Chin. Ann. Math. Ser. B
46(3), 2025, 407–414
DOI: 10.1007/s11401-025-0022-3

Chinese Annals of Mathematics, Series B

A Characterization of Finite Blaschke Products with Degree n

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Abstract In this paper, the authors give a characterization of finite Blaschke products with degree n. The main results are: (1) An n-dimensional complex vector can be the first n Taylor coefficients of a finite Blaschke product with degree no more than $n-1$ if and only if the vector induces a lower triangular Toeplitz matrix with norm 1; (2) an n-dimensional complex vector can be the first n Taylor coefficients of an inner function if and only if the vector induces a lower triangular Toeplitz matrix with norm no more than 1. Möbius transformations acting on contraction matrices play an important role in the proofs.

Keywords Finite Blaschke products, Toeplitz matrices, Contractions, Inner functions, Taylor coefficients
2020 MR Subject Classification 30J10, 30J05, 47B35, 47A60, 46E50

1 Introduction and Preliminaries

Let $\mathbb{D}$ be the unit open disc. In the study of analytic functions on $\mathbb{D}$, it plays an important role of inner functions (we refer to the books [6, 9]), which are analytic functions on $\mathbb{D}$ with unimodular radial limits almost everywhere on the boundary of $\mathbb{D}$. Moreover, a Blaschke product is an inner function of the form

$$B(z) = \lambda z^m \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z},$$

where m is a nonnegative integer, λ is a complex number with $|\lambda| = 1$, and $\{z_n\}$ is a finite or infinite sequence of points in $\mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum\limits_n (1 - |z_n|) < \infty$. In particular, if $\lambda = 1$, we say that B is normalized. The number of zeros of the Blaschke product is called its degree. For convenience, we say that a constant inner function $f(z) \equiv \mathrm{e}^{\mathrm{i}\theta}$ is a Blaschke product with degree 0. Denote by $\mathfrak{B}$ the collection of all Blaschke products, and denote by $\mathfrak{B}^{\mathrm{fin}}$ the collection of all finite Blaschke products.

Denoted by $\mathfrak{I}$ the set of all inner functions on $\mathbb{D}$. For each $\varphi \in \mathfrak{I}$, we could write

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Manuscript received March 3, 2023. Revised April 27, 2023.
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Then the Taylor coefficients $\{c_n\}$ determine the function φ. The inner function φ also induces a Toeplitz operator T_φ on the classical Hardy space H^2. Under the orthonormal base $\{z^n\}_{n=0}^{\infty}$, the Toeplitz operator T_φ could be written as following lower triangular infinite dimensional matrix

$$\begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 & \cdots \\ c_1 & c_0 & 0 & \cdots & 0 & \cdots \\ c_2 & c_1 & c_0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

As well known, Beurling's celebrated theorem (see [2]) states that every nonzero invariant subspace $\mathcal{M}$ of the multiplication operator $f(z) \to zf(z)$ on H^2 is of the form φH^2 for some inner function φ. Moreover, one can see that the orthogonal projection $p_{\mathcal{M}}$ is just $T_\varphi T_\varphi^*$. Notice that $T_\varphi^* T_\varphi$ is the identity. Then for any $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$, $\varphi \in \mathfrak{I}$ if and only if the following two conditions hold:

(1) $\sum_{n=0}^{\infty} |c_n|^2 = 1$,

(2) $\sum_{n=0}^{\infty} c_n \overline{c_{n+k}} = 0$ for all $k = 1, 2, \cdots$.

This provides a characterization of inner functions by the corresponding Toeplitz operators or Taylor coefficients. Furthermore, a natural question is how to characterize Blaschke products by the corresponding Toeplitz operators or Taylor coefficients.

Newman and Shapiro gave some descriptions of the Taylor coefficients of inner functions in [15]. Moreover, Ahern and Kim [1], Verbitskii [18], Dallakyan and Hovhannisyan [4] considered the Taylor coefficients of Blaschke products, respectively. This topic is also related to the reducing subspace of corresponding Toeplitz operator. Zhu [19] showed that for each Blaschke product B of degree 2, T_B has precisely two different minimal reducing subspaces. Furthermore, a conjecture is that for a finite Blaschke product of degree n, T_B has at most n minimal reducing subspaces. It has been known that the conjecture is true for $n = 3, 4$ and in these cases there is a characterization of the minimal reducing subspaces (see [13, 17] for instance). Following from [5, 11], one can see that the conjecture is also true for $n = 5, 6, 7, 8$. In [12], Guo and Huang showed that the Toeplitz operators of thin Blaschke products are irreducible under some mild conditions, and constructed an example of such products. They also provided a geometric characterization of those thin Blaschke products for which the corresponding multiplication operator has a nontrivial reducing subspace.

In this paper, we will use finite truncations of the corresponding Toeplitz operators to give a characterization of finite Blaschke products with degree n. Note that an inner function could be seemed as a power series, a vector and a Toeplitz operator. Let us introduce some notations to represent the finite truncations of inner functions.

Let $\mathfrak{P}$ be the set of all formal power series with complex coefficients. For every $n \in \mathbb{N}$, define $P_n : \mathfrak{P} \to \mathbb{C}^n$ by

$$P_n(f) = (c_0, c_1, \cdots, c_{n-1}) \quad \text{for every } f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathfrak{P}.$$

Denote by $\mathfrak{T}_n$ the set of all n-dimensional lower triangular Toeplitz matrices. Moreover, define $Q_n : \mathfrak{P} \to \mathfrak{T}_n$ by

$$Q_n(f) = Q_n\Big(\sum_{n=0}^{\infty} c_n z^n\Big) = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix},$$

which is the n-truncations of the Toeplitz operator T_f. Obviously, there is a natural bijection $h : \mathbb{C}^n \to \mathfrak{T}_n$ defined by

$$h(c_0, c_1, c_2, \cdots, c_{n-1}) = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix},$$

and we have

$$Q_n(f) = h(P_n(f)).$$

2 Finite Blaschke Products and Toeplitz Matrices

In this section, we will give a characterization of finite Blaschke products with degree n by the norms of the n-truncations of the corresponding Toeplitz operators. Denote by $\mathfrak{T}_n^{\mathrm{contr}}$ the set of all lower triangular Toeplitz contraction matrices, i.e.,

$$\mathfrak{T}_n^{\mathrm{contr}} := \{T \in \mathfrak{T}_n; \|T\| \le 1\}.$$

For convenience, given any $\Phi_n = (\varphi_0, \varphi_1, \cdots, \varphi_{n-1}) \in \mathbb{C}^n$, we always denote

$$\Phi_m = (\varphi_0, \varphi_1, \cdots, \varphi_{m-1}) \in \mathbb{C}^m \quad \text{for any } 1 \le m \le n.$$

We also use T_{Φ_n} to denote the lower triangular Toeplitz matrix corresponding to Φ_n, i.e., $T_{\Phi_n} = h(\Phi_n)$.

Firstly, one can see that each element in $Q_n(\mathfrak{I})$ is a contraction from [3, 16].

Theorem 2.1 *Let* $\varphi(z) = \sum\limits_{n=0}^{\infty} \varphi_n z^n$ *be an analytic function on the unit disk with* $\sum\limits_{n=0}^{\infty} |\varphi_n|^2 =$

1. *Then $\varphi(z)$ is an inner function if and only if the corresponding Toeplitz operator*

$$T_\varphi = \begin{bmatrix} \varphi_0 & 0 & 0 & \cdots & 0 & \cdots \\ \varphi_1 & \varphi_0 & 0 & \cdots & 0 & \cdots \\ \varphi_2 & \varphi_1 & \varphi_0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \varphi_{n-3} & \cdots & \varphi_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has norm 1 on ℓ^2. Furthermore, $Q_n(\mathfrak{I}) \subseteq \mathfrak{T}_n^{\text{contr}}$.

Next, we will obtain $Q_n(\mathfrak{I}) = \mathfrak{T}_n^{\text{contr}}$ and some more results. We need to make some preparations with regard to Möbius transformations acting on the spaces of inner functions and contraction matrices, respectively.

For any $\alpha \in \mathbb{D}$, the Frostman shift (see [8]) is a map $F_\alpha : \mathfrak{I} \to \mathfrak{I}$ defined by

$$F_\alpha(\varphi)(z) = \frac{\alpha - \varphi(z)}{1 - \overline{\alpha}\varphi(z)} \quad \text{for any } \varphi \in \mathfrak{I}.$$

Similarly, we can define an analogue of the Frostman shift on $\mathfrak{T}_n^{\text{contr}}$,

$$\widetilde{F}_\alpha(T) = \frac{\alpha - T}{1 - \overline{\alpha}T} \quad \text{for any } T \in \mathfrak{T}_n^{\text{contr}}.$$

The map $\widetilde{F}_\alpha : \mathfrak{T}_n^{\text{contr}} \to \mathfrak{T}_n^{\text{contr}}$ being well defined is based on the study of the norms of analytic functions of a contraction. The research of this aspect dates back to the von Neumann inequality in 1951. A certain analytic function of a contraction is again a contraction (see [14]), and it was further refined by Fan to an analogous assertion for strict contractions (see [7]). In particular, if A is a contraction and $B(z)$ is a finite Blaschke product of degree n, then $B(A)$ is also a contraction. Moreover, to consider when the norm of $B(A)$ is equal to 1, Gau and Wu obtained a result in [10] as follows.

Theorem 2.2 (see [10]) *Let A be a contraction on Hilbert space H and let $B(z)$ be a Blaschke product with k zeros counting multiplicity. Then*

(a) $\dim\ker(I - B(A)^*B(A)) = \dim\ker(I - (A^k)^*A^k)$, *and*

(b) $\|B(A)\| = 1$ *if and only if* $\|A^k\| = 1$.

In particular, by (b) in the above theorem, one can see when the norm of $\widetilde{F}_\alpha(T)$ is 1 for $T \in \mathfrak{T}_n^{\text{contr}}$.

Lemma 2.1 *Let $\alpha \in \mathbb{D}$ and $T \in \mathfrak{T}_n^{\text{contr}}$. Then $\|\widetilde{F}_\alpha(T)\| = 1$ if and only if $\|T\| = 1$.*

Notice that $Q_n : \mathfrak{P} \to \mathfrak{T}_n$ is a homomorphism between two rings, and Q_n maps $\mathfrak{I}$ into $\mathfrak{T}_n^{\text{contr}}$ by Theorem 2.1. Since

$$F_\alpha(\varphi)(z) = \alpha - (1 - |\alpha|^2)\sum_{k=1}^{\infty} \overline{\alpha}^{k-1}(\varphi(z))^k$$

and

$$\widetilde{F}_\alpha(T_{\Phi_n}) = \alpha I - (1-|\alpha|^2)\sum_{k=1}^{\infty}\overline{\alpha}^{k-1}T_{\Phi_n}^k,$$

we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{I} & \xrightarrow{F_\alpha} & \mathfrak{I} \\ \downarrow Q_n & & \downarrow Q_n \\ \mathfrak{T}_n^{\mathrm{contr}} & \xrightarrow{\widetilde{F}_\alpha} & \mathfrak{T}_n^{\mathrm{contr}}. \end{array}$$

In addition, F_α is a bijection since $F_\alpha^{-1} = F_\alpha$, and $\widetilde{F}_\alpha$ is a bijection since $\widetilde{F}_\alpha^{-1} = \widetilde{F}_\alpha$.

Denote

$$\mathfrak{T}_n^{\alpha,\mathrm{contr}} = \{T_{\Phi_n}; \|T_{\Phi_n}\| \le 1, \text{where } \Phi_n = (\alpha,\varphi_1,\cdots,\varphi_{n-1}) \in \mathbb{C}^n\}$$

and

$$\mathfrak{T}_n^{\alpha,u} = \{T_{\Phi_n}; \|T_{\Phi_n}\| = 1, \text{where } \Phi_n = (\alpha,\varphi_1,\cdots,\varphi_{n-1}) \in \mathbb{C}^n\}.$$

Then $\widetilde{F}_\alpha$ is a bijection from $\mathfrak{T}_n^{\alpha,\mathrm{contr}}$ to $\mathfrak{T}_n^{0,\mathrm{contr}}$. Furthermore, by Lemma 2.1, $\widetilde{F}_\alpha$ is also a bijection from $\mathfrak{T}_n^{\alpha,u}$ to $\mathfrak{T}_n^{0,u}$.

Now, let us consider finite Blaschke products.

Theorem 2.3 *Let $B(z)$ be a finite Blaschke product with degree $m-1$. Then for any $n \ge m$,*

$$\|Q_n(B)\| = 1.$$

Proof Let $B(z)$ be a finite Blaschke product with degree $m-1$ and

$$P_m(B) = \Phi_m = (\varphi_0,\varphi_1,\cdots,\varphi_{m-1}).$$

Obviously, when $m=1$, $B(z)$ is just the constant inner function and consequently,

$$\|T_{\Phi_n}\| = 1 \quad \text{for any } n \ge 1.$$

Suppose that the above conclusion holds when the degree is less than m. Now let $\widetilde{B}(z)$ be a finite Blaschke product with degree m and

$$P_{m+1}(\widetilde{B}) = \Phi_{m+1} = (\varphi_0,\varphi_1,\cdots,\varphi_m).$$

Then by the definition of Frostman shift, $F_{\widetilde{B}(0)}(\widetilde{B})$ is also a finite Blaschke product with degree m and it has a factor z, i.e.,

$$F_{\widetilde{B}(0)}(\widetilde{B})(z) = z \cdot A(z),$$

where $A(z)$ is a finite Blaschke product with degree $m-1$. It follows from the assumption in mathematical induction with respect to the degree of the Blaschke product that

$$\|Q_{m+1}(F_{\widetilde{B}(0)}(\widetilde{B}))\| = \|Q_m(A)\| = 1.$$

Then, by Lemma 2.1,

$$\|Q_{m+1}(\widetilde{B})\| = 1.$$

Furthermore, by Theorem 2.1,

$$\|Q_n(\widetilde{B})\| = 1 \quad \text{for any } n \geq m+1.$$

This finishes the proof.

Theorem 2.4 *For any $n \in \mathbb{N}$, let*

$$\Phi_n = (\varphi_0, \varphi_1, \cdots, \varphi_{n-1}).$$

If $\|T_{\Phi_n}\| = 1$, then there exists a unique inner function $B(z)$ such that $P_n(B) = \Phi_n$. More precisely, if $m \leq n$ is the first positive integer such that $\|T_{\Phi_m}\| = 1$, then the unique inner function $B(z)$ is a finite Blaschke product whose degree is $m-1$.

Proof Without loss of generality, it suffices to consider $\Phi_n = (\varphi_0, \varphi_1, \cdots, \varphi_{n-1})$ with

$$\|T_{\Phi_n}\| = 1 \quad \text{and} \quad \|T_{\Phi_k}\| < 1 \quad \text{for } 1 \leq k \leq n-1.$$

Obviously, when $n = 1$, there exists a unique finite Blaschke product $B(z) \equiv \varphi_0$ whose degree is 0 such that $P_n(B) = \Phi_n$.

Suppose that the above conclusion holds for all $k \leq n-1$. Denote

$$\widetilde{F}_{\varphi_0}(T_{\Phi_n}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \psi_0 & 0 & 0 & 0 & \cdots & 0 \\ \psi_1 & \psi_0 & 0 & 0 & \cdots & 0 \\ \psi_2 & \psi_1 & \psi_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n-2} & \psi_{n-3} & \psi_{n-4} & \psi_{n-5} & \cdots & 0 \end{bmatrix}$$

and

$$\Psi_{n-1} = (\psi_0, \psi_1, \cdots, \psi_{n-2}).$$

By Lemma 2.1,

$$\|T_{\Psi_{n-1}}\| = \|\widetilde{F}_{\varphi_0}(T_{\Phi_n})\| = 1.$$

Consequently, there exists a unique finite Blaschke product $\widehat{B}(z)$ whose degree is no more than $n-2$ such that

$$P_{n-1}(\widehat{B}) = \Psi_{n-1}.$$

Furthermore, the finite Blaschke product $z\widehat{B}(z)$, whose degree is no more than $n-1$, satisfies

$$Q_n(z\widehat{B}) = \widetilde{F}_{\varphi_0}(T_{\Phi_n}).$$

Therefore, $F_{\varphi_0}^{-1}(z\widehat{B}(z))$ is the unique finite Blaschke product with degree no more than $n-1$ such that

$$P_n(B) = \Phi_n.$$

Moreover, by Theorem 2.3 and Lemma 2.1,

$$\|T_{\Phi_k}\| < 1 \quad \text{for } 1 \le k \le n-1$$

implies that the degree of the unique finite Blaschke product $F_{\varphi_0}^{-1}(z\widehat{B}(z))$ is just $n-1$. This finishes the proof.

Remark 2.1 Together with Theorems 2.3–2.4, we establish the relationship between a finite Blaschke product with degree $n-1$ and the n-truncations of the corresponding Toeplitz operators (or its first n Taylor coefficients). In particular, it is computable to determine whether an inner function is a finite Blaschke product with degree $n-1$ by its Taylor coefficients.

By Theorems 2.1 and 2.3, one can see that an n-dimensional complex vector is the first n Taylor coefficients of an inner function, if and only if it induces a lower triangular Toeplitz matrix with norm no more than 1. More precisely, by Theorem 2.4, we obtain the following result.

Theorem 2.5 *For any $n \in \mathbb{N}$,*

$$P_n(\mathfrak{I}) = h^{-1}(\mathfrak{T}_n^{\text{contr}}) = P_n(\mathfrak{B}) = P_n(\mathfrak{B}^{\text{fin}}).$$

Proof It is obvious that $P_n(\mathfrak{B}^{\text{fin}}) \subseteq P_n(\mathfrak{B}) \subseteq P_n(\mathfrak{I})$. By Theorem 2.1, one can see that $P_n(\mathfrak{I}) \subseteq h^{-1}(\mathfrak{T}_n^{\text{contr}})$. For each element $\Phi_n = (\varphi_0, \varphi_1, \cdots, \varphi_{n-1}) \in h^{-1}(\mathfrak{T}_n^{\text{contr}})$, there is a complex number $\varphi_n \in \mathbb{C}$ such that $\|T_{\Phi_{n+1}}\| = 1$, where $\Phi_{n+1} = (\varphi_0, \varphi_1, \cdots, \varphi_{n-1}, \varphi_n)$. Then, by Theorem 2.4, there exists a finite Blaschke product $B(z)$ such that $P_{n+1}(B) = \Phi_{n+1}$. Consequently, we have $P_n(B) = \Phi_n$, which means $h^{-1}(\mathfrak{T}_n^{\text{contr}}) \subseteq P_n(\mathfrak{B}^{\text{fin}})$. This finishes the proof.

Acknowledgement We thank the referees for their valuable comments and suggestions.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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