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Structural Stability of 3D Axisymmetric Steady Subsonic Euler Flows in Finitely Long Nozzles with Variable End Pressures*

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Abstract As a continuation of [Li, J. and Wang, Y. N., Structural stability of steady subsonic Euler flows in 2D finitely long nozzles with variable end pressure, *J. Differential Equations*, **413**, 2014, 70–109], in this paper, the authors study the structural stability of three dimensional axisymmetric steady subsonic Euler flows in finitely long curved nozzles. The reference flow is a general subsonic shear flow in a three dimensional regular cylindrical nozzle with general size of vorticity and without stagnation points. The problem is described by the well-known steady compressible Euler system. With a class of admissible physical conditions and prescribed pressure at the entrance and the exit of the nozzle respectively, they establish the structural stability of this kind of axisymmetric subsonic shear flow with no-zero swirl velocity. Due to the hyperbolic-elliptic coupled form of the Euler system in subsonic regions, the problem is reformulated via a twofold normalized process, including straightening the lateral boundary of the nozzle under the natural Cartesian coordinates and reformulating the problem under the cylindrical coordinates. Accordingly, the Euler system is decoupled into an elliptic mode and three hyperbolic modes with some artificial singular terms under the cylindrical coordinates. The elliptic mode is a mixed type boundary value problem of first order elliptic system for the pressure and the radial velocity angle. Meanwhile, the hyperbolic modes are transport type to control the total energy, the specific entropy and the swirl velocity, respectively. The estimates as well as well-posedness are executed in a Banach space with optimal regularity under the natural Cartesian coordinates in place of the cylindrical coordinates. The authors develop a systematic framework to deal with the artificial singularity and the non-zero swirl velocity in three dimensional axisymmetric case. Their strategy is helpful for other three dimensional problems under axisymmetry.

Keywords Steady compressible Euler system, Subsonic shear flow, First order elliptic system, Structural stability, Axisymmetry
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1 Introduction

As the continuation of [20], in this paper, we consider the unique existence and structural stability of three dimensional axisymmetric steady compressible subsonic flows in finitely long and slightly curved nozzles with a class of admissible physical boundary conditions. The flow

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is governed by the following three dimensional steady compressible Euler system

$$\begin{cases}\partial_{x_1}(\rho u_1)+\partial_{x_2}(\rho u_2)+\partial_{x_3}(\rho u_3)=0,\\ \partial_{x_1}(\rho u_1u_1)+\partial_{x_2}(\rho u_1u_2)+\partial_{x_3}(\rho u_1u_3)+\partial_{x_1}\boldsymbol{P}=0,\\ \partial_{x_1}(\rho u_1u_2)+\partial_{x_2}(\rho u_2u_2)+\partial_{x_3}(\rho u_2u_3)+\partial_{x_2}\boldsymbol{P}=0,\\ \partial_{x_1}(\rho u_1u_3)+\partial_{x_2}(\rho u_2u_3)+\partial_{x_3}(\rho u_3u_3)+\partial_{x_2}\boldsymbol{P}=0,\\ \partial_{x_1}(\rho u_1\boldsymbol{E})+\partial_{x_2}(\rho u_2\boldsymbol{E})+\partial_{x_3}(\rho u_3\boldsymbol{E})=0,\end{cases}\tag{1.1}$$

where $u=(u_1,u_2,u_3)$, ρ and $\boldsymbol{s}$ are independent unknowns of $x=(x_1,x_2,x_3)$, standing for the velocity field, the density and the specific entropy of the flow, respectively. The pressure $\boldsymbol{P}=\boldsymbol{P}(\rho,\boldsymbol{s})$, the internal energy $\boldsymbol{e}=\boldsymbol{e}(\rho,\boldsymbol{s})$ and the total energy

$$\boldsymbol{E}=\frac{1}{2}|u|^2+\boldsymbol{e}+\frac{\boldsymbol{P}}{\rho}\tag{1.2}$$

are smooth with respect to their arguments and $\partial_\rho\boldsymbol{P}>0$, $\partial_\rho\boldsymbol{e}>0$ for $\rho>0$. As usual, we use the quantity $\boldsymbol{c}(\rho,\boldsymbol{s})=\sqrt{\partial_\rho\boldsymbol{P}}$ to denote the local sound speed. It is well known that, in regions where $|u|<\boldsymbol{c}(\rho,\boldsymbol{s})$, the flow is subsonic and the Euler system (1.1) is hyperbolic-elliptic coupled.

In this paper, we just consider the flow as polytropic gas and the corresponding equations of state are

$$\boldsymbol{P}(\rho,\boldsymbol{s})=A(\boldsymbol{s})\rho^\gamma,\quad \boldsymbol{e}(\rho,\boldsymbol{s})=\frac{\boldsymbol{P}(\rho,\boldsymbol{s})}{(\gamma-1)\rho},\quad \rho>0,\tag{1.3}$$

where $\gamma\in(1,3)$ is the adiabatic exponent and $A(\boldsymbol{s})$ is a positive smooth function of $\boldsymbol{s}$.

1.1 Structural stability issue in 3D axisymmetric case

For the aim of structural stability issue, the reference steady subsonic flow is considered as shear flow with general size of vorticity, moving in a three dimensional regular cylindrical nozzle N_b as

$$N_b=\{(x_1,x')\in\mathbb{R}^3: x'=(x_2,x_3)\in B_1(0),\ 0<x_1<\ell\}$$

for any fixed $\ell>0$ and $B_1(0)$ being the unit disk in $\mathbb{R}^2$. Here, we use

$$\Sigma_0=\{(0,x'):x'\in B_1(0)\},\quad \Sigma_\ell=\{(\ell,x'):x'\in B_1(0)\},\quad \Gamma_0=\partial N_b\backslash(\Sigma_0\cup\Sigma_\ell)\tag{1.4}$$

to denote the entrance, the exit and the lateral boundary of N_b, respectively.

We use $u_b=(u_{1b},u_{2b},u_{3b})$, $\boldsymbol{s}_b$ and ρ_b to denote the velocity, the specific entropy and the density of the reference subsonic shear flow respectively, which satisfy

$$\begin{cases}u_{1b}=v_{zb}(|x'|),\quad u_{2b}=u_{3b}=0,\\ \boldsymbol{s}_b,\rho_b \text{ are positive constants}\end{cases}\tag{1.5}$$

with the subsonic restriction $0<v_{zb}(|x'|)<\boldsymbol{c}(\rho_b,\boldsymbol{s}_b)$ for $x'\in B_1(0)$. Here, $v_{zb}(|x'|)>0$ means that the reference subsonic flow has no stagnation point.

Structural stability issue in 3D axisymmetric case Is the reference subsonic flow (1.5) structurally stable under the perturbations on at least one of the following parts: (1) The total energy, the specific entropy and the velocity angle at Σ_0, (2) the pressure at Σ_ℓ, (3) the lateral boundary Γ_0?

1.2 Mathematical formulation and main results

We consider the possible perturbed flows moving in the following three dimensional nozzle

$$N = \{(x_1, x') \in \mathbb{R}^3 : \sqrt{x_2^2 + x_3^2} \leq g(x_1),\ 0 < x_1 < \ell\}. \tag{1.6}$$

Here, the nozzle N is a small perturbation of N_b, i.e.,

$$g(0) = g(\ell) = 1, \quad \|g - 1\|_{2,\alpha} \leq \varepsilon \tag{1.7}$$

for some $\alpha \in (0,1)$ and $\varepsilon > 0$ to be specific later. Hereafter, $\|\cdot\|_{k,\alpha}$ stands for the usual $C^{k,\alpha}$ norm on the related domain. We still use Σ_0, Σ_ℓ in (1.4) and $\Gamma = \partial N \backslash (\Sigma_0 \cup \Sigma_\ell)$ to denote the entrance, the exit and the lateral boundary of the nozzle N, respectively.

On the lateral boundary Γ, the flow satisfies the usual slip boundary condition for compressible flow

$$-g(x_1)g'(x_1)u_1(x) + x_2 u_2(x) + x_3 u_3(x) = 0, \quad x \in \Gamma. \tag{1.8}$$

At the entrance Σ_0, we pose a class of admissible physical boundary conditions for the axisymmetric case

$$\begin{cases} \Big(\dfrac{x_2}{|x'|}\dfrac{u_2}{u_1} + \dfrac{x_3}{|x'|}\dfrac{u_3}{u_1}\Big)(0, x') = v_{r0}(|x'|), \\ \Big(-\dfrac{x_3}{|x'|}u_2 + \dfrac{x_3}{|x'|}u_3\Big)(0, x') = v_{\theta 0}(|x'|), \quad x' \in B_1(0) \\ \boldsymbol{E}(0, x') = \boldsymbol{E}_0(|x'|), \quad \boldsymbol{s}(0, x') = \boldsymbol{s}_0(|x'|), \end{cases} \tag{1.9}$$

with the naturally continuous constraint $v_{r0}(1) = g'(0)$ due to (1.7)–(1.8). Here, $(1.9)_1$ is the flow angle of the radial component of the velocity and $(1.9)_2$ is the swirl component of the velocity while u_1 is usual called as the axis component of the velocity.

At the exit Σ_ℓ, the end pressure is prescribed as

$$\boldsymbol{P}(\ell, x') = P_\ell(|x'|), \quad x' \in B_1(0). \tag{1.10}$$

Additionally, in order to avoid the elaborate framework of weight Hölder spaces, we pose the following conditions to simplify the proof process appropriately,

$$v_{\theta 0}(1) = g'(\ell) = g''(\ell) = P'_\ell(1) = 0. \tag{1.11}$$

Under the presentations above, the structural stability issue in 3D axisymmetric case can be formulated mathematically as the well-posedness of the problem (1.1) with (1.2)–(1.3) and (1.8)–(1.10) in N, which can be roughly stated as the following result.

Theorem 1.1 *Under the assumption* (1.11)*, when* $\|v_{zb}\|_{2,\alpha}$ *is bounded, there exist two positive constants* ε_0 *and* c_0 *depending on the state of the reference subsonic flow in* (1.5)*, if*

$$\|(\boldsymbol{E}_0, \boldsymbol{s}_0, v_{r0}, v_{\theta 0}) - (\boldsymbol{E}_b, \boldsymbol{s}_b, 0, 0)\|_{1,\alpha;\Sigma_0} + \|P_\ell - \boldsymbol{P}_b\|_{1,\alpha;\Sigma_\ell} \leq \varepsilon \tag{1.12}$$

with $\boldsymbol{P}_b = \boldsymbol{P}(\rho_b, \boldsymbol{s}_b)$ *and* $0 < \varepsilon \leq \varepsilon_0$*, then the problem* (1.1) *with* (1.2)–(1.3) *and* (1.8)–(1.11) *has a unique axisymmetric subsonic solution* $(u, \rho, \boldsymbol{s}) \in [C^{1,\alpha}(\overline{N})]^5$*, which satisfies*

$$\|(u, \rho, \boldsymbol{s}) \circ \boldsymbol{m}^{-1} - (u_b, \rho_b, \boldsymbol{s}_b)\|_{1,\alpha;\overline{N_b}} \leq c_0 \varepsilon, \tag{1.13}$$

where the invertible transformation $\boldsymbol{m} : N \to N_b$ *is defined as*

$$\boldsymbol{m}(x) = \Big(x_1, \frac{x_2}{g(x_1)}, \frac{x_3}{g(x_1)}\Big), \quad x \in N,$$

and its inverse is denoted as $\boldsymbol{m}^{-1} : N_b \to N$*.*

Remark 1.1 The assumption (1.11) can be compared with the following two facts: (1) In three dimensional axisymmetriy, one can consider the vanish of swirl velocity. The assumption $v_{\theta 0}(1) = 0$ is the more general case than $v_{\theta 0}(|x'|) \equiv 0$. (2) In the case of the straight nozzle of N, i.e., $g(x_1) = 1$, the $P_\ell'(1) = 0$ is the necessary compatible condition for $C^{1,\alpha}$-Hölder regular solution. Thus, the assumption $g'(\ell) = g''(\ell) = P_\ell'(1) = 0$ is also the more general case. Actually, the assumption (1.11) can be removed if we consider Theorem 1.1 in the weight Hölder spaces as in [20].

Remark 1.2 Except (1.11) for the simplification, Theorem 1.1 (Theorem 2.1) does not need any more compatible conditions on the boundary conditions. In this situation, we establish the main result in $C^{1,\alpha}(\overline{N})$ space, which is with the optimal regularity.

Remark 1.3 To realize the optimal regularity, we reformulate the main problem via a twofold normalized process. The first process is to straight the lateral boundary of the nozzle under the natural Cartesian coordinates and translate the problem to the domain N_b. Continuously, the second process is to formulate the problem under the cylindrical coordinates. In this way, the Euler system can be decoupled into a first order elliptic system for the pressure $\boldsymbol{P}$ and the flow angle of the radial velocity, and three transport equations for the total energy, the specific entropy and the swirl velocity. This decomposition has a good intrinsic structure for the construction of the contractive iteration scheme for the nonlinear problem in a Banach space with optimal regularity.

Remark 1.4 In our twofold normalized process, the cylindrical coordinates are an auxiliary process to decompose the hyperbolic-elliptic coupled form of the Euler system in a good manner. As a price, the artificial singularity $\frac{1}{r}$ will appear in the new nonlinear problem. Therefore, we use the natural Cartesian coordinates as working space instead of the cylindrical coordinates. This is our main strategy, which will be also benefit for other three dimensional problems under axisymmetry.

1.3 Literatures, comments and organization

In the steady Euler system (1.1), $M = \frac{|u|}{c(\rho, s)}$ is usually called as Mach number. In regions where $M > 1$, the flow is supersonic and the Euler system (1.1) is hyperbolic. In this situation, when posed smooth enough "initial data" away from vaccum on the space-like surface, the system (1.1) is at least locally well-posed. In regions where $M < 1$, the flow is subsonic and the Euler system (1.1) is hyperbolic-elliptic coupled. In this case, the nature question is: What kind of admissible boundary conditions can guarantee the well-posedness of boundary value problems of the system (1.1)?

This question was answered in a great variety of settings, such as subsonic outflows, subsonic flows in infinitely long nozzles, subsonic flow in finitely long nozzles, and so on. The subsonic outflows were studied in [1, 9, 15–16, 25] for the potential flow equation when the flow is irrotational, which show that the well-posedness of the subsonic outflow problem described by the potential flow equation can be determined by the constant subsonic velocity at infinity. With respect to case of subsonic Euler flow, one can see [3, 6] for the well-posedness results when the flows past a wall or are in half plane with the give subsonic states at infinity. Another interesting setting is subsonic flows in infinitely long nozzles. In [2], Bers conjectured that a global irrotational subsonic flow uniquely exists in an infinite-long nozzle as long as the incoming mass flus is less than a critical value. This conjecture was achieved in [8, 10, 12, 29–30] for two dimensional case and [5, 13, 31–32] for three dimensional case. The detailed explanation for this part can be found in [20] and the references therein.

For subsonic flow in bounded nozzle, as introduced in [7], it is expected the end pressure condition. This is very different from the case in the infinitely long nozzle. In [11], Du-Weng-Xin established the well-posedness of subsonic potential flow in two dimensional finitely long flat nozzle with the given mass flux as well as the end pressure and the zero vertical velocity at the inlet. In [21, 23] and the references therein, in the subsonic region, the two dimensional Euler system is decoupled into the elliptic mode and the two hyperbolic modes. The elliptic mode is a quasilinear first order elliptic system to control the flow angle and the pressure. The hyperbolic modes are transport types to control the total energy and the entropy. Due to this hyperbolic-elliptic coupled form, the admissible conditions for the subsonic Euler flow in finitely long nozzle can be posed as the end pressure and the velocity angle, the total energy and the entropy and the inlet. One can see [17, 20, 26] for dealing with this kind of admissible boundary conditions.

In current paper, we devote to establishing the structural stability of the reference shear flow in the three dimensional finitely long and slightly curved nozzle under the above mentioned admissible boundary conditions and axisymmetry. One can see the related works [4, 14, 22, 24, 27–28] for three dimensional Euler flow in finitely long nozzles with and without axisymmetry. In [22], Li and his coauthors considered three transonic Euler flows in finitely long nozzles under axisymmetry in the natural Cartesian coordinates. The Euler system is decoupled into an elliptic mode and other hyperbolic modes under the spherical coordinates. Our current paper extends this idea to deal with subsonic axisymmetric flow. One can see [14, 24] for treating the transonic shock under the three dimensional symmetric case via the methods of Lagrange transformation and/or stream function. In [4, 27], the authors used the div-curl decomposition to decouple the Euler system and established the structural stability results of subsonic flow in three dimensional finitely long nozzle with periodic cross section. Very recently, the result in [4] with the mass flux condition was extended to three dimensional finitely long nozzle with compact cross section by [28].

Motivated by [22], our strategy is to formulate the nonlinear problem via a twofold normalized process, including straightening the lateral boundary of the nozzle under the natural Cartesian coordinates $y = (y_1, y') \in \mathbb{R}^3$ and reformulating the problem under the cylindrical coordinates (z, r, θ). Accordingly, the Euler system is decoupled into an elliptic mode and three hyperbolic modes with some artificial singular terms under the cylindrical coordinates. The elliptic mode is a mixed type boundary value problem of first order elliptic system for the pressure and the radial velocity. Meanwhile, the hyperbolic modes are transport type to control the total energy, the specific entropy and the swirl velocity. With the delicate analysis on the possible artificial singularity and the characteristic of the transport operator, the iteration scheme is executed under the natural Cartesian coordinates (y-coordiantes). Based on this strategy, our main result is established with the optimal $C^{1,\alpha}$ regularity and without additional compatible conditions.

The rest is organized as follows. In Section 2, we reformulate the main nonlinear problem via a twofold normalized process, including straightening the lateral boundary of the nozzle under a new natural Cartesian coordinates and reformulating the problem under the three dimensional cylindrical coordinates. Section 3 prepares some preliminary conceptions and estimates, related to the methods to deal with the artificial singularities, the methods of the specific characteristics and the Campanato spaces and its relevance to the Hölder spaces. In Section 4, we establish the well-posedness for a kind of boundary value problems of a first order elliptic system, which is an fundamental model in our analysis. Finally, in Section 5, we construct a contractive iteration scheme for the nonlinear problem and establish the main stability result.

2 Reformulation Under 3D Axisymmetry

Our reformulation is executed in the following twofold normalized process: The first step is to straighten the lateral boundary under the new Cartesian coordinate. Actually, the solvability and the estimates for the nonlinear problem are obtained under this new coordinate. For the consideration of three dimensional axisymmetry, the second step, as an auxiliary process, is to reduce the problem under the cylindrical coordinate. In this way, we can construct an iteration scheme naturally and effectively.

2.1 Straighten the lateral boundary

The transformation $\boldsymbol{m}$ in (1.13) is to straighten the lateral boundary Γ of N, i.e.,

$$\boldsymbol{m}: N \to N_b, \quad x \mapsto y = \boldsymbol{m}(x) := (y_1, y') = \Big(x_1, \frac{x_2}{g(x_1)}, \frac{x_3}{g(x_1)}\Big). \tag{2.1}$$

It is not difficult to find that the transformation $\boldsymbol{m}$ is invertible since $\|g-1\|_{2,\alpha} \le \varepsilon$ in (1.7) for $\varepsilon > 0$ suitably small. Then, under the transformation (2.1), the Euler system (1.1) is converted equivalently into

$$\begin{cases} [g(y_1)\partial_1 - y_2 g'(y_1)\partial_2 - y_3 g'(y_1)\partial_3](\rho u_1) + \partial_2(\rho u_2) + \partial_3(\rho u_3) = 0, \\ [g(y_1)\partial_1 - y_2 g'(y_1)\partial_2 - y_3 g'(y_1)\partial_3](\rho u_1 u_1 + \boldsymbol{P}) + \partial_2(\rho u_1 u_2) + \partial_3(\rho u_1 u_3) = 0, \\ [g(y_1)\partial_1 - y_2 g'(y_1)\partial_2 - y_3 g'(y_1)\partial_3](\rho u_1 u_2) + \partial_2(\rho u_2 u_2 + \boldsymbol{P}) + \partial_3(\rho u_2 u_3) = 0, \\ [g(y_1)\partial_1 - y_2 g'(y_1)\partial_2 - y_3 g'(y_1)\partial_3](\rho u_1 u_3) + \partial_2(\rho u_2 u_3) + \partial_3(\rho u_3 u_3 + \boldsymbol{P}) = 0, \\ [g(y_1)\partial_1 - y_2 g'(y_1)\partial_2 - y_3 g'(y_1)\partial_3](\rho u_1 \boldsymbol{E}) + \partial_2(\rho u_2 \boldsymbol{E}) + \partial_3(\rho u_3 \boldsymbol{E}) = 0 \end{cases} \tag{2.2}$$

with $\partial_i = \partial_{y_i}$ $(i = 1, 2, 3)$. Hereafter, we always use the notation $f(y) := f \circ \boldsymbol{m}^{-1}(y)$ for any function $f(x)$ defined in N.

Meanwhile, the condition (1.8) on the lateral boundary Γ of N becomes

$$-g'(y_1)u_1(y) + y_2 u_2(y) + y_3 u_3(y) = 0, \quad y \in \Gamma_0. \tag{2.3}$$

The physical conditions (1.9)–(1.10) at the entrance Σ_0 and the exit Σ_ℓ of N are changed into

$$\begin{cases} \Big(\frac{y_2}{|y'|}\frac{u_2}{u_1} + \frac{y_3}{|y'|}\frac{u_3}{u_1}\Big)(0, y') = v_{r0}(|y'|), \\ \Big(-\frac{y_3}{|y'|}u_2 + \frac{y_2}{|y'|}u_3\Big)(0, y') = v_{\theta 0}(|y'|), \quad y' \in B_1(0), \\ \boldsymbol{E}(0, y') = \boldsymbol{E}_0(|y'|), \quad \boldsymbol{s}(0, y') = \boldsymbol{s}_0(|y'|) \end{cases} \tag{2.4}$$

with $v_{r0}(1) = g'(0)$ and

$$\boldsymbol{P}(\ell, y') = P_\ell(|y'|), \quad y' \in B_1(0). \tag{2.5}$$

Based on the transformation (2.1), Theorem 1.1 can be restated equivalently as the following result.

Theorem 2.1 *Under the assumptions in Theorem* 1.1 *and* (1.11)*, the problem* (2.2) *with* (2.3)–(2.5) *has a unique axisymmetric solution, which satisfies*

$$\|(u, \rho, \boldsymbol{s}) - (u_b, \rho_b, \boldsymbol{s}_b)\|_{1,\alpha;\overline{N_b}} \le c_0 \varepsilon. \tag{2.6}$$

Here, we use the notation $(u_b, \boldsymbol{E}_b)(y) = (u_b, \boldsymbol{E}_b) \circ \boldsymbol{m}_0^{-1}(y)$ *with* $\boldsymbol{m}_0 : N_b \to N_b, x \mapsto y = x$ *as the identity mapping associated with the reference subsonic flow* (1.5)*.*

2.2 Auxiliary transformation via the cylindrical coordinate

Since the system (2.2) is just the Euler system (1.1) in the new Cartesian coordinate y, it is also the hyperbolic-elliptic coupled in subsonic region. In three dimensional axisymmetric case, we use the cylindrical coordinate to decompose the corresponding hyperbolic modes and elliptic modes. To this end, we define the cylindrical transformation

$$\boldsymbol{T}: N_b \to [0,\ell]\times[0,1]\times\mathbb{T}, \quad y \mapsto (z,r,\theta) \tag{2.7}$$

with $y_1 = z, y_2 = r\cos\theta, y_3 = r\sin\theta$. The inverse of $\boldsymbol{T}$ is denoted as $\boldsymbol{T}^{-1}$. Meanwhile, we set

$$\begin{cases} (v_z, v_r, v_\theta)(z,r,\theta) = \Big(u_1, \dfrac{y_2}{r}u_2 + \dfrac{y_3}{r}u_3, -\dfrac{y_3}{r}u_2 + \dfrac{y_2}{r}u_3\Big)\circ \boldsymbol{T}^{-1}(z,r,\theta), \\ (\boldsymbol{E}, \boldsymbol{s}, \rho, \boldsymbol{P})(z,r,\theta) := (\boldsymbol{E}, \boldsymbol{s}, \rho, \boldsymbol{P})\circ \boldsymbol{T}^{-1}(z,r,\theta). \end{cases} \tag{2.8}$$

In general, the three dimensional axisymmetric assumption means that

$$(v_z, v_r, v_\theta) = (v_z, v_r, v_\theta)(z,r), \quad (\boldsymbol{E}, \boldsymbol{s}, \rho, \boldsymbol{P}) = (\boldsymbol{E}, \boldsymbol{s}, \rho, \boldsymbol{P})(z,r). \tag{2.9}$$

We introduce the set notation {axisymmetry} to denote the collection of all the vector functions $\{(u_1, u_2, u_3, \boldsymbol{P}, \boldsymbol{E}, \boldsymbol{s})(y)\}$ satisfying (2.8)–(2.9).

Under the transformation (2.7) and the notations in (2.8)–(2.9), the system (2.2) can be rewritten as

$$\begin{cases} (g(z)\partial_z - g'(z)r\partial_r)(\rho v_z) + \partial_r(\rho v_r) + \dfrac{1}{r}\rho v_r = 0, \\ (g(z)\partial_z - g'(z)r\partial_r)(\rho v_z v_z + \boldsymbol{P}) + \partial_r(\rho v_z v_r) + \dfrac{1}{r}\rho v_z v_r = 0, \\ (g(z)\partial_z - g'(z)r\partial_r)(\rho v_z v_r) + \partial_r(\rho v_r v_r + \boldsymbol{P}) + \dfrac{1}{r}\rho(v_r^2 - v_\theta^2) = 0, \\ (g(z)\partial_z - g'(z)r\partial_r)(\rho v_z v_\theta) + \partial_r(\rho v_r v_\theta) + \dfrac{2}{r}\rho v_r v_\theta = 0, \\ (g(z)\partial_z - g'(z)r\partial_r)(\rho v_z \boldsymbol{E}) + \partial_r(\rho v_r \boldsymbol{E}) + \dfrac{1}{r}\rho v_r \boldsymbol{E} = 0. \end{cases} \tag{2.10}$$

With the notation $v = \frac{v_r}{v_z}$, the computations

$$(2.10)_3 - v_r \times (2.10)_1, \quad (2.10)_2 - v_z \times (2.10)_1, \quad v_z \times (2.10)_2 + v_r \times (2.10)_3 + v_\theta \times (2.10)_4$$

yield

$$\begin{cases} (g(z)\partial_z - g'(z)r\partial_r)v + \dfrac{1}{\rho v_z^2}\partial_r \boldsymbol{P} \\ -\dfrac{v}{\gamma \boldsymbol{P}}(g(z)\partial_z - g'(z)r\partial_r + v\partial_r)\boldsymbol{P} - \dfrac{1}{r}\Big(v^2 + \dfrac{v_\theta^2}{v_z^2}\Big) = 0, \\ -\partial_r v + \Big(\dfrac{1}{\rho v_z^2} - \dfrac{1}{\gamma \boldsymbol{P}}\Big)(g(z)\partial_z - g'(z)r\partial_r)\boldsymbol{P} - \dfrac{1}{\gamma \boldsymbol{P}} v\partial_r \boldsymbol{P} - \dfrac{1}{r}v = 0, \\ ((g(z)\partial_z - g'(z)r\partial_r) + v\partial_r)v_\theta + \dfrac{1}{r}v v_\theta = 0, \\ ((g(z)\partial_z - g'(z)r\partial_r) + v\partial_r)\boldsymbol{s} = 0, \\ ((g(z)\partial_z - g'(z)r\partial_r) + v\partial_r)\boldsymbol{E} = 0. \end{cases} \tag{2.11}$$

Simultaneously, (2.3)–(2.5) have the forms

$$v(z,1) = g'(z), \quad z \in [0,\ell], \tag{2.12}$$

$$\begin{cases} v(0,r) = v_{r0}(r), & v_\theta(0,r) = v_{\theta 0}(r), \\ \boldsymbol{E}(0,r) = \boldsymbol{E}_0(r), & \boldsymbol{s}(0,r) = \boldsymbol{s}_0(r), \end{cases} \quad r \in [0,1] \tag{2.13}$$

and

$$\boldsymbol{P}(\ell,r) = P_\ell(r), \quad r \in [0,1]. \tag{2.14}$$

Remark 2.1 In (r,z,θ)-coordinate, the reference shear flow has the form

$$\begin{cases} (v_z, v, v_\theta)\big|_{\text{reference flow}} = (v_{zb}(r), 0, 0), \\ \boldsymbol{E}\big|_{\text{reference flow}} = \boldsymbol{E}_b(r), \\ \boldsymbol{s}_b, \rho_b, \boldsymbol{P}_b \text{ are positive constants.} \end{cases} \tag{2.15}$$

Remark 2.2 With the relations in (2.8), u can be expressed as

$$u = (u_1,u_2,u_3) = \Big(v_z, \frac{y_2}{r}v_r - \frac{y_3}{r}v_\theta, \frac{y_3}{r}v_r + \frac{y_2}{r}v_\theta\Big) \circ \boldsymbol{T}. \tag{2.16}$$

Further properties about {axisymmetry} are arranged in Subsection 3.1.

Remark 2.3 The merit of the system (2.11) is that $(2.11)_1$–$(2.11)_2$ is the first order elliptic system for $(v,\boldsymbol{P})$ and the other three equations are transport equations for $v_\theta, \boldsymbol{s}$ and $\boldsymbol{E}$. This means that in three dimensional axisymmetrical case, the hyperbolic-elliptic form of the Euler system (1.1) can be decoupled under the cylindrical coordinate. As a price, we should treat with the artificial singularity caused by the additional $\frac{1}{r}$ factor along the symmetrical axis. Due to (2.16), our strategy is to analyse the system (2.11) under the y-coordinate and establish the $C^{1,\alpha}$ estimates for $\frac{y_i}{r}(v,v_\theta)$ $(i=2,3)$ and other quantities.

2.3 Linearized form

The linearized form is built up due to the good form of the system (2.11) as pointed out in Remark 2.3. Consequently, the main result will be established via the contraction mapping principle in Section 5.

Based on the notations in (2.8) and (2.15), set $w = (w_1,\cdots,w_6)$ as

$$w = (v_z - v_{zb}, v-0, v_\theta - 0, \boldsymbol{P} - \boldsymbol{P}_b, \boldsymbol{E} - \boldsymbol{E}_b, \boldsymbol{s} - \boldsymbol{s}_b). \tag{2.17}$$

It derives from (1.2)–(1.3) that

$$\rho - \rho_b = \mathscr{G}_1 w_4 + \mathscr{G}_2 w_6, \tag{2.18}$$

where $\mathscr{G}_i = \mathscr{G}_i(w)$ $(i=1,2)$ satisfies

$$\begin{cases} \mathscr{G}_1 = \Big(\gamma A(\boldsymbol{s}) \displaystyle\int_0^1 [\rho_b + t(\rho-\rho_b)]^{\gamma-1}\mathrm{d}t\Big)^{-1}, \\ \mathscr{G}_2 = -\rho_b^\gamma \displaystyle\int_0^1 A'(\boldsymbol{s}_b + t(\boldsymbol{s}-\boldsymbol{s}_b))\mathrm{d}t \cdot \mathscr{G}_1. \end{cases} \tag{2.19}$$

In a similar way, we obtain from (1.2), (2.18) and the definition of $v = \frac{v_r}{v_z}$ that

$$w_1 = \mathscr{G}_3 w_2 + \mathscr{G}_4 w_3 + \mathscr{G}_5 w_4 + \mathscr{G}_6 w_5 + \mathscr{G}_7 w_6, \tag{2.20}$$

where $\mathscr{G}_i = \mathscr{G}_i(w)$ $(3 \le i \le 7)$ satisfies

$$\begin{cases} \mathscr{G}_3 = -\dfrac{v_{zb}^2 v}{(1+v^2)(v_z+v_{zb})}, \\ \mathscr{G}_4 = -\dfrac{v_\theta}{(1+v^2)(v_z+v_{zb})}, \\ \mathscr{G}_5 = -\dfrac{2\gamma}{\rho(\gamma-1)(1+v^2)(v_z+v_{zb})}\Big(1 - \dfrac{\boldsymbol{P}_b}{\rho_b}\mathscr{G}_1\Big), \\ \mathscr{G}_6 = \dfrac{2}{(1+v^2)(v_z+v_{zb})}, \\ \mathscr{G}_7 = \dfrac{2\gamma \boldsymbol{P}_b}{\rho\rho_b(\gamma-1)(1+v^2)(v_z+v_{zb})}\mathscr{G}_2. \end{cases}$$

Under the notations in (2.17), we derive from (2.11) that

$$\begin{cases} \partial_z w_2 + a_1(r)\partial_r w_4 = \mathcal{F}_1(w), \\ -\partial_r w_2 - \dfrac{1}{r} w_2 + a_2(r)\partial_z w_4 = \mathcal{F}_2(w), \end{cases} \tag{2.21}$$

where

$$a_1(r) = \frac{1}{\rho_b v_{zb}^2}(r) > 0, \quad a_2(r) = \frac{1}{\rho_b v_{zb}^2}(r) - \frac{1}{\gamma \boldsymbol{P}_b}(r) > 0 \tag{2.22}$$

and

$$\begin{aligned} \mathcal{F}_1(w) =& -\partial_r w_4\Big(\frac{1}{\rho v_z^2} - \frac{1}{\rho_b v_{zb}^2}\Big) + \frac{1}{r}\frac{w_3^2}{v_z^2} \\ & + (1-g(z))\partial_z w_2 + g'(z) r \partial_r w_2 \\ & + \frac{1}{\gamma \boldsymbol{P}} w_2 (g(z)\partial_z - g'(z) r\partial_r + w_2 \partial_r) w_4 + \frac{1}{r} w_2^2, \\ \mathcal{F}_2(w) =& \frac{1}{\gamma \boldsymbol{P}}\partial_r w_4 w_2 + \Big(a_2(r) - \Big(\frac{1}{\rho v_z^2} - \frac{1}{\gamma \boldsymbol{P}}\Big) g(z)\Big)\partial_z w_4 \\ & + \Big(\frac{1}{\rho v_z^2} - \frac{1}{\gamma \boldsymbol{P}}\Big) g'(z) r \partial_r w_4. \end{aligned} \tag{2.23}$$

Meanwhile, we derive from $(2.11)_3$–$(2.11)_4$ and (2.17) that

$$((g(z)\partial_z - g'(z) r\partial_r) + v\partial_r) w_3 + \frac{1}{r} v w_3 = 0 \tag{2.24}$$

and

$$((g(z)\partial_z - g'(z) r\partial_r) + v\partial_r) w_6 = 0. \tag{2.25}$$

With respect to w_5, for any $(0, \beta)$ with $0 \le \beta \le 1$, we define the rightward characteristics $r = r(z; \beta)$ decided by the operator $L_v = g(z)\partial_z - g'(r) r\partial_r + v\partial_r$ in $(2.11)_5$, which starts from $(0,\beta)$. Then, for any $(z,r)\in[0,\ell]\times[0,1]$, one can define the corresponding leftward characteristics $r=r(z;\beta(z,r))$ going through (z,r) and ending at $(0,\beta(z,r))$. In this way, we can define formally

$$w_5=\boldsymbol{E}_0(\beta(z,r))-\boldsymbol{E}_b(r). \tag{2.26}$$

The detailed analysis of w_5 will be carried out by the method of characteristics in Section 5.

3 Preliminary

Before dealing with the main problem, we arrange some basic preparations in this section.

3.1 Estimates for artificial singularity

In this subsection, we give the analysis to deal with the possible artificial singularity $\frac{1}{r}$ caused by the cylindrical transformation $\boldsymbol{T}$ in (2.7).

Lemma 3.1 *When $G(y)\in C^\alpha(\overline{N_b})$ with $G(y_1,0,0)=0$, then for $i=2,3$,*

$$\Big\|\frac{y_i}{|y'|}G\Big\|_{0,\alpha}\le 3\|G\|_{0,\alpha}. \tag{3.1}$$

Proof First, we have

$$\Big\|\frac{y_i}{|y'|}G\Big\|_{0,0}\le \|G\|_{0,\alpha}. \tag{3.2}$$

For any $y=(y_1,y')\in N_b$ and $z=(z_1,z')\in N_b$ with $|z'|\le|y'|$, one has

$$\begin{aligned}
&\Big|\frac{y_i}{|y'|}G(y_1,y')-\frac{z_i}{|z'|}G(z_1,z')\Big|\\
&\le\Big|\frac{y_i}{|y'|}\Big||G(y_1,y')|-G(z_1,z')|+|G(z_1,z')|\Big(\Big|\frac{y_i-z_i}{|y'|}\Big|+|z_i|\frac{|(y'-z')\cdot(y'+z')|}{|y'||z'|(|y'|+|z'|)}\Big)\\
&\le[G]_\alpha(|y-z|^\alpha+2|z'|^\alpha(|y'|^{-\alpha}|y'-z'|^\alpha)\\
&\le 3[G]_\alpha||y-z|^\alpha.
\end{aligned}$$

Combining this with (3.2) yields (3.1), we complete the proof of Lemma 3.1.

Lemma 3.2 *For any axisymmetry function $W(y_1,|y'|)\in C^\alpha(\overline{N_b})$, define*

$$\overline{W}(y):=\overline{W}(y_1,r)=\frac{1}{r}\int_0^r sW(y_1,s)\mathrm{d}s \tag{3.3}$$

with $r=|y'|$, then $\partial_j\big(\frac{y_i}{r}\overline{W}\big)\in C^\alpha(\overline{N_b})$ $(i,j=2,3)$ satisfying

$$\Big\|\partial_j\Big(\frac{y_i}{r}\overline{W}\Big)\Big\|_{0,\alpha}\le C_0\|W\|_{0,\alpha}. \tag{3.4}$$

Proof Note that

$$\frac{y_i}{r}\overline{W}(y)=\frac{y_i}{r^2}\int_0^r s(W(y_1,s)-W(y_1,0))\mathrm{d}s+\frac{y_i}{2}W(y_1,0). \tag{3.5}$$

Without loss of generation, we can assume that $W(y_1, 0) = 0$.

A direct computation yields

$$\begin{aligned}\partial_j\Big(\frac{y_i}{r}\overline{W}\Big) &= \frac{y_iy_j}{r^2}W(y_1,r) + \frac{\delta_{ij}}{r^2}\int_0^r sW(y_1,s)\mathrm{d}s - \frac{2y_iy_j}{r^2}\frac{1}{r^2}\int_0^r sW(y_1,s)\mathrm{d}s \\ &=: I_1 + \delta_{ij}I_2 + I_3. \end{aligned} \tag{3.6}$$

By using Lemma 3.1 twice, one has $I_1 \in C^\alpha(\overline{N_b})$ with

$$\|I_1\|_{0,\alpha} \le 9\|W\|_{0,\alpha}. \tag{3.7}$$

With respect to I_2, one has

$$|I_2| \le \|W\|_{0,\alpha}. \tag{3.8}$$

In addition, for any $y = (y_1, y') \in \overline{N_b}$ and $z = (z_1, z') \in \overline{N_b}$ with $|z'| \le |y'|$, one has

$$\begin{aligned} & I_2(y) - I_2(z) \\ =& \frac{1}{|y'|^2}\int_0^{|y'|} sW(y_1,s)\mathrm{d}s - \frac{1}{|z'|^2}\int_0^{|z'|} sW(z_1,s)\mathrm{d}s \\ =& \frac{1}{|y'|^2}\int_{|z'|}^{|y'|} sW(y_1,s)\mathrm{d}s + \frac{1}{|y'|^2}\int_0^{|z'|} s(W(y_1,s) - W(z_1,s))\mathrm{d}s \\ & + \Big(\frac{1}{|y'|^2} - \frac{1}{|z'|^2}\Big)\int_0^{|z'|} sW(z_1,s)\mathrm{d}s \\ =:& I_{21} + I_{22} + I_{23} \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} |I_{21}| &\le \frac{|y'|^{1+\alpha}}{|y'|^2}(|y'| - |z'|)[W]_\alpha \le 2|y-z|^\alpha[W]_\alpha, \\ |I_{22}| &\le \frac{|z'|^2}{2|y'|^2}|y_1 - z_1|^\alpha[W]_\alpha \le |y-z|^\alpha[W]_\alpha, \\ |I_{23}| &\le \frac{|(y'+z')\cdot(y'-z')|}{|y'|^2|z'|^2}\frac{|z'|^{2+\alpha}}{2+\alpha}[W]_\alpha \le 2|y-z|^\alpha[W]_\alpha. \end{aligned}$$

Combining this with (3.8)–(3.9) shows

$$\|I_2\|_{0,\alpha} \le 6\|W\|_{0,\alpha}. \tag{3.10}$$

Since $\lim\limits_{r\to 0}\frac{1}{r^2}\int_0^r sW(y_1,s)\mathrm{d}s = \frac{1}{2}W(y_1,0) = 0$, then with the proof of (3.10) and Lemma 3.1, one has

$$\|I_3\|_{0,\alpha} \le 54\|W\|_{0,\alpha}. \tag{3.11}$$

Finally, (3.4) comes from (3.6)–(3.7) and (3.10)–(3.11). The proof of Lemma 3.2 is completed.

Lemma 3.3 *Three dimensional axisymmetric functions have the following properties:*
(1) *For any axisymmetric function* $= F(y_1, |y'|) \in C^1(\overline{N_b})$, *one has*

$$\partial_2 F = \partial_3 F = \partial_r F = 0 \quad on\ |y'| = 0. \tag{3.12}$$

(2) *Under the axisymmetric and continuous assumptions of* v_r, v_θ *in* (2.8)–(2.9) *and the corresponding* u_2 *and* u_3, *one has*

$$u_2 = u_3 = v_r = v_\theta = 0 \quad on\ |y'| = 0. \tag{3.13}$$

Proof Since the proof is just a routine, we omit the details here.

3.2 The analysis of the specific characteristics

To treat the transport-type problems (2.24)–(2.25), we establish the systematic analysis of the specific characteristic defined by the first order operator

$$\begin{aligned} L_v &= g(z)\partial_z - g'(z)r\partial_r + v\partial_r \\ &= g(y_1)\partial_1 - g'(y_1)(y_2\partial_2 + y_3\partial_3) + \frac{y_2}{|y'|}v\partial_2 + \frac{y_3}{|y'|}v\partial_3 \\ :&= g(y_1)(\partial_1 + Q_2(y)\partial_2 + Q_3(y)\partial_3) \end{aligned} \tag{3.14}$$

with the forms both in cylindrical coordinate and y-coordinate (the new Cartesian coordinate), where v is a function defined in N_b and

$$Q_i(y) = \frac{1}{g(y_1)}\Big(\frac{y_i}{|y'|}v(y) - y_i g'(y_1)\Big), \quad i = 2, 3.$$

For any $y \in N_b$, the characteristic

$$Y(t; y) = (Y_1(t; y), Y_2(t; y), Y_3(t; y) := (Y_1(t), Y_2(t), Y_3(t)), \quad t \in [0, \ell]$$

of the operator L_v starting from some $(0, \beta_v) \in \Sigma_0$ and going through y can be formally defined as

$$\begin{cases} Y_1'(t) = 1, & t \in [0, \ell], \\ Y_2'(t) = Q_2(Y(t; y)), & t \in [0, \ell], \\ Y_3'(t) = Q_3(Y(t; y)), & t \in [0, \ell], \\ Y(y_1) = y, \quad Y(0) = (0, \beta_v). \end{cases} \tag{3.15}$$

Here, we use the notation $Y(t; y)$ to denote the characteristic defined by (3.15) with t and y as its variable and parameters respectively. The parameters y will be dropped where there is no ambiguity.

It derives from (3.15) that

$$\begin{cases} Y_1(t) = t, \quad Y_i(t) - y_i = \displaystyle\int_{y_1}^{t} Q_i(Y(t; y))\mathrm{d}t, \quad t \in [0, \ell], \\ y_i - \beta_{vi} = \displaystyle\int_0^{y_1} Q_i(Y(t; y))\mathrm{d}t. \end{cases} \tag{3.16}$$

Thus, if $Y(t;y)$ in (3.15) is well defined, we can determine the functions $\beta_v(y) = (\beta_{v2}(y), \beta_{v3}(y))$ from (3.16) as

$$\beta_{vi}(y) = y_i - \int_0^{y_1} Q_i(Y(t))\mathrm{d}t, \quad i = 2, 3. \tag{3.17}$$

Lemma 3.4 *For any axisymmetric $v(y_1, |y'|)$ with $\frac{y_i}{|y'|}v(y_1, |y'|) \in C^{1,\alpha}(\overline{N_b})$ $(i = 2, 3)$ and $v(y_1, 0) = 0, v(y_1, 1) = g'(y_1)$, the characteristic $Y(t;y)$ and $\beta_v(y)$ in (3.16)–(3.17) are well defined and satisfies*

$$\begin{cases} Y(t;y) \in \overline{N}_b, \\ Y(t;y) \cap \Gamma_b = \emptyset, \quad |y'| < 1, \quad t \in [0, \ell], \\ Y(t;y) \in \Gamma_b, \qquad |y'| = 1 \end{cases} \tag{3.18}$$

and

$$\begin{cases} \|Y_1(t;y)\|_{1,\alpha} \le C, \quad \|Y_i(t;y) - y_i\|_{1,\alpha} \le C\|Q_i\|_{1,\alpha}, \quad t \in [0, \ell], \\ \|\beta_{vi}(y) - y_i\|_{1,\alpha} \le C\|Q_i\|_{1,\alpha}. \end{cases} \tag{3.19}$$

In addition, if $\|v\|_{1,0} \le \varepsilon$ for some $\varepsilon > 0$ small, one has

$$\frac{1}{2}|y_i| \le |Y_i(t;y)| \le 2|y_i|, \quad t \in [0, \ell], \quad \frac{1}{2}|y_i| \le |\beta_{vi}(y)| \le 2|y_i|. \tag{3.20}$$

Proof With (3.12) in Lemma 3.3 and Lemma 3.1, we have that

$$\|Q_i\|_{1,\alpha} \le C\Big(\Big\|\frac{y_i}{|y'|}v\Big\|_{1,\alpha} + \|g'\|_{1,\alpha}\Big) \tag{3.21}$$

and

$$Q_i(y) = 0, \quad y \in \Gamma_b. \tag{3.22}$$

Thus, deriving from (3.15) and (3.21), we know that $Y(t)$ is locally well-posed in the interval $I_{y,\lambda} = [y_1 - \lambda, y_1 + \lambda] \cap [0, \ell]$ for some $\lambda > 0$ small. It comes from $(3.15)_2$–$(3.15)_3$ and (3.21) that

$$\begin{aligned} |Y_2(t)| + |Y_3(t)| &\le |y_2| + |y_3| + \|Q_2\|_{L^\infty} + \|Q_3\|_{L^\infty} \\ &\le 2 + C\Big(\Big\|\frac{y_2}{|y'|}v\Big\|_{1,\alpha} + \Big\|\frac{y_3}{|y'|}v\Big\|_{1,\alpha} + \|g\|_{2,\alpha}\Big), \quad t \in I_{y,\lambda}. \end{aligned} \tag{3.23}$$

In addition, due to (3.22) and the theory of locally unique existence, $Y(t) \in \overline{N}_b$ satisfies

$$\begin{cases} Y(t;y) \in \overline{N}_b, \\ Y(t;y) \cap \Gamma_b = \emptyset, \quad |y'| < 1, \quad t \in I_{y,\lambda}. \\ Y(t;y) \in \Gamma_b, \qquad |y'| = 1, \end{cases} \tag{3.24}$$

Based on (3.23)–(3.24), the problem (3.15) is uniquely solved on $[0, \ell]$ and $Y(t)$ satisfies (3.18).

The regularity and estimates in (3.19) comes from the theorem of implicit function and (3.16)–(3.17). The property (3.20) is obtained from (3.16)–(3.17), the expression of $Q_i(y)$ in (3.14) and $\|w\|_{1,0} \le \varepsilon$ with $v(y_1, 0) = 0$. Finally, we complete the proof of Lemma 3.4.

3.3 Short introduction of Morrey and Campanato spaces

The notations and results in this subsection are mainly quoted from [18].

Definition 3.1 (A-type domain) *A domain $\Omega \subset \mathbb{R}^3$ is called as an A-type domain if there exists a generic constant $C > 0$ such that for all $y_0 \in \Omega, 0 < \tau < diam(\Omega)$, we have $|B_\tau^+(y_0)| \geq C\tau^3$ where $B_\tau^+(y_0) = B_\tau(y_0) \cap \Omega$.*

Definition 3.2 (Morrey space) *For any $1 \leq p \leq +\infty, \lambda \geq 0$ and A-type domain $\Omega \subset \mathbb{R}^3$, define the Morrey space $L^{p,\lambda}(\Omega)$ as*

$$L^{p,\lambda}(\Omega) := \{u \in L^P(\Omega) : \|u\|_{L^{p,\lambda}(\Omega)} < +\infty\}$$

with

$$\|u\|_{L^{p,\lambda}(\Omega)}^p := \sup_{y_0 \in \Omega, \tau > 0} \tau^{-\lambda} \int_{B_\tau^+(y_0)} |u|^P \mathrm{d}y.$$

Definition 3.3 (Campanato space) *For any $1 \leq p \leq +\infty, \lambda \geq 0$ and A-type domain $\Omega \subset \mathbb{R}^3$, define the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ as*

$$\mathcal{L}^{p,\lambda}(\Omega) := \{u \in L^P(\Omega) : \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} < +\infty\}$$

with

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{\mathcal{L}^{p,\lambda}(\Omega)}$$

and

$$[u]_{\mathcal{L}^{p,\lambda}(\Omega)}^p := \sup_{y_0 \in \Omega, \tau > 0} \tau^{-\lambda} \int_{B_\tau^+(y_0)} |u - u_{y_0,\tau}|^p \mathrm{d}y < +\infty.$$

Here, $u_{y_0,\tau} = \frac{1}{|B_\tau^+(y_0)|} \int_{B_\tau^+(y_0)} u(y)\mathrm{d}y$.

Based on the Definitions 3.1–3.3, one has the following results.

Proposition 3.1 *If $\lambda_1 \leq \lambda_2$, then $\mathcal{L}^{p,\lambda_2}(\Omega) \subseteq \mathcal{L}^{p,\lambda_1}(\Omega)$.*

Proof This can be shown from the definition of Campanato space, we omit the proof here.

Proposition 3.2 (see [18, Proposition 5.4]) *For any $0 \leq \lambda < 3$, we have $L^{p,\lambda}(\Omega) \cong \mathcal{L}^{p,\lambda}(\Omega)$.*

Proposition 3.3 (see [18, Proposition 5.5]) *For any $\alpha \in (0, 1)$, we have $\mathcal{L}^{p,3+p\alpha}(\Omega) \cong C^\alpha(\overline{\Omega})$. Moreover, $[u]_{\alpha,\Omega}$ is equivalent to $[u]_{\mathcal{L}^{p,3+p\alpha}(\Omega)}$.*

Proposition 3.3 shows that, if we want to proof that $u \in C^\alpha(\overline{\Omega})$, we just need to prove $u \in \mathcal{L}^{p,3+p\alpha}(\Omega)$. Moreover, with Propositions 3.1–3.3, for any $\mu > 0$, one has

$$C^\alpha \cong \mathcal{L}^{2,3+2\alpha} \subseteq \mathcal{L}^{2,3-\mu} \cong L^{2,3-\mu}. \tag{3.25}$$

Proposition 3.4 (see [18, Lemma 5.13]) *Let Φ be a non-decreasing function, A, a, b, R_0 are positive constants with $a > b$, satisfying*

$$\Phi(\tau) \leq A\Big[\Big(\frac{\tau}{R}\Big)^a + \sigma\Big]\Phi(R) + BR^b$$

for all $0 < \tau < R < r_0$, then there exist constants $\sigma_0 = \sigma_0(A, a, b)$ and $C = C(A, a, b)$ such that if $\sigma < \sigma_0$, then

$$\Phi(\tau) \leq C\Big[\Big(\frac{\tau}{R}\Big)^b \Phi(R) + B\Big]\tau^b.$$

4 Model Problem

In this section, we study the following model problem

$$\begin{cases} \partial_z d + a_1(r)\partial_r e = f_1, & (z, r) \in (0, \ell) \times (0, 1), \\ -\partial_r d - \dfrac{1}{r} d + a_2(r)\partial_z e = f_2, & (z, r) \in (0, \ell) \times (0, 1), \\ d(0, r) = d_0(r), & r \in [0, 1], \\ e(\ell, r) = e_\ell(r), & r \in [0, 1], \\ d(z, 0) = 0, \quad d(z, 1) = m(z), & z \in [0, \ell]. \end{cases} \tag{4.1}$$

The well-posedness of the problem (4.1) can be presented as the following result.

Theorem 4.1 *When $f_i \circ \boldsymbol{T} \in C^\alpha(\overline{N_b})$, $a_i \circ \boldsymbol{T} \in C^{1,\alpha}(\overline{B_1(0)})$ $(i = 1, 2)$ satisfy*

$$a_i > \lambda, \quad \|a_i \circ \boldsymbol{T}\|_{C^{1,\alpha}(\overline{B_1(0)})} < \Lambda$$

for some positive constants $\lambda < \Lambda$, $\frac{y_i}{r} d_0 \circ \boldsymbol{T}, e_\ell \circ \boldsymbol{T} \in C^{1,\alpha}(\overline{B_1(0)})$, $m(z) \in C^{1,\alpha}([0, \ell])$ with the natural admissible conditions

$$f_1(z, 0) = 0, \quad 0 \leq z \leq \ell, \tag{4.2a}$$

$$d_0(0) = 0, \quad d_0(1) = m(0), \tag{4.2b}$$

$$d_0'(0) = e_\ell'(0) = 0, \quad m'(\ell) + a_1(1)e_\ell'(1) = f_1(\ell, 1), \tag{4.2c}$$

then the model problem (4.1) has a unique solution (d, e) such that $\big(\frac{y_i}{r} d \circ \boldsymbol{T}, e \circ \boldsymbol{T}\big) \in C^{1,\alpha}(\overline{N_b})$, satisfying

$$\begin{aligned} &\Big\|\frac{y_2}{r} d \circ \boldsymbol{T}\Big\|_{1,\alpha} + \Big\|\frac{y_3}{r} d \circ \boldsymbol{T}\Big\|_{1,\alpha} + \|e \circ \boldsymbol{T}\|_{1,\alpha} \\ &\leq C_0\Big(\|f_1 \circ \boldsymbol{T}\|_{0,\alpha} + \|f_2 \circ \boldsymbol{T}\|_{0,\alpha} + \Big\|\frac{y_2}{r} d_0 \circ \boldsymbol{T}\Big\|_{1,\alpha} \\ &\quad + \Big\|\frac{y_3}{r} d_0 \circ \boldsymbol{T}\Big\|_{1,\alpha} + \|e_\ell \circ \boldsymbol{T}\|_{1,\alpha} + \|m\|_{1,\alpha}\Big). \end{aligned} \tag{4.3}$$

For simplicity of notations, we drop "$\circ \boldsymbol{T}$" hereafter in this section where there is no ambiguity.

Remark 4.1 (Reduce to the case $d_0 = e_\ell = m = 0$) In Theorem 4.1, without loss of generality, we can assume $u_0 = v_\ell = m = 0$. Otherwise, set

$$\begin{cases} D = d - d_0(r) - r^2 m(z) + r^2 m(0), \\ E = e - e_\ell(r), \end{cases} \tag{4.4}$$

then the problem (4.1) is equivalent converted into

$$\begin{cases} \partial_z D + a_1(r)\partial_r E = F_1, & (z,r) \in (0,\ell)\times(0,1), \\ -\partial_r D - \dfrac{1}{r}D + a_2(r)\partial_z E = F_2, & (z,r) \in (0,\ell)\times(0,1), \\ D(0,r) = 0, & r \in [0,1], \\ E(\ell,r) = 0, & r \in [0,1], \\ D(z,0) = 0, \quad D(z,1) = 0, & z \in [0,\ell], \end{cases} \tag{4.5}$$

where

$$\begin{cases} F_1 = f_1 - r^2 m'(z) - a_1(r)e_\ell'(r), \\ F_2 = f_2 + d_0'(r) + \dfrac{d_0(r)}{r} + 3r(m(z) - m(0)). \end{cases} \tag{4.6}$$

Since $e_\ell'(0) = 0$, one has $\partial_i e_\ell(0) = \frac{y_i}{r} e_\ell'(0) = 0$ for $i = 2,3$ by Lemma 3.3. Thus, with Lemma 3.1, one has

$$e_\ell'(r) = \frac{y_2}{r}\partial_2 e_\ell + \frac{y_3}{r}\partial_3 e_\ell \in C^\alpha(\overline{N_b}).$$

Combining this with

$$d_0'(r) + \frac{d_0(r)}{r} = \partial_2\Big(\frac{y_2}{r}d_0\Big) + \partial_3\Big(\frac{y_3}{r}d_0\Big) \in C^\alpha(\overline{N_b}),$$

arrives at $F_i \in C^\alpha(\overline{N_b})$ $(i = 1,2)$ with $F_1(z,0) = 0$ and $F_1(\ell,1) = 0$.

Based on Remark 4.1, we assume $d_0 = e_\ell = m = 0$ in the problem (4.1) with $f_1(z,0) = 0$ and $f_1(\ell,1) = 0$. In this way, the unique solvability of the problem (4.1) can be reduced to the unique solvability of the following two problems

$$\begin{cases} \partial_z d_1 + a_1(r)\partial_r e_1 = f_1, & (z,r) \in (0,\ell)\times(0,1), \\ -\partial_r d_1 - \dfrac{1}{r}d_1 + a_2(r)\partial_z e_1 = 0, & (z,r) \in (0,\ell)\times(0,1), \\ d_1(0,r) = 0, & r \in [0,1], \\ e_1(\ell,r) = 0, & r \in [0,1], \\ d_1(z,0) = 0, \quad d_1(z,1) = 0, & z \in [0,\ell] \end{cases} \tag{4.7}$$

and

$$\begin{cases} \partial_z d_2 + a_1(r)\partial_r e_2 = 0, & (z,r) \in (0,\ell) \times (0,1), \\ -\partial_r d_2 - \dfrac{1}{r} d_2 + a_2(r)\partial_z e_2 = f_2, & (z,r) \in (0,\ell) \times (0,1), \\ d_2(0,r) = 0, & r \in [0,1], \\ e_2(\ell,r) = 0, & r \in [0,1], \\ d_2(z,0) = 0, \quad d_2(z,1) = 0, & z \in [0,\ell] \end{cases} \tag{4.8}$$

with $(d,e) = (d_1,e_1) + (d_2,e_2)$.

4.1 Solvability of (d_1, e_1)

It derives from $(4.7)_2$ that

$$d_1 = \frac{1}{r}\int_0^r a_2(s) s \partial_z e_1(z,s) \mathrm{d}s. \tag{4.9}$$

Substituting (4.9) into (4.7) implies

$$\begin{cases} \quad a_2(r)\partial_z^2 e_1 + a_1(r)\Big(\partial_r^2 e_1 + \dfrac{1}{r}\partial_r e_1\Big) + a_1'(r)\partial_r e_1 & \\ = \partial_r f_1 + \dfrac{1}{r} f_1, & (z,r) \in (0,\ell) \times (0,1), \\ \partial_r e_1 = \dfrac{f_1}{a_1}, & (z,r) \in [0,\ell] \times \{0,1\}, \\ \partial_z e_1(0,r) = e_1(\ell,r) = 0, & r \in [0,1]. \end{cases} \tag{4.10}$$

With the help of the transformation (2.7), the problem (4.10) has the form as

$$\begin{cases} \partial_1(a_2(|y'|)\partial_1 e_1) + \displaystyle\sum_{i=2}^3 \partial_i(a_1(|y'|)\partial_i e_1) = \sum_{i=2}^3 \partial_i\Big(\frac{y_i}{|y'|} f_1\Big), & y \in N_b, \\ \partial_n e_1 = \dfrac{f_1}{a_1(1)}, & y \in [0,\ell] \times \partial B_1(0), \\ \partial_1 e_1(0,y') = e_1(\ell,y') = 0, & y' \in B_1(0). \end{cases} \tag{4.11}$$

With $e_1 = e_{11} + e_{12}$, the problem (4.11) can be split into the following two problems

$$\begin{cases} Le_{11} = \displaystyle\sum_{i=2}^3 \partial_i\Big(\frac{y_i}{|y'|} f_1\Big), & y \in N_b, \\ \partial_1 e_{11}(0,y') = \partial_1 e_{11}(\ell,y') = 0, & y' \in B_1(0), \\ \partial_n e_{11}(y_1,y') = \dfrac{f_1}{a_1(1)}, & |y'| = 1, \\ \dfrac{1}{|N_b|}\displaystyle\int_{N_b} e_{11} \mathrm{d}y = 0 & \end{cases} \tag{4.12}$$

and

$$\begin{cases} Le_{12} = 0, & y \in N_b, \\ \partial_1 e_{12}(0, y') = 0, & y' \in B_1(0), \\ e_{12}(\ell, y') = -e_{11}(\ell, y'), & y' \in B_1(0), \\ \partial_n e_{12}(y_1, y') = 0, & |y'| = 1, \end{cases} \tag{4.13}$$

where the divergence operator $L = \partial_1(a_2(|y'|)\partial_1 \cdot) + \sum\limits_{i=2}^{3} \partial_i(a_1(|y'|)\partial_i \cdot)$.

With respect to the problem (4.12), one has the following result.

Lemma 4.1 *When $f_1(y_1, |y'|) \in C^\alpha(\overline{N_b})$ $(i = 2, 3)$ with $f_1(z, 0) = 0$ and $f_1(\ell, 1) = 0$, the problem* (4.12) *has a unique H^1-weak solution e_{11} such that*

$$\|e_{11}\|_{H^1(N_b)} \le C\|f_1\|_{0,\alpha}. \tag{4.14}$$

Proof Set

$$\widetilde{e} = \begin{cases} e_{11}(y_1, y'), & y_1 \in [0, \ell], \\ e_{11}(-y_1, y'), & y_1 \in [-\ell, 0], \\ e_{11}(2\ell - y_1, y'), & y_1 \in [\ell, 2\ell] \end{cases}$$

and

$$\widetilde{f}(y_1, y') = \begin{cases} f_1(y_1, y'), & y_1 \in [0, \ell], \\ f_1(-y_1, y'), & y_1 \in [-\ell, 0], \\ f_1(2\ell - y_1, y'), & y_1 \in [\ell, 2\ell]. \end{cases}$$

Then the problem (4.12) can be equivalently converted into the following problem of $\widetilde{e}$

$$\begin{cases} L\widetilde{e} = \sum\limits_{i=2}^{3} \partial_i\Big(\frac{y_i}{|y'|}\widetilde{f}\Big), & y \in N_b^1, \\ \partial_1\widetilde{e}(-\ell, y') = \partial_1\widetilde{e}(2\ell, y') = 0, & y' \in B_1(0), \\ \partial_n\widetilde{e}(y_1, y') = \dfrac{\widetilde{f}}{a_1(1)}, & |y'| = 1, \\ \dfrac{1}{|N_b^1|}\displaystyle\int_{N_b^1} \widetilde{e}\mathrm{d}y = 0 \end{cases} \tag{4.15}$$

with $\widetilde{f}(z, 0) = 0$ for $z \in [-\ell, 2\ell]$, $\widetilde{f}(\ell, y') = 0$ for $|y'| = 1$ and $N_b^1 = [-\ell, 2\ell] \times B_1(0)$. Since the boundary conditions of the problem (4.15) are Neumann type, thus, $\widetilde{e}$ being the H^1-weak solution of (4.15) means for any test function $\phi \in H^1(N_b^1)$,

$$\int_{N_b^1} (a_2\partial_1\widetilde{e}\partial_1\phi + a_1\partial_2\widetilde{e}\partial_2\phi + a_1\partial_3\widetilde{e}\partial_3\phi)\mathrm{d}y = \int_{N_b^1} \Big(\frac{y_2}{|y'|}\widetilde{f}\partial_2\phi + \frac{y_3}{|y'|}\widetilde{f}\partial_3\phi\Big)\mathrm{d}y. \tag{4.16}$$

By Lax-Milgram theorem (see [19, Theorem 5.8]) and the argument of uniqueness for the axisymmetric case, the problem (4.15) has a unique axisymmetric weak solution $\widetilde{e} \in H^1(N_b^1)$.

Together with the Poincaré inequality and $(4.15)_4$, one has

$$\|\widetilde{e}\|_{H^1(N_b^1)} \le C\sum_{i=2}^{3}\Big\|\Big(\frac{y_i}{|y'|}\widetilde{f}\Big)\Big\|_{L^2(N_b^1)} \le C\|f_1\|_{0,\alpha}.$$

Thus, we obtain (4.14) and finish the proof of Lemma 4.1.

In addition, the regularity of e_{11} can be improved into $C^{1,\alpha}(\overline{N}_b)$. To this end, we first establish the following lemma to homogenize the boundary condition on $|y'|=1$.

Lemma 4.2 *Under the assumption in Lemma 4.1, there exists a bounded operator*

$$\mathcal{T}: C^\alpha(\partial N_b^1 \cap \{|y'|=1\}) \to C^{1,\alpha}(\overline{N}_b^1),$$

such that for each $\varphi(y_1,|y'|)$ with $\varphi \in C^\alpha(\partial N_b^1 \cap \{|y'|=1\})$,

$$\partial_n(\mathcal{T}\varphi)|_{[-\frac{\ell}{2},\frac{3\ell}{2}]\times\{|y'|=1\}} = \varphi, \quad \|\mathcal{T}\varphi\|_{1,\alpha} \le C\|\varphi\|_{0,\alpha} \tag{4.17}$$

for some generic positive constant C.

Proof First, we choose a monotonically increasing cut-off function $h(r) \in C^\infty([0,1])$, $0 \le h'(r) \le 2$ with

$$h(r) = \begin{cases} r, & r \in \Big[\dfrac{3}{4}, 1\Big], \\ 0, & r \in \Big[0, \dfrac{1}{4}\Big]. \end{cases}$$

Thus $h(|y'|) \in C^\infty(\overline{B_1(0)})$. In addition, we choose two non-negative cut-off functions $\chi_0(t) \in C_0^\infty(\mathbb{R})$ and $\chi_1(t) \in C_0^\infty((-\ell,2\ell))$, such that $\int_{\mathbb{R}} \chi_0(t)\mathrm{d}t = 1$ and

$$0 \le \chi_1(t) \le 1; \quad \chi_1(t) = 1, \quad t \in \Big[-\frac{\ell}{2}, \frac{3\ell}{2}\Big]; \quad \chi_1(t) = 0, \quad t \in \Big(-\infty, -\frac{3}{4}\ell\Big] \cup \Big[\frac{7}{4}\ell, +\infty\Big).$$

For each $\varphi(y_1,|y'|)$ with $\varphi \in C^\alpha(\partial N_b^1 \cap \{|y'|=1\})$, we define

$$\mathcal{T}\varphi(y) = (h(|y'|)-1)\int_{\mathbb{R}} [\varphi(\cdot,1)\chi_1(\cdot)](y_1 - (1-h(|y'|))t)\chi_0(t)\mathrm{d}t.$$

A direct computation shows that

$$\partial_n \mathcal{T}\varphi = \varphi \quad \text{on } \Big[-\frac{\ell}{2}, \frac{3\ell}{4}\Big] \times \{|y'|=1\}, \quad \|\mathcal{T}\varphi\|_{1,\alpha} \le C\|\varphi\|_{0,\alpha}.$$

This yields (4.17) and the proof of Lemma 4.2 is finished.

Lemma 4.3 *Under the assumptions in Lemma 4.1, $e_{11} \in C^{1,\alpha}(\overline{N_b})$ and*

$$\|e_{11}\|_{1,\alpha} \le C\|f_1\|_{0,\alpha}. \tag{4.18}$$

Proof We also start from the notation $\widetilde{e}$ in the problem (4.15). For any $y_0 \in N_b$, let $B_\tau^+(y_0) = B_\tau(y_0) \cap N_b^1$. To see that $\widetilde{e} \in C^{1,\alpha}(\overline{N_b})$, from (4.14), the Definition 3.3 of Campanato space and Proposition 3.3, we just need to prove that

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \le C\|f_1\|_{0,\alpha}^2 \tau^{3+2\alpha} \tag{4.19}$$

for any $0 < \tau < \frac{1}{8}$ where

$$(\nabla\widetilde{e})_{y_0,\tau} = \frac{1}{|B_\tau^+(y_0)|} \int_{B_\tau^+(y_0)} D\widetilde{e}\mathrm{d}y. \tag{4.20}$$

From the interior estimates in [18, Theorem 5.14] and (4.14), (4.19) holds for $y_0 \in [0,\ell] \times B_1(0)$ with

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \le C\|f_1\|_{0,\alpha}^2 \tau^{3+2\alpha}, \tag{4.21}$$

where $C = C(\lambda,\Lambda)$ and $\partial_\tau^+(y_0) \cap \{|y'| = 1\} = \emptyset$.

For $y_0 \in [0,\ell] \times (B_1(0) \setminus B_{\frac{7}{8}}(0))$ and $B_R^+(y_0) \cap \{|y'| = 1\} \neq \emptyset$ with $0 < R < \frac{1}{8}$, the process of proving (4.19) is divided into the following three steps.

Step 1 Homogeneous equation with constant coefficient in $B_R^+(y_0)$.

In this step, we namely consider the case that in (4.15), a_1, a_2 are positive constants and $\widetilde{f} = 0$ in $B_R^+(y_0)$ for $0 < \tau < R < \frac{1}{8}$.

We first prove the following Caccioppoli inequality

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y \le C\frac{1}{(R-\tau)^2} \int_{B_R^+(y_0)} |\widetilde{e} - H|^2 \mathrm{d}y, \tag{4.22}$$

where $C = C(\lambda,\Lambda)$ and H can be any positive constant.

In fact, choose the test function $\phi = \eta^2(\widetilde{e} - H)$ in (4.16) with $\eta \in H_0^1(B_R(y_0) \cap \overline{N_b^1})$ being a cut-off function as: $0 \le \eta \le 1$ in $B_R^+(y_0)$, $\eta \equiv 1$ in $B_\tau^+(y_0)$, and $|D\eta| \le \frac{C}{R-\tau}$. Then one has

$$\begin{aligned}&\int_{N_b^1} \eta^2(a_2(\partial_1\widetilde{e})^2 + a_1(\partial_2\widetilde{e})^2 + a_1(\partial_3\widetilde{e})^2)\mathrm{d}y \\ &= -\int_{N_b^1} 2\eta(\widetilde{e} - H)(a_2\partial_1\widetilde{e}\partial_1\eta + a_1\partial_2\widetilde{e}\partial_2\eta + \partial_3\widetilde{e}\partial_3\eta)\mathrm{d}y,\end{aligned}$$

which yields

$$\lambda \int_{B_R^+(y_0)} (\eta D\widetilde{e})^2 \mathrm{d}y \le \varepsilon \int_{B_R^+(y_0)} (\eta D\widetilde{e})^2 \mathrm{d}y + C(\varepsilon) \int_{B_R^+(y_0)} |D\eta|^2 |\widetilde{e} - H|^2 \mathrm{d}y.$$

Thus, (4.22) is proved with $\varepsilon = \frac{\lambda}{2}$ and the definition of η.

Next, we prove the following two inequalities

$$\int_{B_\tau^+(y_0)} |\widetilde{e}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^3 \int_{B_R^+(y_0)} |\widetilde{e}|^2 \mathrm{d}y \tag{4.23}$$

and

$$\int_{B_\tau^+(y_0)} |\widetilde{e}-(\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5 \int_{B_R^+(y_0)} |\widetilde{e}-(\widetilde{e})_{y_0,R}|^2 \mathrm{d}y. \tag{4.24}$$

When $\frac{R}{2} \le \tau \le R$, (4.23) holds clearly. When $0<\tau<\frac{R}{2}$, in the way similar to [19, Theorem 8.10, Theorem 8.13], one has $\widetilde{e} \in H^2(B^+_{\frac{3R}{4}}(y_0))$ with

$$\|\widetilde{e}\|_{H^2(B^+_{\frac{3R}{4}}(y_0))} \le C\|\widetilde{e}\|_{L^2(B^+_R(y_0))}. \tag{4.25}$$

Here, one should deal with the Neumann boundary condition $\partial_n \widetilde{e}\big|_{|y'|=1}=0$ when $B^+_R(y_0)\backslash N_b^1 \neq \emptyset$ and can use the estimation skill of the tangential-normal derivatives to obtain (4.25).

Combining (4.25) with Sobolev imbedding Theorem and the scaling skill shows that

$$\sup_{B_{\frac{R}{2}}(y_0)} |\widetilde{e}| \le CR^{-\frac{3}{2}}\|\widetilde{e}\|_{H^2(B^+_{\frac{R}{2}}(y_0))} \le CR^{-\frac{3}{2}}\|\widetilde{e}\|_{L^2(B^+_R(y_0))}. \tag{4.26}$$

Therefore, it derives from (4.26) that

$$\int_{B_\tau(y_0)} |\widetilde{e}|^2 \mathrm{d}y \le C\tau^3 \sup_{B_{\frac{R}{2}}(y_0)} |\widetilde{e}|^2 \le C\Big(\frac{\tau}{R}\Big)^3 \int_{B_R^+(y_0)} |\widetilde{e}|^2 \mathrm{d}y.$$

This shows (4.23).

Now, we prove (4.24).

For $0<\tau<\frac{R}{2}$, with (4.25), $\partial_z \widetilde{e} \in H^1(B_{\frac{3R}{4}}(y_0))$ also satisfies the homogeneous equation with constant coefficients in $B_R^+(y_0)$ with $\partial_n\partial_z\widetilde{e}=0$ on $|y'|=1$. Thus following the process of the interior estimates and the estimates near the lateral boundary of N_b^1 routinely as in the proof of (4.23), we have

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^3 \int_{B^+_{\frac{3R}{4}}(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y. \tag{4.27}$$

By use of the Poincaré inequality and the Caccioppoli inequality (4.22) for the left-hand side and the right-hand side of (4.27), respectively, (4.24) is proved for $0<\tau<\frac{R}{2}$.

For $\frac{R}{2}<\tau<R$, one has

$$\begin{aligned} &\Big(\frac{\tau}{R}\Big)^5 \int_{B_R^+(y_0)} |\widetilde{e}-(\widetilde{e})_{y_0,R}|^2 \mathrm{d}y \\ \ge& \Big(\frac{1}{2}\Big)^5 \int_{B_R^+(y_0)} |\widetilde{e}-(\widetilde{e})_{y_0,R}|^2 \mathrm{d}y \\ \ge& \Big(\frac{1}{2}\Big)^5 \int_{B_\tau^+(y_0)} |\widetilde{e}-(\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y, \end{aligned} \tag{4.28}$$

where the last inequality holds since $\Phi(\xi) := \int_{B_\xi^+(y_0)} |\widetilde{e}-(\widetilde{e})_{y_0,\xi}|^2 \mathrm{d}y$ is monotonically increasing with respect to ξ. So we obtain (4.24).

In the similay way to deal with (4.27)–(4.28), one also has

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^3 \int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y$$

and

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}-(\nabla\widetilde{e})_{y_0,\tau}|^2\mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5\int_{B_R^+(y_0)} |\nabla\widetilde{e}-(\nabla\widetilde{e})_{y_0,R}|^2\mathrm{d}y.$$

Together with Lemma 4.1, one has

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}-(\nabla\widetilde{e})_{y_0,\tau}|^2\mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5\int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2\mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5\|f_1\|_{0,\alpha} \le C\tau^{3+2\alpha}\|f_1\|_{0,\alpha},$$

since R is a fixed constant. Therefore, (4.19) is proved for this case.

Step 2 Inhomogeneous equation with constant coefficient in $B_R^+(y_0)$.

In this step, we consider that in (4.15), a_1, a_2 are positive constants and $\widetilde{f}\ne 0$ in $B_R^+(y_0)$ for $0<\tau<R<\frac{1}{8}$. By use of Lemma 4.2, without loss of generality, we assume $\widetilde{f}(z,y')=0$ on $|y'|=1$, otherwise, we use $\widetilde{e}-\frac{\mathcal{T}\widetilde{f}}{a_1(1)}$ to replace $\widetilde{e}$ itself.

Let $\widehat{e}$ be the solution to the following homogeneous problem

$$\begin{cases} L\widehat{e}=0 & \text{in } B_R^+(y_0),\\ \widehat{e}=\widetilde{e} & \text{on } \partial B_R(y_0)\cap N_b^1,\\ \partial_n\widehat{e}=\partial_n\widetilde{e}=0 & \text{on } B_R(y_0)\cap\partial N_b^1. \end{cases} \tag{4.29}$$

Based on Lemme 4.1, by Lax-Milgram theorem, the problem (4.29) has a unique $H^1(B_R^+(y_0))$ weak solution $\widehat{e}$ satisfying

$$\|\widehat{e}\|_{H^1(B_R^+(y_0))} \le C\|\widetilde{e}\|_{H^1(B_R^+(y_0))} \le C\|f_1\|_{0,\alpha}. \tag{4.30}$$

Similar to Step 1, one has

$$\int_{B_\tau^+(y_0)} |\nabla\widehat{e}-(\nabla\widehat{e})_{y_0,\tau}|^2\mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5\int_{B_R^+(y_0)} |\nabla\widehat{e}-(\nabla\widehat{e})_{y_0,R}|^2\mathrm{d}y. \tag{4.31}$$

Let $\check{e}=\widetilde{e}-\widehat{e}$, thus it derives from (4.31), (4.24) and the monotonicity of $\Phi(\xi)$ defined after (4.28) that

$$\begin{aligned}
&\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}-(\nabla\widetilde{e})_{y_0,\tau}|^2\mathrm{d}y\\
&=\int_{B_\tau^+(y_0)} |\nabla\widehat{e}-(\nabla\widehat{e})_{y_0,\tau}+\nabla\check{e}-(\nabla\check{e})_{y_0,\tau}|^2\mathrm{d}y\\
&\le 2\int_{B_\tau^+(y_0)} |\nabla\widehat{e}-(\nabla\widehat{e})_{y_0,\tau}|^2\mathrm{d}y+2\int_{B_\tau^+(y_0)} |\nabla\check{e}-(\nabla\check{e})_{y_0,\tau}|^2\mathrm{d}y\\
&\le C\Big(\frac{\tau}{R}\Big)^5\int_{B_R^+(y_0)} |\nabla\widehat{e}-(\nabla\widehat{e})_{y_0,R}|^2\mathrm{d}y+C\int_{B_R^+(y_0)} |\nabla\check{e}-(\nabla\check{e})_{y_0,R}|^2\mathrm{d}y\\
&\le C\Big(\frac{\tau}{R}\Big)^5\int_{B_R^+(y_0)} |\nabla\widetilde{e}-(\nabla\widetilde{e})_{y_0,R}|^2\mathrm{d}y+C\int_{B_R^+(y_0)} |\nabla\check{e}|^2\mathrm{d}y+C\int_{B_R^+(y_0)} |(\nabla\check{e})_{y_0,R}|^2\mathrm{d}y\\
&\le C\Big(\frac{\tau}{R}\Big)^5\int_{B_R^+(y_0)} |\nabla\widetilde{e}-(\nabla\widetilde{e})_{y_0,R}|^2\mathrm{d}y+C\int_{B_R^+(y_0)} |\nabla\check{e}|^2\mathrm{d}y,
\end{aligned} \tag{4.32}$$

where the last inequality comes from Hölder inequality.

Now we estimate $\int_{B_R^+(y_0)} |\nabla \check{e}|^2 \mathrm{d}y$. From the definition of $\check{e}$, $\check{e}$ is the unique weak solution of the following problem

$$\begin{cases} L_1 \check{e} = \sum_{i=2}^{3} \partial_i \Big(\frac{y_i}{|y'|} \widetilde{f}_1 \Big) & \text{in} B_R^+(y_0), \\ \check{e} = 0 & \text{on } \partial B_R(y_0) \cap N_b^1, \\ \partial_n \check{e} = 0 & \text{on } B_r(y_0) \cap \partial N_b^1, \end{cases} \tag{4.33}$$

namely, for any test function $\phi \in H_0^1(B_R^+(y_0))$,

$$\int_{B_R^+(y_0)} (a_2 \partial_1 \check{e} \partial_1 \phi + a_1 \partial_2 \check{e} \partial_2 \phi + a_1 \partial_3 \check{e} \partial_3 \phi) \mathrm{d}y = \int_{B_R^+(y_0)} \Big(\frac{y_2}{|y'|} \widetilde{f}_1 \partial_2 \phi + \frac{y_3}{|y'|} \widetilde{f}_1 \partial_3 \phi \Big) \mathrm{d}y. \tag{4.34}$$

In (4.34), choose $\phi = \check{e}$, together with Hölder inequality and the ellipticity of the system, one has

$$\int_{B_R^+(y_0)} |\nabla \check{e}|^2 \mathrm{d}y \le C \int_{B_R^+(y_0)} |\widetilde{f}_1|^2 \mathrm{d}y \le C \|f_1\|^2_{C^\alpha(\overline{N_b})} R^{3+2\alpha}. \tag{4.35}$$

Here, the last estimate yields from the facts $B_R^+(y_0) \cap \{|y'| = 1\} \neq \emptyset$ and $\widetilde{f}_1\big|_{|y'|=1} = 0$.

Substituting (4.35) into (4.32) leads to

$$\begin{aligned} &\int_{B_\tau^+(y_0)} |\nabla \widetilde{e} - (\nabla \widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \\ &\le C \Big(\frac{\tau}{R} \Big)^5 \int_{B_R^+(y_0)} |\nabla \widetilde{e} - (\nabla \widetilde{e})_{y_0,R}|^2 \mathrm{d}y + C \|f_1\|^2_{C^\alpha(\overline{N_b})} R^{3+2\alpha}. \end{aligned} \tag{4.36}$$

Applying Proposition 3.4 to (4.36), we obtain

$$\begin{aligned} &\int_{B_\tau^+(y_0)} |\nabla \widetilde{e}_1 - (\nabla \widetilde{e}_1)_{y_0,\tau}|^2 \mathrm{d}y \\ &\le C \Big(\frac{1}{R^{3+2\alpha}} \int_{B_R^+(y_0)} |\nabla \widetilde{e}_1 - (\nabla \widetilde{e}_1)_{y_0,R}|^2 \mathrm{d}y + \|f_1\|^2_{C^\alpha(\overline{N_b})} \Big) \tau^{3+2\alpha}, \end{aligned}$$

thus

$$\int_{B_\tau^+(y_0)} |\nabla \widetilde{e}_1 - (\nabla \widetilde{e}_1)_{y_0,\tau}|^2 \mathrm{d}y \le C (\|\nabla \widetilde{e}_1\|^2_{L^2(N_b)} + \|f_1\|^2_{C^\alpha(\overline{N_b})}) \tau^{3+2\alpha}.$$

Together with (4.30) and (4.21), (4.19) is proved for this case.

Step 3 General case in $B_R^+(y_0)$.

By use of Lemma 4.2, without loss of generality, we also assume $\widetilde{f}(z, y') = 0$ on $|y'| = 1$, otherwise, we use $\widetilde{e} - \frac{\mathcal{T} \widetilde{f}}{a_1(1)}$ to replace $\widetilde{e}$ itself.

Fix $y_0 \in N_b$, rewrite the first equation in (4.15) as

$$\widehat{L} \widetilde{e} = \partial_1 (a_2(|y_0'|) \partial_1 \widetilde{e}) + \partial_2 (a_1(|y_0'|) \partial_2 \widetilde{e}) + \partial_3 (a_1(|y_0'|) \partial_3 \widetilde{e}) = \sum_{i=1}^{3} \partial_i \mathbf{F}_1^i, \tag{4.37}$$

where $\mathbf{F}_1 = (\mathbf{F}_1^1, \mathbf{F}_1^2, \mathbf{F}_1^3)$ with

$$\begin{cases} \mathbf{F}_1^1 = (a_2(|y_0'|) - a_2(|y'|)) \partial_1 \widetilde{e}, \\ \mathbf{F}_1^i = (a_1(|y_0'|) - a_1(|y'|)) \partial_i \widetilde{e} + \frac{y_i}{|y'|} \widetilde{f}, \quad i = 2, 3. \end{cases} \tag{4.38}$$

For any $0 < \tau < R < \frac{1}{8}$, with a little bit of conceptual confusion, we still define $\widehat{e}$ as the solution to the following homogeneous problem

$$\begin{cases} \widehat{L}\widehat{e} = 0 & \text{in } B_R^+(y_0), \\ \widehat{e} = \widetilde{e} & \text{on } \partial B_R(y_0) \cap N_b^1, \\ \partial_n \widehat{e} = 0 & \text{on } B_R(y_0) \cap \partial N_b^1 \end{cases} \tag{4.39}$$

and $\check{e} = \widetilde{e} - \widehat{e}$.

Similarly, based on Lemma 4.1, by Lax-Milgram theorem, there exists a unique $H^1(B_R^+(y_0))$ solution $\widehat{e}$ to the problem (4.39), which satisfies

$$\|\widehat{e}\|_{H^1(B_R^+(y_0))} \le C\|\widetilde{e}\|_{H^1(B_R^+(y_0))} \le C\|f_1\|_{0,\alpha}. \tag{4.40}$$

First, from (3.25), we have $\widetilde{f} \in L^{2,3-\mu}$ with $0 < \mu \le 3$. Then we claim $\nabla\widetilde{e} \in L^{2,3-\mu}$ for any $0 < \mu \le 3$ with

$$\|\nabla\widetilde{e}\|_{L^{2,3-\mu}(N_b)} \le C(\|\nabla\widetilde{e}\|_{L^2(N_b)} + \|f_1\|_{\alpha;N_b}). \tag{4.41}$$

Together with Lemma 4.1, we obtain

$$\|\nabla\widetilde{e}\|_{L^{2,3-\mu}(N_b)} \le C\|f_1\|_{\alpha;N_b}. \tag{4.42}$$

Now, we prove (4.41).

In fact, from (4.27), one has

$$\int_{B_\tau^+(y_0)} |\nabla\widehat{e}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^3 \int_{B_R^+(y_0)} |\nabla\widehat{e}|^2 \mathrm{d}y.$$

This yields

$$\begin{aligned} \int_{B_\tau^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y &\le 2\int_{B_\tau^+(y_0)} |\nabla\widehat{e}|^2 \mathrm{d}y + 2\int_{B_\tau^+(y_0)} |\nabla\check{e}|^2 \mathrm{d}y \\ &\le C\Big(\frac{\tau}{R}\Big)^3 \int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y + 2\int_{B_R^+(y_0)} |\nabla\check{e}|^2 \mathrm{d}y. \end{aligned} \tag{4.43}$$

From Lemma 4.1, (4.37) and (4.39), $\check{e} = \widetilde{e} - \widehat{e}$ is the unique $H^1(B_R^+(y_0))$ weak solution of

$$\begin{cases} \widehat{L}\check{e} = \displaystyle\sum_{i=1}^3 \partial_i \mathbf{F}_1^i & \text{in } B_R^+(y_0), \\ \check{e} = 0 & \text{on } \partial B_R(y_0) \cap N_b^1, \\ \partial_n \check{e} = 0 & \text{on } B_R(y_0) \cap \partial N_b^1. \end{cases} \tag{4.44}$$

The energy estimate of the problem (4.44) with the test function $\check{e}$ yields

$$\begin{aligned} \int_{B_R^+(y_0)} |\nabla\check{e}|^2 \mathrm{d}y &\le C\int_{B_R^+(y_0))} |\mathbf{F}_1|^2 \mathrm{d}y \\ &\le C\int_{B_R^+(y_0))} |f_1|^2 \mathrm{d}y + C\omega^2(R)\int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y, \end{aligned} \tag{4.45}$$

where $\omega^2(R) = \sup\limits_{y\in B_R^+(y_0)} \sum\limits_{i=1}^{2} |a_i(|y_0'|) - a_i(|y'|)|^2$.

Note that $a_i \in C^{1,\alpha}(\overline{N_b})$, then for any $\zeta > 0$, there exists a positive constant R_0, such that for any $0 < \tau < R < R_0$, we have $\omega^2(R) < \zeta$. In this situation, substituting (4.45) into (4.43) yields

$$\begin{aligned}\int_{B_\tau^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y &\le C\Big[\Big(\frac{\tau}{R}\Big)^3 + \zeta\Big] \int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y + C\int_{B_R^+(y_0))} |f_1|^2 \mathrm{d}y \\ &\le C\Big[\Big(\frac{\tau}{R}\Big)^3 + \sigma\Big] \int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y + C\|f_1\|^2_{\alpha;N_b} R^{3-\mu}. \qquad (4.46)\end{aligned}$$

Choose $\Phi(\tau) = \int_{B_\tau^+(y_0)} |D\widetilde{e}|^2 \mathrm{d}y$, $A = C$, $B = C\|f_1\|^2_{C^\alpha(\overline{N_b})}$, $a = 3$, $b = 3+2\alpha$ in Proposition 3.4, then there exists a positive constant $\widetilde{R}_0 \in (0, R_0)$, such that for any $0 < \tau < R < \widetilde{R}_0$, (4.41) has been proved from (4.46).

With the same argument in Step 2 used to obtain (4.32) and (4.35), we get

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5 \int_{B_R^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,R}|^2 \mathrm{d}y + C\int_{B_R^+(y_0)} |\nabla\check{e}|^2 \mathrm{d}y \quad (4.47)$$

and

$$\int_{B_R^+(y_0)} |\nabla\check{e}|^2 \mathrm{d}y \le C\sum_{i=1}^{3} \int_{B_R^+(y_0)} |\mathbf{F}_1 - (\mathbf{F}_1)_{y_0,R}|^2 \mathrm{d}y. \qquad (4.48)$$

From the definition of $\mathbf{F}_1$ in (4.38), one has

$$\begin{aligned}&\int_{B_R^+(y_0)} |\mathbf{F}_1 - (\mathbf{F}_1)_{y_0,R}|^2 \mathrm{d}y \\ &\le \int_{B_R^+(y_0)} |\widetilde{f} - (\widetilde{f})_{y_0,R}|^2 \mathrm{d}y + \sum_{i=1}^{3} \big|\widehat{a}_i(|y_0'|) - \widehat{a}_i(|y'|)\big|^2 \times \int_{B_R^+(y_0)} |D\widetilde{e}|^2 \mathrm{d}y, \qquad (4.49)\end{aligned}$$

where $\widehat{a}_1 = a_2$, $\widehat{a}_2 = \widehat{a}_3 = a_1$. Note that $a_i \in C^{1,\alpha}(\overline{N_b})$, we have

$$\sup_{y\in B_R^+(y_0)} |\widehat{a}_i(|y_0'|) - \widehat{a}_i(|y'|)|^2 \le [a_i]^2_{0,\alpha} R^{2\alpha}. \qquad (4.50)$$

Since $f_1 \in C^\alpha(\overline{N}_b)$ and $f_1(y_1, y')\big|_{|y'|=1} = 0$, substituting (4.48)–(4.50) into (4.47) yields

$$\begin{aligned}&\int_{B_\tau^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \\ &\le C\Big(\frac{\tau}{R}\Big)^5 \int_{B_R^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,R}|^2 \mathrm{d}y \\ &\quad + C[f_1]^2_\alpha R^{3+2\alpha} + C[a_i]^2_{0,\alpha} R^{2\alpha} \int_{B_R^+(y_0)} |D\widetilde{e}|^2 \mathrm{d}y \\ &\le C\Big(\frac{\tau}{R}\Big)^5 \int_{B_R^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,R}|^2 \mathrm{d}y + C\|f_1\|_{\alpha;N_b} R^{3+2\alpha-\mu}, \quad 0 < \mu < 2\alpha, \qquad (4.51)\end{aligned}$$

where the last inequality comes from (4.42). Together with Proposition 3.4, one has

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \le C\|f_1\|^2_{\alpha;N_b}\tau^{3+2\alpha-\mu},$$

which means $\nabla\widetilde{e} \in C^{\alpha-\frac{\mu}{2}}(\overline{N_b})$ for any $0<\mu<2\alpha$ with

$$\|\nabla\widetilde{e}\|_{\alpha-\frac{\mu}{2};N_b} \le C\|f_1\|_{\alpha;N_b}.$$

This yields (4.41) with the help of (3.25). Thus $\nabla\widetilde{e}$ is bounded and

$$\int_{B_R^+(y_0)} |\nabla\widetilde{e}|^2 \mathrm{d}y \le C\|f_1\|^2_{\alpha;N_b}R^3. \tag{4.52}$$

Substituting (4.52) into the first inequality of (4.51) arrives at

$$\int_{B_\tau^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,\tau}|^2 \mathrm{d}y \le C\Big(\frac{\tau}{R}\Big)^5 \int_{B_R^+(y_0)} |\nabla\widetilde{e} - (\nabla\widetilde{e})_{y_0,R}|^2 \mathrm{d}y + C\|f_1\|_{\alpha;N_b}R^{3+2\alpha}. \tag{4.53}$$

Using Proposition 3.4 again, we can infer that $\nabla\widetilde{e} \in C^\alpha(\overline{N_b})$ with the estimate (4.19). Finally, we finish the proof of Lemma 4.3.

With respect to the problem (4.13), we have the following result.

Lemma 4.4 *Under the assumptions in Lemma* 4.1*, the problem* (4.13) *has a unique solution* $e_{12} \in C^{1,\alpha}(\overline{N_b})$ *with*

$$\|e_{12}\|_{1,\alpha} \le C\|f_1\|_{1,\alpha}. \tag{4.54}$$

Proof By Lemma 4.1 and Lemma 4.3, the problem (4.12) has a unique solution $e_{11} \in C^{1,\alpha}(\overline{N}_b)$ with $\partial_n e_{11}(\ell,y)|_{|y'|=1} = 0$. This yields $\partial_n e_{12}(\ell,y')|_{|y'|=1} = 0$, which matches the lateral boundary condition of e_{12} at the points $y=(\ell,y')|_{|y'|=1}$. Therefore, Lemma 4.4 can be derived from in [19, Theorems 6.30–6.31] with the local extension skill.

Lemma 4.5 *Under the assumption of Theorem* 4.1*, the problem* (4.7) *has a unique solution* (d_1,e_1) *with*

$$\Big\|\frac{y_2}{r}d_1\circ\boldsymbol{T}\Big\|_{1,\alpha} + \Big\|\frac{y_3}{r}d_1\circ\boldsymbol{T}\Big\|_{1,\alpha} + \|e_1\circ\boldsymbol{T}\|_{1,\alpha} \le C\|f_1\|_{0,\alpha},$$

where $C=C(\lambda,\Lambda)$.

Proof This lemma can be directly derived from Lemmas 4.3–4.4, Lemma 3.2 and the expression of $\partial_z d_1$ in $(4.7)_1$.

4.2 Solvability of (d_2,e_2)

Set

$$\partial_z\psi = e_2, \quad \partial_r\psi = -\frac{1}{a_1(r)}d_2, \quad \psi(0,0)=0. \tag{4.55}$$

Substituting (4.55) into (4.8) shows

$$\begin{cases} a_2(r)\partial_z^2\psi + a_1(r)\Big(\partial_r^2\psi + \dfrac{1}{r}\partial_r\psi\Big) + a_1'(r)\partial_r\psi = f_2, & (z,r)\in(0,\ell)\times(0,1),\\ \psi(0,r)=0, & r\in[0,1],\\ \partial_z\psi(\ell,r)=0, & r\in[0,1],\\ \partial_r\psi(z,0)=\partial_r\psi(z,1)=0, & z\in[0,\ell]. \end{cases} \tag{4.56}$$

With the inverse of the transformation (2.7), the problem (4.56) can be written as

$$\begin{cases} L\psi = f_2, & y\in N_b,\\ \partial_n\psi = 0, & y\in[0,\ell]\times\partial B_1(0),\\ \psi(0,y')=\partial_1\psi(\ell,y')=0, & y'\in B_1(0). \end{cases} \tag{4.57}$$

We have the following well-posedness result of the problem (4.57).

Lemma 4.6 *Under the assumptions of Theorem* 4.1*, the problem* (4.57) *has a unique solution* $\psi\in C^{2,\alpha}(\overline{N_b})$ *with*

$$\|\psi\|_{2,\alpha}\le C\|f_2\|_{0,\alpha}, \tag{4.58}$$

where $C=C(\lambda,\Lambda)$.

Proof Set

$$\widetilde{\psi}(y_1,y')=\begin{cases} \psi(y_1,y'), & y_1\in[0,\ell],\\ \psi(2\ell-y_1,y'), & y_1\in[\ell,2\ell] \end{cases} \tag{4.59}$$

and

$$\widetilde{f_2}(y_1,y')=\begin{cases} f_2(y_1,y'), & y_1\in[0,\ell],\\ f_2(2\ell-y_1,y'), & y_1\in[\ell,2\ell], \end{cases} \tag{4.60}$$

then the problem (4.57) is equivalently converted into the following problem of $\widetilde{\psi}$ as

$$\begin{cases} L\widetilde{\psi} = \widetilde{f_2}, & y\in N_b^2,\\ \partial_n\widetilde{\psi} = 0, & y\in[0,2\ell]\times\partial B_1(0),\\ \widetilde{\psi}(0,y')=\widetilde{\psi}(2\ell,y')=0, & y'\in B_1(0) \end{cases} \tag{4.61}$$

with $N_b^2=[0,2\ell]\times B_1(0)$ and the divergence type operator L defined in (4.12).

By Lax-Milgram theorem, when $f_2\in C^{\alpha}(\overline{N}_b)$, the problem (4.61) has a unique axisymmetric solution $\widetilde{\psi}\in H^1(N_b^2)$. According to [19, Chapter 6, Chapter 8], one has

$$\widetilde{\psi}\in C^{2,\alpha}([0,2\ell]\times B_1(0))\cap C^0(\overline{N_b^2}). \tag{4.62}$$

It remains to show that $\widetilde{\psi}$ admits the regularity estimate (4.58).

Step 1 Estimate of $\widetilde{\psi}$ in $[0,2\ell]\times\overline{B_{\frac{3}{4}}(0)}$.

Set

$$\varphi(y) = \frac{\boldsymbol{F}}{\lambda}(y_1(2\ell - y_1) + k(y_2^2 + y_3^2)),$$

where $\boldsymbol{F} = \|\widetilde{f}_2\|_{0,\alpha}$ and $k = \frac{\lambda}{8\Lambda}$. Without loss of generality, we assume $\boldsymbol{F} \neq 0$.

Then a direct computation yields

$$\begin{cases} L(\pm\widetilde{\psi} + \varphi) \leq 0 & \text{in } N_b^2, \\ \partial_n(\pm\widetilde{\psi} + \varphi) > 0 & \text{on } |y'| = 1, \\ \pm\widetilde{\psi} + \varphi \geq 0 & \text{on } y_1 = 0, 2\ell. \end{cases} \tag{4.63}$$

By use of the comparison principle, it derives from (4.63) that

$$\|\widetilde{\psi}\|_{L^\infty} \leq C\|\widetilde{f}_2\|_{0,\alpha}, \tag{4.64}$$

where $C = C(\lambda, \Lambda)$.

Then it follows from the Schauder interior and boundary estimates in [19, Theorem 6.2, Corollary 6.7] that

$$\|\widetilde{\psi}\|_{C^{2,\alpha}([0,2\ell]\times\overline{B_{\frac{3}{4}}(0)})} \leq C(\|\widetilde{\psi}\|_{L^\infty} + \|\widetilde{f}_2\|_{0,\alpha}) \leq C\|\widetilde{f}_2\|_{0,\alpha}. \tag{4.65}$$

Step 2 Estimate of $\widetilde{\psi}$ near $\{r = 1\}$.

To estimate ψ in $[0, 2\ell] \times \big\{y' : |y'| \in \big[\frac{3}{4}, 1\big]\big\}$, we consider the following problem under the (z, r) coordinate

$$\begin{cases} a_2(r)\partial_z^2\widetilde{\psi} + a_1(r)\partial_r^2\widetilde{\psi} + \Big(\dfrac{a_1(r)}{r} + a_1'(r)\Big)\partial_r\widetilde{\psi} = \widetilde{f}_2, & (z, r) \in (0, 2\ell) \times \Big(\dfrac{1}{2}, 1\Big), \\ \widetilde{\psi}(0, r) = \widetilde{\psi}(2\ell, r) = 0, & r \in [0, 1], \\ \widetilde{\psi}\Big(z, \dfrac{1}{2}\Big) = \widetilde{\psi}(y_1, y') \quad \text{with } |y'| = \dfrac{1}{2}, & z \in [0, 2\ell], \\ \partial_r\widetilde{\psi}(z, 1) = 0, & z \in [0, \ell]. \end{cases}$$

Set

$$\overline{a}_i(r) = \begin{cases} a_i(r), & r \in \Big[\dfrac{1}{2}, 1\Big], \\ a_i(2 - r), & r \in \Big[1, \dfrac{3}{2}\Big], \end{cases} \quad i = 1, 2,$$

$$\overline{\psi} = \begin{cases} \widetilde{\psi}(z, r), & r \in \Big[\dfrac{1}{2}, 1\Big], \\ \widetilde{\psi}(z, 2 - r), & r \in \Big[1, \dfrac{3}{2}\Big] \end{cases}$$

and

$$\overline{f}_2 = \begin{cases} \Big(\widetilde{f}_2 - \Big(\dfrac{a_1(r)}{r} + a_1'(r)\Big)\partial_r\widetilde{\psi}\Big)(z, r), & r \in \Big[\dfrac{1}{2}, 1\Big], \\ \Big(\widetilde{f}_2 - \Big(\dfrac{a_1(r)}{r} + a_1'(r)\Big)\partial_r\widetilde{\psi}\Big)(z, 2 - r), & r \in \Big[1, \dfrac{3}{2}\Big], \end{cases}$$

then

$$\begin{cases} \overline{a}_2(r)\partial_z^2\overline{\psi} + \overline{a}_1(r)\partial_r^2\overline{\psi} = \overline{f}_2, & (z,r) \in (0,2\ell) \times \Big(\dfrac{1}{2}, \dfrac{3}{2}\Big), \\ \overline{\psi}(0,r) = \overline{\psi}(2\ell, r) = 0, & r \in \Big[0, \dfrac{3}{2}\Big], \\ \overline{\psi}\Big(z, \dfrac{1}{2}\Big) = \widetilde{\psi}(y_1, y') \quad \text{with } |y'| = \dfrac{1}{2}, & z \in [0, 2\ell], \\ \overline{\psi}\Big(z, \frac{3}{2}\Big) = \widetilde{\psi}(y_1, y') \quad \text{with } |y'| = \dfrac{1}{2}, & z \in [0, 2\ell], \\ \partial_r\overline{\psi}(z, 1) = 0, & z \in [0, \ell]. \end{cases} \tag{4.66}$$

Similar to (4.65) in Step 1, with the definitions of $\overline{\psi}, \overline{f}_2$ and (4.64), (4.66), one has

$$\begin{aligned} &\|\widetilde{\psi}\|_{C^{2,\alpha}([0,2\ell]\times\{\frac{3}{4}\le|y'|\le 1\})} \\ &\le \|\overline{\psi}\|_{C^{2,\alpha}([0,2\ell]\times\{\frac{3}{4}\le|y'|\le\frac{5}{4}\})} \\ &\le C(\|\overline{\psi}\|_{L^\infty} + \|\overline{f}_2\|_{0,\alpha}) \\ &\le C\Big(\widetilde{f}_2\|_{0,\alpha} + \sum_{i=2}^{3} \|\partial_i\widetilde{\psi}\|_{0,\alpha}\Big). \end{aligned}$$

Combining this with (4.64)–(4.65) and (4.59)–(4.60) yields (4.58) and the proof of Lemma 4.6 is completed.

Lemma 4.7 *Under the assumption of Theorem* 4.1*, the problem* (4.8) *has a unique solution* (d_2, e_2) *with*

$$\Big\|\frac{y_2}{r}d_2 \circ \boldsymbol{T}\Big\|_{1,\alpha} + \Big\|\frac{y_3}{r}d_2 \circ \boldsymbol{T}\Big\|_{1,\alpha} + \|e_2 \circ \boldsymbol{T}\|_{1,\alpha} \le C\|f_2\|_{0,\alpha}, \tag{4.67}$$

where $C = C(\lambda, \Lambda)$.

Proof From (4.55) and Lemma 4.6, one has

$$e_2 \circ \boldsymbol{T} = \partial_{y_1}\psi \in C^{1,\alpha}(\overline{N_b}) \tag{4.68}$$

and

$$\frac{y_i}{r}d_2 \circ \boldsymbol{T} = -a_2(r)\frac{y_i}{r}\partial_r\psi = -a_2(|y'|)\partial_{y_i}\psi \in C^{1,\alpha}(\overline{N_b}), \quad i = 2, 3. \tag{4.69}$$

Finally, (4.67) comes from (4.68)–(4.69) and (4.58). We complete the proof of Lemma 4.7.

Proof of Theorem 4.1 Theorem 4.1 can be established directly from Lemma 4.5 and Lemma 4.7.

5 Proof of the Main Theorem

Based on the linearized form for the nonlinear problem constructed in Section 2, Theorem 2.1 as well as the main Theorem 1.1 will be proved via the Banach fixed point theorem. To this end, in the next Subsection 5.1, we construct an iteration scheme determined by the linearized form.

5.1 Iteration scheme

Based on (2.20)–(2.21) and (2.24)–(2.26), we can construct the iteration scheme.

With the notations

$$U = (u_1, u_2, u_3, \boldsymbol{P}, \boldsymbol{E}, \boldsymbol{s})(y) \in \{\text{axisymmetry}\},$$

$$U_b = (v_{zb}(|y'|), 0, 0, \boldsymbol{P}_b, \boldsymbol{E}_b(|y'|), \boldsymbol{s}_b)$$

with $\boldsymbol{P}_b = A(\boldsymbol{s}_b)\rho_b^\gamma$ where $\rho_b, \boldsymbol{s}_b$ are positive constants, one can define $w = w(z,r)$ by (2.8)–(2.9) and (2.17). Thus, we introduce a Banach space Ξ_δ as

$$\begin{aligned}\Xi_\delta =\Big\{& w(z,r) \in C^{1,\alpha}(\overline{N_b})^5 : [[w]]_{1,\alpha} := \|(w_1, w_4, w_5, w_6)\|_{1,\alpha} + \sum_{i,j=2}^{3} \Big\|\frac{y_i}{|y'|} w_j\Big\|_{1,\alpha} < \delta,\\ & w_2(z,0) = w_3(z,0) = 0, w_2(z,1) = g'(z), w_3(\ell,1) = \partial_r w_4(\ell,1) = 0\Big\}\end{aligned} \tag{5.1}$$

with the positive constant δ to be specific later.

Based on the reduction in Subsection 2.3, we can formally define a mapping

$$\mathbb{T}:\ \Xi_\delta \to \Xi_\delta, \quad \overline{w} \mapsto w = \mathbb{T}(\overline{w}) \tag{5.2}$$

in the following way.

Due to (2.21) with (2.12)–(2.14), w_2 and w_4 are defined by the following problem

$$\begin{cases} \partial_z w_2 + a_1(r)\partial_r w_4 = \mathcal{F}_1(\overline{w}), & (z,r) \in (0,\ell) \times (0,1),\\ -\partial_r w_2 - \dfrac{1}{r} w_2 + a_2(r)\partial_z w_4 = \mathcal{F}_2(\overline{w}), & (z,r) \in (0,\ell) \times (0,1),\\ w_2(0,r) = v_{r0}(r), & r \in (0,1),\\ w_2(z,0) = 0, \quad w_2(z,1) = g'(z), & z \in [0,\ell],\\ w_4(\ell,r) = P_\ell(r) - \boldsymbol{P}_b. \end{cases} \tag{5.3}$$

Once w_2 and w_4 are obtained, we can define $v = w_2 + 0$.

According to (2.24), w_3 is determined as

$$\begin{cases} (g(z)\partial_z - g'(z) r\partial_r + v\partial_r) w_3 + \dfrac{1}{r} v w_3 = 0, & (z,r) \in (0,\ell) \times (0,1),\\ w_3(0,r) = v_{\theta 0}(r), & r \in [0,1]. \end{cases} \tag{5.4}$$

Similarly, according to (2.25), w_6 is defined as

$$\begin{cases} (g(z)\partial_z - g'(z) r\partial_r + v\partial_r) w_6 = 0, & (z,r) \in (0,\ell) \times (0,1),\\ w_6(0,r) = \boldsymbol{s}_0(r) - \boldsymbol{s}_b, & r \in [0,1]. \end{cases} \tag{5.5}$$

As for the definition of w_5, in the way of deriving (2.26), for any $(z,r) \in [0,\ell] \times [0,1]$, we define the leftward characteristics $r = r(z; \beta(r,z))$ with respect to the operator $L_v = g(z)\partial_z - g'(r) r\partial_r + v\partial_r$, going through (z,r) and ending at $(0, \beta(r,z))$. Thus, we define

$$w_5 = \boldsymbol{E}_0(\beta(r,z)) - \boldsymbol{E}_b(r). \tag{5.6}$$

Finally, with the identity 2.20, w_1 is defined as

$$w_1 = \mathscr{G}_3(\overline{w})w_2 + \mathscr{G}_4(\overline{w})w_3 + \mathscr{G}_5(\overline{w})w_4 + \mathscr{G}_6(\overline{w})w_5 + \mathscr{G}_7(\overline{w})w_6. \tag{5.7}$$

Consequently, if the problems (5.3)–(5.7) are well solved, then the mapping $w = \mathbb{T}(\overline{w})$ can be formally defined. Therefore, the proof of the main Theorem 2.1 is reduced to show that the mapping $\mathbb{T}$ is well-defined and has a unique fixed point in Ξ_δ for suitable positive δ.

5.2 Apriori estimates

For the aims of boundness and contraction of the mapping $\mathbb{T}$, at first, we establish some necessary apriori estimates of $\mathcal{F}_i(w)$ $(i = 1, 2)$ defined in (2.23). For the convenience, hereafter, we introduce a generic positive constant $\overline{C}$ as: For any functions $F = F(w, \partial w)$ and $G = G(w)$ smooth with respect to their arguments with any $w \in \Xi_\delta$, we define

$$\|F(w, \partial w)\|_{0,\alpha} + \|G(w)\|_{1,\alpha} \le \overline{C}. \tag{5.8}$$

For the aim of the boundness of the mapping $\mathbb{T}$, we have the following result.

Lemma 5.1 *For $\mathcal{F}_i(w) (i = 1, 2)$ defined in (2.23) with $w \in \Xi_\delta$, one has*

$$\sum_{i=1}^{2} \|\mathcal{F}_i(w)\|_{0,\alpha} \le \overline{C}(\delta + \varepsilon)\delta. \tag{5.9}$$

Proof From (2.18) and (2.20), one has

$$\begin{aligned} \|\rho - \rho_b\|_{1,\alpha} &\le \overline{C}(\|w_4\|_{1,\alpha} + \|w_6\|_{1,\alpha}) \le \overline{C}\delta, \\ \|w_1\|_{1,\alpha} &\le \overline{C}\sum_{i=2}^{6} \|w_i\|_{1,\alpha} \le \overline{C}\delta. \end{aligned} \tag{5.10}$$

Due to the axisymmetry, it derives from Lemma 3.3 that $\partial_i w_4(z, 0) = 0$ $(i = 2, 3)$. Combining this with $w \in \Xi_\delta$ yields

$$\begin{aligned} \mathcal{F}_1(w) = {} & \frac{1}{\rho\rho_b v_z v_{zb}}[v_{zb}^2(\rho_b - \rho) + \rho(v_{zb} + v_z)(v_{zb} - v_z)]\Big(\frac{y_2}{r}\partial_2 w_4 + \frac{y_3}{r}\partial_3 w_4\Big) \\ & + (1 - g(z))\partial_z w_2 + g'(z)(y_2\partial_2 w_2 + y_3\partial_3 w_2) \\ & + \frac{1}{\gamma P}w_2(g(z)\partial_z w_4 - g'(z)(y_2\partial_2 w_4 + y_3\partial_3 w_4)) + w_2\Big(\frac{y_2}{r}\partial_2 w_4 + \frac{y_3}{r}\partial_3 w_4\Big) \\ & + \frac{w_2}{r}\int_0^r \partial_r w_2(z, t)\mathrm{d}t + \frac{w_3}{v_z^2 r}\int_0^r \partial_r w_3(z, t)\mathrm{d}t, \\ \mathcal{F}_2(w) = {} & \frac{1}{\gamma P}w_2\Big(\frac{y_2}{r}\partial_2 w_4 + \frac{y_3}{r}\partial_3 w_4\Big) + \Big(a_2(r) - \Big(\frac{1}{\rho v_z^2} - \frac{1}{\gamma P}\Big)g(z)\Big)\partial_z w_4 \\ & + \Big(\frac{1}{\rho v_z^2} - \frac{1}{\gamma P}\Big)g'(z)(y_2\partial_2 w_4 + y_3\partial_3 w_4) \end{aligned} \tag{5.11}$$

with

$$\mathcal{F}_1(w)(z, 0) = 0, \quad \mathcal{F}_1(w)(\ell, 1) = 0. \tag{5.12}$$

Since $w_2(z,0) = w_3(z,0) = \partial_i w_4(z,0) = 0$ $(i = 2,3)$, by use of Lemma 3.1, it derives from (5.10)–(5.11) that

$$\begin{aligned}
&\|\mathcal{F}_1(\overline{w})\|_{0,\alpha} + \|\mathcal{F}_2(\overline{w})\|_{0,\alpha}\\
&\le \overline{C}(\|\rho-\rho_b\|_{0,\alpha} + \|v_z - v_{zb}\|_{0,\alpha} + \|1-g\|_{2,\alpha} + \|w\|_{1,\alpha})(\|\partial w_2\|_{0,\alpha} + \|\partial w_4\|_{0,\alpha})\\
&\le \overline{C}(\delta+\varepsilon)\delta.
\end{aligned}$$

This shows (5.9) and the proof of Lemma 5.1 is completed.

Next, we establish the estimates for the aim of the contraction of the mapping $\mathbb{T}$. To this end, for any $w \in \Xi_\delta$, we denote the new norm $\langle\cdot\rangle_\alpha$ as

$$\langle w\rangle_\alpha = \|(w_1,w_5,w_6)\|_{0,\alpha} + \|w_4\|_{1,\alpha} + \sum_{i=2}^{2}\Big(\Big\|\frac{y_i}{|y'|}w_2\Big\|_{1,\alpha} + \Big\|\frac{y_i}{|y'|}w_3\Big\|_{0,\alpha}\Big). \tag{5.13}$$

For any $w^i \in \Xi_\delta$ $(i = 1,2)$, with the notations in (2.17)–(2.18), one can define ρ^i and $\boldsymbol{P}^i$ accordingly. Then the necessary estimates for the contraction of the mapping $\mathbb{T}$ can be stated as the following result.

Lemma 5.2 *With the notations above, we have*

$$\|\mathcal{F}_i(w^1) - \mathcal{F}_i(w^2)\|_{0,\alpha} \le \overline{C}(\delta+\varepsilon)\langle w^1 - w^2\rangle_\alpha, \quad i = 1,2. \tag{5.14}$$

Proof From (2.18), one has

$$\begin{aligned}
\|\rho_1-\rho_2\|_{0,\alpha} &\le \sum_{i=1}^{2}\|\mathscr{G}_i(w^1) - \mathscr{G}_i(w^2)\|_{0,\alpha}(\|w_4^1\|_{0,\alpha} + \|w_6^1\|_{0,\alpha})\\
&\quad + \sum_{i=1}^{2}\|\mathscr{G}_i(w^2)\|_{0,\alpha}(\|w_4^1 - w_4^2\|_{0,\alpha} + \|w_6^1 - w_6^2\|_{0,\alpha})\\
&\le \overline{C}\langle w^1 - w^2\rangle_\alpha.
\end{aligned}$$

Combining this with (5.11) arrives

$$\begin{aligned}
&\|\mathcal{F}_1(w^1) - \mathcal{F}_1(w^2)\|_{0,\alpha}\\
&\le \overline{C}[[w^1]]_{1,\alpha}(\|\rho^1-\rho^2\|_{0,\alpha} + \langle w^1 - w^2\rangle_\alpha)\\
&\quad + \overline{C}(\|\rho^2-\rho_b\|_{1,\alpha} + \|v_z^2 - v_{zb}\|_{1,\alpha} + \|1-g\|_{2,\alpha} + [[w^2]]_{1,\alpha})\langle w^1 - w^2\rangle_\alpha\\
&\le \overline{C}(\delta+\varepsilon)\langle w^1 - w^2\rangle_\alpha.
\end{aligned} \tag{5.15}$$

In a similar way, we can directly compute that

$$\|\mathcal{F}_2(w^1) - \mathcal{F}_2(w^2)\|_{0,\alpha} \le \overline{C}(\delta+\varepsilon)\langle w^1 - w^2\rangle_\alpha. \tag{5.16}$$

Finally, (5.14) comes from (5.15)–(5.16). And we finish the proof of Lemma 5.2.

5.3 Well-posedness of the mapping $\mathbb{T}$

With the notations in (5.2), for any $\overline{W} \in \Xi_\delta$, we solve $w = \mathbb{T}(\overline{W})$ in the following four steps.

Step 1 Solvability of w_2 and w_4.

From (5.12), (1.9) and Lemma 3.3, one has

$$\begin{aligned} &\mathcal{F}_1(\overline{w})(z,0) = 0, \quad \mathcal{F}_1(\overline{w})(\ell,1) = 0, \\ &v_{r0}(0) = 0, \quad v_{r0}(1) = g'(0). \end{aligned} \tag{5.17}$$

Meanwhile, it derives from (1.11) that

$$g''(\ell) + a_1(1)P_\ell'(1) = 0 = \mathcal{F}_1(\overline{W})(\ell,1) = 0. \tag{5.18}$$

Combining (5.17)–(5.18) with Theorem 4.1 and (5.9) shows that the problem (5.3) has a unique solution (w_2, w_4) with

$$\begin{aligned} &\Big\|\frac{y_2}{r}w_2\circ\boldsymbol{T}\Big\|_{1,\alpha} + \Big\|\frac{y_3}{r}w_2\circ\boldsymbol{T}\Big\|_{1,\alpha} + \|w_4\circ\boldsymbol{T}\|_{1,\alpha} \\ \le\,& \overline{C}\Big(\sum_{i=1}^{2}\|\mathcal{F}_i(\overline{w})\circ\boldsymbol{T}\|_{0,\alpha} + \sum_{j=2}^{3}\Big\|\frac{y_j}{r}v_{r0}\circ\boldsymbol{T}\Big\|_{1,\alpha} + \|(P_\ell - P_b)\circ\boldsymbol{T}\|_{1,\alpha} + \|g'\|_{1,\alpha}\Big) \\ \le\,& \overline{C}(\delta^2 + \varepsilon). \end{aligned} \tag{5.19}$$

Step 2 Solvability of w_5 and w_6.

For $v = w_2 + 0$, we define the operator L_v and $\beta_{vi}(y)$ $(i = 2, 3)$ by (3.14) and (3.17). Thus, w_5 in (5.6) can be expressed as

$$w_5 = \boldsymbol{E}_0(|\beta_{v2}(y), \beta_{v3}(y))|) - E_b(r).$$

Combining this with $v = w_2 + 0$ and (1.12), (3.21), (3.19), (5.19) yields

$$\|w_5\|_{1,\alpha} \le \overline{C}(\varepsilon + \|Q_1\|_{1,\alpha} + \|Q_2\|_{1,\alpha}) \le \overline{C}(\delta^2 + \varepsilon). \tag{5.20}$$

In the similar way, the problem (5.5) has a unique solution w_6 satisfying

$$w_6 = \boldsymbol{s}_0(|(\beta_{v2}(y), \beta_{v3}(y))|) - \boldsymbol{s}_b \tag{5.21}$$

and

$$\|w_6\|_{1,\alpha} \le \overline{C}(\delta^2 + \varepsilon). \tag{5.22}$$

Step 3 Solvability of w_3.

With the notations of L_v and $\beta_{vi}(y)$ $(i = 2, 3)$ in the above Step 2 and the characteristic in Lemma 3.4, the problem (5.4) has a unique solution w_3 as

$$w_3(y_1, |y'|) = v_{\theta 0}(|(\beta_{v2}(y), \beta_{v3}(y))|)\exp\Big\{-\int_0^{y_1}\frac{v(t, R_v(t;y))}{g(t)R_v(t;y)}\mathrm{d}t\Big\} \tag{5.23}$$

with $R_v(t;y) = |(Y_2(t;y), Y_3(t;y)|$.

Here, for $i = 2,3$ and $j = 1,2,3$, a direct computation shows

$$\partial_j\Big(\frac{y_i}{|y'|}w_3\Big) = \Big(\frac{\delta_{ij}}{|y'|} - (1-\delta_1^j)\frac{y_iy_j}{|y'|^3} + \delta_1^j\frac{y_i}{|y'|}\Big)w_3 + \frac{y_i}{|y'|}(J_1^j(y) + J_2^j(y) + J_3^j(y) + J_4^j(y))\exp\Big\{-\int_0^{y_1}\frac{v(t,R_v(t;y))}{g(t)R_v(t;y)}\mathrm{d}t\Big\} \tag{5.24}$$

with

$$J_1^j(y) = \partial_r v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|)\frac{\beta_{vi}(y)\partial_j\beta_{vi}(y)}{|(\beta_{v2}(y),\beta_{v3}(y))|},$$
$$J_2^j(y) = -\delta_j^1 v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|)\frac{v(y_1,R_v(y_1;y))}{g(y_1)R_v(y_1;y)},$$
$$J_3^j(y) = -v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|)\int_0^{y_1}\frac{\partial_r v(t,R_v(t;y))}{g(t)R_v(t;y)}\frac{Y_i(t;y)\partial_jY_i(t;y)}{R_v(t;y)}\mathrm{d}t,$$
$$J_4^j(y) = v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|)\int_0^{y_1}\frac{v(t,R_v(t;y))}{g(t)R_v^2(t;y)}\frac{Y_i(t;y)\partial_jY_i(t;y)}{R_v(t;y)}\mathrm{d}t.$$

Now, we only estimate $\big[\frac{y_i}{|y'|}J_4^j(y)\big]_\alpha$ since other cases are easier. Due to $v_{\theta 0}(0) = 0$, we denote

$$\begin{aligned}J_4^j(y) &= v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|)\int_0^{y_1}\frac{1}{R_v^2(t;y)}J_{41}^i(t;y)\partial_jY_i(t;y)\mathrm{d}t\\ &= \int_0^1\partial_r v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|t)\mathrm{d}t\cdot|(\beta_{v2}(y),\beta_{v3}(y))|\cdot\int_0^{y_1}\frac{1}{g(t)R_v^2(t;y)}J_{41}^i(t;y)\partial_jY_i(t;y)\mathrm{d}t\\ &:= J_{42}^1(y)\cdot J_{42}^2(y)\cdot J_{42}^3(y)\end{aligned} \tag{5.25}$$

with

$$J_{41}^i(t;y) = \frac{Y_i(t;y)}{R_v(t;y)}v(t;R_v(t;y)) = \frac{z_i}{|z'|}v(z)\big|_{z=(t,Y_2(t;y),Y_3(t;y))}.$$

For any $y,z\in N_b$ with $|z'|\le|y'|$, one has

$$\begin{aligned}&J_4^j(y) - J_4^j(z)\\ &= \int_0^1(\partial_r v_{\theta 0}(|(\beta_{v2}(y),\beta_{v3}(y))|t) - \partial_r v_{\theta 0}(|(\beta_{v2}(z),\beta_{v3}(z))|t))\mathrm{d}t\cdot J_{42}^2(y)\cdot J_{42}^3(y)\\ &\quad + J_{42}^1(z)\cdot(|(\beta_{v2}(y),\beta_{v3}(y))| - |(\beta_{v2}(z),\beta_{v3}(z))|)\cdot J_{42}^3(y)\\ &\quad + J_{42}^1(z)\cdot J_{42}^2(z)\cdot(J_{42}^3(y) - J_{43}^3(z))\\ &:= J_{43}^1(y,z) + J_{43}^2(y,z) + J_{43}^3(y,z).\end{aligned} \tag{5.26}$$

With respect to $J_{43}^1(y,z)$, by use of (3.19)–(3.20) and $v(y_1,0) = 0$, one has

$$\begin{aligned}|J_{43}^1(y,z)| &\le \overline{C}[\partial_r v_{\theta 0}]_\alpha|(\beta_{v2}(y),\beta_{v3}(y)) - (\beta_{v2}(z),\beta_{v3}(z))|^\alpha\sup_{t\in[0,\ell]}\|\partial_jY_i(t)\|_{0,\alpha}\|\partial_r v\|_{0,\alpha}\\ &\le \overline{C}\|\partial_r v_{\theta 0}\|_{0,\alpha}\|\partial_r v\|_{0,\alpha}|y-z|^\alpha\\ &\le \overline{C}(\delta^2+\varepsilon)|y-z|^\alpha,\end{aligned} \tag{5.27}$$

where the last inequality is due to (5.19) and (1.12). With respect to $J_{43}^2(y,z)$, as for (5.27), by use of $\partial_r v_{\theta 0}(0)=0$, one has

$$\begin{aligned}|J_{43}^2(y,z)| &\le \overline{C}[\partial_r v_{\theta 0}]_\alpha |(\beta_{v2}(z),\beta_{v3}(z))|^\alpha |(\beta_{v2}(y),\beta_{v3}(y))|^{1-\alpha}|y-z|^\alpha |J_{42}^3(y)|\\ &\le \overline{C}\|\partial_r v_{\theta 0}\|_{0,\alpha}\|\partial_r v\|_{0,\alpha}|y-z|^\alpha\\ &\le \overline{C}(\delta^2+\varepsilon)|y-z|^\alpha. \end{aligned}\tag{5.28}$$

With respect to $J_{43}^3(y,z)$, in a similar way, one has

$$\begin{aligned}&|J_{43}^3(y,z)|\\ \le{}& \overline{C}\|\partial_r v_{\theta 0}\|_{0,\alpha}|(\beta_{v2}(z),\beta_{v3}(z))|^{1+\alpha}\\ &\times\Big\{\|\partial_r v\|_{0,\alpha}\Big(\Big|\int_{z_1}^{y_1}\frac{1}{R_v^{1-\alpha}(t;y)}\mathrm{d}t\Big|+\int_0^{z_1}\frac{1}{R_v^{1-\alpha}(t;y)}\mathrm{d}t\Big)\Big\}|y-z|^\alpha\\ &+\overline{C}\|\partial_r v_{\theta 0}\|_{0,\alpha}|(\beta_{v2}(z),\beta_{v3}(z))|^{1+\alpha}\\ &\times\|\partial_r v\|_{0,\alpha}\int_0^{z_1}\frac{R_v^{1+\alpha}(t;z)(R_v(t;y)+R_v(t;z))}{R_v^2(t;y)R_v^2(t;z)}|(Y_2(t;y),Y_3(t;y))-(Y_2(t,z),Y_3(t;z))|\mathrm{d}t\\ \le{}& \overline{C}(\delta^2+\varepsilon)|y-z|^\alpha. \end{aligned}\tag{5.29}$$

In addition,

$$\begin{aligned}&\Big|\Big(\frac{y_i}{|y'|}-\frac{z_i}{|z'|}\Big)J_4^j(z)\Big|\\ &\le \overline{C}|y-z|^\alpha|y'|^{-\alpha}\|\partial_r v_{\theta 0}\|_{0,\alpha}\|\partial_r v\|_{0,\alpha}|z|^{1+\alpha}|z|^{-1+\alpha}\\ &\le \overline{C}(\delta^2+\varepsilon)|y-z|^\alpha. \end{aligned}\tag{5.30}$$

Substituting (5.27)–(5.29) into (5.26) arrives at

$$[J_4^j]_\alpha \le \overline{C}(\delta^2+\varepsilon).$$

Combining this with, (5.23)–(5.25), (5.31) and a routine verification yields

$$\Big\|\frac{y_i}{|y'|}w_3\Big\|_{1,\alpha}\le \overline{C}(\delta^2+\varepsilon),\quad i=2,3.\tag{5.31}$$

Step 4 Solvability of w_1.

Since w_i ($i=2,\cdots,6$) is obtained, by (5.7), we define w_1 as

$$\begin{aligned}w_1 &= \mathscr{G}_3(\overline{w})w_2+\mathscr{G}_4(\overline{w})w_3+\mathscr{G}_5(\overline{w})w_4+\mathscr{G}_6(\overline{w})w_5+\mathscr{G}_7(\overline{w})w_6\\ &=\sum_{i=2}^3\Big(\frac{y_i}{|y'|}\mathscr{G}_3(\overline{w})\frac{y_i}{|y'|}w_2+\frac{y_i}{|y'|}\mathscr{G}_4(\overline{w})\frac{y_i}{|y'|}w_3\Big)\\ &\quad+\mathscr{G}_5(\overline{w})w_4+\mathscr{G}_6(\overline{w})w_5+\mathscr{G}_7(\overline{w})w_6. \end{aligned}\tag{5.32}$$

Combining this with $\overline{w}\in\Xi_\delta$, (5.19)–(5.20), (5.22) and (5.31) shows

$$\|w_1\|_{1,\alpha}\le \overline{C}(\delta^2+\varepsilon).\tag{5.33}$$

Finally, we define $w = \mathbb{T}(\overline{w})$ by Steps 1–4. It derives from (5.19)–(5.20), (5.22), (5.31) and (5.33) that

$$[[w]]_{1,\alpha} \le \overline{C}(\delta^2 + \varepsilon). \tag{5.34}$$

We choose $\delta = 2\overline{C}\varepsilon$ for $0 < \varepsilon \le \varepsilon_0$ with some $\varepsilon_0 > 0$ such that $0 < \delta < \frac{1}{2}$. Combining this with (5.34) shows

$$[[w]]_{1,\alpha} < \delta. \tag{5.35}$$

This yields that $w = \mathbb{T}(\overline{w}) \in \Xi_\delta$ for $\delta = 2\overline{C}\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$, namely, $\mathbb{T}$ maps Ξ_δ into itself when $\delta = 2\overline{C}\varepsilon$.

5.4 Contraction of the mapping $\mathbb{T}$

For any given two states $\overline{w}^i \in \Xi_\delta$ $(i = 1, 2)$, we define $w^i = \mathbb{T}(\overline{w}^i)$ by Subsection 5.3. We will frequently use the norm $\langle \cdot \rangle_\alpha$ defined in (5.13) in this section without further declaration.

It derives from (5.3) that

$$\begin{cases} \partial_z(w_2^1 - w_2^2) + a_1(r)\partial_r(w_4^1 - w_4^2) = \mathcal{F}_1(\overline{w}^1) - \mathcal{F}_1(\overline{w}^2), \\ -\partial_r(w_2^1 - w_2^2) - \dfrac{1}{r}(w_2^1 - w_2^2) + a_2(r)\partial_z(w_4^1 - w_4^2) = \mathcal{F}_2(\overline{w}^1) - \mathcal{F}_2(\overline{w}^2), \\ (w_2^1 - w_2^2)(0, r) = 0, & r \in (0, 1), \\ (w_2^1 - w_2^2)(z, 0) = 0, \quad (w_2^1 - w_2^2)(z, 1) = 0, & z \in [0, \ell], \\ (w_4^1 - w_4^2)(\ell, r) = 0. \end{cases}$$

Then similar to Step 1 in Subsection 5.3, with Lemma 5.2, one has

$$\begin{aligned} &\Big\| \frac{y_i}{r}(w_2^1 - w_2^2) \circ \boldsymbol{T} \Big\|_{1,\alpha} + \|(w_4^1 - w_4^2) \circ \boldsymbol{T}\|_{1,\alpha} \\ &\le \overline{C}(\|(\mathcal{F}_1(\overline{w}^1) - \mathcal{F}_1(\overline{w}^2)) \circ \boldsymbol{T}\|_{0,\alpha} + \|(\mathcal{F}_2(\overline{w}^1) - \mathcal{F}_2(\overline{w}^2)) \circ \boldsymbol{T}\|_{0,\alpha}) \\ &\le \overline{C}\varepsilon \left\langle \overline{w}^1 - \overline{w}^2 \right\rangle_\alpha . \end{aligned} \tag{5.36}$$

With $v^i = w_2^i + 0$ for $i = 2, 3$, we can define the operator L_{v^i} with the coefficient $Q_j^i(y)$, the characteristic $Y^i(t; y)$ and $\beta_{v^i j}(y)$ $(j = 2, 3)$ by (3.14) and (3.16)–(3.17), respectively. By use of (5.6), one has

$$w_5^1 - w_5^2 = \boldsymbol{E}_0(|(\beta_{v^1 2}(y), \beta_{v^1 3}(y))|) - \boldsymbol{E}_0(|(\beta_{v^2 2}(y), \beta_{v^2 3}(y))|). \tag{5.37}$$

It comes from (3.16), (1.7) and the definition of Ξ_δ that

$$\begin{aligned} &\|Y_2^1(t; y) - Y_2^2(t; y)\|_{0,\alpha} + \|Y_3^1(t; y) - Y_3^2(t; y)\|_{0,\alpha} \\ &\le \overline{C}(\|Q_2^1(Y^1) - Q_2^2(Y^2)\|_{0,\alpha} + \|Q_3^1(Y^1) - Q_3^2(Y^2)\|_{0,\alpha}) \\ &\le \overline{C} \sum_{i=2}^{3} \Big\| \frac{y_i}{|y'|}(w_2^1 - w_2^2)(y) \Big\|_{1,\alpha} \end{aligned}$$

$$+\varepsilon(\|Y_2^1(t;y)-Y_2^2(t;y)\|_{0,\alpha}+\|Y_3^1(t;y)-Y_3^2(t;y)\|_{0,\alpha}).$$

Combining this with (5.36), (1.12) and (3.16) yields

$$\|Y_2^1(t;y)-Y_2^2(t;y)\|_{0,\alpha}+\|Y_3^1(t;y)-Y_3^2(t;y)\|_{0,\alpha}\le \overline{C}\varepsilon\langle \overline{w}^1-\overline{w}^2\rangle_\alpha, \tag{5.38}$$

and then

$$\begin{aligned}&\|w_5^1-w_5^2\|_{0,\alpha}\\ \le{}& \overline{C}(\|\beta_{v^12}(y)-\beta_{v_2^2}(y)\|_{0,\alpha}+\|\beta_{v^13}(y)-\beta_{v^23}(y)\|_{0,\alpha})\\ \le{}& \overline{C}(\|Q_2^1(Y^1)-Q_2^2(Y^2)\|_{0,\alpha}+\|Q_3^1(Y^1)-Q_3^2(Y^2)\|_{0,\alpha})\\ \le{}& \overline{C}\varepsilon\langle \overline{w}^1-\overline{w}^2\rangle_\alpha.\end{aligned} \tag{5.39}$$

In the same way, one has

$$\|w_6^1-w_6^2\|_{0,\alpha}\le \overline{C}\varepsilon\langle \overline{w}^1-\overline{w}^2\rangle_\alpha. \tag{5.40}$$

With respect to $w_3^1-w_3^2$, tracking the similar estimates in Step 3 in Subsection 5.3 and (5.38), one has

$$\Big\|\frac{y_2}{r}(w_3^1-w_3^2)\Big\|_{0,\alpha}+\Big\|\frac{y_3}{r}(w_3^1-w_3^2)\Big\|_{0,\alpha}\le \overline{C}\varepsilon\langle \overline{w}^1-\overline{w}^2\rangle_\alpha. \tag{5.41}$$

By use of (5.32), we also have

$$\|w_1^1-w_1^2\|_{0,\alpha}\le \overline{C}\varepsilon\langle \overline{w}^1-\overline{w}^2\rangle_\alpha. \tag{5.42}$$

Combining (5.36) and (5.39)–(5.42) yields

$$\langle w^1-w^2\rangle_\alpha\le \overline{C}\varepsilon\langle \overline{w}^1-\overline{w}^2\rangle_\alpha\le \frac{1}{2}\langle \overline{w}^1-\overline{w}^2\rangle_\alpha$$

with $0<\varepsilon\le\varepsilon_0$ for some $\varepsilon_0>0$. From this, we obtain that $\mathbb{T}$ is a contraction mapping from Ξ_δ to itself.

5.5 Proofs of Theorem 2.1 and Theorem 1.1

Proof of Theorem 2.1 For any $U^0=(u_1^0,u_2^0,u_3^0,\boldsymbol{P}^0,\boldsymbol{E}^0,\boldsymbol{s}^0)\in\{\text{axisymmetry}\}$ with w^0 being the combination of $U^0-U_b\in\Xi_\delta$ by (2.8)–(2.9), we define a series $\{U^n\}_{n=1}^\infty$ with $U^n=(u_1^n,u_2^n,u_3^n,\boldsymbol{P}^n,\boldsymbol{E}^n,\boldsymbol{s}^n)$ as well as w^n as

$$w^n=\mathbb{T}(w^{n-1}),\quad n\ge 1. \tag{5.43}$$

From Subsections 5.3–5.4, we have $w^n\in\Xi_\delta$ $(n\ge1)$ for some certain $\delta=O(\varepsilon)$ small determined in (5.35) and

$$\langle w^n-w^{n-1}\rangle_\alpha\le\frac{1}{2}\langle w^{n-1}-w^{n-2}\rangle_\alpha \tag{5.44}$$

shown by Subsection 5.4 with the norm $\langle\cdot\rangle$ defined in (5.13).

By (5.43)–(5.44), there exists a unique $w \in \Xi_\delta$ and the corresponding U defined by (2.8)–(2.9), solving the problem (2.2) with (2.3)–(2.5) and satisfying the estimates (2.6). Therefore, we finally complete the proof of Theorem 2.1.

Proof of Theorem 1.1 Since the transformation $\boldsymbol{m}$ defined in (1.13) is reversible due to $\|g-1\|_{2,\alpha} \le \varepsilon$ in (1.7), thus, Theorem 1.1 comes from Theorem 2.1 immediately and the proof is finished.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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