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Exact Controllability for a Refined Stochastic Plate Equation*

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Abstract The classical stochastic plate equation suffers from a lack of exact controllability, even with controls that are effective in both the drift and diffusion terms and on the boundary. To address this issue, a one-dimensional refined stochastic plate equation was previously proposed and established as exactly controllable in [Yu, Y. and Zhang, J. -F., Carleman estimates of refined stochastic beam equations and applications, *SIAM J. Control Optim.*, **60**, 2022, 2947–2970]. In this paper, the authors establish the exact controllability of the multidimensional refined stochastic plate equation with two interior controls and two boundary controls by a new global Carleman estimate. Furthermore, they show that at least one boundary control and the action of two interior controls are necessary for exact controllability.

Keywords Stochastic plate equation, Exact controllability, Observability estimate, Carleman estimate
2020 MR Subject Classification 93B05, 93B07

1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Write $\mathbb{F}$ for the progressive σ-field with respect to $\mathbf{F}$. Let H be a Banach space. Denote by $L^2_{\mathcal{F}_t}(\Omega; H)$ the space of all $\mathcal{F}_t$-measurable random variables ξ such that $\mathbb{E}|\xi|^2_H < \infty$; by $L^2_{\mathbb{F}}(0, T; H)$ the space consisting of all H-valued $\mathbf{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$; by $L^\infty_{\mathbb{F}}(0, T; H)$ the space consisting of all H-valued $\mathbf{F}$-adapted bounded processes; and by $C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$ the space consisting of all H-valued $\mathbf{F}$-adapted processes $X(\cdot)$ such that $X(\cdot) : [0, T] \to L^2_{\mathcal{F}_\cdot}(\Omega; H)$ is continuous. All these spaces are Banach spaces with the canonical norms (see [25, Section 2.6]).

Let $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with a C^4 boundary Γ. Set $Q = (0, T) \times G$ and $\Sigma = (0, T) \times \Gamma$. Denote by $\nu(x) = (\nu^1(x), \cdots, \nu^n(x))$ the unit outward normal vector of Γ at point x.

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Consider the following refined stochastic plate equation

$$\begin{cases} \mathrm{d}y = \widehat{y}\mathrm{d}t + (a_3 y + f)\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\widehat{y} + \Delta^2 y \mathrm{d}t = (a_1 y + a_2 \cdot \nabla y + a_5 g)\mathrm{d}t + (a_4 y + g)\mathrm{d}W(t) & \text{in } Q, \\ y = h_1, \quad \dfrac{\partial y}{\partial \nu} = h_2 & \text{on } \Sigma, \\ (y(0), \widehat{y}(0)) = (y_0, \widehat{y}_0) & \text{in } G. \end{cases} \tag{1.1}$$

Here, $(y_0, \widehat{y}_0) \in H^{-1}(G) \times (H^3(G) \cap H_0^2(G))^*$ (where $(H^3(G) \cap H_0^2(G))^*$ is the dual space of $H^3(G) \cap H_0^2(G)$ with respect to the pivot space $L^2(G)$), the coefficients

$$a_1, a_3, a_4 \in L_{\mathbb{F}}^{\infty}(0,T; W^{1,\infty}(G)), \quad a_2 \in L_{\mathbb{F}}^{\infty}(0,T; W^{2,\infty}(G;\mathbb{R}^n)), \quad a_5 \in L_{\mathbb{F}}^{\infty}(0,T; W^{3,\infty}(G))$$

and the controls

$$\begin{aligned}(f, g, h_1, h_2) \in\ & L_{\mathbb{F}}^2(0,T; H^{-1}(G)) \times L_{\mathbb{F}}^2(0,T; (H^3(G) \cap H_0^2(G))^*) \times L_{\mathbb{F}}^2(0,T; L^2(\Gamma)) \\ & \times L_{\mathbb{F}}^2(0,T; H^{-1}(\Gamma)).\end{aligned}$$

Remark 1.1 The influence of the control g on the drift term is reflected by the term $a_5 g$. Specifically, if a control g is introduced in the diffusion term, the term $a_5 g$ will appear as a side effect. This leads to technical difficulties in the study of the exact controllability of (1.1). It is worth noting that based on the same reasoning, an additional term of $a_6 f$ could be included in the drift term of the first equation of (1.1). However, we do not know how to handle this term now.

The control system (1.1) is a nonhomegeneous boundary value problem. Its solution is understood in the sense of transposition. For the readers' convenience, we recall it briefly below. A systematic introduction to that can be found in [25, Section 7.2].

First, we introduce the following reference equation

$$\begin{cases} \mathrm{d}z = \widehat{z}\mathrm{d}t + (Z - a_5 z)\mathrm{d}W(t) & \text{in } Q_\tau, \\ \mathrm{d}\widehat{z} + \Delta^2 z \mathrm{d}t = [(a_1 - \operatorname{div} a_2 - a_4 a_5) z - a_2 \cdot \nabla z - a_3 \widehat{Z} + a_4 Z]\mathrm{d}t + \widehat{Z}\mathrm{d}W(t) & \text{in } Q_\tau, \\ z = \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma_\tau, \\ (z(\tau), \widehat{z}(\tau)) = (z^\tau, \widehat{z}^\tau) & \text{in } G, \end{cases} \tag{1.2}$$

where $\tau \in (0,T]$, $Q_\tau \overset{\Delta}{=} (0,\tau) \times G$, $\Sigma_\tau \overset{\Delta}{=} (0,\tau) \times \Gamma$ and $(z^\tau, \widehat{z}^\tau) \in L_{\mathcal{F}_\tau}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G))$. By the classical well-posedness result for backward stochastic evolution equations (see [25, Section 4.2]), we know that (1.2) admits a unique weak solution

$$\begin{aligned}(z, Z, \widehat{z}, \widehat{Z}) \in\ & L_{\mathbb{F}}^2(\Omega; C([0,\tau]; (H^3(G) \cap H_0^2(G)))) \times L_{\mathbb{F}}^2(0,\tau; (H^3(G) \cap H_0^2(G))) \\ & \times L_{\mathbb{F}}^2(\Omega; C([0,\tau]; H_0^1(G))) \times L_{\mathbb{F}}^2(0,\tau; H_0^1(G)).\end{aligned}$$

Furthermore, for $0 \le s, t \le \tau$, it holds that

$$|(z(t), \widehat{z}(t))|_{L_{\mathcal{F}_t}^2(\Omega; H^3(G) \cap H_0^2(G)) \times L_{\mathcal{F}_t}^2(\Omega; H_0^1(G))}$$

$$
\begin{aligned}
&\leq C\big(|(z(s),\widehat{z}(s))|_{L^2_{\mathcal{F}_s}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_s}(\Omega;H_0^1(G))}\\
&\quad+|(Z,\widehat{Z})|_{L^2_{\mathbb{F}}(0,\tau;H^3(G)\cap H_0^2(G))\times L^2_{\mathbb{F}}(0,\tau;H_0^1(G))}\big).
\end{aligned}
\tag{1.3}
$$

Here and in what follows, we denote by C a generic positive constant depending on G, T, τ and a_i, $i=1,\cdots,5$, whose value may vary from line to line.

Next, we give the following hidden regularity for solutions to (1.2).

Proposition 1.1 *Let $(z^\tau,\widehat{z}^\tau)\in L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))$. Then the solution $(z,Z,\widehat{z},\widehat{Z})$ of* (1.2) *satisfies $|\nabla\Delta z||_\Gamma\in L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))$. Furthermore,*

$$
|\nabla\Delta z|_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))}\leq C|(z^\tau,\widehat{z}^\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))}.
$$

Proof of Proposition 1.1 is put in Section 2.

Now we are in a position to give the definition of the transposition solution to (1.1).

Definition 1.1 *A pair of stochastic processes $(y,\widehat{y})\in C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))\times C_{\mathbb{F}}([0,T];L^2(\Omega;(H^3(G)\cap H_0^2(G))^*))$ is a transposition solution to* (1.1) *if for any $\tau\in(0,T]$ and $(z^\tau,\widehat{z}^\tau)\in L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))$, we have*

$$
\begin{aligned}
&\mathbb{E}\langle\widehat{y}(\tau),z^\tau\rangle_{(H^3(G)\cap H_0^2(G))^*,H^3(G)\cap H_0^2(G)}-\langle\widehat{y}_0,z(0)\rangle_{(H^3(G)\cap H_0^2(G))^*,H^3(G)\cap H_0^2(G)}\\
&\quad-\mathbb{E}\langle y(\tau),\widehat{z}^\tau\rangle_{H^{-1}(G),H_0^1(G)}+\langle y_0,\widehat{z}(0)\rangle_{H^{-1}(G),H_0^1(G)}\\
&=-\mathbb{E}\int_0^\tau\langle f,\widehat{Z}\rangle_{H^{-1}(G),H_0^1(G)}\mathrm{d}t+\mathbb{E}\int_0^\tau\langle g,Z\rangle_{(H^3(G)\cap H_0^2(G))^*,H^3(G)\cap H_0^2(G)}\mathrm{d}t\\
&\quad+\mathbb{E}\int_0^\tau\int_\Gamma\frac{\partial\Delta z}{\partial\nu}h_1\mathrm{d}\Gamma\mathrm{d}t-\mathbb{E}\int_0^\tau\langle h_2,\Delta z\rangle_{H^{-1}(\Gamma),H^1(\Gamma)}\mathrm{d}t.
\end{aligned}
$$

Here, $(z,Z,\widehat{z},\widehat{Z})$ solves (1.2).

Combining Proposition 1.1 and the well-posedness for stochastic evolution equation with an unbounded control operator in the sense of transposition solution (see [25, Theorem 7.12]), we immediately get the following well-posedness result for (1.1).

Proposition 1.2 *For each $(y_0,\widehat{y}_0)\in H^{-1}(G)\times(H^3(G)\cap H_0^2(G))^*$, the system* (1.1) *admits a unique transposition solution $(y,\widehat{y})$. Moreover,*

$$
\begin{aligned}
&|(y,\widehat{y})|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))\times C_{\mathbb{F}}([0,T];L^2(\Omega;(H^3(G)\cap H_0^2(G))^*))}\\
&\leq C\big(|y_0|_{H^{-1}(G)}+|\widehat{y}_0|_{(H^3(G)\cap H_0^2(G))^*}+|f|_{L^2_{\mathbb{F}}(0,T;H^{-1}(G))}+|g|_{L^2_{\mathbb{F}}(0,T;(H^3(G)\cap H_0^2(G))^*)}\\
&\quad+|h_1|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma))}+|h_2|_{L^2_{\mathbb{F}}(0,T;H^{-1}(\Gamma))}\big).
\end{aligned}
$$

Now we give the definition of the exact controllability for (1.1).

Definition 1.2 *The system* (1.1) *is called exactly controllable at time T if for any $(y_0,\widehat{y}_0)\in H^{-1}(G)\times(H^3(G)\cap H_0^2(G))^*$ and $(y_1,\widehat{y}_1)\in L^2_{\mathcal{F}_T}(\Omega;H^{-1}(G))\times L^2_{\mathcal{F}_T}(\Omega;(H^3(G)\cap H_0^2(G))^*)$, there exist controls*

$$
\begin{aligned}
(f,g,h_1,h_2)\in{}&L^2_{\mathbb{F}}(0,T;H^{-1}(G))\times L^2_{\mathbb{F}}(0,T;(H^3(G)\cap H_0^2(G))^*)\times L^2_{\mathbb{F}}(0,T;L^2(\Gamma))\\
&\times L^2_{\mathbb{F}}(0,T;H^{-1}(\Gamma))
\end{aligned}
$$

such that the solution $(y,\widehat{y})$ to (1.1) *satisfies that $(y(T,\cdot),\widehat{y}(T,\cdot))=(y_1,\widehat{y}_1)$, $\mathbb{P}$-a.s.*

Remark 1.2 In the definition of exact controllability for (1.1), we set the state space to be $L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G)) \times L^2_{\mathcal{F}_T}(\Omega; (H^3(G) \cap H^2_0(G))^*)$. However, it would be more natural to choose the state space as $L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-2}(G))$. Furthermore, the controls $g \in L^2_{\mathbb{F}}(0,T;(H^3(G) \cap H^2_0(G))^*)$ and $h_2 \in L^2_{\mathbb{F}}(0,T;H^{-1}(\Gamma))$ are highly irregular. It is expected that more regular controls will be needed to achieve the desired goal, although the method to do so is currently unknown. Notably, even for the deterministic plate equation, the existing results, such as [15] only prove the exact controllability in the space $H^{-1}(G) \times (H^3(G) \cap H^2_0(G))^*$ with controls in the Dirichlet and Neumann boundary conditions.

The main result of this paper is the following theorem.

Theorem 1.1 *The system* (1.1) *is exactly controllable at any time* $T > 0$.

Remark 1.3 Similar to the derivation process in [18, 24, 29], the refined stochastic plate equation (1.1) can be obtained from the classical stochastic plate equation

$$\begin{cases} \mathrm{d}y_t + \Delta^2 y \mathrm{d}t = (a_1 y + a_2 \cdot \nabla y + f)\mathrm{d}t + (a_4 y + g)\mathrm{d}W(t) & \text{in } Q, \\ y = h_1, \quad \dfrac{\partial y}{\partial \nu} = h_2 & \text{on } \Sigma, \\ (y(0), y_t(0)) = (y_0, y_1) & \text{in } G. \end{cases} \tag{1.4}$$

Here, (y_0, y_1) are the initial data, and f, g, h_1, h_2 are controls.

The system (1.4) is widely used in structural engineering for modeling beams, bridges and other structures, as demonstrated in various studies such as [2–4, 12]. For instance, (1.4) can be utilized to describe fluttering or large-amplitude vibration of an elastic panel induced by aerodynamic forces that are perturbed by random fluctuations, as explored in [3]. However, as proven in [29, Theorem 4.1] and [24, Theorem 2.1], the system (1.4) cannot be exactly controllable for any $T > 0$. Therefore, motivated by the negative controllability result and following a similar derivation process as in [24], we investigate a refined stochastic plate equation, namely (1.1).

We put four controls in the system (1.1). Similar to [24, Theorem 2.3], one can find that boundary controls h_1 and h_2 in (1.1) cannot be dropped simultaneously, and internal controls f and g must be acted on the whole domain G. More precisely, we have the following result.

Theorem 1.2 *The system* (1.1) *is not exactly controllable at any time* $T > 0$ *provided that one of the following three conditions is satisfied*:

(1) $a_3 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega))$, $G \backslash \overline{G_0} \neq \emptyset$ *and* $\operatorname{supp} f \subset G_0$;

(2) $a_4 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega))$, $G \backslash \overline{G_0} \neq \emptyset$ *and* $\operatorname{supp} g \subset G_0$;

(3) $h_1 = h_2 = 0$.

Remark 1.4 It is worth investigating whether one of the boundary controls can be removed for the system (1.1), as has been done for the deterministic plate equation in [15, 19]. However, we currently do not have a method for achieving this for the stochastic case.

Remark 1.5 By letting $G = (0, 1)$, we can deduce from Theorem 3.2 that the system (1.1) is exactly controllable with controls in any nonempty subset of the boundary Γ. In fact, thanks to Remark 1.7, for any $(y_0, \widehat{y}_0)$ and $(y_1, \widehat{y}_1)$ satisfying Definition 1.2, one can find

$(f, g, h_1, h_2) \in L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; (H^3(G) \cap H^2_0(G))^*) \times (L^2_{\mathbb{F}}(0, T))^2$ such that the solution $(y, \widehat{y})$ to (1.1), where the boundary conditions are

$$y(\cdot, 0) = h_1, \quad y_x(\cdot, 0) = h_2, \quad y(\cdot, 1) = 0, \quad y_x(\cdot, 1) = 0 \quad \text{on } (0, T)$$

satisfies that $(y(T, \cdot), \widehat{y}(T, \cdot)) = (y_1, \widehat{y}_1)$, $\mathbb{P}$-a.s.. In the multidimensional case, it is worth to studying whether (1.1) is still exactly controllable under the assumption that $(h_1, h_2) \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$, where Γ_0 is a nonempty subset of Γ.

By a standard duality argument, Theorem 1.1 is equivalent to the following observability estimate (see [25, Theorem 7.17]).

Theorem 1.3 *There exists a constant $C > 0$ such that for every $(z^T, \widehat{z}^T) \in L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))$, it holds that*

$$\begin{aligned}
&|(z^T, \widehat{z}^T)|^2_{L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))} \\
&\leq C\mathbb{E} \int_\Sigma (|\nabla \Delta z|^2 + |\Delta z|^2) \mathrm{d}\Gamma \mathrm{d}t + C|(Z, \widehat{Z})|^2_{L^2_{\mathbb{F}}(0,T; H^3(G) \cap H^2_0(G)) \times L^2_{\mathbb{F}}(0,T; H^1_0(G))},
\end{aligned}$$

where $(z, Z, \widehat{z}, \widehat{Z})$ is the solution to (1.2) with $\tau = T$, $z(T) = z^T$ and $\widehat{z}(T) = \widehat{z}^T$.

Remark 1.6 In order to estimate $|\nabla \Delta z|$ on the boundary of the dual system (1.2), we may need better regularity than $H^2_0(G) \times L^2(G)$ for the initial data, as suggested by the sharp trace estimate established for the deterministic plate equation in [17]. Therefore, we choose the final data of (1.2) in $L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))$.

Remark 1.7 Let $G = (0, 1)$. By choosing $x_0 > 1$ in (3.4), from the proof of Theorem 3.2, we can deduce that

$$\begin{aligned}
&\mathbb{E} \int_Q \theta^2 (s\lambda\xi |\widehat{v}_x|^2 + s^3 \lambda^{\frac{7}{2}} \xi^3 |\widehat{v}|^2 + \lambda |v_{xxx}|^2 + s^3 \lambda^4 \xi^3 |v_{xx}|^2 + s^4 \lambda^6 \xi^4 |v_x|^2 \\
&\quad + s^6 \lambda^8 \xi^6 |v|^2) \mathrm{d}x \mathrm{d}t \\
&\leq C\mathbb{E} \int_Q \theta^2 (s^6 \lambda^6 \xi^6 f_1^2 + s^4 \lambda^4 \xi^4 |f_{1x}|^2 + s^2 \lambda^2 \xi^2 |f_{1xx}|^2 + f_2^2 + s^6 \lambda^6 \xi^6 g_1^2 \\
&\quad + s^4 \lambda^4 \xi^4 |g_{1x}|^2 + s^2 \lambda^2 \xi^2 |g_{1xx}|^2 + |g_{1xxx}|^2 + s^2 \lambda^2 \xi^2 |g_2|^2) \mathrm{d}x \mathrm{d}t \\
&\quad + C\mathbb{E} \int_0^T \theta^2 (s\lambda\xi |v_{xxx}(0)|^2 + s^3 \lambda^3 \xi^3 |v_{xx}(0)|^2) \mathrm{d}t.
\end{aligned}$$

In fact, on Σ, we have

$$\begin{aligned}
V_1 \cdot \nu &= -2s\lambda\xi \eta_x \nu w^2_{xxx} - 10 s^3 \lambda^3 \xi^3 \eta^3_x \nu w^2_{xx} \\
&\geq Cs\lambda\xi (w^2_{xxx}(1) - w^2_{xxx}(0)) + Cs^3 \lambda^3 \xi^3 (w^2_{xx}(1) - w^2_{xx}(0))
\end{aligned}$$

and

$$V_2 \cdot \nu \geq O(\mathrm{e}^{C\lambda})(w^2_{xxx}(1) + w^2_{xxx}(0)) + s^2 O(\mathrm{e}^{C\lambda})(w^2_{xx}(1) + w^2_{xx}(0)).$$

The remainder of the proof follows a similar approach to that of Theorem 1.3. Hence, we arrive the following observability inequality

$$|(z^T, \widehat{z}^T)|^2_{L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H^2_0(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))}$$

$$\leq C\mathbb{E}\int_0^T (|z_{xxx}(0)|^2 + |z_{xx}(0)|^2)\mathrm{d}t + C|(Z,\widehat{Z})|^2_{L^2_{\mathbb{F}}(0,T;H^3(G)\cap H^2_0(G))\times L^2_{\mathbb{F}}(0,T;H^1_0(G))}.$$

Remark 1.8 One can also consider the exact controllability of a refined stochastic plate equation when the boundary controls are as follows:

$$y = h_1 \quad \text{and} \quad \Delta y = h_2 \quad \text{on } \Sigma.$$

By the standard duality argument and the technique of transforming the controllability of the forward stochastic equation into the controllability of a backward equation (see [25, Section 7.5]), we need to prove the observability estimate for the following equation

$$\begin{cases} \mathrm{d}z = \widehat{z}\mathrm{d}t - a_5 z\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\widehat{z} + \Delta^2 z\mathrm{d}t = [(a_1 - \operatorname{div} a_2)z - a_2\cdot\nabla z - a_4 a_5 z]\mathrm{d}t & \text{in } Q, \\ z = \Delta z = 0 & \text{on } \Sigma, \\ (z(0),\widehat{z}(0)) = (z_0,\widehat{z}_0) & \text{in } G. \end{cases} \tag{1.5}$$

Following the idea in [30], we can rewrite (1.5) as two coupled stochastic Schrödinger equations, and the desired observability estimate can be obtained from the Carleman estimate for the latter. In fact, letting $u = \mathrm{i}\widehat{z} + \Delta z$, we have

$$\mathrm{i}\mathrm{d}z + \Delta z\mathrm{d}t = u\mathrm{d}t - \mathrm{i}a_5 z\mathrm{d}W(t)$$

and

$$-\mathrm{i}\mathrm{d}u + \Delta u\mathrm{d}t = [(a_1 - \operatorname{div} a_2)z - a_2\cdot\nabla z - a_4 a_5 z]\mathrm{d}t + \mathrm{i}\Delta(a_5 z)\mathrm{d}W(t).$$

Clearly, it holds that $z = u = 0$ on Σ. Thanks to [21, Theorem 1.2], for any $(z_0,\widehat{z}_0) \in \{\eta \in H^3(G) | \Delta\eta \in H^1_0(G)\} \times H^1_0(G)$, and under the assumption that $a_2 \equiv 0$, we can obtain the following inequality

$$|(\Delta z_0,\widehat{z}_0)|^2_{H^1_0(G)\times H^1_0(G)} \leq C\mathbb{E}\int_0^T\int_\Gamma \Big(\Big|\frac{\partial\Delta z}{\partial\nu}\Big|^2 + \Big|\frac{\partial z}{\partial\nu}\Big|^2 + \Big|\frac{\partial\widehat{z}}{\partial\nu}\Big|^2\Big)\mathrm{d}\Gamma\mathrm{d}t,$$

where $(z,\widehat{z})$ is the solution to (1.5). It should be noted that the above observability estimate has been proven for the deterministic plate equation in [16].

A plenty of works on exact controllability for deterministic plate equations exist in the literature, including works such as [1, 5, 9–11, 15, 18, 20, 27, 30] and the references therein. However, to the best of our knowledge, the work in [29] is the only published study that investigates the exact controllability of stochastic beam equations. In their study, the authors demonstrate that the equation given in (1.1) is exactly controllable when G is an interval. In this article, we employ a stochastic Carleman estimate to prove Theorem 1.3. This type of estimate is one of the most useful tools in studying the controllability of stochastic partial differential equations, as evident from previous works such as [6, 8, 21–23, 25, 28] and references therein. However, we note that [29] is the only published study to use this method for investigating the exact controllability of stochastic beam equations.

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries. Section 3 is devoted to establishing a Carleman estimate for the adjoint equation (1.2). By means of that Carleman estimate, we prove Theorem 1.3 in Section 4. At last, Section 5 is addressed to the proof of Theorem 1.2.

2 Some Preliminary Results

This section provides some preliminary results. In the rest of this paper, the notation $y_{x_i} \equiv y_{x_i}(x) = \frac{\partial y(x)}{\partial x_i}$ will be used for simplicity, where x_i is the i-th coordinate of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we use notations z_{x_j}, v_{x_j}, etc. for the partial derivatives of z and v with respect to x_j.

We first prove the hidden regularity for solutions to (1.2).

Proof of Proposition 1.1 For any $\rho \stackrel{\Delta}{=} (\rho^1, \cdots, \rho^n) \in C^2(\mathbb{R}^{n+1}; \mathbb{R}^n)$, by Itô's formula and (1.2), we have

$$
\begin{aligned}
&\sum_{j=1}^{n} (2\rho \cdot \nabla\Delta z \Delta z_{x_j} - \rho^j |\nabla\Delta z|^2)_{x_j} \mathrm{d}t \\
&= -\operatorname{div}\rho |\nabla\Delta z|^2 \mathrm{d}t + 2\rho \cdot \nabla\Delta z[(a_1 - \operatorname{div} a_2 - a_4 a_5) z - a_2 \cdot \nabla z + a_4 Z - a_3 \widehat{Z}]\mathrm{d}t \\
&\quad + 2\rho \cdot \nabla\Delta z \widehat{Z} \mathrm{d}W(t) + 2\nabla\Delta z D\rho \cdot \nabla\Delta z \mathrm{d}t - 2\mathrm{d}(\rho \cdot \nabla\Delta z \widehat{z}) + 2\rho_t \cdot \nabla\Delta z \widehat{z} \mathrm{d}t \\
&\quad + 2\rho \cdot \nabla\Delta(Z - a_5 z)\widehat{Z}\mathrm{d}t + 2\rho \cdot \nabla\Delta(Z - a_5 z)\widehat{z}\mathrm{d}W(t) + 2\operatorname{div}(\widehat{z}\rho\nabla^2\widehat{z})\mathrm{d}t \\
&\quad - 2\operatorname{div}(\widehat{z}\nabla\widehat{z} D\rho)\mathrm{d}t + 2\Delta\rho \cdot \nabla\widehat{z}\widehat{z}\mathrm{d}t + 2\nabla\widehat{z} D\rho \cdot \nabla\widehat{z}\mathrm{d}t - \operatorname{div}(\rho|\nabla\widehat{z}|^2)\mathrm{d}t + \operatorname{div}\rho|\nabla\widehat{z}|^2\mathrm{d}t. \qquad (2.1)
\end{aligned}
$$

Since $\Gamma \in C^3$, there exists a vector field $\zeta \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ such that $\zeta = \nu$ on Γ (see [13, Lemma 2.1]). Setting $\rho = \zeta$, integrating (2.1) in Q_τ, and taking expectation on Ω, we have

$$
\begin{aligned}
&\mathbb{E}\int_{\Sigma_\tau} \Big(2\Big|\frac{\partial\Delta z}{\partial\nu}\Big|^2 - |\nabla\Delta z|^2\Big)\mathrm{d}\Gamma\mathrm{d}t \\
&= \mathbb{E}\int_{\Sigma_\tau} \sum_{j=1}^{n} (2\rho \cdot \nabla\Delta z \Delta z_{x_j} - \rho^j |\nabla\Delta z|^2)\nu^j \mathrm{d}\Gamma\mathrm{d}t \\
&= -2\mathbb{E}\int_G \rho \cdot \nabla\Delta z^\tau \widehat{z}^\tau \mathrm{d}x + 2\mathbb{E}\int_G \rho \cdot \nabla\Delta z(0)\widehat{z}(0)\mathrm{d}x \\
&\quad + \mathbb{E}\int_{Q_\tau} \{-\operatorname{div}\rho|\nabla\Delta z|^2 + 2\rho \cdot \nabla\Delta z[(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2 \cdot \nabla z + a_4 Z - a_3\widehat{Z}] \\
&\quad + 2\nabla\Delta z D\rho \cdot \nabla\Delta z + 2\rho_t \cdot \nabla\Delta z\widehat{z} + 2\rho \cdot \nabla\Delta(Z - a_5 z)\widehat{Z} + 2\Delta\rho \cdot \nabla\widehat{z}\widehat{z} \\
&\quad + 2\nabla\widehat{z} D\rho \cdot \nabla\widehat{z} + \operatorname{div}\rho|\nabla\widehat{z}|^2\}\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&2\Big|\frac{\partial\Delta z}{\partial\nu}\Big|^2_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))} - |\nabla\Delta z|^2_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))} \\
&\le C(|z|^2_{L^2_{\mathbb{F}}(\Omega;C([0,\tau];H^3(G)\cap H_0^2(G)))} + |\widehat{z}|^2_{L^2_{\mathbb{F}}(\Omega;C([0,\tau];H_0^1(G)))} + |Z|^2_{L^2_{\mathbb{F}}(0,\tau;H^3(G)\cap H_0^2(G))} \\
&\quad + |\widehat{Z}|^2_{L^2_{\mathbb{F}}(0,\tau;L^2(G))}) \\
&\le C|(z^\tau, \widehat{z}^\tau)|^2_{L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))}. \qquad (2.2)
\end{aligned}
$$

Denote by ∇_σ the tangential gradient on Γ. We have

$$
|\nabla\Delta z|^2 = \Big|\frac{\partial\Delta z}{\partial\nu}\Big|^2 + |\nabla_\sigma\Delta z|^2,
$$

which, together with (2.2), implies that

$$\Big|\frac{\partial \Delta z}{\partial \nu}\Big|^2_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))} \le |\nabla_\sigma \Delta z|^2_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))} + C|(z^\tau,\widehat{z}^\tau)|^2_{L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))}.$$

Now we are going to prove

$$|\nabla_\sigma \Delta z|^2_{L^2_{\mathbb{F}}(0,\tau;L^2(\Gamma))} \le C|(z^\tau,\widehat{z}^\tau)|^2_{L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H_0^2(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))}. \tag{2.3}$$

As the proof of [14, Theorem 2.2], we introduce the following operator

$$\begin{cases} \mathscr{B} = \displaystyle\sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} = \text{ a first-order differential operator (time independent)} \\ \text{with coefficients } b_i \in C^4(\overline{G}) \text{ and such that } \mathscr{B} \text{ is tangential to } \Gamma, \text{ i.e.,} \\ \displaystyle\sum_{i=1}^n b_i(x)\nu^i = 0 \quad \text{ on } \Gamma. \end{cases} \tag{2.4}$$

The operator $\mathscr{B}$ can be thought of as the pre-image, under the diffeomorphism via partitions of unity from G onto half-space $\{(x,y)\in\mathbb{R}^n \mid x>0, y\in\mathbb{R}^{n-1}\}$ of the tangential derivative on the boundary $x=0$.

Define

$$p \stackrel{\Delta}{=} \mathscr{B}z \in C_{\mathbb{F}}([0,\tau];L^2(\Omega;H^2(G))), \quad \widehat{p} \stackrel{\Delta}{=} \mathscr{B}\widehat{z} \in C_{\mathbb{F}}([0,\tau];L^2(\Omega;L^2(G))).$$

From (1.2), we have

$$\begin{cases} \mathrm{d}p = \widehat{p}\mathrm{d}t + g_1\mathrm{d}W(t) & \text{in } Q_\tau, \\ \mathrm{d}\widehat{p} + \Delta^2 p\mathrm{d}t = f_2\mathrm{d}t + g_2\mathrm{d}W(t) & \text{in } Q_\tau, \\ p = 0, \quad \dfrac{\partial p}{\partial \nu} = \Big[\dfrac{\partial}{\partial \nu},\mathscr{B}\Big]z & \text{on } \Sigma_\tau, \\ (p(\tau),\widehat{p}(\tau)) = (\mathscr{B}z^\tau,\mathscr{B}\widehat{z}^\tau) & \text{in } G, \end{cases} \tag{2.5}$$

where

$$g_1 = \mathscr{B}(Z - a_5 z), \quad g_2 = \mathscr{B}\widehat{Z}$$

and

$$f_2 = \mathscr{B}[(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2\cdot\nabla z - a_3\widehat{Z} + a_4 Z] + [\Delta^2,\mathscr{B}]z,$$

and $[\cdot,\cdot]$ denotes the commutator of two operators.

From (2.4), we get that

$$|[\Delta^2,\mathscr{B}]z|_{L^2_{\mathbb{F}}(0,\tau;H^{-1}(G))} \le C|(z^\tau,\widehat{z}^\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega;(H^3(G)\cap H_0^2(G)))\times L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G))} \tag{2.6}$$

and that

$$\mathbb{E}\int_{\Sigma_\tau} |\nabla_\sigma \Delta z|^2 \mathrm{d}\Gamma\mathrm{d}t = \mathbb{E}\int_{\Sigma_\tau} |\mathscr{B}\Delta z|^2\mathrm{d}\Gamma\mathrm{d}t = \mathbb{E}\int_{\Sigma_\tau}|\Delta p|^2\mathrm{d}\Gamma\mathrm{d}t + \mathbb{E}\int_{\Sigma_\tau}|[\mathscr{B},\Delta]z|^2\mathrm{d}\Gamma\mathrm{d}t. \tag{2.7}$$

By (2.4) again, we find that

$$\mathbb{E}\int_{\Sigma_\tau}|[\mathscr{B},\Delta]z|^2\mathrm{d}\Gamma\mathrm{d}t \le C\mathbb{E}\int_0^\tau |z|^2_{H^3(G)}\mathrm{d}t$$

$$\leq C|(z^\tau, \widehat{z}^\tau)|^2_{L^2_{\mathcal{F}_\tau}(\Omega;H^3(G)\cap H^2_0(G))\times L^2_{\mathcal{F}_\tau}(\Omega;H^1_0(G))}. \tag{2.8}$$

Combining (2.7) and (2.8), to show (2.3), we only need to prove

$$\mathbb{E}\int_{\Sigma_\tau} |\Delta p|^2 \mathrm{d}\Gamma\mathrm{d}t \leq C|(z^\tau, \widehat{z}^\tau)|^2_{L^2_{\mathcal{F}_\tau}(\Omega;(H^3(G)\cap H^2_0(G)))\times L^2_{\mathcal{F}_\tau}(\Omega;H^1_0(G))}. \tag{2.9}$$

For any $\rho \stackrel{\Delta}{=} (\rho^1, \cdots, \rho^n) \in C^2(\mathbb{R}^{n+1};\mathbb{R}^n)$, by Itô's formula and (2.5), we have

$$\begin{aligned}
&\sum_{i,k,l=1}^n (\rho^i p_{x_kx_k}p_{x_lx_l})_{x_i}\mathrm{d}t \\
=& \sum_{i,k,l=1}^n [\rho^i_{x_i} p_{x_kx_k}p_{x_lx_l} + (2\rho^i p_{x_ix_kx_k}p_{x_l})_{x_l} - (2\rho^i_{x_l} p_{x_ix_k}p_{x_l})_{x_k} + 2\rho^i_{x_lx_k} p_{x_ix_k}p_{x_l} \\
&+ 2\rho^i_{x_l} p_{x_ix_k}p_{x_lx_k} - (2\rho^i p_{x_lx_kx_k}p_{x_l})_{x_i} + (2\rho^i_{x_i} p_{x_lx_k}p_{x_l})_{x_k} - 2\rho^i_{x_ix_k} p_{x_lx_k}p_{x_l} \\
&- 2\rho^i_{x_i} p_{x_lx_k}p_{x_lx_k} + (2\rho^i p_{x_lx_kx_k}p_{x_i})_{x_l} - (2\rho^i_{x_l} p_{x_lx_k}p_{x_i})_{x_k} + 2\rho^i_{x_lx_k} p_{x_lx_k}p_{x_i} \\
&+ 2\rho^i_{x_l} p_{x_lx_k}p_{x_ix_k}]\mathrm{d}t - 2\rho\cdot\nabla p(f_2\mathrm{d}t + g_2\mathrm{d}W(t)) + \mathrm{d}(2\rho\cdot\nabla p\widehat{p}) - 2\rho_t\cdot\nabla p\widehat{p}\mathrm{d}t \\
&+ \operatorname{div}\rho\widehat{p}^2\mathrm{d}t - 2\rho\cdot d\nabla p\mathrm{d}\widehat{p} - \operatorname{div}(\rho\widehat{p}^2)\mathrm{d}t - 2\rho\cdot\nabla g_1\widehat{p}\mathrm{d}W(t).
\end{aligned} \tag{2.10}$$

Setting $\rho = \zeta$, integrating (2.10) in Q_τ, and taking expectation on Ω, we have

$$\begin{aligned}
&\mathbb{E}\int_{\Sigma_\tau} |\Delta p|^2\mathrm{d}\Gamma\mathrm{d}t \\
=& \mathbb{E}\int_{\Sigma_\tau} \sum_{i,k,l=1}^n \rho^i\nu^i p_{x_kx_k}p_{x_lx_l}\mathrm{d}\Gamma\mathrm{d}t \\
=& 2\mathbb{E}\int_G \rho\cdot\nabla p^\tau\widehat{p}^\tau\mathrm{d}x - 2\mathbb{E}\int_G \rho\cdot\nabla p(0)\widehat{p}(0)\mathrm{d}x \\
&+ \mathbb{E}\int_{Q_\tau} (\operatorname{div}\rho\widehat{p}^2 - 2\rho\cdot\nabla p f_2 - 2\rho_t\cdot\nabla p\widehat{p} - 2\rho\cdot\nabla g_1 g_2)\mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}\int_{Q_\tau} \sum_{i,k,l=1}^n (\rho^i_{x_i} p_{x_kx_k}p_{x_lx_l} + 2\rho^i_{x_lx_k} p_{x_ix_k}p_{x_l} + 2\rho^i_{x_l} p_{x_ix_k}p_{x_lx_k} - 2\rho^i_{x_ix_k} p_{x_lx_k}p_{x_l} \\
&- 2\rho^i_{x_i} p_{x_lx_k}p_{x_lx_k} + 2\rho^i_{x_lx_k} p_{x_lx_k}p_{x_i} + 2\rho^i_{x_l} p_{x_lx_k}p_{x_ix_k})\mathrm{d}x\mathrm{d}t \\
&+ 2\mathbb{E}\int_{\Sigma_\tau} \sum_{i,k,l=1}^n (\rho^i_{x_i} p_{x_lx_k}p_{x_l}\nu^k - \rho^i_{x_l} p_{x_ix_k}p_{x_l}\nu^k \\
&+ \rho^i p_{x_lx_kx_k}p_{x_i}\nu^l - \rho^i_{x_l} p_{x_lx_k}p_{x_i}\nu^k)\mathrm{d}\Gamma\mathrm{d}t,
\end{aligned} \tag{2.11}$$

which, together with (2.4)–(2.6), implies (2.9). Then we complete the proof.

Next, we give a pointwise weighted identity, which will play an important role in the proof of Theorem 3.1.

We have the following fundamental identity.

Theorem 2.1 *Let v be an $H^4(G)$-valued Itô process and $\widehat{v}$ be an $H^2(G)$-valued Itô process such that*

$$\mathrm{d}v = (\widehat{v} + f_1)\mathrm{d}t + g_1\mathrm{d}W(t)$$

for some $f_1 \in L^2_{\mathbb{F}}(0,T;H^2_0(G))$ *and* $g_1 \in L^2_{\mathbb{F}}(0,T;H^4(G)\cap H^2_0(G))$. *Let* $\eta \in C^2(\mathbb{R}\times\mathbb{R}^n)$. *Set* $\theta = \mathrm{e}^{\ell}$, $\ell = s\xi$, $\xi = \mathrm{e}^{\lambda\eta}$, $w = \theta v$ *and* $\widehat{w} = \theta\widehat{v} + \ell_t w$. *Then, for any* $t \in [0,T]$ *and a.e.* $(x,\omega) \in G\times\Omega$,

$$
\begin{aligned}
&2\theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t) - 2\,\mathrm{div}(V_1+V_2)\mathrm{d}t\\
={}& 2I_2^2\mathrm{d}t + 2I_2I_3 + 2(M_1+M_2)\mathrm{d}t + 2\sum_{i,j,k,l=1}^n \Lambda_1^{ijkl} w_{x_ix_j}w_{x_kx_l}\mathrm{d}t + 2\sum_{i,j=1}^n \Lambda_2^{ij} w_{x_i}w_{x_j}\mathrm{d}t\\
&+ 2\Lambda_3 w^2\mathrm{d}t + 2\Lambda_4 + 2\mathrm{d}(I_2\widehat{w}) - \sum_{i=1}^n (\Phi_1^i \widehat{w}\mathrm{d}\Delta w)_{x_i}.
\end{aligned}
\tag{2.12}
$$

Here

$$
I_1 = \Delta^2 w\mathrm{d}t + \Psi_2\Delta w\mathrm{d}t + \sum_{i,j=1}^n \Psi_3^{ij} w_{x_ix_j}\mathrm{d}t + \sum_{i=1}^n \Psi_4^i w_{x_i}\mathrm{d}t + \sum_{i=1}^n \Psi_5^i w_{x_i}\mathrm{d}t + \Psi_6 w\mathrm{d}t + \mathrm{d}\widehat{w} \tag{2.13}
$$

with

$$
\begin{cases}
\Psi_2 = 2s^2\lambda^2\xi^2|\nabla\eta|^2, \quad \Psi_3^{ij} = 4s^2\lambda^2\xi^2\eta_{x_i}\eta_{x_j},\\
\Psi_4^i = 12s^2\lambda^3\xi^2|\nabla\eta|^2\eta_{x_i} + 4s^2\lambda^2\xi^2\Delta\eta\eta_{x_i},\\
\Psi_5^i = \displaystyle\sum_{j=1}^n 8s^2\lambda^2\xi^2\eta_{x_ix_j}\eta_{x_j}, \quad \Psi_6 = s^4\lambda^4\xi^4|\nabla\eta|^4,
\end{cases}
\qquad i,j = 1,\cdots,n, \tag{2.14}
$$

$$
I_2 = \sum_{i=1}^n \Phi_1^i \Delta w_{x_i} + \Phi_2\Delta w + \sum_{i,j=1}^n \Phi_3^{ij} w_{x_ix_j} + \sum_{i=1}^n \Phi_4^i w_{x_i} + \Phi_5 w \tag{2.15}
$$

with

$$
\begin{cases}
\Phi_1^i = -4s\lambda\xi\eta_{x_i}, \quad \Phi_2 = -2s\lambda^2\xi|\nabla\eta|^2 - 2s\lambda\xi\Delta\eta - \lambda,\\
\Phi_3^{ij} = 4s\lambda\xi\eta_{x_ix_j} - 4s\lambda^2\xi\eta_{x_i}\eta_{x_j}, \quad \Phi_4^i = -4s^3\lambda^3\xi^3|\nabla\eta|^2\eta_{x_i},\\
\Phi_5 = -6s^3\lambda^4\xi^3|\nabla\eta|^4 - 12s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)\\
\qquad -2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta - s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4,
\end{cases}
\qquad i,j = 1,\cdots,n, \tag{2.16}
$$

$$
\begin{aligned}
I_3 ={}& -8s\lambda\xi\sum_{i,j=1}^n \eta_{x_ix_j}w_{x_ix_j}\mathrm{d}t - 4\nabla\Delta\ell\cdot\nabla w\mathrm{d}t + 4(\nabla\ell\cdot\nabla\Delta\ell)w\mathrm{d}t + 2|\nabla^2\ell|^2 w\mathrm{d}t\\
&- \Delta^2\ell w\mathrm{d}t + |\Delta\ell|^2 w\mathrm{d}t + 8s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)w\mathrm{d}t + s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4 w\mathrm{d}t + \lambda\Delta w\mathrm{d}t\\
&- \ell_t\theta f_1\mathrm{d}t - \ell_t\theta g_1\mathrm{d}W(t) + (\ell_t^2 - \ell_{tt})w\mathrm{d}t - 2\ell_t\widehat{w}\mathrm{d}t,
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
&V_1 = [V_1^1, V_1^2, \cdots, V_1^n], \quad V_2 = [V_2^1, V_2^2, \cdots, V_2^n],\\
&V_1^j = \sum_{i,k=1}^n \Big[\sum_{l=1}^n \Phi_1^l w_{x_kx_kx_l}w_{x_ix_ix_j} - \frac{1}{2}\sum_{l=1}^n \Phi_1^j w_{x_kx_kx_l}w_{x_ix_ix_l} + \frac{1}{2}\Psi_2\Phi_1^j w_{x_ix_i}w_{x_kx_k}\\
&\qquad + \sum_{l=1}^n \Psi_3^{ik}\Phi_1^l w_{x_ix_k}w_{x_lx_j} - \frac{1}{2}\sum_{l=1}^n \Psi_3^{ij}\Phi_1^l w_{x_ix_k}w_{x_kx_l} + \Phi_4^k w_{x_ix_ix_j}w_{x_k} - \Phi_4^k w_{x_ix_j}w_{x_ix_k}\\
&\qquad + \frac{1}{2}\Phi_4^j w_{x_ix_k}^2 + \Big(\Psi_2\Phi_4^k\delta_{ij} - \frac{1}{2}\Psi_2\Phi_4^j\delta_{ik} - \frac{1}{2}\Psi_6\Phi_1^j\delta_{ik} + \Psi_3^{ij}\Phi_4^k\Big)w_{x_i}w_{x_k}\Big],
\end{aligned}
$$

$$
\begin{aligned}
V_2^j &= \sum_{i,l,r,m=1}^{n} \Theta_1^{ijlrm} w_{x_ix_ix_l} w_{x_rx_m} + \sum_{i,k,l,r=1}^{n} \Theta_2^{ijklr} w_{x_ix_k} w_{x_lx_r} + \sum_{i,k,l=1}^{n} \Theta_3^{ijkl} w_{x_ix_k} w_{x_l} \\
&\quad + \sum_{i,k=1}^{n} \Theta_4^{ijk} w_{x_i} w_{x_k} + \sum_{i=1}^{n} \Theta_5 w_{x_ix_ix_j} w + \sum_{i,k=1}^{n} \Theta_6^{ijk} w_{x_ix_k} w + \sum_{i=1}^{n} \Theta_7^{ij} w_{x_i} w + \Theta_8^j w^2 + \Theta_9^j, \\
M_1 &= 8s\lambda^2\xi|\nabla\Delta w\cdot\nabla\eta|^2 + 32s^3\lambda^4\xi^3|\nabla^2 w\nabla\eta\nabla\eta|^2 + 48s^3\lambda^3\xi^3\nabla^2\eta(\nabla^2 w\nabla\eta)(\nabla^2 w\nabla\eta) \\
&\quad + 16s^3\lambda^3\xi^3(\nabla^2 w\nabla\eta\nabla\eta)\sum_{i,j=1}^{n}\eta_{x_ix_j}w_{x_ix_j} - 16s^3\lambda^4\xi^3|\nabla\eta|^2|\nabla^2 w\nabla\eta|^2 \\
&\quad + 6s^3\lambda^4\xi^3|\nabla\eta|^4|\nabla^2 w|^2 + 4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)|\nabla^2 w|^2 + 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta|\nabla^2 w|^2 \\
&\quad + 2s^3\lambda^4\xi^3|\nabla\eta|^4|\Delta w|^2 - 4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)|\Delta w|^2 - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta|\Delta w|^2 \\
&\quad + 40s^5\lambda^6\xi^5|\nabla\eta|^4|\nabla w\cdot\nabla\eta|^2 + 64s^5\lambda^5\xi^5(\nabla^2\eta\nabla\eta\nabla\eta)|\nabla w\cdot\nabla\eta|^2 \\
&\quad - 16s^5\lambda^6\xi^5|\nabla\eta|^6|\nabla w|^2 + 8s^7\lambda^8\xi^7|\nabla\eta|^8 w^2 + \lambda|\nabla\Delta w|^2 - s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4|\Delta w|^2 \\
&\quad + 4s^5\lambda^{\frac{11}{2}}\xi^5|\nabla\eta|^4|\nabla w\cdot\nabla\eta|^2 + 2s^5\lambda^{\frac{11}{2}}\xi^5|\nabla\eta|^6|\nabla w|^2 - s^7\lambda^{\frac{15}{2}}\xi^7|\nabla\eta|^8 w^2, \\
M_2 &= \Big(-\frac{1}{2}\sum_{i=1}^{n}\Phi_{1x_i}^i + \Phi_2\Big)|\nabla\widehat{w}|^2 + \sum_{i,j=1}^{n}(-\Phi_{1x_j}^i + \Phi_3^{ij})\widehat{w}_{x_i}\widehat{w}_{x_j} \\
&\quad + \frac{1}{2}\Big[\sum_{i,j=1}^{n}(\Phi_{1x_ix_jx_j}^i - \Phi_{2x_ix_i}\delta_{ij} - \Phi_{3x_ix_j}^{ij} + \Phi_{4x_i}^i\delta_{ij}) - 2\Phi_5\Big]\widehat{w}^2 \\
&\quad - \sum_{i,k,j=1}^{n}(\Psi_5^i\Phi_1^k)_{x_j} w_{x_i} w_{x_kx_j} - 4s^2\lambda^3\xi^2|\nabla^2 w\nabla\eta|^2 - 2s^2\lambda^3\xi^2|\nabla\eta|^2|\Delta w|^2 \\
&\quad + s^4\lambda^5\xi^4|\nabla\eta|^4|\nabla w|^2, \\
\Lambda_1^{ijkl} &= \Phi_{3x_ix_j}^{kl} + \Phi_{3x_ix_l}^{kj} + \sum_{r=1}^{n}\Big(\frac{1}{2}\Phi_{2x_rx_r}\delta_{ij}\delta_{kl} - \Phi_{3x_jx_r}^{kr}\delta_{il} - \Phi_{3x_ix_r}^{kr}\delta_{lj} + \frac{1}{2}\sum_{m=1}^{n}\Phi_{3x_rx_m}^{rm}\delta_{ik}\delta_{lj}\Big), \\
\Lambda_2^{ij} &= \sum_{k=1}^{n}\Big[-\Phi_{4x_ix_kx_k}^j + \frac{1}{2}\Phi_{4x_ix_jx_k}^k - 2\Phi_{5x_ix_j} + (\Psi_2\Phi_3^{jk})_{x_ix_k} - \frac{1}{2}(\Psi_2\Phi_3^{ij})_{x_kx_k} \\
&\quad - \sum_{l=1}^{n}\frac{1}{2}(\Psi_2\Phi_3^{kl}\delta_{ij})_{x_kx_l} + (\Psi_3^{ik}\Phi_2)_{x_kx_j} - \sum_{l=1}^{n}\frac{1}{2}(\Psi_3^{kl}\Phi_2\delta_{ij})_{x_kx_l} - \frac{1}{2}(\Psi_3^{ij}\Phi_2)_{x_kx_k} \\
&\quad + \sum_{l=1}^{n}(\Psi_3^{ik}\Phi_3^{jl})_{x_kx_l} - \sum_{l=1}^{n}\frac{1}{2}(\Psi_3^{ij}\Phi_3^{kl})_{x_kx_l} - \frac{1}{2}(\Psi_3^{kl}\Phi_3^{ij})_{x_kx_l} + \frac{1}{2}(\Psi_4^i\Phi_1^j)_{x_kx_k} \\
&\quad + (\Psi_4^i\Phi_2)_{x_j} + \frac{1}{2}(\Psi_4^k\Phi_2\delta_{ij})_{x_k} - (\Psi_4^i\Phi_3^{jk})_{x_k} + \frac{1}{2}(\Psi_4^k\Phi_3^{ij})_{x_k} + (\Psi_5^i\Phi_2)_{x_j} \\
&\quad + \frac{1}{2}(\Psi_5^k\Phi_2\delta_{ij})_{x_k} - (\Psi_5^i\Phi_3^{jk})_{x_k} + \frac{1}{2}(\Psi_5^k\Phi_3^{ij})_{x_k}\Big], \\
\Lambda_3 &= \sum_{i=1}^{n}\Big[\sum_{j=1}^{n}\frac{1}{2}\Phi_{5x_ix_ix_jx_j} + \sum_{j=1}^{n}\frac{1}{2}(\Psi_2\Phi_5)_{x_jx_j} + \sum_{j=1}^{n}\frac{1}{2}(\Psi_3^{ij}\Phi_5)_{x_ix_j} - \frac{1}{2}(\Psi_4^i\Phi_5)_{x_i} - \frac{1}{2}(\Psi_5^i\Phi_5)_{x_i} \\
&\quad - \sum_{j=1}^{n}\frac{1}{2}(\Psi_6\Phi_1^i)_{x_ix_jx_j} + \frac{1}{2}(\Psi_6\Phi_2)_{x_ix_i}\Big], \\
\Lambda_4 &= \sum_{i,j=1}^{n}[-\Phi_{1t}^i w_{x_ix_jx_j} + (-\Phi_{2t}\delta_{ij} - \Phi_{3t}^{ij})w_{x_ix_j}]\widehat{w}\mathrm{d}t - \sum_{i=1}^{n}\Phi_{4t}^i w_{x_i}\widehat{w}\mathrm{d}t - \Phi_{5t}w\widehat{w}\mathrm{d}t
\end{aligned}
$$

$$+\sum_{i,j=1}^{n}(\Phi^i_{1x_i}\widehat{w}+\Phi^i_1\widehat{w}_{x_i}-\Phi_2\widehat{w}\delta_{ij})[(\theta f_1)_{x_jx_j}\mathrm{d}t+(\theta g_1)_{x_jx_j}\mathrm{d}W(t)]-\Phi^i_5\widehat{w}(\theta f_1\mathrm{d}t+\theta g_1\mathrm{d}W(t))$$
$$-\sum_{i,j=1}^{n}\Phi^{ij}_3\widehat{w}[(\theta f_1)_{x_ix_j}\mathrm{d}t+(\theta g_1)_{x_ix_j}\mathrm{d}W(t)]-\sum_{i=1}^{n}\Phi^i_4\widehat{w}[(\theta f_1)_{x_i}\mathrm{d}t+(\theta g_1)_{x_i}\mathrm{d}W(t)]$$
$$-\sum_{i,j=1}^{n}(\Phi^i_1\mathrm{d}w_{x_ix_ix_j}+\Phi_2\delta_{ij}\mathrm{d}w_{x_ix_i}+\Phi^{ij}_3\mathrm{d}w_{x_ix_j}+\Phi^i_4\delta_{ij}\mathrm{d}w_{x_i})\mathrm{d}\widehat{w}+\Phi_5\mathrm{d}w\mathrm{d}\widehat{w}$$

and

$$\begin{aligned}
\Theta^{ijklrm}_1&=\Phi_2\delta_{lj}\delta_{rm}+\Phi^{rm}_3\delta_{lj}-\Phi^{mj}_3\delta_{lr}+\Phi^{lr}_3\delta_{mj},\\
\Theta^{ijklr}_2&=-\frac{1}{2}\Phi_{2x_j}\delta_{ik}\delta_{lr}-\Phi^{lr}_{3x_i}\delta_{kj}+\Phi^{rj}_{3x_k}\delta_{il}+\sum_{m=1}^{n}\Phi^{rm}_{3x_m}\delta_{kj}\delta_{il}-\frac{1}{2}\sum_{m=1}^{n}\Phi^{jm}_{3x_m}\delta_{il}\delta_{kr}-\Phi^{kl}_{3x_i}\delta_{rj},\\
\Theta^{ijkl}_3&=-\Phi^l_{4x_i}\delta_{kj}-\Phi_5\delta_{ik}\delta_{lj}+\Psi_2\Phi^{ik}_3\delta_{lj}-\Psi_2\Phi^{ij}_3\delta_{kl}+\Psi_3\Phi_2\delta_{ik}-\Psi_3\Phi_2\delta_{ij}+\Psi^{ik}_3\Phi^{lj}_3\\
&\quad-\Psi^{ij}_3\Phi^{kl}_3+\Psi^l_4\Phi^k_1\delta_{ij}+\Psi^l_5\Phi^k_1\delta_{ij},\\
\Theta^{ijk}_4&=\Phi^k_{4x_ix_j}-\frac{1}{2}\Phi^j_{4x_ix_k}+2\Phi_{5x_i}\delta_{kj}-\Phi_{5x_j}\delta_{ik}-(\Psi_2\Phi^{jk}_3)_{x_i}+\frac{1}{2}(\Psi_2\Phi^{ik}_3)_{x_j}\\
&\quad+\sum_{l=1}^{n}\frac{1}{2}(\Psi_2\Phi^{jl}_3)_{x_l}\delta_{ik}-\sum_{l=1}^{n}(\Psi^{il}_3\Phi_2)_{x_l}\delta_{kj}+\frac{1}{2}\sum_{l=1}^{n}(\Psi^{lj}_3\Phi_2)_{x_l}\delta_{ik}+\frac{1}{2}(\Psi^{ik}_3\Phi_2)_{x_j}\\
&\quad-\sum_{l=1}^{n}(\Psi^{ij}_3\Phi^{kl}_3)_{x_l}+\frac{1}{2}\sum_{l=1}^{n}(\Psi^{ik}_3\Phi^{jl}_3)_{x_l}+\frac{1}{2}\sum_{l=1}^{n}(\Psi^{lj}_3\Phi^{ik}_3)_{x_l}-\frac{1}{2}(\Psi^i_4\Phi^k_1)_{x_j}+\Psi^i_4\Phi_2\delta_{kj}\\
&\quad-\frac{1}{2}\Psi^j_4\Phi_2\delta_{ik}+\Psi^i_4\Phi^{kj}_3-\frac{1}{2}\Psi^j_4\Phi^{ik}_3+\Psi^i_5\Phi_2\delta_{kj}-\frac{1}{2}\Psi^j_5\Phi_2\delta_{ik}+\Psi^i_5\Phi^{kj}_3-\frac{1}{2}\Psi^j_5\Phi^{ik}_3,\\
\Theta_5&=\Phi_5,\\
\Theta^{ijk}_6&=-\Phi_{5x_j}\delta_{ik}+\Psi_6\Phi^k_1\delta_{ij},\\
\Theta^{ij}_7&=\sum_{k=1}^{n}\Phi_{5x_kx_k}\delta_{ij}+\Psi_2\Phi_5\delta_{ij}+\Psi^{ij}_3\Phi_5-(\Psi_6\Phi^i_1)_{x_j}+\Psi_6\Phi_2\delta_{ij}+\Psi_6\Phi^{ij}_3,\\
\Theta^j_8&=-\frac{1}{2}\sum_{i=1}^{n}\Phi_{5x_ix_ix_j}-\frac{1}{2}(\Psi_2\Phi_5)_{x_j}-\frac{1}{2}\sum_{i=1}^{n}(\Psi^{ij}_3\Phi_5)_{x_i}+\frac{1}{2}\Psi^j_4\Phi_5+\frac{1}{2}\Psi^j_5\Phi_5\\
&\quad+\frac{1}{2}\sum_{i=1}^{n}(\Psi_6\Phi^j_1)_{x_ix_i}-\frac{1}{2}(\Psi_6\Phi_2)_{x_j}-\frac{1}{2}\sum_{i=1}^{n}(\Psi_6\Phi^{ij}_3)_{x_i}+\frac{1}{2}\Psi_6\Phi^j_4,\\
\Theta^j_9&=\sum_{i=1}^{n}\Big(\Phi^i_{1x_i}\widehat{w}_{x_j}\widehat{w}+\Phi^i_1\widehat{w}_{x_i}\widehat{w}_{x_j}-\frac{1}{2}\Phi^i_{1x_ix_j}\widehat{w}^2-\frac{1}{2}\Phi^j_1\widehat{w}^2_{x_i}-\Phi^{ij}_3\widehat{w}\widehat{w}_{x_i}+\frac{1}{2}\Phi^{ij}_{3x_i}\widehat{w}^2\Big)\\
&\quad-\Phi_2\widehat{w}\widehat{w}_{x_j}+\frac{1}{2}\Phi_{2x_j}\widehat{w}^2-\frac{1}{2}\Phi^j_4\widehat{w}^2.
\end{aligned}$$

Remark 2.1 Since we do not put any further assumptions on v and $\widehat{v}$, the identity (2.12) seems to be very complicated. For solutions to (1.2) or (1.5), many terms, such as V_1 and V_2, will merge or vanish by means of the boundary conditions. Furthermore, compared with energy terms, such as $8s\lambda^2\xi|\nabla\Delta w\cdot\nabla\eta|^2$, $2s^3\lambda^4\xi^3|\nabla\eta|^4|\Delta w|^2$, $40s^5\lambda^6\xi^5|\nabla\eta|^4|\nabla w\cdot\nabla\eta|^2$, $\lambda|\nabla\Delta w|^2$, many terms in (2.12) is of no great importance. We only need to estimate their orders with respect to s and λ. Hence, an effective way to simplify (2.12) is that we do not write these terms explicitly and just claim that the order of s and λ for them are lower than the terms with a "good sign". However, we do not do this since we want to provide the full details for readers, and particularly for beginners.

Although the form of the identity (2.12) is very complex, its proof follows from some basic computations. To avoid defocusing the main theorem of this paper, we put the proof of Theorem 2.1 in the appendix.

3 Carleman Estimate for the Adjoint Equation

This section is devoted to establishing a Carleman estimate for a backward stochastic plate equation. To this end, we first introduce the weight function η.

For any $\delta > 0, T > 0$ and $0 < \varepsilon_1 < \frac{1}{2}$, we choose $x_0 \in \mathbb{R}^n \setminus \overline{G}$ such that

$$R_0 \stackrel{\Delta}{=} \min_{x \in \overline{G}} |x - x_0|^2 > 2\delta, \tag{3.1}$$

and choose sufficiently large β satisfying

$$R_1 \stackrel{\Delta}{=} \max_{x \in \overline{G}} |x - x_0|^2 \leq \beta \varepsilon_1^2 T^2 - \delta. \tag{3.2}$$

We also choose sufficiently small ε_0 with $0 < \varepsilon_0 < \varepsilon_1$ such that

$$R_0 - \beta \varepsilon_0^2 T^2 \geq \delta. \tag{3.3}$$

Let

$$\eta(t, x) = |x - x_0|^2 - \beta\Big(t - \frac{T}{2}\Big)^2. \tag{3.4}$$

From (3.1)–(3.4), it is easy to see that η satisfies the following conditions.

Condition 3.1

(1) $|\eta(t, x)|_{C^2(\overline{Q})} \leq C_1$.

(2) $|\nabla \eta(t, x)| \geq C_2 > 0, \ \forall \ (t, x) \in \overline{Q}$.

(3) *For all* (t, x) *in* $J_1 \stackrel{\Delta}{=} \big[(0, \frac{T}{2} - \varepsilon_1 T) \cup \big(\frac{T}{2} + \varepsilon_1 T, T\big)\big] \times G$, *it holds that* $\eta(t, x) \leq -\delta$.

(4) *For all* (t, x) *in* $J_2 \stackrel{\Delta}{=} \big(\frac{T}{2} - \varepsilon_0 T, \frac{T}{2} + \varepsilon_0 T\big) \times G$, *it holds that* $\eta(t, x) \geq \delta$.

Recall that $\theta = \mathrm{e}^{\ell}$, $\ell = s\xi$ and $\xi = \mathrm{e}^{\lambda\eta}$. With η given by (3.4), the functions ℓ and θ are also defined.

We also need the following known result.

Lemma 3.1 (see [7, Theorem 2.1]) *Let* $q \in H_0^2(G)$. *Then there exists a constant* $C > 0$ *independent of* s *and* λ, *and parameter* $\widehat{\lambda} > 1$ *and* $\widehat{s} > 1$ *such that, for all* $\lambda \geq \widehat{\lambda}$ *and* $s \geq \widehat{s}$,

$$s^4\lambda^6 \int_Q \xi^4\theta^2(s^2\lambda^2\xi^2|q|^2 + |\nabla q|^2)\mathrm{d}x\mathrm{d}t \leq C \int_Q s^3\lambda^4\xi^3\theta^2|\Delta q|^2\mathrm{d}x\mathrm{d}t.$$

We have the following Carleman estimate.

Theorem 3.1 *There exist constants* $C > 0$ *and* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$, *one can find* $s_0 = s_0(\lambda) > 0$ *so that for any* $s \geq s_0$,

$$(v, \widehat{v}) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; H^4(G) \cap H_0^2(G))) \times L^2_{\mathbb{F}}(\Omega; C([0, T]; H_0^2(G)))$$

and

$$f_1, f_2, g_2 \in L^2_{\mathbb{F}}(0,T;H^2_0(G)), \quad g_1 \in L^2_{\mathbb{F}}(0,T;H^4(G)\cap H^2_0(G))$$

satisfying

$$\begin{cases} \mathrm{d}v = (\widehat{v} + f_1)\mathrm{d}t + g_1\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t = f_2\mathrm{d}t + g_2\mathrm{d}W(t) & \text{in } Q, \\ v = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma \end{cases} \tag{3.5}$$

and

$$v(0,\cdot) = v(T,\cdot) = \widehat{v}(0,\cdot) = \widehat{v}(T,\cdot) = 0 \quad \text{in } G, \quad \mathbb{P}\text{-a.s.}, \tag{3.6}$$

it holds that

$$\begin{aligned} &\mathbb{E}\int_Q \theta^2 (s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2 + \lambda|\nabla\Delta v|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2 \\ &+ s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)\mathrm{d}x\mathrm{d}t \\ \le{}& C\mathbb{E}\int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2 \\ &+ s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t \\ &+ C\mathbb{E}\int_\Sigma \theta^2(s\lambda\xi|\nabla\Delta v|^2 + s^3\lambda^3\xi^3|\Delta v|^2)\mathrm{d}\Gamma\mathrm{d}t. \end{aligned} \tag{3.7}$$

Proof In what follows, for a positive integer r, we denote by $O(\lambda^r)$ a function of order λ^r for large λ. Similarly, we use the notation $O(\mathrm{e}^{C\lambda})$.

In order to shorten the formulae, we define

$$\begin{aligned} &\mathcal{A}_1 \triangleq \mathbb{E}\int_Q (s^5 O(\mathrm{e}^{C\lambda}) + s^6\xi^6 O(\lambda^7))|w|^2\mathrm{d}x\mathrm{d}t, \\ &\mathcal{A}_2 \triangleq \mathbb{E}\int_Q (s^3 O(\mathrm{e}^{C\lambda}) + s^4\xi^4 O(\lambda^5))|\nabla w|^2\mathrm{d}x\mathrm{d}t, \\ &\mathcal{A}_3 \triangleq \mathbb{E}\int_Q (s O(\mathrm{e}^{C\lambda}) + s^2\xi^2 O(\lambda^3))|\nabla^2 w|^2\mathrm{d}x\mathrm{d}t, \\ &\mathcal{A}_4 \triangleq \mathbb{E}\int_Q O(1)|\nabla\Delta w|^2\mathrm{d}x\mathrm{d}t, \\ &\mathcal{B}_2 \triangleq \mathbb{E}\int_\Sigma (s^2 O(\mathrm{e}^{C\lambda}) + s^3\xi^3 O(\lambda^2))|\nabla^2 w|^2\mathrm{d}\Gamma\mathrm{d}t, \\ &\widehat{\mathcal{A}}_1 \triangleq \mathbb{E}\int_Q (s^2 O(\mathrm{e}^{C\lambda}) + s^3\xi^3 O(\lambda^3))|\widehat{w}|^2\mathrm{d}x\mathrm{d}t, \\ &\widehat{\mathcal{A}}_2 \triangleq \mathbb{E}\int_Q (O(\lambda) + s\xi O(1))|\nabla\widehat{w}|^2\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.8}$$

and

$$\mathcal{A} \triangleq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \widehat{\mathcal{A}}_1 + \widehat{\mathcal{A}}_2.$$

Integrating (2.12) on Q, taking mathematical expectation in both sides, and noting (3.6), we obtain

$$2\mathbb{E}\int_Q \theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t)\mathrm{d}x - 2\mathbb{E}\int_Q \mathrm{div}(V_1 + V_2)\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= 2\mathbb{E}\int_Q I_2^2 \mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q I_2 I_3 \mathrm{d}x + 2\mathbb{E}\int_Q (M_1 + M_2)\mathrm{d}x\mathrm{d}t \\
&\quad + 2\sum_{i,j,k,l=1}^{n} \mathbb{E}\int_Q \Lambda_1^{ijkl} w_{x_i x_j} w_{x_k x_l} \mathrm{d}x\mathrm{d}t + 2\sum_{i,j=1}^{n} \mathbb{E}\int_Q \Lambda_2^{ij} w_{x_i} w_{x_j} \mathrm{d}x\mathrm{d}t \\
&\quad + 2\mathbb{E}\int_Q \Lambda_3 w^2 \mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q \Lambda_4 \mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

By Condition 3.1, (2.14), (2.16) and (3.8), we have

$$
\sum_{i,j,k,l=1}^{n} \mathbb{E}\int_Q \Lambda_1^{ijkl} w_{x_i x_j} w_{x_k x_l} \mathrm{d}x\mathrm{d}t \geq -C\mathbb{E}\int_Q s\lambda^4 \xi |\nabla^2 w|^2 \mathrm{d}x\mathrm{d}t \geq -\mathcal{A}_3, \qquad (3.10)
$$

$$
\sum_{i,j=1}^{n} \mathbb{E}\int_Q \Lambda_2^{ij} w_{x_i} w_{x_j} \mathrm{d}x\mathrm{d}t \geq -C\mathbb{E}\int_Q (s^3\lambda^6\xi^3 + s^2\lambda^6\xi^2)|\nabla w|^2 \mathrm{d}x\mathrm{d}t \geq -\mathcal{A}_2, \qquad (3.11)
$$

$$
\mathbb{E}\int_Q \Lambda_3 w^2 \mathrm{d}x\mathrm{d}t \geq -C\mathbb{E}\int_Q s^3\lambda^8\xi^3(1 + s\xi + s^2\xi^2)|w|^2 \mathrm{d}x\mathrm{d}t \geq -\mathcal{A}_1 \qquad (3.12)
$$

and

$$
\begin{aligned}
\mathbb{E}\int_Q \Lambda_4 \mathrm{d}x &\geq -C\mathbb{E}\int_Q [|\nabla\Delta w|^2 + sO(\mathrm{e}^{C\lambda})|\nabla^2 w|^2 + s^3 O(\mathrm{e}^{C\lambda})|\nabla w|^2 + s^5 O(\mathrm{e}^{C\lambda})|w|^2 \\
&\quad + s^2 O(\mathrm{e}^{C\lambda})|\widehat{w}|^2 + s^3\lambda^3\xi^3|\widehat{w}|^2 + (s\xi + \lambda)|\nabla\widehat{w}|^2]\mathrm{d}x\mathrm{d}t \\
&\quad - C\mathbb{E}\int_Q \theta^2 (s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + s^6\lambda^6\xi^6 g_1^2 \\
&\quad + s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t \\
&\geq -C\mathbb{E}\int_Q \theta^2 (s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + s^6\lambda^6\xi^6 g_1^2 \\
&\quad + s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t - \mathcal{A}. \qquad (3.13)
\end{aligned}
$$

It follows from (3.9)–(3.13) that

$$
\begin{aligned}
&2\mathbb{E}\int_Q \theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t)\mathrm{d}x - 2\mathbb{E}\int_Q \mathrm{div}(V_1 + V_2)\mathrm{d}x\mathrm{d}t \\
&\geq 2\mathbb{E}\int_Q I_2^2 \mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q I_2 I_3 \mathrm{d}x + 2\mathbb{E}\int_Q (M_1 + M_2)\mathrm{d}x\mathrm{d}t \\
&\quad - C\mathbb{E}\int_Q \theta^2 (s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + s^6\lambda^6\xi^6 g_1^2 + s^4\lambda^4\xi^4|\nabla g_1|^2 \\
&\quad + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t - \mathcal{A}. \qquad (3.14)
\end{aligned}
$$

We will estimate the terms in (3.14) one by one. The procedure is divided into three steps.

Step 1 In this step, we consider the divergence terms.

Thanks to the boundary conditions satisfied by v, it is easy to check that

$$
w = 0, \quad \nabla w = 0, \quad \nabla^2 w = \theta\nabla^2 v \quad \text{on } \Sigma. \qquad (3.15)
$$

We also have

$$
\frac{\partial w_{x_i}}{\partial\nu}\nu^j = w_{x_i x_j} = w_{x_j x_i} = \frac{\partial w_{x_j}}{\partial\nu}\nu^i, \qquad (3.16)
$$

which implies

$$
\begin{aligned}
|\Delta w|^2 &= \sum_{i,j=1}^{n} w_{x_ix_i}w_{x_jx_j} = \sum_{i,j=1}^{n} \frac{\partial w_{x_i}}{\partial \nu}\nu^i \frac{\partial w_{x_j}}{\partial \nu}\nu^j \\
&= \sum_{i,j=1}^{n} \frac{\partial w_{x_i}}{\partial \nu}\nu^j \frac{\partial w_{x_j}}{\partial \nu}\nu^i = \sum_{i,j=1}^{n} w_{x_ix_j}^2 \\
&= |\nabla^2 w|^2 = \sum_{i=1}^{n} \Big(\frac{\partial w_i}{\partial \nu}\Big)^2 \quad \text{on } \Sigma.
\end{aligned} \tag{3.17}
$$

Thanks to (3.15), we obtain

$$
\begin{aligned}
V_1\cdot\nu = &\sum_{i,j,k,l=1}^{n} \Phi_1^l w_{x_kx_kx_l}w_{x_ix_ix_j}\nu^j - \frac{1}{2}\sum_{i,j,k,l=1}^{n} \Phi_1^j w_{x_kx_kx_l}w_{x_ix_ix_l}\nu^j \\
&+ \frac{1}{2}\sum_{i,j,k=1}^{n} \Psi_2\Phi_1^j w_{x_ix_i}w_{x_kx_k}\nu^j + \sum_{i,j,k,l=1}^{n} \Psi_3^{ik}\Phi_1^l w_{x_ix_k}w_{x_lx_j}\nu^j \\
&- \frac{1}{2}\sum_{i,j,k,l=1}^{n} \Psi_3^{ij}\Phi_1^l w_{x_ix_k}w_{x_kx_l}\nu^j - \sum_{i,j,k=1}^{n} \Phi_4^k w_{x_ix_j}w_{x_ix_k}\nu^j \\
&+ \frac{1}{2}\sum_{i,j,k=1}^{n} \Phi_4^j w_{x_ix_k}^2\nu^j \quad \text{on } \Sigma.
\end{aligned} \tag{3.18}
$$

By Condition 3.1 and (2.16), we have

$$
\sum_{i,j,k,l=1}^{n} \Phi_1^l w_{x_kx_kx_l}w_{x_ix_ix_j}\nu^j - \frac{1}{2}\sum_{i,j,k,l=1}^{n} \Phi_1^j\nu^j w_{x_kx_kx_l}w_{x_ix_ix_l} \geq -Cs\lambda\xi|\nabla\Delta w|^2 \quad \text{on } \Sigma. \tag{3.19}
$$

From (2.14) and (2.16), we get

$$
\frac{1}{2}\sum_{i,j,k=1}^{n} \Psi_2\Phi_1^j\nu^j w_{x_ix_i}w_{x_kx_k} = -4s^3\lambda^3\xi^3|\nabla\eta|^2\frac{\partial\eta}{\partial\nu}|\Delta w|^2 \quad \text{on } \Sigma. \tag{3.20}
$$

Combining (2.14), (2.16) and (3.16)–(3.17), we find

$$
\begin{aligned}
\sum_{i,j,k,l=1}^{n} \Psi_3^{ik}\Phi_1^l w_{x_ix_k}w_{x_lx_j}\nu^j &= -16\sum_{i,j,k,l=1}^{n} s^3\lambda^3\xi^3\eta_{x_i}\eta_{x_k}\eta_{x_l}\nu^j w_{x_ix_k}w_{x_lx_j} \\
&= -16\sum_{i,j,k,l=1}^{n} s^3\lambda^3\xi^3\eta_{x_i}\eta_{x_k}\nu^j w_{x_ix_k}\frac{\partial w_j}{\partial\nu}\frac{\partial\eta}{\partial\nu} \\
&= -16\sum_{i,j,k,l=1}^{n} s^3\lambda^3\xi^3\Big(\frac{\partial\eta}{\partial\nu}\Big)^3\Big(\frac{\partial w_j}{\partial\nu}\Big)^2 \\
&= -16\sum_{i,j,k,l=1}^{n} s^3\lambda^3\xi^3\Big(\frac{\partial\eta}{\partial\nu}\Big)^3|\Delta w|^2 \quad \text{on } \Sigma
\end{aligned} \tag{3.21}
$$

and

$$
-\frac{1}{2}\sum_{i,j,k,l=1}^{n} \Psi_3^{ij}\Phi_1^l w_{x_ix_k}w_{x_kx_l}\nu^j - \sum_{i,j,k=1}^{n} \Phi_4^k w_{x_ix_j}w_{x_ix_k}\nu^j + \frac{1}{2}\sum_{i,j,k=1}^{n} \Phi_4^j w_{x_ix_k}^2\nu^j
$$

$$= 8 \sum_{i,j,k,l=1}^{n} s^3\lambda^3\xi^3\Big(\frac{\partial \eta}{\partial \nu}\Big)^3|\Delta w|^2 + 2s^3\lambda^3\xi^3|\nabla\eta|^2\frac{\partial\eta}{\partial\nu}|\Delta w|^2 \quad \text{on } \Sigma. \tag{3.22}$$

Hence, combining Condition 3.1 and (3.18)–(3.22), we obtain

$$\begin{aligned} V_1\cdot\nu &\geq -Cs\lambda\xi|\nabla\Delta w|^2 - 8\sum_{i,j,k,l=1}^{n} s^3\lambda^3\xi^3\Big(\frac{\partial \eta}{\partial \nu}\Big)^3|\Delta w|^2 - 2s^3\lambda^3\xi^3|\nabla\eta|^2\frac{\partial\eta}{\partial\nu}|\Delta w|^2 \\ &\geq -Cs\lambda\xi|\nabla\Delta w|^2 - Cs^3\lambda^3\xi^3|\Delta w|^2 \quad \text{on } \Sigma. \end{aligned}$$

This implies that

$$\mathbb{E}\int_Q \operatorname{div} V_1 \mathrm{d}x\mathrm{d}t \geq -C\mathbb{E}\int_\Sigma s^3\lambda^3\xi^3|\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t - C\mathbb{E}\int_\Sigma s\lambda\xi|\nabla\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t. \tag{3.23}$$

Thanks to Condition 3.1, (2.16), (3.8) and (3.15), we have

$$\begin{aligned} &\mathbb{E}\int_Q \operatorname{div} V_2\mathrm{d}x\mathrm{d}t = \mathbb{E}\int_\Sigma V_2\cdot\nu\mathrm{d}\Gamma\mathrm{d}t \\ &= \mathbb{E}\int_\Sigma \sum_{j=1}^{n}\Big(\sum_{i,k,l,r,m=1}^{n}\Theta_1^{ijklrm}w_{x_ix_kx_l}w_{x_rx_m} + \sum_{i,k,l,r=1}\Theta_2^{ijklr}w_{x_ix_k}w_{x_lx_r} + \Theta_9^j\Big)\nu^j\mathrm{d}\Gamma\mathrm{d}t \\ &\geq \mathbb{E}\int_\Sigma [s\xi O(\lambda)|\nabla\Delta w|^2 + sO(\mathrm{e}^{C\lambda})|\nabla^2 w|^2]\mathrm{d}\Gamma\mathrm{d}t \\ &\geq -C\mathbb{E}\int_\Sigma s\lambda\xi|\nabla\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t + \mathcal{B}_2. \end{aligned} \tag{3.24}$$

Combining (3.14) and (3.23)–(3.24), we obtain

$$\begin{aligned} &2\mathbb{E}\int_Q \theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t)\mathrm{d}x + C\mathbb{E}\int_\Sigma s^3\lambda^3\xi^3|\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t \\ &+ C\mathbb{E}\int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + s^6\lambda^6\xi^6 g_1^2 + s^4\lambda^4\xi^4|\nabla g_1|^2 \\ &+ s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t \\ &+ C\mathbb{E}\int_\Sigma s\lambda\xi|\nabla\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t + \mathcal{A} + \mathcal{B}_2 \\ &\geq 2\mathbb{E}\int_Q I_2^2\mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q I_2I_3\mathrm{d}x + 2\mathbb{E}\int_Q (M_1+M_2)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.25}$$

Step 2 In this step, we study $2\mathbb{E}\int_Q (M_1+M_2)\mathrm{d}x\mathrm{d}t$ via integration by parts.

From Condition 3.1, (3.8) and (3.15), we get

$$\begin{aligned} &-\mathbb{E}\int_Q 16s^3\lambda^4\xi^3|\nabla\eta|^2|\nabla^2 w\nabla\eta|^2\mathrm{d}x\mathrm{d}t \\ &= -\mathbb{E}\int_Q 16s^3\lambda^4\xi^3\sum_{i,j,k,l=1}^{n}\eta_{x_i}^2\eta_{x_k}\eta_{x_l}w_{x_kx_j}w_{x_lx_j}\mathrm{d}x\mathrm{d}t \\ &= \mathbb{E}\int_\Sigma 8s^3\lambda^4\sum_{i,j,k,l=1}^{n}[-2\xi^3\eta_{x_i}^2\eta_{x_k}\eta_{x_l}w_{x_kx_j}w_{x_l} + (\xi^3\eta_{x_i}^2\eta_{x_k}\eta_{x_l})_{x_j}w_{x_k}w_{x_l}]\nu^j\mathrm{d}\Gamma\mathrm{d}t \end{aligned}$$

$$+\mathbb{E}\int_Q 8s^3\lambda^4\sum_{i,j,k,l=1}^n[2\xi^3\eta_{x_i}^2\eta_{x_k}\eta_{x_l}w_{x_kx_jx_j}w_{x_l}-(\xi^3\eta_{x_i}^2\eta_{x_k}\eta_{x_l})_{x_jx_j}w_{x_k}w_{x_l}]\mathrm{d}x\mathrm{d}t$$
$$\geq -4\mathbb{E}\int_Q s\lambda^2\xi|\nabla\Delta w\nabla\eta|^2\mathrm{d}x\mathrm{d}t-16\mathbb{E}\int_Q s^5\lambda^6\xi^5|\nabla\eta|^4|\nabla w\nabla\eta|^2\mathrm{d}x\mathrm{d}t-\mathcal{A}_2. \quad (3.26)$$

Thanks to (3.4) and Condition 3.1, we know that there esists $\lambda_1>0$ such that for all $\lambda\geq\lambda_1$, it holds that

$$\begin{aligned}&64\mathbb{E}\int_Q s^5\lambda^5\xi^5(\nabla^2\eta\nabla\eta\nabla\eta)|\nabla w\cdot\nabla\eta|^2\mathrm{d}x\mathrm{d}t\\&=128\mathbb{E}\int_Q s^5\lambda^5\xi^5|\nabla\eta|^2|\nabla w\cdot\nabla\eta|^2\mathrm{d}x\mathrm{d}t\\&\geq -\mathbb{E}\int_Q s^5\lambda^6\xi^5|\nabla\eta|^4|\nabla w\cdot\nabla\eta|^2\mathrm{d}x\mathrm{d}t. \quad (3.27)\end{aligned}$$

Combining Condition 3.1, (3.4), (3.8) and (3.15), we get

$$\begin{aligned}&\mathbb{E}\int_Q[4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)+2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta](|\nabla^2 w|^2-|\Delta w|^2)\mathrm{d}x\mathrm{d}t\\&=\frac{1}{2}\mathbb{E}\int_Q\Delta[4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)+2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta]|\nabla w|^2\mathrm{d}x\mathrm{d}t\\&\quad+\mathbb{E}\int_Q\nabla[4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)+2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta]\cdot\nabla w\Delta w\mathrm{d}x\mathrm{d}t\\&\geq-\mathbb{E}\int_Q s^2\xi^2O(\lambda^3)|\nabla^2 w|^2\mathrm{d}x\mathrm{d}t-\mathbb{E}\int_Q(s^3O(e^{C\lambda})+s^4\xi^4O(\lambda^5))|\nabla w|^2\mathrm{d}x\mathrm{d}t\\&\geq-\mathcal{A}_2-\mathcal{A}_3, \quad (3.28)\end{aligned}$$

and

$$\begin{aligned}&32\mathbb{E}\int_Q s^3\lambda^3\xi^3(\nabla^2 w\nabla\eta\nabla\eta)\sum_{i,j=1}^n\eta_{x_ix_j}w_{x_ix_j}\mathrm{d}x\mathrm{d}t\\&=32\sum_{i,j,k,l=1}^n\mathbb{E}\int_Q s^3\lambda^3\xi^3\eta_{x_kx_l}\eta_{x_i}\eta_{x_j}w_{x_ix_j}w_{x_kx_l}\mathrm{d}x\mathrm{d}t\\&=32\sum_{i,j,k,l=1}^n\mathbb{E}\int_\Sigma s^3\lambda^3\xi^3\eta_{x_kx_l}\eta_{x_i}\eta_{x_j}(w_{x_i}w_{x_kx_l}\nu^j-w_{x_i}w_{x_jx_l}\nu^k)\mathrm{d}\Gamma\mathrm{d}t\\&\quad-32\sum_{i,j,k,l=1}^n\mathbb{E}\int_Q s^3\lambda^3(\xi^3\eta_{x_kx_l}\eta_{x_i}\eta_{x_j})_{x_j}w_{x_i}w_{x_kx_l}\mathrm{d}x\mathrm{d}t\\&\quad+32\sum_{i,j,k,l=1}^n\mathbb{E}\int_Q s^3\lambda^3(\xi^3\eta_{x_kx_l}\eta_{x_i}\eta_{x_j})_{x_k}w_{x_i}w_{x_jx_l}\mathrm{d}x\mathrm{d}t\\&\quad+32\mathbb{E}\int_Q s^3\lambda^3\xi^3\nabla^2\eta(\nabla^2 w\nabla\eta)(\nabla^2 w\nabla\eta)\mathrm{d}x\mathrm{d}t\\&\geq 32\mathbb{E}\int_Q s^3\lambda^3\xi^3\nabla^2\eta(\nabla^2 w\nabla\eta)(\nabla^2 w\nabla\eta)\mathrm{d}x\mathrm{d}t-\mathcal{A}_2-\mathcal{A}_3\\&=64\mathbb{E}\int_Q s^3\lambda^3\xi^3|\nabla^2 w\nabla\eta|^2\mathrm{d}x\mathrm{d}t-\mathcal{A}_2-\mathcal{A}_3. \quad (3.29)\end{aligned}$$

From (3.8) and (3.15), we see

$$
\begin{aligned}
&\mathbb{E}\int_Q [(8s^3\lambda^4\xi^3 - s^3\lambda^{\frac{7}{2}}\xi^3)|\nabla\eta|^4|\Delta w|^2 - 2(8s^3\lambda^4\xi^3 - s^3\lambda^{\frac{7}{2}}\xi^3)|\nabla\eta|^6 s^2\lambda^2\xi^2|\nabla w|^2 \\
&\quad + (8s^3\lambda^4\xi^3 - s^3\lambda^{\frac{7}{2}}\xi^3)|\nabla\eta|^8 s^4\lambda^4\xi^4|w|^2]\mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q [(8s^3\lambda^4\xi^3 - s^3\lambda^{\frac{7}{2}}\xi^3)|\nabla\eta|^4(\Delta w + s^2\lambda^2\xi^2|\nabla\eta|^2 w)^2 \\
&\quad - 2(8s^3\lambda^4\xi^3 - s^3\lambda^{\frac{7}{2}}\xi^3)|\nabla\eta|^6 s^2\lambda^2\xi^2(|\nabla w|^2 + w\Delta w)]\mathrm{d}x\mathrm{d}t \\
&\geq \mathbb{E}\int_Q (8s^3\lambda^4\xi^3 - s^3\lambda^{\frac{7}{2}}\xi^3)|\nabla\eta|^4(\Delta w + s^2\lambda^2\xi^2|\nabla\eta|^2 w)^2\mathrm{d}x\mathrm{d}t - \mathcal{A}_1.
\end{aligned}
\tag{3.30}
$$

Noting that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for $a, b, c \in \mathbb{R}$, we find

$$
\begin{aligned}
&\mathbb{E}\int_Q s^3\lambda^4\xi^3\theta^2|\nabla\eta|^4|\Delta v|^2\mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q s^3\lambda^4\xi^3|\nabla\eta|^4(\Delta w - 2s\lambda\xi\nabla\eta\nabla w + s^2\lambda^2\xi^2|\nabla\eta|^2 w - s\lambda^2\xi|\nabla\eta|^2 w - s\lambda\xi|\Delta\eta|w)^2\mathrm{d}x\mathrm{d}t \\
&\leq 3\mathbb{E}\int_Q s^3\lambda^4\xi^3|\nabla\eta|^4(\Delta w + s^2\lambda^2\xi^2|\nabla\eta|^2 w)^2\mathrm{d}x\mathrm{d}t \\
&\quad + 12\mathbb{E}\int_Q s^5\lambda^6\xi^5|\nabla\eta|^4|\nabla w\cdot\nabla\eta|^2\mathrm{d}x\mathrm{d}t + \mathcal{A}_1.
\end{aligned}
\tag{3.31}
$$

Combining (3.26)–(3.31), we know there exists $\lambda_2 \geq \lambda_1$ such that for all $\lambda \geq \lambda_2$, it holds that

$$
\mathbb{E}\int_Q M_1\mathrm{d}x\mathrm{d}t + \mathcal{A} \geq \mathbb{E}\int_Q \lambda|\nabla\Delta w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q s^3\lambda^4\xi^3\theta^2|\nabla\eta|^4|\Delta v|^2\mathrm{d}x\mathrm{d}t. \tag{3.32}
$$

It follows from Condition 3.1, (2.14), (2.16), (3.4) and (3.8), that

$$
\mathbb{E}\int_Q M_2\mathrm{d}x\mathrm{d}t + \mathcal{A} \geq \mathbb{E}\int_Q (16s\lambda\xi|\nabla\widehat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\nabla\eta|^4\widehat{w}^2)\mathrm{d}x\mathrm{d}t. \tag{3.33}
$$

Thanks to (3.25) and (3.32)–(3.33), for $\lambda \geq \lambda_2$, we have

$$
\begin{aligned}
&2\mathbb{E}\int_Q \theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t)\mathrm{d}x + C\mathbb{E}\int_\Sigma s^3\lambda^3\xi^3|\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t \\
&\quad + C\mathbb{E}\int_\Sigma s\lambda\xi|\nabla\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t + \mathcal{A} + \mathcal{B}_2 \\
&\quad + C\mathbb{E}\int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + s^6\lambda^6\xi^6 g_1^2 \\
&\quad + s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t \\
&\geq 2\mathbb{E}\int_Q I_2^2\mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q I_2 I_3\mathrm{d}x \\
&\quad + \mathbb{E}\int_Q (\lambda|\nabla\Delta w|^2 + s^3\lambda^4\xi^3\theta^2|\Delta v|^2 + s\lambda\xi|\nabla\widehat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\widehat{w}^2)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.34}
$$

Step 3 In this step, we get the estimate of v.

From (3.5), we have

$$
2\mathbb{E}\int_Q \theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t)\mathrm{d}x \leq \mathbb{E}\int_Q I_2^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 f_2^2\mathrm{d}x\mathrm{d}t. \tag{3.35}
$$

Thanks to Condition 3.1, (2.17) and (3.8), we find

$$2\mathbb{E}\int_Q I_2I_3\mathrm{d}x \geq -\mathbb{E}\int_Q I_2^2\mathrm{d}x\mathrm{d}t - \mathcal{A} - C\mathbb{E}\int_Q s^2\lambda^2\xi^2\theta^2 f_1^2\mathrm{d}x\mathrm{d}t. \tag{3.36}$$

Noting that

$$|\nabla\Delta w|^2 \leq C\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\nabla^2 v|^2) \quad \text{ on } \Sigma,$$

by (3.15) and (3.17), there exists $\lambda_3 > 0$ such that for all $\lambda \geq \lambda_3$, there is $s_1 = s_1(\lambda) > 0$, such that for all $s \geq s_1$, we have

$$\begin{aligned}&\mathcal{B}_2 + \mathbb{E}\int_\Sigma s^3\lambda^3\xi^3|\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t + \mathbb{E}\int_\Sigma s\lambda\xi|\nabla\Delta w|^2\mathrm{d}\Gamma\mathrm{d}t\\ &\leq C\mathbb{E}\int_\Sigma s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2)\mathrm{d}\Gamma\mathrm{d}t.\end{aligned} \tag{3.37}$$

Thanks to (3.34)–(3.37), for $\lambda \geq \lambda_3$ and $s \geq s_1$, we get

$$\begin{aligned}&C\mathbb{E}\int_\Sigma s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2)\mathrm{d}\Gamma\mathrm{d}t + \mathcal{A}\\ &+ C\mathbb{E}\int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2\\ &+ s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t\\ \geq\ &\mathbb{E}\int_Q(\lambda|\nabla\Delta w|^2 + s^3\lambda^4\xi^3\theta^2|\Delta v|^2 + s\lambda\xi|\nabla\widehat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\widehat{w}^2)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.38}$$

By Lemma 3.1, for $\lambda \geq \max\{\lambda_3, \widehat{\lambda}\}$ and $s \geq \max\{s_1, \widehat{s}\}$, we obtain

$$\mathbb{E}\int_Q(s^6\lambda^8\xi^6\theta^2|v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2)\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q s^3\lambda^4\xi^3\theta^2|\Delta v|^2\mathrm{d}x\mathrm{d}t,$$

which, together with (3.38), implies

$$\begin{aligned}&C\mathbb{E}\int_\Sigma s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2)\mathrm{d}\Gamma\mathrm{d}t + \mathcal{A}\\ &+ C\mathbb{E}\int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2\\ &+ s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t\\ \geq\ &\mathbb{E}\int_Q(\lambda|\nabla\Delta w|^2 + s^3\lambda^4\xi^3\theta^2|\Delta v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2 + s^6\lambda^8\xi^6\theta^2|v|^2\\ &+ s\lambda\xi|\nabla\widehat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\widehat{w}^2)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.39}$$

Let $\widetilde{v} = s\lambda^2\xi\mathrm{e}^{s\xi}v$. Then, we have

$$\begin{aligned}&\mathbb{E}\int_Q s^2\lambda^4\xi^2\theta^2|\nabla^2 v|^2\mathrm{d}x\mathrm{d}t\\ =\ &\mathbb{E}\int_Q s^2\lambda^4\xi^2\theta^2|\nabla^2(s^{-1}\lambda^{-2}\xi^{-1}\theta^{-1}\widetilde{v})|^2\mathrm{d}x\mathrm{d}t\\ \leq\ &C\mathbb{E}\int_Q s^2\lambda^4\xi^2(s^{-2}\lambda^{-4}\xi^{-2}|\nabla^2\widetilde{v}|^2 + \lambda^{-2}|\nabla\widetilde{v}|^2 + s^2\xi^2|\widetilde{v}|^2)\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq C|\widetilde{v}|^2_{L^2_{\mathbb{F}}(0,T;H^2(G))} + C\mathbb{E}\int_Q s^2\lambda^2\xi^2|\nabla\widetilde{v}|^2\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_Q s^4\lambda^4\xi^4|\widetilde{v}|^2\mathrm{d}x\mathrm{d}t \\
&\leq C|\widetilde{v}|^2_{L^2_{\mathbb{F}}(0,T;H^2(G))} + C\mathbb{E}\int_Q s^2\lambda^2\xi^2(s^4\lambda^6\xi^4\theta^2v^2 + s^2\lambda^4\xi^2\theta^2|\nabla v|^2)\mathrm{d}x\mathrm{d}t \\
&\quad + C\mathbb{E}\int_Q s^6\lambda^8\xi^6\theta^2|v|^2\mathrm{d}x\mathrm{d}t \\
&\leq C|\widetilde{v}|^2_{L^2_{\mathbb{F}}(0,T;H^2(G))} + C\mathbb{E}\int_Q (s^6\lambda^8\xi^6\theta^2|v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2)\mathrm{d}x\mathrm{d}t. \qquad (3.40)
\end{aligned}
$$

It follows from $\widetilde{v} = 0$ on Σ that

$$
\begin{aligned}
&|\widetilde{v}|^2_{L^2_{\mathbb{F}}(0,T;H^2(G))} \\
&\leq C|\Delta\widetilde{v}|^2_{L^2_{\mathbb{F}}(0,T;L^2(G))} \\
&\leq C\mathbb{E}\int_Q (s^6\lambda^8\xi^6\theta^2|v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2 + s^2\lambda^4\xi^2\theta^2|\Delta v|^2)\mathrm{d}x\mathrm{d}t. \qquad (3.41)
\end{aligned}
$$

Combining (3.40) and (3.41), we obtain

$$
\begin{aligned}
&\mathbb{E}\int_Q s^2\lambda^4\xi^2\theta^2|\nabla^2 v|^2\mathrm{d}x\mathrm{d}t \\
&\leq C\mathbb{E}\int_Q (s^6\lambda^8\xi^6\theta^2|v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2 + s^2\lambda^4\xi^2\theta^2|\Delta v|^2)\mathrm{d}x\mathrm{d}t. \qquad (3.42)
\end{aligned}
$$

From (3.39) and (3.42), there exists $\lambda_4 \geq \max\{\lambda_3, \widehat{\lambda}\}$ such that for all $\lambda \geq \lambda_4$, there is an $s_2 = s_2(\lambda) > \max\{s_1, \widehat{s}\}$, such that for all $s \geq s_2$, we have that

$$
\begin{aligned}
&C\mathbb{E}\int_\Sigma s\lambda\xi\theta^2(|\nabla\Delta v|^2 + s^2\lambda^2\xi^2|\Delta v|^2)\mathrm{d}\Gamma\mathrm{d}t + \mathcal{A} \\
&+ C\mathbb{E}\int_Q \theta^2(s^6\lambda^6\xi^6 f_1^2 + s^4\lambda^4\xi^4|\nabla f_1|^2 + s^2\lambda^2\xi^2|\nabla^2 f_1|^2 + f_2^2 + s^6\lambda^6\xi^6 g_1^2 \\
&+ s^4\lambda^4\xi^4|\nabla g_1|^2 + s^2\lambda^2\xi^2|\nabla^2 g_1|^2 + |\nabla\Delta g_1|^2 + s^2\lambda^2\xi^2|g_2|^2)\mathrm{d}x\mathrm{d}t \\
\geq\ &\mathbb{E}\int_Q (\lambda|\nabla\Delta w|^2 + s^3\lambda^4\xi^3\theta^2|\Delta v|^2 + s^2\lambda^4\xi^2\theta^2|\nabla^2 v|^2 + s^4\lambda^6\xi^4\theta^2|\nabla v|^2 + s^6\lambda^8\xi^6\theta^2|v|^2 \\
&+ s\lambda\xi|\nabla\widehat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\widehat{w}^2)\mathrm{d}x\mathrm{d}t. \qquad (3.43)
\end{aligned}
$$

Recalling $v = \theta^{-1}w$ and $\widehat{v} = \theta^{-1}(\widehat{w} - \ell_t w)$, we get

$$
\begin{aligned}
&\mathbb{E}\int_Q \theta^2(\lambda|\nabla\Delta v|^2 + s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2)\mathrm{d}x\mathrm{d}t \\
=\ &\mathbb{E}\int_Q \theta^2(\lambda|\nabla\Delta(\theta^{-1}w)|^2 + s\lambda\xi|\nabla[\theta^{-1}(\widehat{w} - \ell_t w)]|^2 + s^3\lambda^{\frac{7}{2}}\xi^3\theta^{-2}|\widehat{w} - \ell_t w|^2)\mathrm{d}x\mathrm{d}t \\
\leq\ &C\mathbb{E}\int_Q [\lambda(|\nabla\Delta w|^2 + s^2\lambda^2\xi^2|\nabla^2 w|^2 + s^4\lambda^4\xi^4|\nabla w|^2 + s^6\lambda^6\xi^6|w|^2) + s^3\lambda^{\frac{7}{2}}\xi^3(|\widehat{w}|^2 + s^2\lambda^2\xi^2|w|^2) \\
&+ s\lambda\xi(|\nabla\widehat{w}|^2 + s^2\lambda^2\xi^2|\widehat{w}|^2 + s^2\lambda^2\xi^2|\nabla w|^2 + s^4\lambda^4\xi^4|w|^2)]\mathrm{d}x\mathrm{d}t \\
\leq\ &C\mathbb{E}\int_Q (\lambda|\nabla\Delta w|^2 + s\lambda\xi|\nabla\widehat{w}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{w}|^2)\mathrm{d}x\mathrm{d}t + \mathcal{A}. \qquad (3.44)
\end{aligned}
$$

Thanks to (3.8) and Condition 3.1, there exists $\lambda_5 > 0$ such that for all $\lambda \geq \lambda_5$, there is an $s_3 = s_3(\lambda) > 0$, such that for all $s \geq s_3$, we have

$$\mathcal{A} \leq \frac{1}{C}\mathbb{E}\int_Q \theta^2 (s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2 + s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)\mathrm{d}x\mathrm{d}t. \tag{3.45}$$

Let us choose $\lambda_0 \geq \max\{\lambda_4, \lambda_5\}$. Combining (3.43)–(3.45), for all $\lambda \geq \lambda_0$, one can find $s_0 = s_0(\lambda) \geq \max\{s_2, s_3\}$ so that for any $s \geq s_0$, inequality (3.7) holds.

4 Proof of the Observability Estimate

Proof of Theorem 1.3 Let $\chi \in C_0^\infty([0,T])$ satisfy

$$\chi = 1 \text{ in } \Big(\frac{T}{2} - \varepsilon_1 T, \frac{T}{2} + \varepsilon_1 T\Big).$$

Put $v = \chi z$ and $\widehat{v} = \chi\widehat{z} + \chi_t z$ for $(z, \widehat{z})$ satisfying (1.2), then $(v, \widehat{v})$ fulfills $v(0,\cdot) = v(T,\cdot) = \widehat{v}(0,\cdot) = \widehat{v}(T,\cdot) = 0$ in G, and solves

$$\begin{cases} \mathrm{d}v = \widehat{v}\mathrm{d}t + \chi(Z - a_5 z)\mathrm{d}W(t) & \text{in } Q,\\ \mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t = \widetilde{f}_2\mathrm{d}t + \widetilde{g}_2\mathrm{d}W(t) & \text{in } Q,\\ v = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \end{cases} \tag{4.1}$$

where

$$\widetilde{f}_2 = \chi[(a_1 - \operatorname{div} a_2 - a_4 a_5)z - a_2\nabla z - a_3\widehat{Z} + a_4 Z] + 2\chi_t\widehat{z} + \chi_{tt}z$$

and

$$\widetilde{g}_2 = \chi\widehat{Z} + \chi_t(Z - a_5 z).$$

By Theorem 3.1, for $\lambda \geq \lambda_0$ and $s \geq s_0$, we have

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2\chi^2(\lambda|\nabla\Delta z|^2 + s^2\lambda^4\xi^2|\nabla^2 z|^2 + s^3\lambda^4\xi^3|\Delta z|^2 + s^4\lambda^6\xi^4|\nabla z|^2 + s^6\lambda^8\xi^6|z|^2)\mathrm{d}x\mathrm{d}t\\
&\leq \mathbb{E}\int_Q \theta^2(s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2 + \lambda|\nabla\Delta v|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2\\
&\quad + s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)\mathrm{d}x\mathrm{d}t\\
&\leq C\mathbb{E}\int_Q \theta^2\chi^2(s^6\lambda^6\xi^6 z^2 + s^4\lambda^4\xi^4|\nabla z|^2 + s^2\lambda^2\xi^2|\nabla^2 z|^2 + |\nabla\Delta z|^2 + |z|^2 + |\nabla z|^2)\mathrm{d}x\mathrm{d}t\\
&\quad + C\mathbb{E}\int_{J_1} \theta^2(|\widehat{z}|^2 + z^2 + s^2\lambda^2\xi^2 z^2)\mathrm{d}x\mathrm{d}t + C(s,\lambda)|(Z,\widehat{Z})|^2_{L^2_{\mathbb{F}}(0,T;H^3(G))\times L^2_{\mathbb{F}}(0,T;L^2(G))}\\
&\quad + C\mathbb{E}\int_\Sigma \theta^2(s\lambda\xi|\nabla\Delta z|^2 + s^3\lambda^3\xi^3|\Delta z|^2)\mathrm{d}\Gamma\mathrm{d}t.
\end{aligned}$$

This, together with Condition 3.1, implies that there exists $\widetilde{\lambda}_1 \geq \lambda_0$ such that for all $\lambda \geq \widetilde{\lambda}_1$, there is $\widetilde{s}_1 = \widetilde{s}_1(\lambda) \geq s_0$, so that for any $s \geq \widetilde{s}_1$, it holds

$$\mathbb{E}\int_Q \theta^2(s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2 + \lambda|\nabla\Delta v|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2$$

$$
\begin{aligned}
&+ s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)\mathrm{d}x\mathrm{d}t\\
&\leq C\mathbb{E}\int_{J_1}\theta^2(|\widehat{z}|^2 + z^2 + s^2\lambda^2\xi^2 z^2)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_{\Sigma}\theta^2(s\lambda\xi|\nabla\Delta z|^2 + s^3\lambda^3\xi^3|\Delta z|^2)\mathrm{d}\Gamma\mathrm{d}t\\
&\quad + C(s,\lambda)|(Z,\widehat{Z})|^2_{L^2_{\mathbb{F}}(0,T;H^3(G))\times L^2_{\mathbb{F}}(0,T;H^1(G))}. \qquad (4.2)
\end{aligned}
$$

Thanks to Condition 3.1, we obtain

$$
\begin{aligned}
&\mathrm{e}^{2s\mathrm{e}^{\lambda\delta}-C\lambda-6\ln s}\mathbb{E}\int_{J_2}(|\nabla\Delta z|^2 + |\nabla^2 z|^2 + |\nabla z|^2 + z^2 + |\nabla\widehat{z}|^2 + |\widehat{z}|^2)\mathrm{d}x\mathrm{d}t\\
&\leq \mathrm{e}^{2s\mathrm{e}^{\lambda\delta}}\mathbb{E}\int_{J_2}(s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2 + \lambda|\nabla\Delta v|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2\\
&\quad + s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)\mathrm{d}x\mathrm{d}t\\
&\leq \mathbb{E}\int_Q\theta^2(s\lambda\xi|\nabla\widehat{v}|^2 + s^3\lambda^{\frac{7}{2}}\xi^3|\widehat{v}|^2 + \lambda|\nabla\Delta v|^2 + s^2\lambda^4\xi^2|\nabla^2 v|^2 + s^3\lambda^4\xi^3|\Delta v|^2\\
&\quad + s^4\lambda^6\xi^4|\nabla v|^2 + s^6\lambda^8\xi^6|v|^2)\mathrm{d}x\mathrm{d}t. \qquad (4.3)
\end{aligned}
$$

From (1.3) and Condition 3.1, we see

$$
\begin{aligned}
&|(z^{\mathrm{T}},\widehat{z}^{\mathrm{T}})|_{L^2_{\mathcal{F}_T}(\Omega;H^3(G)\cap H^2_0(G))\times L^2_{\mathcal{F}_T}(\Omega;H^1_0(G))}\\
&\leq C\mathbb{E}\int_{J_2}(|\nabla\Delta z|^2 + |\nabla^2 z|^2 + |\nabla z|^2 + z^2 + |\nabla\widehat{z}|^2 + |\widehat{z}|^2)\mathrm{d}x\mathrm{d}t\\
&\quad + C(s,\lambda)|(Z,\widehat{Z})|^2_{L^2_{\mathbb{F}}(0,T;H^3(G)\cap H^2_0(G))\times L^2_{\mathbb{F}}(0,T;H^1_0(G))}, \qquad (4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_{J_1}\theta^2(|\widehat{z}|^2 + z^2 + s^2\lambda^2\xi^2 z^2)\mathrm{d}x\mathrm{d}t\\
&\leq \mathrm{e}^{2s\mathrm{e}^{-\lambda\delta}+C\lambda+2\ln s}\mathbb{E}\int_{J_1}(|\widehat{z}|^2 + z^2)\mathrm{d}x\mathrm{d}t\\
&\leq C\mathrm{e}^{2s\mathrm{e}^{-\lambda\delta}+C\lambda+2\ln s}|(z^{\mathrm{T}},\widehat{z}^{\mathrm{T}})|_{L^2_{\mathcal{F}_T}(\Omega;H^3(G)\cap H^2_0(G))\times L^2_{\mathcal{F}_T}(\Omega;H^1_0(G))}\\
&\quad + C(s,\lambda)|(Z,\widehat{Z})|^2_{L^2_{\mathbb{F}}(0,T;H^3(G)\cap H^2_0(G))\times L^2_{\mathbb{F}}(0,T;H^1_0(G))}. \qquad (4.5)
\end{aligned}
$$

Combining (4.2)–(4.5), choosing $\lambda \geq \widetilde{\lambda}_1$ and $s \geq \widetilde{s}_1$ such

$$C\exp(2s\mathrm{e}^{-\lambda\delta} - 2s\mathrm{e}^{\lambda\delta} + C\lambda + 8\ln s) \leq \frac{1}{2},$$

we get the desired observability estimate.

5 Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to that for [24, Theorem 2.3]. We provide it here for the convenience of the readers. To begin with, we recall the following result.

Lemma 5.1 (see [26, Lemma 2.1]) *There exists a random variable $\zeta \in L^2_{\mathcal{F}_T}(\Omega)$ such that it is impossible to find $\varsigma_1, \varsigma_2 \in L^2_{\mathbb{F}}(0,T) \times C_{\mathbb{F}}([0,T];L^2(\Omega))$ and $\alpha \in \mathbb{R}$ satisfying*

$$\zeta = \alpha + \int_0^T \varsigma_1(t)\mathrm{d}t + \int_0^T \varsigma_2(t)\mathrm{d}W(t).$$

Proof of Theorem 1.2 We employ a contradiction argument and divide the proof into three cases.

Case 1 $a_3 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega))$, $G\backslash\overline{G_0} \neq \emptyset$ and $\operatorname{supp} f \subset G_0$.

Let $\rho \in C_0^\infty(G\backslash G_0)$ satisfying $|\rho|_{L^2(G)} = 1$. Suppose that (1.1) was exactly controllable. By Definition 1.2, for $(y_0, \widehat{y}_0) = (0,0)$, there exist controls (f, g, h_1, h_2) with $\operatorname{supp} f \subset G_0$ a.e. $(t,\omega) \in (0,T) \times \Omega$ such that the solution to (1.1) fulfills $(y(T), \widehat{y}(T)) = (\rho\zeta, 0)$, where ζ is given in Lemma 5.1. Hence,

$$\rho\zeta = \int_0^T \widehat{y}\mathrm{d}t + \int_0^T (a_3 y + f)\mathrm{d}W(t). \tag{5.1}$$

Multiplying (5.1) by ρ and integrating it in G, we arrive that

$$\zeta = \int_0^T \langle \widehat{y}, \rho\rangle_{(H^3(G)\cap H_0^2(G))^*, H^3(G)\cap H_0^2(G)}\mathrm{d}t + \int_0^T \langle a_3 y, \rho\rangle_{H^{-1}(G), H_0^1(G)}\mathrm{d}W(t),$$

which contradicts Lemma 5.1.

Case 2 $a_4 \in C_{\mathbb{F}}([0,T]; L^\infty(\Omega))$, $G\backslash\overline{G_0} \neq \emptyset$ and $\operatorname{supp} g \subset G_0$.

Choose ρ as in Case 1. Assume that (1.1) was exactly controllable. Then, for $(y_0, \widehat{y}_0) = (0,0)$, there exist controls (f, g, h_1, h_2) with $\operatorname{supp} g \subset G_0$ a.e. $(t,\omega) \in (0,T) \times \Omega$ such that the solution to (1.1) fulfills $(y(T), \widehat{y}(T)) = (0, \zeta)$.

Clearly, $(\phi, \widehat{\phi}) \overset{\Delta}{=} (\rho y, \rho\widehat{y})$ satisfies

$$\begin{cases} \mathrm{d}\phi = \widehat{\phi}\mathrm{d}t + (a_3\phi + \rho f)\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\widehat{\phi} + \Delta^2\phi\mathrm{d}t = \widetilde{f}_2\mathrm{d}t + a_4\phi\mathrm{d}W(t) & \text{in } Q, \\ \phi = \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Sigma, \\ (\phi(0), \widehat{\phi}(0)) = (0,0) & \text{in } G, \end{cases}$$

where $\widetilde{f}_2 = [\Delta^2, \rho]y + a_1\phi + \rho a_2 \cdot \nabla\phi$. Furthermore, we have $(\phi(T), \widehat{\phi}(T)) = (0, \rho\zeta)$. Hence, we have

$$\zeta = -\int_0^T (\langle \Delta^2\phi, \rho\rangle_{H^{-5}(G), H_0^5(G)} + \langle \widetilde{f}_2, \rho\rangle_{H^{-4}(G), H_0^4(G)})\mathrm{d}t + \int_0^T \langle a_3\phi, \rho\rangle_{H^{-1}(G), H_0^1(G)}\mathrm{d}W(t),$$

which contradicts Lemma 5.1.

Case 3 $h_1 = h_2 = 0$.

Assume that (1.1) was exactly controllable. Then, from the equivalence between the exact controllability of (1.1) and the observability estimate of (1.2), we get that for any $(z^T, \widehat{z}^T) \in L^2_{\mathcal{F}_T}(\Omega; H^3(G) \cap H_0^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H_0^1(G))$, the solution $(z, Z, \widehat{z}, \widehat{Z})$ to (1.2) (with $\tau = T$ and $(z(T), \widehat{z}(T)) = (z^T, \widehat{z}^T)$) satisfies

$$\begin{aligned} &|(z^T, \widehat{z}^T)|_{L^2_{\mathcal{F}_T}(\Omega; H^3(G)\cap H_0^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H_0^1(G))} \\ &\leq C(|Z|_{L^2_{\mathbb{F}}(0,T; H^3(G)\cap H_0^2(G))} + |\widehat{Z}|_{L^2_{\mathbb{F}}(0,T; H_0^1(G))}). \end{aligned} \tag{5.2}$$

For any nonzero $(\Phi_0, \Phi_1) \in (H^3(G) \cap H_0^2(G)) \times H_0^1(G)$, let $(\Phi, \widehat{\Phi})$ solve the equation

$$\begin{cases} \mathrm{d}\Phi = \widehat{\Phi}\mathrm{d}t - a_5\Phi\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}\widehat{\Phi} + \Delta^2\Phi\mathrm{d}t = [(a_1 - \operatorname{div} a_2 - a_4a_5)\Phi - a_2 \cdot \nabla\Phi]\mathrm{d}t & \text{in } Q, \\ \Phi = \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on } \Sigma, \\ (\Phi(0), \widehat{\Phi}(0)) = (\Phi_0, \Phi_1) & \text{in } G. \end{cases}$$

Clearly, $(\Phi, 0, \widehat{\Phi}, 0)$ solves (1.2) with the final datum $(z^T, \widehat{z}^T) = (\Phi(T), \widehat{\Phi}(T))$, a contradiction to (5.2).

Appendix A Proof of the Weighted Identity

Proof of Theorem 2.1 It is clear that

$$\mathrm{d}w = \mathrm{d}(\theta v) = \theta\mathrm{d}v + \ell_t\theta v\mathrm{d}t = \widehat{w}\mathrm{d}t + \theta f_1\mathrm{d}t + \theta g_1\mathrm{d}W(t)$$

and

$$\begin{aligned} \theta\mathrm{d}\widehat{v} &= \theta\mathrm{d}[\theta^{-1}(\widehat{w} - \ell_t w)] = \mathrm{d}\widehat{w} - \ell_t\mathrm{d}w - \ell_{tt}w\mathrm{d}t - \ell_t\widehat{w}\mathrm{d}t + \ell_t^2 w\mathrm{d}t \\ &= \mathrm{d}\widehat{w} - 2\ell_t\widehat{w}\mathrm{d}t + (\ell_t^2 - \ell_{tt})w\mathrm{d}t - \ell_t\theta f_1\mathrm{d}t - \ell_t\theta g_t\mathrm{d}W(t). \end{aligned} \tag{A.1}$$

We also have

$$\begin{aligned} \theta\Delta^2 v = {} & \Delta^2 w - 4s\lambda\xi\nabla\eta \cdot \nabla\Delta w - 4s\lambda^2\xi(\nabla^2 w\nabla\eta\nabla\eta) - 4s\lambda\xi\sum_{i,j=1}^n \eta_{x_ix_j}w_{x_ix_j} \\ & + 2s^2\lambda^2\xi^2|\nabla\eta|^2\Delta w - 2s\lambda^2\xi|\nabla\eta|^2\Delta w - 2s\lambda\xi\Delta\eta\Delta w + 4s^2\lambda^2\xi^2(\nabla^2 w\nabla\eta\nabla\eta) \\ & - 4\nabla\Delta\ell \cdot \nabla w + 12s^2\lambda^3\xi^2|\nabla\eta|^2\nabla\eta\nabla w + 8s^2\lambda^2\xi^2(\nabla^2\eta\nabla\eta\nabla w) \\ & - 4s^3\lambda^3\xi^3|\nabla\eta|^2\nabla\eta\nabla w + 4s^2\lambda^2\xi^2\Delta\eta\nabla\eta\nabla w + 4(\nabla\ell \cdot \nabla\Delta\ell)w + 2|\nabla^2\ell|^2 w - \Delta^2\ell w \\ & - 6s^3\lambda^4\xi^3|\nabla\eta|^4 w - 4s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)w + s^4\lambda^4\xi^4|\nabla\eta|^4 w \\ & - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta w + |\Delta\ell|^2 w. \end{aligned} \tag{A.2}$$

From (A.1)–(A.2) and (2.13)–(2.17), we have

$$2\theta I_2(\mathrm{d}\widehat{v} + \Delta^2 v\mathrm{d}t) = 2I_2(I_1 + I_2\mathrm{d}t + I_3).$$

We will compute I_1I_2 under the form $\sum\limits_{i=1}^7\sum\limits_{j=1}^5 I_{ij}$, where I_{ij} is the product of the i-th term of I_1 with the j-th term of I_2. Note that I_{ij} are the same as [23, Appendix A] for $i = 1, \cdots, 6$ and $j = 1, \cdots, 5$, except for I_{13}.

We have

$$\begin{aligned} I_{13} &= \sum_{i,j,k,l=1}^n \Phi_3^{kl} w_{x_ix_ix_jx_j} w_{x_kx_l}\mathrm{d}t \\ &= \sum_{i,j,k,l=1}^n \Big(\Phi_3^{kl} w_{x_ix_ix_j} w_{x_kx_l} - \Phi_3^{kj} w_{x_ix_ix_l} w_{x_kx_l} + \Phi_3^{kl} w_{x_ix_ix_l} w_{x_kx_j} - \Phi_{3_{x_i}}^{kl} w_{x_ix_j} w_{x_kx_l} \end{aligned}$$

$$+ \Phi_{3_{x_l}}^{kj} w_{x_i x_l} w_{x_i x_k} + \Phi_{3_{x_l}}^{kl} w_{x_i x_j} w_{x_i x_k} - \frac{1}{2}\Phi_{3_{x_l}}^{jl} w_{x_i x_k}^2 - \Phi_{3_{x_i}}^{kl} w_{x_j x_l} w_{x_i x_k}\Big)_{x_j} \mathrm{d}t$$
$$- \sum_{i,j,k,l=1}^{n} \Phi_3^{kl} w_{x_i x_i x_l} w_{x_k x_j x_j} \mathrm{d}t + \sum_{i,j,k,l=1}^{n} \Phi_{3x_i x_j}^{kl} w_{x_i x_j} w_{x_k x_l} \mathrm{d}t - \sum_{i,j,k,l=1}^{n} \Phi_{3x_l x_j}^{kl} w_{x_i x_j} w_{x_i x_k} \mathrm{d}t$$
$$- \sum_{i,j,k,l=1}^{n} \Phi_{3x_i x_l}^{kl} w_{x_i x_j} w_{x_k x_j} \mathrm{d}t + \sum_{i,j,k,l=1}^{n} \Phi_{3x_i x_j}^{kl} w_{x_i x_l} w_{x_k x_j} \mathrm{d}t + \frac{1}{2}\sum_{i,j,k,l=1}^{n} \Phi_{3x_k x_l}^{kl} w_{x_i x_j}^2 \mathrm{d}t.$$

We have

$$\begin{aligned}
I_{71} &= \sum_{i,j=1}^{n} \Phi_1^i w_{x_i x_j x_j} \mathrm{d}\widehat{w} \\
&= \sum_{i,j=1}^{n} \Big(- \Phi_1^j \widehat{w} \mathrm{d}w_{x_i x_i} + \Phi_{1x_i}^i \widehat{w}\widehat{w}_{x_j} - \frac{1}{2}\Phi_{1x_i x_j}^i \widehat{w}^2 + \Phi_1^i \widehat{w}_{x_i}\widehat{w}_{x_j} - \frac{1}{2}\Phi_1^j \widehat{w}_{x_i}^2 \Big)_{x_j} \mathrm{d}t \\
&\quad + \sum_{i,j=1}^{n} \mathrm{d}(\Phi_1^i w_{x_i x_j x_j}\widehat{w}) + \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{1x_i x_j x_j}^i \widehat{w}^2 \mathrm{d}t - \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{1x_i}^i \widehat{w}_{x_j}^2 \mathrm{d}t - \sum_{i,j=1}^{n} \Phi_{1x_j}^i \widehat{w}_{x_i}\widehat{w}_{x_j} \mathrm{d}t \\
&\quad - \sum_{i,j=1}^{n} \Phi_{1t}^i w_{x_i x_j x_j}\widehat{w} \mathrm{d}t + \sum_{i,j=1}^{n} (\Phi_{1x_i}^i \widehat{w} + \Phi_1^i \widehat{w}_{x_i})[(\theta f_1)_{x_j x_j} \mathrm{d}t + (\theta g_1)_{x_j x_j} \mathrm{d}W(t)] \\
&\quad - \sum_{i,j=1}^{n} \Phi_1^i \mathrm{d}w_{x_i x_j x_j} \mathrm{d}\widehat{w}, \\
I_{72} &= \sum_{i=1}^{n} \Phi_2 w_{x_i x_i} \mathrm{d}\widehat{w} \\
&= \sum_{j=1}^{n} \Big(- \Phi_2 \widehat{w}\widehat{w}_{x_j} + \frac{1}{2}\Phi_{2x_j}\widehat{w}^2 \Big)_{x_j} \mathrm{d}t + \sum_{i=1}^{n} \mathrm{d}(\Phi_2 w_{x_i x_i}\widehat{w}) - \sum_{i=1}^{n} \Phi_{2t} w_{x_i x_i}\widehat{w} \mathrm{d}t \\
&\quad - \frac{1}{2}\sum_{i=1}^{n} \Phi_{2x_i x_i}\widehat{w}^2 \mathrm{d}t + \sum_{i=1}^{n} \Phi_2 \widehat{w}_{x_i}^2 \mathrm{d}t - \sum_{i=1}^{n} \Phi_2 \mathrm{d}w_{x_i x_i} \mathrm{d}\widehat{w} \\
&\quad - \sum_{i=1}^{n} \Phi_2 \widehat{w}[(\theta f_1)_{x_i x_i} \mathrm{d}t + (\theta g_1)_{x_i x_i} \mathrm{d}W(t)], \\
I_{73} &= \sum_{i,j=1}^{n} \Phi_3^{ij} w_{x_i x_j} \mathrm{d}\widehat{w} \\
&= \sum_{i,j=1}^{n} \Big(- \Phi_3^{ij} \widehat{w}\widehat{w}_{x_i} + \frac{1}{2}\Phi_{3x_i}^{ij}\widehat{w}^2 \Big)_{x_j} \mathrm{d}t + \sum_{i,j=1}^{n} \mathrm{d}(\Phi_3^{ij} w_{x_i x_j}\widehat{w}) - \sum_{i,j=1}^{n} \Phi_{3t}^{ij} w_{x_i x_j}\widehat{w} \mathrm{d}t \\
&\quad - \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{3x_i x_j}^{ij}\widehat{w}^2 \mathrm{d}t - \sum_{i,j=1}^{n} \Phi_3^{ij} \mathrm{d}w_{x_i x_j} \mathrm{d}\widehat{w} + \sum_{i,j=1}^{n} \Phi_3^{ij} \widehat{w}_{x_i}\widehat{w}_{x_j} \mathrm{d}t \\
&\quad - \sum_{i,j=1}^{n} \Phi_3^{ij} \widehat{w}[(\theta f_1)_{x_i x_j} \mathrm{d}t + (\theta g_1)_{x_i x_j} \mathrm{d}W(t)], \\
I_{74} &= \sum_{i=1}^{n} \Phi_4^i w_{x_i} \mathrm{d}\widehat{w}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \Big(-\frac{1}{2}\Phi_4^j \widehat{w}^2 \Big)_{x_j} \mathrm{d}t + \sum_{i=1}^{n} \mathrm{d}(\Phi_4^i w_{x_i} \widehat{w}) - \sum_{i=1}^{n} \Phi_{4t}^i w_{x_i} \widehat{w} \mathrm{d}t + \frac{1}{2}\sum_{i=1}^{n} \Phi_{4x_i}^i \widehat{w}^2 \mathrm{d}t \\
&\quad - \sum_{i=1}^{n} \Phi_4^i \mathrm{d}w_{x_i} \mathrm{d}\widehat{w} - \sum_{i=1}^{n} \Phi_4^i \widehat{w}[(\theta f_1)_{x_i} \mathrm{d}t + (\theta g_1)_{x_i} \mathrm{d}W(t)],
\end{aligned}
$$

$$
I_{75} = \Phi_5 w \mathrm{d}\widehat{w} = \mathrm{d}(\Phi_5 w \widehat{w}) - \Phi_{5t} w \widehat{w} \mathrm{d}t - \Phi_5 \widehat{w}^2 \mathrm{d}t - \Phi_5 \mathrm{d}w \mathrm{d}\widehat{w} - \Phi_5 \widehat{w}(\theta f_1 \mathrm{d}t + \theta g_1 \mathrm{d}W(t)).
$$

By summing all the I_{ij}, we get (2.12).

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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