times \mathbb{T}^{1+d}_{\frac{s_0}{3}}) \subset \Gamma \times \mathbb{T}^{1+d}_{\frac{s_0}{2}}$. Without loss of generality, we still write $\Gamma_1 = \Gamma$.

Again by (3.27)–(3.28), we have

$$R^* := \{R, F\}(I,\theta,\varphi) + \frac{1}{2}\{\{H,F\},F\} \circ X_F^1(I,\theta,\varphi) = O_{\frac{s_0}{4}}((\log A)^C A^{n-2}). \tag{3.29}$$

Write

$$R^* = \sum_{\substack{|k|+|l|\le K \\ l\neq 0}} \widehat{R}^*(I,k,l)e^{i(k\theta+\langle l,\varphi\rangle)} + \sum_{\substack{|k|+|l|> K \\ l\neq 0}} \widehat{R}^*(I,k,l)e^{i(k\theta+\langle l,\varphi\rangle)} := R^{(1)}_{\le} + \overline{R}^{(1)}_{>}.$$

Letting $R^{(1)}_{>} = R_{>} + \overline{R}^{(1)}_{>}$, we have

$$H^{(1)} = \omega\cdot J + H_0(I) + \widehat{R}_{\le}(I,0,\varphi) + R^{(1)}_{\le} + R^{(1)}_{>}. \tag{3.30}$$

By (3.29), following the proof of (3.8), we have $\overline{R}^{(1)}_{>} = O_{\frac{s_0}{4}}(A^{-c_0 n})$. By (3.8), we furthermore have

$$R^{(1)}_{>} = O_{\frac{s_0}{4}}(A^{-c_0 n}). \tag{3.31}$$

By (3.29),

$$R^{(1)}_{\le} = O_{\frac{s_0}{4}}((\log A)^C A^{n-2}). \tag{3.32}$$

Recall $H(I,\theta,\varphi)$ is real analytic. In particular, $H(I,\theta,\varphi)$ is real for real argument (I,θ,ϕ). It follows that $\widehat{R}(I,-k,-l) = \overline{\widehat{R}(I,k,l)}$, for $I \in \Gamma\cap\mathbb{R}$. By (3.16), we have $\widehat{F}(I,-k,-l) = \overline{\widehat{F}(I,k,l)}$, for $I \in \Gamma\cap\mathbb{R}$. It follows that $F(I,\theta,\varphi)$ is real analytic. Moreover, all of $\widehat{R}_1(I,0,\varphi)$, $R^{(1)}_{\le}$ and $R^{(1)}_{>}$ are real analytic in $\Gamma\times\mathbb{T}^{1+d}_{\frac{s_0}{4}}$.

Now we search for a symplectic coordinate change to eliminate $R^{(1)}_{\le}$. We can repeat the previous procedure to eliminate $R^{(1)}_{\le}$. Comparing (3.6) with (3.30), we see that $\widehat{R}_1(I,0,\varphi)$ is a new term in Hamiltonian $H^{(1)}$. We must be careful about that the influence of $\widehat{R}_{\le}(I,0,\varphi)$ in eliminating $R^{(1)}_{\le}$. To check the effect of $\widehat{R}_{\le}(I,0,\varphi)$, we introduce a new Hamiltonian of the form

$$F^{(1)} = F^{(1)}(I,\theta,\varphi) = \sum_{\substack{|k|+|l|\le K \\ (k,l)\in\mathbb{Z}^{1+d} \\ k\neq 0}} \widehat{F}^{(1)}(I,k,l)e^{i(k\theta+l\cdot\varphi)}.$$

Then

$$H^{(2)} = H^{(1)}\circ X_F^{(1),1} = \omega\cdot J + H_0(I) + \widehat{R}_{\le}(I,0,\varphi) + R^{(1)}_{>} \tag{3.33}$$

$$+ \{\omega\cdot J + H_0(I), F\} + R^{(1)}_{\le} \tag{3.34}$$

$$+ \{\widehat{R}_{\le}, F\}\circ X_F^{(1),1} \tag{3.35}$$

$$+ \{R^{(1)}_{\le} + R^{(1)}_{>}, F\}\circ X_F^{(1),1} + \frac{1}{2}\{\omega\cdot J + H_0(I) + R^{(1)}_{\le} R^{(1)}_{>}, F^{(1)}\}\circ X_F^{(1),1}, \tag{3.36}$$

where $X_F^{(1),1} = X_F^{(1)t}\,|_{t=1}$ is its time-1 map.

Let

$$\{\omega \cdot J + H_0(I), F\} + R_{\leq}^{(1)}(I,\theta,\varphi) = \widehat{R}_{\leq}^{(1)}(I,0,\varphi). \tag{3.37}$$

In view of (3.32), with the same procedure as the previous, we have

$$F^{(1)} = O_{\frac{s_0}{8}}((\log A)^C A^{-2}), \tag{3.38}$$

furthermore,

$$(3.30) = O_{\frac{s_0}{7}}((\log A)^C A^{n-4}) = O_{\frac{s_0}{7}}((\log A)^C A^{n-3}). \tag{3.39}$$

Observe (3.33)–(3.34) and (3.36), the effect of the new term $\widehat{R}_{\leq}(I,0,\varphi)$ is $\{\widehat{R}_{\leq}, F^{(1)}\}$. Note $\widehat{R}_{\leq} = O_{\frac{s_0}{5}}((\log A)^C A^{n-1})$. It follows from (3.38), then

$$\{\widehat{R}_{\leq}(I,0,\varphi), F^{(1)}\} = O_{\frac{s_0}{7}}((\log A)^C A^{n-3}). \tag{3.40}$$

Write

$$(3.35) + (3.36) = R_{\leq}^{(2)} + R_{>}^{(2)}$$

and

$$\widehat{R}_{\leq}^{(2)}(I,0,\varphi) = \widehat{R}_{\leq}(I,0,\varphi) + R_{\leq}^{(1)}(I,0,\varphi).$$

Then

$$H^{(2)} = \omega \cdot J + H_0(I) + \widehat{R}_{\leq}^{(2)}(I,0,\varphi) + R_{\leq}^{(2)} + R_{>}^{(2)}.$$

Repeat the previous procedure $M \gg 1$ many times, then

$$H^{(M)} = \omega \cdot J + H_0(I) + \widehat{R}_{\leq}^{(M)}(I,0,\varphi) + R^{(M)}(I,\theta,\varphi), \tag{3.41}$$

where

$$\widehat{R}_{\leq}^{(M)} = O_{\frac{s_0}{10M}}((\log A)^C A^{n-1}), \tag{3.42}$$

$$R^{(M)} = R_{\leq}^{(M)} + R_{>}^{(M)} = O_{\frac{s_0}{10M}}((\log A)^C A^{-c_0 n}). \tag{3.43}$$

By the previews proof, we have that $\widehat{R}_{\leq}^{(M)}(I,0,\varphi)$, $R_{\leq}^{(M)}$ and $R_{>}^{(M)}$ are real analytic in $\Gamma \times \mathbb{T}^{1+d}_{\frac{s_0}{10M}}$.

4 KAM Theorem

In this section, we will search for a symplectic coordinate change to remove the dependence on $\varphi = \omega t$ in $\widehat{R}_{\leq}(I,0,\varphi)$. To this end, let

$$F = F(I,0,\varphi) = \sum_{l \in \mathbb{Z}^d \setminus \{0\}} \widehat{F}(I,l) \mathrm{e}^{\mathrm{i}\langle l,\varphi\rangle}.$$

Let X_F^t be the flow of the Hamiltonian system defined by F. Then

$$H^{(M+1)} = H^{(M)} \circ X_F^1 = \omega \cdot J + H_0(I) + \widehat{R}_{\leq}^{(M)} + R^{(M)}$$

$$+\{\omega\cdot J+H_0(I),F\}+\{\widehat{R}^{(M)}_{\le},F\}+\{R^{(M)},F\}$$
$$+\frac{1}{2}\{\{H^{(M)},F\},F\}\circ X^1_F. \tag{4.1}$$

Let

$$\{\omega\cdot J+H_0(I),F\}+\widehat{R}^{(M)}_{\le}=[\widehat{R}^{(M)}_{\le}], \tag{4.2}$$

where

$$[\widehat{R}^{(M)}_{\le}]=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\widehat{R}^{(M)}_{\le}(I,0,\phi)\mathrm{d}\phi.$$

Rewriting (4.2) by advantage of Poisson bracket (3.11), we have

$$-\omega\cdot\partial_\varphi F=[\widehat{R}^{(M)}_{\le}]-\widehat{R}^{(M)}_{\le}. \tag{4.3}$$

Passing to Fourier coefficients, we have

$$F=\sum_{l\in\mathbb{Z}^d\setminus\{0\}}\frac{\widehat{R}(I,0,\varphi)}{-\mathrm{i}\langle l,\omega\rangle}\mathrm{e}^{\mathrm{i}\langle l,\varphi\rangle}. \tag{4.4}$$

Since $\omega\in DC_{\gamma_0}$, we have

$$F=O_{\frac{s_0}{20M}}((\log A)^C A^{n-1}). \tag{4.5}$$

Note that the size of F is not small, we should verify that the flow X^t_F exists for $t\in[0,1]$. Observing that F does not depends on θ, we have that the system defined by F reads

$$\dot{\theta}=\frac{\partial F}{\partial I},\quad \dot{I}=-\frac{\partial F}{\partial\theta}=0,\quad \dot{\varphi}=\omega,\quad \dot{J}=-\frac{\partial F}{\partial\varphi}.$$

Denote by $(\theta(0),I(0),\varphi(0),J(0))=(\theta,I,\varphi,J)$ the initial values. Then

$$I(t)=I,\quad \varphi(t)=\varphi+\omega t,\quad \theta(t)=\theta+\int_0^t\frac{\partial f}{\partial I}(I,0,\varphi+\omega\tau)\mathrm{d}\tau. \tag{4.6}$$

It implies that the solution can be given out explicitly. Furthermore, the solution does exist for $t\in R$, so X^1_F is well-defined. We claim

$$X^1_F(\Gamma\times\mathbb{T}^{1+d}_{\frac{s_0}{30M}})\subset\Gamma\times\mathbb{T}^{1+d}_{\frac{s_0}{20M}}. \tag{4.7}$$

Recall $\widehat{R}^{(M)}_{\le}$, $R^{(M)}_{\le}$ and $R^{(M)}_{>}$ are real analytic. It follows that F is real analytic in $\Gamma\times\mathbb{T}^{1+d}_{\frac{s_0}{30M}}$. So, for $(I,0,\varphi)\in\Gamma\times\mathbb{T}^{1+d}_{\frac{s_0}{30M}}$, we have

$$\begin{aligned}|\Im F(I,0,\theta)|&=|\Im F(I,0,\mathrm{Re}\varphi+\Im\varphi)|\\&\le\Big(\sup_{\Gamma_M\times\mathbb{T}^{1+d}_{\frac{s_0}{20M}}}|\partial F|\Big)(|\Im\varphi|)\\&\le C(\log A)^C A^{n-1}|\Im\varphi|.\end{aligned}$$

Let

$$|\Im\varphi|\le\Big(\frac{s_0}{20M}\Big)[C(\log A)^C A^{n-1}]^{-1}:=N.$$

Then $|\Im F(I,0,\varphi)| \leq \frac{s_0}{20M}$. By (4.7), we have

$$X_F^1(\Gamma \times \mathbb{T}_N^{1+d}) \subset \Gamma \times \mathbb{T}^{1+d}_{\frac{s_0}{20M}}. \tag{4.8}$$

By (4.1)–(4.2),

$$H^{(M+1)} = \omega \cdot J + H_0(I) + [\widehat{R}_{\leq}^{(M)}](I) + R^{(M+1)}, \tag{4.9}$$

where

$$R^{(M+1)} = R^{(M)} + \{\widehat{R}_{\leq}^{(M)}, F\} + \{R^{(M)}, F\} + \frac{1}{2}\{\{H^{(M)}, F\}, F\} \circ X_F^1. \tag{4.10}$$

By the definition of Poisson bracket (see (3.11)),

$$\{\widehat{R}_{\leq}^{(M)}(I,\varphi), F(I,\varphi)\} \equiv 0. \tag{4.11}$$

By (3.43) and (4.5), we have

$$\{R^{(M)}, F\} = O_{\Gamma\times\mathbb{T}_N^{1+d}}((\log A)^C A^{-C_0 n+n}). \tag{4.12}$$

By (4.11),

$$\begin{aligned}\{H^{(M)}, F\} &= \{\omega \cdot J + H_0(I) + \widehat{R}_{\leq}^{(M)} + R^{(M)}, F\}\\ &= \{\omega \cdot J + H_0(I), F\} + \{R^{(m)}, F\}\\ &= \{\widehat{R}^{(M)}(I) - \widehat{R}^{(M)}(I,\varphi), F(I,\varphi)\} + \{R^{(M)}, F\}\\ &= \{R^{(M)}, F\}.\end{aligned}$$

Thus, by (3.43) and (4.5),

$$\frac{1}{2}\{\{H^{(M)}, F\}, F\} \circ X_F^1 = \frac{1}{2}\{\{R^{(M)}, F\}, F\} \circ X_F^1 = O_{\Gamma\times\mathbb{T}_N^{1+d}}((\log A)^C A^{-C_0 n+2n}).$$

Finally,

$$R^{(M+1)} = O_{\Gamma\times\mathbb{T}_N^{1+d}}((\log A)^C A^{-C_0 n+2n}). \tag{4.13}$$

Note that $\widehat{R}_{\leq}^{(M)} = O((\log A)^C A^{n-1})$. We have

$$\sup_{I\in\Gamma} |\partial_I^\alpha [\widehat{R}_{\leq}^{(M)}]| \leq C(\log A)^C A^{n-1}, \quad |\alpha| \leq C. \tag{4.14}$$

Recall $N = ((\log A)^C A^{n-1})^{-1}$, $\Gamma = [\Gamma_-, \Gamma_+]$ with

$$\Gamma_+ - \Gamma_- > (C_+ \log A)^{-50d}, \quad K = C_+ \log A, \quad \gamma = C^{**}(\log A)^{-41d+1}, \quad H_0(I) = dA^n I^{\frac{2n+1}{n+1}}.$$

And recall Diophantine condition (3.23),

$$|\langle l, \omega\rangle + kH_0'(I)| \geq \frac{\gamma}{(1+|l|)^{d+2}}, \quad |k| + |l| \leq K,\ k \neq 0,\ (k,l) \in \mathbb{Z}^{1+d}.$$

Let $N_1 = A^{-10n}$. By Newmann series, we have that $\forall I_0 \in \Gamma$ with $[I_0 - N_1, I_0 + N_1] \in \Gamma$, and for $I \in B(I_0, N_1) := \{I \in \mathbb{C} \mid |I - I_0| \le N_1\}$,

$$|\langle l, \omega\rangle + kH_0'(I)| \ge \frac{\gamma}{2(1+|l|)^{d+2}}, \quad |k| + |l| \le K,\ k \ne 0,\ (k, l) \in \mathbb{Z}^{1+d}. \tag{4.15}$$

Moreover, all the estimates hold for I in the complex domain $B(I_0, N_1)$, in particular,

$$\sup_{B(I_0,N_1)} |\partial_I^\alpha [\widehat{R}_{\le}^{(M)}]| \le C(\log A)^C A^{n-1}, \tag{4.16}$$

$$\sup_{B(I_0,N_1)\times \mathbb{T}_N^{1+d}} |R^{(M+1)}| \le C(\log A)^C A^{-C_0 n + 2n}. \tag{4.17}$$

Recall $H_0''(I) = \frac{2(n+1)nd}{(n+2)^2} I^{-\frac{2}{n+2}} A^n \ge C_{00} A^n$, $I \in \Gamma$, which obeys Kolmogorov's non-degenerate condition. It follows that from a standard measure estimate in KAM exists a subset $\mathcal{O}_0 \subset \Gamma$ with $\mathrm{Leb}\,\mathcal{O}_0 \ge (\mathrm{Leb}\Gamma)(1 - CN_1^{10})$ such that for $\forall I_0 \in \mathcal{O}_0$,

$$|\langle l, \omega\rangle + kH_0'(I_0)| \ge \frac{N_1^{10}}{(1+|l|)^{d+2}}, \quad \forall (k, l) \in \mathbb{Z}^{1+d} \setminus \{0\}, \tag{4.18}$$

where $\omega \in DC_{\gamma_0}$ fixed and $\mathcal{O}_0$ depends on ω. Let $I = I_0 + \rho$, $\rho \in B(N_1) := \{\rho \in \mathbb{C} \mid |\rho| < N_1\}$ and

$$\begin{aligned} \mu = \mu(I_0) &= H_0' + \partial_I [\widehat{R}_{\le}^{(M)}](I_0) \\ &= \Big(d\frac{2(n+1)}{n+2} I_0^{\frac{n}{n+2}}\Big) A^n + O(A^{n-1}) \\ &\sim A^n, \quad A \to +\infty. \end{aligned}$$

By Taylor formula,

$$H^{(M+1)} = H_0(I_0) + [\widehat{R}_{\le}^{(M)}(I_0)] + \omega \cdot J + \mu\rho + \frac{1}{2}\Omega\rho^2 + h(\rho) + R(\rho, \theta, \varphi),$$

where

$$\begin{aligned} \Omega &= \partial_I^2 (H_0(I_3)) + [\widehat{R}_{\le}^{(M)}](I) \mid_{I=I_0}, \\ h(\rho) &= \rho^3 \int_0^1 \int_0^1 \int_0^1 xy \partial_I^3 (H_0(I_0 + \rho xyz) + [\widehat{R}_{\le}^{(M)}](I_0 + \rho xyz)) \mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned}$$

and

$$R(\rho, \theta, \varphi) = R^M(I_0 + \rho, \theta, \varphi).$$

Let $\varepsilon^{-1} = A^n (A \to +\infty)$. Then, by (4.16)–(4.17),

$$\mu \sim \varepsilon^{-1}, \quad \Omega \sim \varepsilon^{-1}, \tag{4.19}$$

$$\sup_{B(N_1)\times \mathbb{T}_N^{1+d}} \|R^{M+1}\| \le C\varepsilon^{c_0 - 2}, \quad \sup_{B(N_1)} |h(\rho)| \le \varepsilon^{-1}\rho^3. \tag{4.20}$$

Recall

$$N_1 = A^{-10n} = \varepsilon^{10} \quad \text{and} \quad c_0 \gg \max\{10, C\}, \tag{4.21}$$

where C is taken over all the previous universal constant.

Theorem 4.1 (KAM theorem) *There exists a symplectic coordinate* Ψ :

$$\Psi : B\Big(\frac{1}{2}N_1\Big) \times \mathbb{T}^{1+d}_{\frac{1}{2}N} \to B(N_1) \times \mathbb{T}^{1+d}_{N},$$

such that for $(\rho, \theta, \varphi) \in B(\frac{1}{2}N_1) \times \mathbb{T}^{1+d}_{\frac{N}{2}}$,

$$\widetilde{H} := H^{M+1} \circ \Psi = \text{Const} + \omega \cdot J + \mu\rho + \frac{1}{2}\widetilde{\Omega}\rho^2 + \widetilde{h}(\rho) + R_\infty(\rho, \theta, \varphi),$$

where

$$\sup_{B(\frac{1}{2}N_1)\times \mathbb{T}^{1+d}_{\frac{N_1}{2}}} |R_\infty(\rho, \theta, \varphi)| \le C|\rho|^3, \qquad \sup_{B(\frac{1}{2}N_1)} |\widetilde{h}(\rho)| \le C|\rho|^3$$

and $|\widetilde{\Omega} - \Omega| \le C\varepsilon$. *In particular,* $\mathbb{T}^{1+d} \times \{\rho = 0\}$ *is an invariant torus of the Hamiltonian system defined by* $\widetilde{H}$.

Proof Note $\varphi = \omega t$ and $h(\rho)$, $R(\rho, \theta, \varphi)$ do not depend on J. While doing KAM iteration, there is no frequency drift from φ. Moreover, the frequency drifts from θ can be counterbalanced by $\Omega \sim \varepsilon^{-1}$.

The remaining proof is standard. See [2, 14], for example. Let Φ =composition of all the previous symplectic coordinate changes. Then $\Phi(\mathbb{T}^{1+d} \times \{\rho = 0\})$ is an invariant torus of (2.10) with (2.11). Observe that (2.6) can be rewritten as

$$\dot{\varphi} = \omega, \quad \dot{\theta} = \frac{\partial H(I, \theta, \varphi)}{\partial I}, \quad \dot{I} = -\frac{\partial H(I, \theta, \varphi)}{\partial \theta}, \tag{4.22}$$

which is a sub-system of (2.10). Thus, $\Phi(\mathbb{T}^{1+d} \times \{\rho = 0\})$ is a KAM torus of dimension $1 + d$ for (4.22) in the phase space $\mathbb{T}^{d+1} \times \mathbb{R}$. One see that the KAM torus is of co-dimension 1. Note $\Phi(\mathbb{T}^{1+d} \times \{\rho = 0\})$ clusters at infinity when $A \to +\infty$. It follows that all solutions (1.1) are bounded.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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