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Ergodic Stochastic Maximum Principle with Markov Regime-Switching*

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Abstract This paper is concerned with the ergodic stochastic optimal control problem with Markov Regime-Switching in a dissipative system. The proposed approach primarily relies on duality techniques. The control system is described by controlled dissipative stochastic differential equations and modulated by a continuous-time, finite-state Markov chain. The cost functional is ergodic, which is the expected long-run mean average type. The control domain is assumed to be convex, and the convex variation technique is used. Both necessary condition version and sufficient condition version of the stochastic maximum principle are established for optimal control. An example is discussed to illustrate the significance of our results.

Keywords Ergodic Stochastic maximum principle, Markov regime-switching, Backward stochastic differential equation, Dissipative systems, Infinite horizon
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1 Introduction

The stochastic optimal control problem is an important problem in stochastic control theory. The stochastic maximum principle (SMP for short) and the dynamic programming principle are two of the most important tools to solve the stochastic optimal control problem. The SMP was firstly introduced by Kushner [13] and extended by Bismut [5] and Peng [18] to more general cases. Bismut [5] introduced the linear backward stochastic differential equations (BSDEs for short) as the adjoint equations, which play an important role in control theory. Pardoux and Peng obtained the existence and uniqueness of the solution for nonlinear BSDEs in [17], which has been widely used in stochastic control and mathematical finance. Peng obtained the general SMP for the stochastic control system in [18] by applying the second-order adjoint equations, which overcame the difficulty that appears when the control enters both the drift and diffusion coefficients and the control domain is non-convex.

In the past few decades, Markov regime-switching models have been extensively researched and received significant attention in finance and stochastic optimal controls. A regime-switching

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model can be formulated as a stochastic differential equation (SDE for short), where the coefficients are modulated by a continuous-time, finite-state Markov chain. Each state of this Markov chain represents a different regime of the system (or a level of an economic indicator), depending on the market mode. The model switches among a finite number of these states. Compared to traditional systems based on diffusion processes, Markov regime-switching models make more sense from the empirical point of view. With the development of stochastic analysis and stochastic control theory, much work has been done on stochastic control problems for the regime-switching system. The regime-switching model in economic and finance fields was first introduced by Hamilton in [11] which described a time series model and then intensively investigated in the past two decades in mathematical finance. Zhou and Yin [25] studied a continuous-time mean-variance model modulated by a Markov chain, which represents the regime-switching. Donnelly [8] proved a sufficient SMP within a regime-switching diffusion model. In our paper, to deal with the regime-switching part, we follow the method in [8]. Yang et al. [22] and Yang et al. [23] also studied the state-dependent switching control. Tao and Wu [20] derived both the necessary and sufficient maximum principle for the forward-backward regime-switching model by using the results about BSDEs with Markov chains. Moreover, Wang and Wu [21] obtained the maximum principle for forward-backward regime-switching systems with impulse controls. Zhang et al. [24] developed a global form SMP for a Markov regime-switching mean-field model which is driven by Brownian motions and Poisson jumps. Bellalah et al. [2] bridged the gap by providing for the first time in the literature a model that accounts explicitly and simultaneously for inflation, information costs, and short sales in the portfolio performance with regime-switching. Recently, Abdallah et al. [3] conducted a study on a stochastic optimal control problem concerning an infinite horizon Markov regime-switching jump-diffusion model, focusing on the investigation of optimal portfolio and consumption strategies within a switching diffusion market.

Furthermore, the concept of the ergodic control system is widely utilized to describe models over long time periods or with an infinite horizon in physics and mathematical finance. Optimal stochastic control problems over such periods, aimed at minimizing the ergodic cost functional, often reflect the controller's ambition to enhance performance on a long-term and average basis. These challenges are known as ergodic stochastic optimal control problems. Over the past few decades, considerable research has focused on these problems for ergodic systems, addressing both finite and infinite dimensional cases. Ghosh et al. [10] explored the ergodic control problem of switching diffusions, which is prevalent in various applications like fault-tolerant control systems and flexible manufacturing systems. Arisawa et al. [1] and Bensoussan et al. [4] studied the ergodic stochastic optimal control problems and the corresponding Hamilton-Jacobi-Bellman (HJB for short) equations in the finite dimensional case using analytic techniques. In the context of infinite dimensions, Fuhrman et al. [9] investigated the existence, uniqueness and regularity of solutions for ergodic BSDEs in Banach space, applying these findings to the optimal ergodic control of a Banach valued stochastic state equation. More recently, Orrieri et al. [14] introduced a version of the SMP tailored to ergodic control problems in finite-dimensional controlled dissipative systems. These seminal works have laid a robust foundation for addressing ergodic stochastic optimal control issues.

However, to our best knowledge so far, the SMP for Markov regime-switching models of ergodic type cost for controlled dissipative systems has not yet been established. This gap in the existing literature motivates our research. In this paper, we develop an ergodic maximum principle for an ergodic stochastic optimal control problem with Markov regime-switching.

Our study focuses on a stochastic control system characterized by an Itô-type SDE as the state equation under dissipative assumptions and over infinite horizon, where the state process is governed by a continuous-time finite state Markov regime-switching model. The cost functional, being of ergodic type, reflects the long-term average behavior of the system, distinguishing it from the general infinite horizon performance criterion (cf. [3]). Our approach primarily relies on duality techniques and we employ convex perturbation techniques, as the ergodic cost functional is expected to remain unchanged after applying the finite time perturbation; the underlying reasons for it will be explained in a subsequent section of this study.

It is worth mentioning that for the ergodic control problem under the Markov regime-switching model, there is an additional jump martingale appearing in the first order adjoint equation. Moreover, the first order adjoint equation is a BSDE over an infinite horizon without a terminal term. To overcome these obstacles, we employ the truncated adjoint equations method outlined in [14]. We construct a family of truncated adjoint equations, which are terminal-value BSDEs with jump martingale term on the finite time horizon $[0,\ T]$. By using Itô's formula with jump martingale from [8], we establish the existence and uniqueness of solutions to these truncated adjoint equations and derive the duality relation, which in turn allows us to obtain and prove the existence and uniqueness of solutions to the original adjoint equation. Finally, leveraging the duality technique, we establish the necessary conditions and the sufficient conditions for the SMP in our model.

The main contributions of this paper can be summarized as follows: We obtain the existence and uniqueness of solutions of a class of BSDEs associated with jump martingale over an infinite horizon without a terminal term in dissipative systems; we establish the ergodic maximum principle for Markov regime-switching models and provide a clear proof of the necessary conditions and the sufficient conditions for the SMP in our model, demonstrating its applicability through a solvable example.

The rest of the paper is organized as follows. We devote Section 2 to presenting the notations and the basic assumptions. We give the preliminaries of ergodic optimal control problem in Markov regime-switching model under the dissipative conditions. In Section 3, we use the convex perturbation of the optimal control, introduce the adjoint equation and give the duality relation which is important to our main theorems. The main results are given in Sections 4–5. We establish two versions of necessary SMPs and one version of sufficient SMP in our model. Finally, in Section 6, we present a solvable example as an application of the maximum principle.

2 Preliminary

In this section, we formulate the stochastic control problem in a regime-switching diffusion model and introduce some assumptions. Firstly, we introduce some notations which would be used in the following. We denote $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$, $\|\cdot\|_2$ the Hilbert-Schmidt norm in $\mathbb{R}^{n\times n}$, $\langle\cdot,\cdot\rangle$ the inner product in some Hilbert space, and class $\mathcal{C}^2$ the set of all twice continuously differentiable function.

Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ be a given complete filtered probability space. On this probability space, there is a standard Brownian motion $B = \{B_t\}_{t\geq 0}$ taking value in $\mathbb{R}^n$ and a continuous-time stationary Markov chain $\alpha = \{\alpha_t\}_{t\geq 0}$ taking value in finite state space $I = \{e_1, \cdots, e_N\}$. We assume that B and α are independent and the completed filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is generated jointly by B and α, i.e.,

$$\mathscr{F}_t := \sigma(\alpha_s, B_s,\ s \in [0,t]) \vee \mathscr{N}(P), \quad t \in [0,\infty),$$

where $\mathscr{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null subsets.

The generator of α is an $N \times N$ matrix $\mathcal{Q} = \{q_{ij}\}_{i,j=1}^N$ and the initial value of α is i_0. For each pair of distinct states (i,j), we define counting process $[Q_{ij}] : \Omega \times [0,\infty) \to \mathbb{N}$ by $[Q_{ij}](\omega,t) := \sum_{0<s\leq t} \mathbf{1}_{[\alpha(s^-)=i]}(\omega)\mathbf{1}_{[\alpha_s=j]}(\omega)$, for all t in $[0,\infty)$, where $\mathbf{1}$ is the indicator function, compensator process $\langle Q_{ij}\rangle : \Omega \times [0,\infty) \to [0,\infty)$ by $\langle Q_{ij}\rangle(\omega,t) := q_{ij}\int_0^t \mathbf{1}_{[\alpha(s^-)=i]}(\omega)\mathrm{d}s$, for all t in $[0,\infty)$, and compensated process $Q_{ij}(\omega,t) : \Omega \times [0,\infty) \to [0,\infty)$ by $Q_{ij}(\omega,t) := [Q_{ij}](\omega,t) - \langle Q_{ij}\rangle(\omega,t)$, for all t in $[0,\infty)$. As detailed in [19], the compensated process is a purely discontinuous square-integrable martingale with initial value zero.

In this paper, we consider the optimal control problem in predictable structure. We denote the predictable σ-algebra on $\Omega \times [0,\infty)$ associated with the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ by $\mathcal{P}^\star$. A stochastic process X is predictable, written as $X \in \mathcal{P}^\star$, if it is $\mathcal{P}^\star$-measurable.

We will use the following notations for any real number $1 \leq p,\ q < \infty$, $T > 0$, continuity interval $\mathbb{T}$ on $\mathbb{R}^+$, such as $[0,T]$ or $[0,+\infty)$, and t in $\mathbb{T}$:

$$\begin{aligned}
L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n) :&= \{\mathbb{R}^n\text{-valued}, \mathscr{F}_t\text{-measurable random variables } \xi \text{ s.t. } \mathbb{E}|\xi|^2 < \infty\},\\
L^p(\Omega\times\mathbb{T};\mathbb{R}^n) :&= \Big\{\mathbb{R}^n\text{-valued}, (\mathscr{F}_t)\text{-progressively measurable processes } X \text{ s.t.}\\
&\qquad \Big(\mathbb{E}\int_{\mathbb{T}} |X_t|^p\,\mathrm{d}t\Big)^{\frac{1}{p}} < \infty\Big\},\\
L^q(\mathbb{T};L^p(\Omega;\mathbb{R}^n)) :&= \Big\{\mathbb{R}^n\text{-valued}, (\mathscr{F}_t)\text{-progressively measurable processes } X \text{ s.t.}\\
&\qquad \int_{\mathbb{T}} (\mathbb{E}|X_t|^p)^{\frac{q}{p}}\mathrm{d}t < \infty\Big\},\\
L^\infty(\mathbb{T};L^p(\Omega;\mathbb{R}^n)) :&= \Big\{\mathbb{R}^n\text{-valued}, (\mathscr{F}_t)\text{-progressively measurable processes } X \text{ s.t.}\\
&\qquad \sup_{t\in\mathbb{T}}(\mathbb{E}|X_t|^p)^{\frac{1}{p}} < \infty\Big\},\\
L^p_{\mathcal{P}^\star}(\Omega\times\mathbb{T};\mathbb{R}^n) :&= \Big\{\mathbb{R}^n\text{-valued, predictable processes } X \text{ s.t.}\\
&\qquad \Big(\mathbb{E}\int_{\mathbb{T}} |X_t|^p\,\mathrm{d}t\Big)^{\frac{1}{p}} < \infty\Big\},\\
L^2_{\mathcal{P}^\star}(Q,\mathbb{T}) :&= \Big\{\mathbb{R}^n\text{-valued, predictable processes } \Gamma = \{(\Gamma^{(1)}_{ij})^N_{i,j=1},\cdots,(\Gamma^{(n)}_{ij})^N_{i,j=1}\},\\
&\qquad \text{where } \Gamma^{(l)}_{ii} = 0\ (\mathbb{P}\otimes \mathrm{d}t)\text{-a.s. in } \Omega\times\mathbb{T},\ \forall i \in I,\\
&\qquad \Gamma^{(l)}_{ij} \in \mathcal{P}^\star,\ \forall i,j \in I, i \neq j,\ \sum_{l=1}^n\sum_{i,j=1}^N \mathbb{E}\int_{\mathbb{T}} \|\Gamma^{(l)}_{ij}\|^2 \mathrm{d}[Q_{ij}](t) < \infty\Big\}.
\end{aligned}$$

We now turn to the mathematical formulation of the problem. For any initial condition $x_0 \in \mathbb{R}^n$, $i_0 \in I$, we consider stochastic control system where the state of the system is governed by an $\mathbb{R}^n$-valued controlled Markovian regime-switching SDE:

$$\begin{cases} \mathrm{d}X_t = b(X_t, u_t, \alpha_t)\mathrm{d}t + \sigma(X_t, u_t, \alpha_t)\mathrm{d}B_t, \quad t \geq 0, \\ X_0 = x_0, \quad \alpha_0 = i_0 \end{cases} \tag{2.1}$$

along with the cost functional (expected long-run average types):

$$J(u(\cdot)) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^T f(X_t, u_t, \alpha_t)\mathrm{d}t\Big], \tag{2.2}$$

where B_t is d-dimensional Brownian motion, and $b : \mathbb{R}^n \times \mathbb{R}^l \times I \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^l \times I \to \mathbb{R}^{n\times d}$ and $f : \mathbb{R}^n \times \mathbb{R}^l \times I \to \mathbb{R}$ are three given continuous functions. The specific assumptions on the coefficients will be given later.

We fix that $m \in \mathbb{N}$, $p > (4m + 2) \vee 4$ and $k > \frac{p-1}{2}$ for the rest of the paper which are three constants involved in the admissible controls and assumptions. We denote U as a given nonempty closed convex subset of $\mathbb{R}^l$, and the set of all admissible controls is denoted by

$$U_{ad} := \Big\{u : \Omega \times \mathbb{R}^+ \to U;\ \text{where } u \in \mathcal{P}^\star \text{ and } \sup_{t\geq 0} \mathbb{E}|u_t|^p < \infty\Big\}.$$

Now, we give the main assumptions of this paper.

(A1) (Drift term) The mapping $b : \mathbb{R}^n \times U \times I \to \mathbb{R}^n$ is a given continuous function such that for any $e \in I$, $b(x, u, e)$ is $\mathscr{B}(\mathbb{R}^n) \otimes \mathscr{B}(U)$-measurable and class $\mathcal{C}^2$ with respect to x and u. Moreover, there exists a constant $C_1 > 0$ such that, for all $x \in \mathbb{R},\ u \in U,\ e \in I$:

$$\begin{aligned} |b(x, u, e)| &\leq C_1(1 + |x|^{2m+1} + |u|); \\ |b_x(x, u, e)| &\leq C_1(1 + |x|^{2m}); \\ |b_u(x, u, e)| &\leq C_1. \end{aligned} \tag{2.3}$$

(A2) (Diffusion term) The mapping $\sigma : \mathbb{R}^n \times U \times I \to \mathbb{R}^{n\times d}$ is a given continuous function such that for any $e \in I$, $\sigma(x, u, e)$ is $\mathscr{B}(\mathbb{R}^n) \otimes \mathscr{B}(U)$ measurable and class $\mathcal{C}^2$ with respect to x and u. Moreover, there exists a constant $C_2 > 0$ such that, for all $x \in \mathbb{R},\ u \in U,\ e \in I$:

$$\begin{aligned} \|\sigma(x, u, e)\|_2 &\leq C_2(1 + |x|^m + |u|); \\ \|\sigma_x(x, u, e)\|_2 &\leq C_2(1 + |x|^m); \\ \|\sigma_u(x, u, e)\|_2 &\leq C_2. \end{aligned} \tag{2.4}$$

(A3) (Joint dissipativity) There exists a constant $C_k < 0$ such that, for all $x, y \in \mathbb{R}^n,\ u \in U,\ e \in I$:

$$\langle b_x(x, u, e)y, y\rangle + k\|\sigma_x(x, u, e)y\|_2^2 \leq C_k|y|^2, \tag{2.5}$$

which implies

$$\langle b(x, u, e) - b(y, u, e), x - y\rangle + k\|\sigma(x, u, e) - \sigma(y, u, e)\|_2^2 \leq C_k|x - y|^2. \tag{2.6}$$

(A4) (Cost) The mapping $f : \mathbb{R}^n \times U \times I \to \mathbb{R}$ is a given continuous function such that for any $e \in I$, $f(x, u, e)$ is $\mathscr{B}(\mathbb{R}^n) \otimes \mathscr{B}(U)$ measurable, bounded from below by a constant f_0 and differentiable in x and u. Moreover, there exists a constant $C > 0$ such that

$$|f_x(x, u, e)| + |f_u(x, u, e)| \leq C(1 + |x| + |u|). \tag{2.7}$$

We claim that under (A1)–(A3), for any given initial condition and admissible control, there exists a unique strong solution of (2.1) and the following estimate holds:

$$\mathbb{E}|X_t|^p \le \mathrm{e}^{-p\beta t}|x_0|^p + C\Big(1 + \sup_{t\ge 0}\mathbb{E}|u_t|^p\Big) \tag{2.8}$$

for some positive constants β and C which only depend on the constants m, p and k appearing in assumptions. A proof of this claim will be given in the following Theorem 2.1. It is easy to check, the cost functional (2.2) is well-defined for any given admissible control. The main purpose of this paper is to study the stochastic optimal control problem, which is stated as follows.

Problem 2.1 Find an admissible control $\overline{u}(\cdot)$ over U_{ad} such that

$$J(\overline{u}(\cdot)) = \inf_{u(\cdot)\in U_{ad}} J(u(\cdot)). \tag{2.9}$$

The admissible control $\overline{u}(\cdot)$ found above is called optimal control, and the aim of this work is to give some necessary and sufficient conditions for optimality of the controlled system.

We conclude this section by giving the solvability and estimate of (2.1).

Theorem 2.1 *Let assumptions* (A1)–(A3) *hold. Then, for any given initial condition* (x_0, i_0) *and admissible control* $u(\cdot) \in U_{ad}$, (2.1) *admits a unique progressively measurable solution* X. *Moreover, the estimate* (2.8) *holds for some positive constants* $C = C(p)$ *and* β.

Proof The proof of existence and uniqueness of a strong solution to (2.1) follows a piecewise method similar to that in [7]. Now we prove the estimate (2.8). Denote $\widetilde{X}_t := \mathrm{e}^{\beta t}X_t$, for any positive constant β. We apply Itô's formula to $\mathrm{e}^{\beta t}X_t$. Then $\widetilde{X}_t$ solves:

$$\begin{cases} \mathrm{d}\widetilde{X}_t = \beta\widetilde{X}_t\mathrm{d}t + \mathrm{e}^{\beta t}b(\mathrm{e}^{-\beta t}\widetilde{X}_t, u_t, \alpha_t)\mathrm{d}t + \mathrm{e}^{\beta t}\sigma(\mathrm{e}^{-\beta t}\widetilde{X}_t, u_t, \alpha_t)\mathrm{d}B_t, \quad t \ge 0, \\ \widetilde{X}_0 = x_0. \end{cases} \tag{2.10}$$

We denote $\widetilde{b}_t(x,u,e) := \mathrm{e}^{\beta t}b(\mathrm{e}^{-\beta t}x, u, e)$, $\widetilde{\sigma}_t(x,u,e) := \mathrm{e}^{\beta t}\sigma(\mathrm{e}^{-\beta t}x,u,e)$, and $\widetilde{b}_{t,0} := \widetilde{b}_t(0,u,e)$, $\widetilde{\sigma}_{t,0} := \widetilde{\sigma}_t(0,u,e)$ for short. Due to (A3), for all $e \in I$, it follows that

$$\langle \widetilde{b}_t(x,u,e) - \widetilde{b}_t(y,u,e), x-y\rangle + k\|\widetilde{\sigma}_t(x,u,e) - \widetilde{\sigma}_t(y,u,e)\|_2^2 \le C_k|x-y|^2 \tag{2.11}$$

and

$$\begin{aligned} |\widetilde{b}_{t,0}| &\le C\mathrm{e}^{\beta t}(1+|u|), \\ \|\widetilde{\sigma}_{t,0}\|_2 &\le C\mathrm{e}^{\beta t}(1+|u|). \end{aligned} \tag{2.12}$$

Denote $p = 2q$, and apply Itô's formula to $|X_t|^{2q}$, we obtain

$$\begin{aligned} \mathbb{E}|\widetilde{X}_t|^{2q} = |x_0|^{2q} &+ \mathbb{E}\Big[\int_0^t \nabla_x\Phi(\widetilde{X}_s, s)\phi_s\mathrm{d}B_s + \int_0^t \nabla_x\Phi(\widetilde{X}_s,s)\psi_s\mathrm{d}s\Big] \\ &+ \mathbb{E}\frac{1}{2}\int_0^t Tr[\Phi_{xx}(\widetilde{X}_s,s)\phi_s\phi_s^{\mathrm{T}}]\mathrm{d}s, \end{aligned} \tag{2.13}$$

where $\phi_t = \widetilde{\sigma}_t,\ \psi_t = \beta \widetilde{X}_t + \widetilde{b}_t$,

$$\nabla_x \Phi(X,t) = \Big(\frac{\partial \Phi}{\partial x_i}\Big)_{i=1,\cdots,n} = 2q|X|^{2(q-1)}X,$$
$$\Phi_{xx}(X,t) = \Big(\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\Big)_{i=1,\cdots,n}^{j=1,\cdots,n} = 4q(q-1)|X|^{2(q-2)}X \circ X + 2q|X|^{2(q-1)}\mathbf{I},$$

where $\mathbf{I}$ is an n-dimension identity matrix. Hence

$$\begin{aligned}\mathbb{E}|\widetilde{X}_t|^{2q} &= |x_0|^{2q} + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\Big(\langle \widetilde{X}_s, \widetilde{b}_s\rangle + \frac{1}{2}\|\widetilde{\sigma}_s\|_2^2\Big)\mathrm{d}s \\ &\quad + 2q\beta\mathbb{E}\int_0^t |\widetilde{X}_s|^{2q}\mathrm{d}s + 2q(q-1)\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-2)} Tr\{\widetilde{\sigma}_s\widetilde{\sigma}_s^{\mathrm{T}}(\widetilde{x}_i\widetilde{x}_j)_{i=1,\cdots,n}^{j=1,\cdots,n}\}\mathrm{d}s \\ &\le |x_0|^{2q} + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\Big(\langle \widetilde{X}_s, \widetilde{b}_s\rangle + \Big(q-\frac{1}{2}\Big)\|\widetilde{\sigma}_s\|_2^2\Big)\mathrm{d}s \\ &\quad + 2q\beta\mathbb{E}\int_0^t |\widetilde{X}_s|^{2q}\mathrm{d}s \\ &= |x_0|^{2q} + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\Big(\langle \widetilde{X}_s, \widetilde{b}_s - \widetilde{b}_{s,0}\rangle + \Big(q-\frac{1}{2}\Big)\|\widetilde{\sigma}_s\|_2^2\Big)\mathrm{d}s \\ &\quad + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\langle \widetilde{X}_s, \widetilde{b}_{s,0}\rangle \mathrm{d}s + 2q\beta\mathbb{E}\int_0^t |\widetilde{X}_s|^{2q}\mathrm{d}s. \end{aligned} \tag{2.14}$$

By Young's inequality $(x+y)^2 \le (1+\theta)x^2 + \big(1+\frac{1}{\theta}\big)y^2$, for any $\theta > 0$, we can obtain that

$$\begin{aligned}\mathbb{E}|\widetilde{X}_t|^{2q} &\le |x_0|^{2q} + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\Big(\langle \widetilde{X}_s, \widetilde{b}_s - \widetilde{b}_{s,0}\rangle + \Big(q-\frac{1}{2}\Big)(1+\theta)\|\widetilde{\sigma}_s - \widetilde{\sigma}_{s,0}\|_2^2\Big)\mathrm{d}s \\ &\quad + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\Big(\langle \widetilde{X}_s, \widetilde{b}_{t,0}\rangle + \Big(q-\frac{1}{2}\Big)\Big(1+\frac{1}{\theta}\Big)\|\widetilde{\sigma}_{s,0}\|_2^2\Big)\mathrm{d}s \\ &\quad + 2q\beta\mathbb{E}\int_0^t |\widetilde{X}_s|^{2q}\mathrm{d}s. \end{aligned} \tag{2.15}$$

Specially, for those $\theta > 0$ such that $\big(q-\frac{1}{2}\big)(1+\theta) \le k$. Thanks to (2.11) and Young's inequality, it follows that

$$\begin{aligned}\mathbb{E}|\widetilde{X}_t|^{2q} &\le |x_0|^{2q} + 2q\Big(C_k + \beta + \frac{\delta}{2}\Big)\mathbb{E}\int_0^t |\widetilde{X}_s|^{2q}\mathrm{d}s \\ &\quad + 2q\mathbb{E}\int_0^t |\widetilde{X}_s|^{2(q-1)}\Big(\frac{1}{2\delta}|\widetilde{b}_{s,0}|^2 + \Big(q-\frac{1}{2}\Big)\Big(1+\frac{1}{\theta}\Big)\|\widetilde{\sigma}_{s,0}\|_2^2\Big)\mathrm{d}s. \end{aligned} \tag{2.16}$$

By (2.12) and Young's inequality $|x|^\alpha|y|^\beta \le \kappa|x|^{(\alpha+\beta)} + \frac{\beta}{\alpha+\beta}\big[\frac{\alpha}{\kappa(\alpha+\beta)}\big]^{\frac{\alpha}{\beta}}|y|^{(\alpha+\beta)}$, for all $\alpha, \beta, \kappa > 0$ we infer that

$$\begin{aligned}\mathbb{E}|\widetilde{X}_t|^{2q} &\le |x_0|^{2q} + 2q\Big(C_k + \beta + \frac{\delta}{2} + 2\kappa\Big)\mathbb{E}\int_0^t |\widetilde{X}_s|^{2q}\mathrm{d}s \\ &\quad + C\mathbb{E}\int_0^t \mathrm{e}^{2q\beta s}(1+|u_s|^{2q})\mathrm{d}s, \end{aligned} \tag{2.17}$$

where the constant C depends only on q, δ, θ and κ. Choosing β, δ and κ small enough and recalling that $C_k < 0$, such that $C_k + \beta + \frac{\delta}{2} + 2\kappa = 0$, then we have the following estimate:

$$\mathbb{E}|X_t|^p \le \mathrm{e}^{-p\beta t}|x|^p + C\Big(1 + \sup_{t\ge 0}\mathbb{E}|u_t|^p\Big). \tag{2.18}$$

We thus complete the proof.

Remark 2.1 The constant C is time-independent; in other words, it is uniform over time. When $\theta > 0$, the inequality $\big(q - \frac{1}{2}\big)(1+\theta) \le k$ holds, and this implies that k must be greater than $\frac{p-1}{2}$.

3 Variational and Adjoint Equations

In this section, we will introduce the variational equation and adjoint equation of the controlled system. Firstly, we elucidate the reasons why perturbations in local time of optimal control cannot be employed to investigate ergodic optimal control problems. Let $\varepsilon > 0$ be given and for any local time interval $E = [t_1, t_2]$ with $t_2 - t_1 = \varepsilon$ and for any admissible control $u(\cdot)$, we define the following:

$$u^\varepsilon(t) = \begin{cases} u(t), & t \in E, \\ \overline{u}(t), & t \in [0,\infty)\backslash E. \end{cases} \tag{3.1}$$

It is clear that $u^\varepsilon(\cdot) \in U_{ad}$ and we denote the corresponding state process by $X^\varepsilon(\cdot)$. Then, we obtain the following inequality:

$$\begin{aligned} |J(\overline{u}(\cdot)) - J(u^\varepsilon(\cdot))| &\le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^{t_2} |f(\overline{X}_t, \overline{u}_t, \alpha_t) - f(X_t^\varepsilon, u_t^\varepsilon, \alpha_t)|\,\mathrm{d}t\Big] \\ &\quad + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_{t_2}^{T} |f(\overline{X}_t, \overline{u}_t, \alpha_t) - f(X_t^\varepsilon, \overline{u}_t, \alpha_t)|\mathrm{d}t\Big] \\ &= 0 + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_{t_2}^{T} |f(\overline{X}_t, \overline{u}_t, \alpha_t) - f(X_t^\varepsilon, \overline{u}_t, \alpha_t)|\mathrm{d}t\Big], \end{aligned} \tag{3.2}$$

where the first term equals zero due to the boundedness of f. Applying Itô's formula to $|\overline{X}_t - X_t^\varepsilon|^2$ over the interval $[t_2, t]$ and assuming the dissipativity condition (A3), we can deduce that $\mathbb{E}[|\overline{X}_t - X_t^\varepsilon|^2]$ decreases exponentially over time. Combining this with (A4), we further deduce that

$$|J(\overline{u}(\cdot)) - J(u^\varepsilon(\cdot))| \le C \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_{t_2}^{T} \mathrm{e}^{-\widehat{\beta}t}\mathrm{d}t\Big] = 0 \tag{3.3}$$

for some $\widehat{\beta} > 0$. This indicates that, in ergodic optimal control problems, methods involving finite-time perturbations of the optimal control, such as needle perturbation technique, are not viable. Therefore, we adopt an alternative approach utilizing convex perturbations. However, for this purpose, it is necessary for the control domain U to be a convex set.

For $\varepsilon \in (0,1]$, and any admissible control $u(\cdot)$, we denote $u^\varepsilon(\cdot) := (1-\varepsilon)\overline{u}(\cdot) + \varepsilon u(\cdot) = \overline{u}(\cdot) + \varepsilon v(\cdot)$, where $v(\cdot) := u(\cdot) - \overline{u}(\cdot)$. Then, $u^\varepsilon(\cdot)$ is an admissible control and we denote the corresponding state process by $X^\varepsilon(\cdot)$, and we use $\overline{X}(\cdot)$ to denote the corresponding state process under the optimal control process $\overline{u}(\cdot)$. Our first goal is to give some estimates of our system which will be used in the subsequent proof.

Lemma 3.1 *Let assumptions* (A1)–(A3) *hold. Then there exists a constant C which only depends on the constants m, p and k appearing in assumptions such that*

$$\sup_{t\ge0} \mathbb{E}|X_t^\varepsilon - \overline{X}_t|^p \le C\varepsilon^2 \sup_{t\ge0} \mathbb{E}|v_t|^p. \tag{3.4}$$

Proof The proof goes through by the same method adopted in Theorem 2.1.

Now, we introduce the variational equation of our model and then provide an estimate of its solution,

$$\begin{cases} \mathrm{d}Y_t = b_x(\overline{X}_t, \overline{u}_t, \alpha_t)Y_t\mathrm{d}t + \sigma_x(\overline{X}_t, \overline{u}_t, \alpha_t)Y_t\mathrm{d}B_t \\ \qquad\quad + b_u(\overline{X}_t, \overline{u}_t, \alpha_t)v_t\mathrm{d}t + \sigma_u(\overline{X}_t, \overline{u}_t, \alpha_t)v_t\mathrm{d}B_t, \quad t \geq 0, \\ Y_0 = 0, \end{cases} \tag{3.5}$$

where $(\overline{X}_t, \overline{u}_t)$ is an optimal pair for the system (2.1).

Lemma 3.2 *Let assumptions* (A1)–(A3) *hold. Then the variational equation* (3.5) *admits a unique adapted solution $Y = \{Y_t\}_{t\geq 0}$, and there exists a constant C which only depends on the constants m, p and k appearing in assumptions such that*

$$\mathbb{E}|Y_t|^p \leq C \sup_{s\in[0,t]} \mathbb{E}|v_s|^p. \tag{3.6}$$

In particular,

$$\sup_{t\geq 0} \mathbb{E}|Y_t|^p \leq C \sup_{s\in[0,t]} \mathbb{E}|v_s|^p < \infty. \tag{3.7}$$

Proof The proof goes through by the same method adopted in Theorem 2.1.

Denote $\widetilde{Y}_t := \mathrm{e}^{\beta t}Y_t$, for any positive constant β. We apply Itô's formula to $\mathrm{e}^{\beta t}Y_t$. Then $\widetilde{Y}_t$ solves:

$$\begin{cases} \mathrm{d}\widetilde{Y}_t = [\beta\widetilde{Y}_t + b_x(\overline{X}_t, \overline{u}_t, \alpha_t)\widetilde{Y}_t + \mathrm{e}^{\beta t}b_u(\overline{X}_t, \overline{u}_t, \alpha_t)v_t]\mathrm{d}t \\ \qquad\quad + [\sigma_x(\overline{X}_t, \overline{u}_t, \alpha_t)\widetilde{Y}_t + \mathrm{e}^{\beta t}\sigma_u(\overline{X}_t, \overline{u}_t, \alpha_t)v_t]\mathrm{d}B_t, \quad t \geq 0, \\ \widetilde{Y}_0 = 0. \end{cases} \tag{3.8}$$

We denote $\widetilde{b}_t(y, v, e) := b_x(\overline{X}_t, \overline{u}_t, e)y + \mathrm{e}^{\beta t}b_u(\overline{X}_t, \overline{u}_t, e)v$, $\widetilde{\sigma}_t(y, v, e) := \sigma_x(\overline{X}_t, \overline{u}_t, e)y + \mathrm{e}^{\beta t}\sigma_u(\overline{X}_t, \overline{u}_t, e)v$, and $\widetilde{b}_{t,0} := \widetilde{b}_t(0, v, e)$, $\widetilde{\sigma}_{t,0} := \widetilde{\sigma}_t(0, v, e)$ for short. Then

$$\mathrm{d}\widetilde{Y}_t = [\beta\widetilde{Y}_t + \widetilde{b}_t(\widetilde{Y}_t, v_t, e)]\mathrm{d}t + \widetilde{\sigma}_t(\widetilde{Y}_t, v_t, e)\mathrm{d}B_t, \quad t \geq 0. \tag{3.9}$$

Due to (A1)–(A3), for all $e \in I$, it follows that

$$\begin{aligned} |\widetilde{b}_{t,0}| &= |\mathrm{e}^{\beta t}b_u(\overline{X}_t, \overline{u}_t, e)v| \leq C_1\mathrm{e}^{\beta t}|v|, \\ \|\widetilde{\sigma}_{t,0}\|_2 &= |\mathrm{e}^{\beta t}\sigma_u(\overline{X}_t, \overline{u}_t, e)v| \leq C_2\mathrm{e}^{\beta t}|v| \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\langle \widetilde{b}_t(y_1, v, e) - \widetilde{b}_t(y_2, v, e), y_1 - y_2\rangle + k\|\widetilde{\sigma}_t(y_1, v, e) - \widetilde{\sigma}_t(y_2, v, e)\|_2^2 \\ &= \langle b_x(\overline{X}_t, \overline{u}_t, e)(y_1 - y_2), y_1 - y_2\rangle + k\|\sigma_x(\overline{X}_t, \overline{u}_t, e)(y_1 - y_2)\|_2^2 \\ &\leq C_k|y_1 - y_2|^2. \end{aligned} \tag{3.11}$$

Let $p=2q$. Apply Itô's formula to $|\widetilde{Y}_t|^{2q}$ over the interval $[0,\ t]$, followed by Young's inequalities. For any $\theta>0$, we can obtain that

$$\begin{aligned}\mathbb{E}|\widetilde{Y}_t|^{2q} &\le 2q\mathbb{E}\int_0^t |\widetilde{Y}_s|^{2(q-1)}\Big(\langle \widetilde{Y}_s, \widetilde{b}_s-\widetilde{b}_{s,0}\rangle + \Big(q-\frac{1}{2}\Big)(1+\theta)\|\widetilde{\sigma}_s-\widetilde{\sigma}_{s,0}\|_2^2\Big)\mathrm{d}s\\ &\quad + 2q\mathbb{E}\int_0^t |\widetilde{Y}_s|^{2(q-1)}\Big(\langle \widetilde{Y}_s, \widetilde{b}_{t,0}\rangle + \Big(q-\frac{1}{2}\Big)\Big(1+\frac{1}{\theta}\Big)\|\widetilde{\sigma}_{s,0}\|_2^2\Big)\mathrm{d}s\\ &\quad + 2q\beta\mathbb{E}\int_0^t |\widetilde{Y}_s|^{2q}\mathrm{d}s. \end{aligned}\tag{3.12}$$

Specially, for those $\theta>0$ such that $\big(q-\frac{1}{2}\big)(1+\theta)\le k$. Thanks to (3.11) and Young's inequality, it follows that

$$\begin{aligned}\mathbb{E}|\widetilde{Y}_t|^{2q} &\le 2q\Big(C_k+\beta+\frac{\delta}{2}\Big)\mathbb{E}\int_0^t |\widetilde{Y}_s|^{2q}\mathrm{d}s\\ &\quad + 2q\mathbb{E}\int_0^t |\widetilde{Y}_s|^{2(q-1)}\Big(\frac{1}{2\delta}|\widetilde{b}_{s,0}|^2 + \Big(q-\frac{1}{2}\Big)\Big(1+\frac{1}{\theta}\Big)\|\widetilde{\sigma}_{s,0}\|_2^2\Big)\mathrm{d}s. \end{aligned}\tag{3.13}$$

By (3.10) and Young's inequality $|x|^{\alpha}|y|^{\beta}\le \kappa|x|^{(\alpha+\beta)}+\frac{\beta}{\alpha+\beta}\big[\frac{\alpha}{\kappa(\alpha+\beta)}\big]^{\frac{\alpha}{\beta}}|y|^{(\alpha+\beta)}$, for all $\alpha,\beta,\kappa>0$, we infer that

$$\mathbb{E}|\widetilde{Y}_t|^{2q} \le 2q\Big(C_k+\beta+\frac{\delta}{2}+2\kappa\Big)\mathbb{E}\int_0^t |\widetilde{Y}_s|^{2q}\mathrm{d}s + C\mathbb{E}\int_0^t \mathrm{e}^{2q\beta s}(|v_s|^{2q})\mathrm{d}s, \tag{3.14}$$

where the constant C depends only on q, δ, θ and κ. Choosing β, δ and κ small enough and recalling that $C_k<0$, such that $C_k+\beta+\frac{\delta}{2}+2\kappa=0$, then we have the following estimate:

$$\mathbb{E}|Y_t|^p \le C\sup_{s\in[0,t]}\mathbb{E}|v_s|^p. \tag{3.15}$$

We thus complete the proof.

In order to obtain the expansion of the cost functional, we give the following proposition which is fundamental.

Proposition 3.1 *Let assumptions* (A1)–(A3) *hold. Define* $\widehat{X}_t^{\varepsilon} := \frac{X_t^{\varepsilon}-\overline{X}_t}{\varepsilon} - Y_t$. *Then we have*

$$\lim_{\varepsilon\to 0_+}\sup_{t\ge 0}\mathbb{E}|\widehat{X}_t^{\varepsilon}|^2 = 0. \tag{3.16}$$

Proof Clearly, $\widehat{X}_t^{\varepsilon}$ satisfies the following SDE:

$$\begin{cases}\mathrm{d}\widehat{X}_t^{\varepsilon} = \dfrac{1}{\varepsilon}[b(X_t^{\varepsilon},u_t^{\varepsilon},\alpha_t) - b(\overline{X}_t,\overline{u}_t,\alpha_t) - \varepsilon b_x(\overline{X}_t,\overline{u}_t,\alpha_t)Y_t - \varepsilon b_u(\overline{X}_t,\overline{u}_t,\alpha_t)v_t]\mathrm{d}t\\ \qquad\qquad + \dfrac{1}{\varepsilon}[(X_t^{\varepsilon},u_t^{\varepsilon},\alpha_t) - \sigma(\overline{X}_t,\overline{u}_t,\alpha_t) - \varepsilon\sigma_x(\overline{X}_t,\overline{u}_t,\alpha_t)Y_t - \varepsilon\sigma_u(\overline{X}_t,\overline{u}_t,\alpha_t)v_t]\mathrm{d}B_t,\\ \widehat{X}_0^{\varepsilon} = 0. \end{cases}\tag{3.17}$$

First, according to Taylor expansion, it follows that

$$\mathrm{d}\widehat{X}_t^{\varepsilon} = (A_t^{\varepsilon,x}\widehat{X}_t^{\varepsilon} + A_t^{\varepsilon,y}Y_t + A_t^{\varepsilon,v}v_t)\mathrm{d}t + (B_t^{\varepsilon,x}\widehat{X}_t^{\varepsilon} + B_t^{\varepsilon,y}Y_t + B_t^{\varepsilon,v}v_t)\mathrm{d}B_t, \tag{3.18}$$

where

$$\begin{aligned}
A_t^{\varepsilon,x} &:= \int_0^1 [b_x(\overline{X}_t + \lambda\varepsilon(Y_t + \widehat{X}_t^\varepsilon), \overline{u}_t + \lambda\varepsilon v_t, \alpha_t)]\mathrm{d}\lambda, \\
A_t^{\varepsilon,y} &:= \int_0^1 [b_x(\overline{X}_t + \lambda\varepsilon(Y_t + \widehat{X}_t^\varepsilon), \overline{u}_t + \lambda\varepsilon v_t, \alpha_t) - b_x(\overline{X}_t, \overline{u}_t, \alpha_t)]\mathrm{d}\lambda, \\
A_t^{\varepsilon,v} &:= \int_0^1 [b_u(\overline{X}_t + \lambda\varepsilon(Y_t + \widehat{X}_t^\varepsilon), \overline{u}_t + \lambda\varepsilon v_t, \alpha_t) - b_u(\overline{X}_t, \overline{u}_t, \alpha_t)]\mathrm{d}\lambda, \\
B_t^{\varepsilon,x} &:= \int_0^1 [\sigma_x(\overline{X}_t + \lambda\varepsilon(Y_t + \widehat{X}_t^\varepsilon), \overline{u}_t + \lambda\varepsilon v_t, \alpha_t)]\mathrm{d}\lambda, \\
B_t^{\varepsilon,y} &:= \int_0^1 [\sigma_x(\overline{X}_t + \lambda\varepsilon(Y_t + \widehat{X}_t^\varepsilon), \overline{u}_t + \lambda\varepsilon v_t, \alpha_t) - \sigma_x(\overline{X}_t, \overline{u}_t, \alpha_t)]\mathrm{d}\lambda, \\
B_t^{\varepsilon,v} &:= \int_0^1 [\sigma_u(\overline{X}_t + \lambda\varepsilon(Y_t + \widehat{X}_t^\varepsilon), \overline{u}_t + \lambda\varepsilon v_t, \alpha_t) - \sigma_u(\overline{X}_t, \overline{u}_t, \alpha_t)]\mathrm{d}\lambda.
\end{aligned} \tag{3.19}$$

Next applying Itô's formula to $\mathrm{e}^{\beta t}|X_t|^2$, we can obtain

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{\beta t}|\widehat{X}_t^\varepsilon|^2) = 2\mathbb{E}\int_0^t \mathrm{e}^{\beta s}\langle A_s^{\varepsilon,x}\widehat{X}_s^\varepsilon + A_s^{\varepsilon,y}Y_s + A_s^{\varepsilon,v}v_s, \widehat{X}_s^\varepsilon\rangle\mathrm{d}s + \beta\mathbb{E}\int_0^t \mathrm{e}^{\beta s}|\widehat{X}_t^\varepsilon|^2\mathrm{d}s \\
+ \mathbb{E}\int_0^t \mathrm{e}^{\beta s}\|B_s^{\varepsilon,x}\widehat{X}_s^\varepsilon + B_s^{\varepsilon,y}Y_s + B_s^{\varepsilon,v}v_s\|_2^2\mathrm{d}s.
\end{aligned} \tag{3.20}$$

Thanks to (A3), we can take $\beta > 0$ small enough such that

$$2\langle A_s^{\varepsilon,x}\widehat{X}_s^\varepsilon, \widehat{X}_s^\varepsilon\rangle + 2k\|B_s^{\varepsilon,x}\widehat{X}_s^\varepsilon\|_2^2 + \beta|\widehat{X}_s^\varepsilon|^2 < 0. \tag{3.21}$$

After repeating the computations from the proof of Theorem 2.1, we obtain

$$\mathbb{E}|\widehat{X}_t^\varepsilon|^2 \leq C\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}[|A_t^{\varepsilon,y}Y_t|^2 + |A_t^{\varepsilon,v}v_t|^2 + \|B_t^{\varepsilon,y}Y_t\|_2^2 + \|B_t^{\varepsilon,v}v_t\|_2^2]\mathrm{d}s. \tag{3.22}$$

For any fixed constant κ, κ' satisfying

$$\kappa := \begin{cases} 2, & m = 0, \\ 1 < \kappa < \dfrac{p}{4m}, & m \geq 1, \end{cases} \qquad \kappa' := \frac{\kappa}{\kappa - 1},$$

it is easy to verify $\frac{1}{\kappa} + \frac{1}{\kappa'} = 1$, $2\kappa' < p$, $4m\kappa < p$ and $2\kappa < p$. By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}|A_s^{\varepsilon,y}Y_s|^2\mathrm{d}s \\
&\leq \int_0^t \mathrm{e}^{-\beta(t-s)}(\mathbb{E}|Y_s|^{2\kappa'})^{\frac{1}{\kappa'}} \cdot \Big(\int_0^1 \mathbb{E}\big|b_x(\overline{X}_s + \lambda\varepsilon(Y_s + \widehat{X}_s^\varepsilon), \overline{u}_s + \lambda\varepsilon v_s, \alpha_s) \\
&\quad - b_x(\overline{X}_s, \overline{u}_s, \alpha_s)|^{2\kappa}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s.
\end{aligned} \tag{3.23}$$

Recalling Lemma 3.2, then we can obtain

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}|A_s^{\varepsilon,y}Y_s|^2\mathrm{d}s\\&\le C\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1 \mathbb{E}|b_x(\overline{X}_s+\lambda\varepsilon(Y_s+\widehat{X}_s^{\varepsilon}),\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)-b_x(\overline{X}_s,\overline{u}_s,\alpha_s)|^{2\kappa}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\le C\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1 \mathbb{E}|b_x(\overline{X}_s+\lambda\varepsilon(Y_s+\widehat{X}_s^{\varepsilon}),\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)\\&\quad -b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\quad +C\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1 \mathbb{E}|b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)-b_x(\overline{X}_s,\overline{u}_s,\alpha_s)|^{2\kappa}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s. \qquad (3.24)\end{aligned}$$

Now we prove the convergence of the first term. According to (A1), for all $e\in I$, $u\in U$, $b_x(\cdot,u,e)$ is a locally Lipschitz function with respect to x. So that for all constant $R\in\mathbb{R}^+$, for all $e\in I$, $u\in U$, there exists a constant positive C_R such that $b_x(\cdot,u,e)$ is Lipschitz function with Lipschitz constant C_R in the ball of radius R. For t and ε, we first define the sets $\mathcal{A}_{t,\varepsilon}(R):=\{\omega\in\Omega\mid |\overline{X}_t|\vee|X_t^{\varepsilon}|>R\}$. By Chebyshev's inequality and previous estimate (2.8), it follows that

$$\mathbb{P}(\mathcal{A}_{t,\varepsilon}(R))\le\frac{\mathbb{E}|\overline{X}_t|^2+\mathbb{E}|X_t^{\varepsilon}|^2}{R^2}\le\frac{C}{R^2}. \qquad (3.25)$$

It is easy to check, for all positive constant θ, exists R_θ large enough such that $\mathcal{A}_{t,\varepsilon}(R_\theta)$ is less than θ. Denote $X_t^{\lambda}:=\overline{X}_t+\lambda\varepsilon(Y_t+\widehat{X}_t^{\varepsilon})=(1-\lambda)\overline{X}_t+\lambda X_t^{\varepsilon}$. Taking a positive constant δ satisfying $4m\kappa(1+\delta)\le p$, then by Hölder's inequality, we can obtain

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1 \mathbb{E}|b_x(X_s^{\lambda},\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)-b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\le C\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1\int_{\mathcal{A}_{t,\varepsilon}(R_\theta)}|b_x(X_s^{\lambda},\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)\\&\quad -b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa}\mathrm{d}\mathbb{P}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\quad +C\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1\int_{\mathcal{A}^c_{t,\varepsilon}(R_\theta)}|b_x(X_s^{\lambda},\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)\\&\quad -b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa}\mathrm{d}\mathbb{P}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\le C\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1\theta^{\frac{\delta}{1+\delta}}(\mathbb{E}\big|b_x(X_s^{\lambda},\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)\\&\quad -b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa(1+\delta)})^{\frac{1}{1+\delta}}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\quad +C\int_0^t \mathrm{e}^{-\beta(t-s)}C_{R_\theta}^{\frac{1}{\kappa}}(\mathbb{E}|X_s^{\varepsilon}-\overline{X}_s|^{2\kappa})^{\frac{1}{\kappa}}\mathrm{d}s. \qquad (3.26)\end{aligned}$$

Thanks to estimate (2.8) and (A1), we have that

$$\begin{aligned}&\mathbb{E}|b_x(X_s^{\lambda},\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)-b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa(1+\delta)}\\&\le C(\mathbb{E}|X_s^{\lambda}|^{4m\kappa(1+\delta)}+\mathbb{E}|\overline{X}_s|^{4m\kappa(1+\delta)}+1)\le\widetilde{C}, \qquad (3.27)\end{aligned}$$

as well as

$$\lim_{\varepsilon\to 0_+}\sup_{s\geq 0}\mathbb{E}|X_s^{\varepsilon}-\overline{X}_s|^p=0. \tag{3.28}$$

Since θ is an arbitrary positive constant, we can obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0_+}\sup_{s\geq 0}\int_0^t \mathrm{e}^{-\beta(t-s)}\Big(\int_0^1\mathbb{E}|b_x(X_s^{\lambda},\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)-b_x(\overline{X}_s,\overline{u}_s+\lambda\varepsilon v_s,\alpha_s)|^{2\kappa}\mathrm{d}\lambda\Big)^{\frac{1}{\kappa}}\mathrm{d}s\\&\leq \lim_{\varepsilon\to 0_+}\sup_{s\geq 0}C\int_0^t \mathrm{e}^{-\beta(t-s)}C_{R_\theta}^{\frac{1}{\kappa}}(\mathbb{E}|X_s^{\varepsilon}-\overline{X}_s|^{2\kappa})^{\frac{1}{\kappa}}\mathrm{d}s=0.\end{aligned} \tag{3.29}$$

The proof of the convergence of the second term is analogous. Consequently, we infer that

$$\lim_{\varepsilon\to 0_+}\sup_{s\geq 0}\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}|A_s^{\varepsilon,y}Y_s|^2\mathrm{d}s=0. \tag{3.30}$$

Following a similar argument, we can obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0_+}\sup_{s\geq 0}\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}|A_s^{\varepsilon,v}v_s|^2\mathrm{d}s=0;\\&\lim_{\varepsilon\to 0_+}\sup_{s\geq 0}\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}|B_s^{\varepsilon,y}Y_s|^2\mathrm{d}s=0;\\&\lim_{\varepsilon\to 0_+}\sup_{s\geq 0}\int_0^t \mathrm{e}^{-\beta(t-s)}\mathbb{E}|B_s^{\varepsilon,v}v_s|^2\mathrm{d}s=0,\end{aligned} \tag{3.31}$$

which implies the required result.

Remark 3.1 The conditions $\frac{1}{\kappa}+\frac{1}{\kappa'}=1$, $2\kappa'<p$, $4m\kappa<p$ and $2\kappa<p$ imply that p must be greater than $(4m+2)\vee 4$.

In the next part of this section we will give the convex perturbation of the cost functional. Firstly, by (A4) and the estimate (2.8), we have

$$\begin{aligned}J(u(\cdot))&=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\Big[\int_0^T f(X_t,u_t,\alpha_t)\mathrm{d}t\Big]\\&\leq \limsup_{T\to\infty}\frac{1}{T}\Big[\int_0^T K\Big(1+\sup_{t\geq 0}\mathbb{E}|X_t|^2+\sup_{t\geq 0}\mathbb{E}|u_t|^2\Big)\mathrm{d}t\Big]<\infty.\end{aligned} \tag{3.32}$$

To simplify the writing in what follows, we denote $J^T(\cdot)$ for the truncated cost functional,

$$J^T(u(\cdot)):=\mathbb{E}\Big[\int_0^T f(X_t,u_t,\alpha_t)\mathrm{d}t\Big]. \tag{3.33}$$

Now we give the expansion of the functional with respect to a convex perturbation of the control.

Lemma 3.3 *Let $\overline{u}(\cdot)$ be an optimal control, and $u(\cdot)$ be any admissible control, then the following estimate holds*:

$$\begin{aligned}&\lim_{\varepsilon\to 0_+}\frac{J(u^{\varepsilon}(\cdot))-J(\overline{u}(\cdot))}{\varepsilon}\\&\leq \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T[\langle f_x(\overline{X}_t,\overline{u}_t,\alpha_t),Y_t\rangle+\langle f_u(\overline{X}_t,\overline{u}_t,\alpha_t),v_t\rangle]\mathrm{d}t.\end{aligned} \tag{3.34}$$

Proof First, by Taylor expansion, we have

$$\frac{J^T(u^\varepsilon(\cdot))-J^T(\overline{u}(\cdot))}{\varepsilon}=\mathbb{E}\int_0^T\int_0^1[\langle \widehat{X}_t^\varepsilon+Y_t,f_x(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle \\ +\langle v_t,f_u(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle]\mathrm{d}\lambda\mathrm{d}t. \tag{3.35}$$

Thus

$$\begin{aligned}&\frac{J(u^\varepsilon(\cdot))-J(\overline{u}(\cdot))}{\varepsilon}\\ &=\frac{1}{\varepsilon}\Big[\limsup_{T\to\infty}\frac{1}{T}J^T(u^\varepsilon(\cdot))-\limsup_{T\to\infty}\frac{1}{T}J^T(\overline{u}(\cdot))\Big]\\ &\le\limsup_{T\to\infty}\frac{J^T(u^\varepsilon(\cdot))-J^T(\overline{u}(\cdot))}{T\varepsilon}\\ &\le\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\int_0^1[\langle \widehat{X}_t^\varepsilon,f_x(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle]\mathrm{d}\lambda\mathrm{d}t\\ &\quad+\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\int_0^1[\langle Y_t,f_x(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle\\ &\quad+\langle v_t,f_u(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle]\mathrm{d}\lambda\mathrm{d}t=:\mathrm{I}+\mathrm{II},\end{aligned} \tag{3.36}$$

where

$$\mathrm{I}:=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\int_0^1[\langle \widehat{X}_t^\varepsilon,f_x(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle]\mathrm{d}\lambda\mathrm{d}t \tag{3.37}$$

and

$$\mathrm{II}:=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\int_0^1[\langle Y_t,f_x(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle \\ +\langle v_t,f_u(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t),\overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)\rangle]\mathrm{d}\lambda\mathrm{d}t. \tag{3.38}$$

By applying Hölder's inequality, we can obtain

$$\mathrm{I}\le\limsup_{T\to\infty}\frac{1}{T}\int_0^T\int_0^1[(\mathbb{E}|\widehat{X}_t^\varepsilon|^2)^{\frac{1}{2}}\mathbb{E}|f_x(\overline{X}_t+\lambda(X_t^\varepsilon-\overline{X}_t), \\ \overline{u}_t+\lambda(u_t^\varepsilon-\overline{u}),\alpha_t)|^2)^{\frac{1}{2}}]\mathrm{d}\lambda\mathrm{d}t. \tag{3.39}$$

Thanks to (A4) and previous estimate of $\widehat{X}_t^\varepsilon$ and Y_t, it can be seen that

$$\lim_{\varepsilon\to0_+}\mathrm{I}=0, \tag{3.40}$$

as well as

$$\lim_{\varepsilon\to0_+}\mathrm{II}=\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T[\langle f_x(\overline{X}_t,\overline{u}_t,\alpha_t),Y_t\rangle+\langle f_u(\overline{X}_t,\overline{u}_t,\alpha_t),v_t\rangle]\mathrm{d}t. \tag{3.41}$$

Based on the above argument, we complete the proof.

Now we introduce the adjoint equation associated to the system, which is an infinite horizon BSDE in $\mathbb{R}^n$. For the given admissible pair (X,u) in the system (2.1), the adjoint equation, characterizing the unknown $\mathbb{R}^n\times\mathbb{R}^{n\times d}\times(\mathbb{R}^{N\times N})^n$-valued adapted processes p_t, $q_t=(q_t^1,\cdots,q_t^d)$, and $z_t = (z_t^{(1)}, \cdots, z_t^{(n)})$, is the following infinite horizon BSDE with Markov regime-switching

$$-\mathrm{d}p_t = \Big[C_t^{\mathrm{T}} p_t + \sum_{k=1}^{d} D_t^{k\,\mathrm{T}} q_t^k - E_t\Big]\mathrm{d}t - \sum_{k=1}^{d} q_t^k \mathrm{d}B_t^k - z_t \cdot \mathrm{d}Q_t. \tag{3.42}$$

Here the coefficients have the form $C_t = b_x(X_t, u_t, \alpha_t)$, $D_t = \sigma_x(X_t, u_t, \alpha_t)$, $E_t = f_x(X_t, u_t, \alpha_t)$ and

$$z_t \cdot \mathrm{d}Q_t := \Big(\sum_{i \neq j} z_{ij}^{(1)}(t)\mathrm{d}Q_{ij}(t), \cdots, \sum_{i \neq j} z_{ij}^{(n)}(t)\mathrm{d}Q_{ij}(t)\Big)^T. \tag{3.43}$$

Since adjoint equation (3.42) is defined on an infinite horizon, we provide the definition of solutions to adjoint equation (3.42).

Definition 3.1 *A solution to the adjoint equation* (3.42) *is an adapted processes* (p_t, q_t, z_t) *such that*:

(1) *For fixed $T > 0$, and for all t in $[0, T]$,*

$$p_t = p_T + \int_t^T \Big[C_s^{\mathrm{T}} p_s + \sum_{k=1}^{d} D_s^{k\,\mathrm{T}} q_s^k - E_s\Big]\mathrm{d}s - \sum_{k=1}^{d} \int_t^T q_s^k \mathrm{d}B_s^k - \int_t^T z_t \cdot \mathrm{d}Q_t, \ \mathbb{P}\text{-}a.s. \tag{3.44}$$

(2) *The adjoint process $\{p_t\}_{t \geq 0}$ has continuous trajectories.*

(3) *For fixed $T > 0$,*

$$\begin{aligned} &\{p_t\}_{t \geq 0} \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^n)), \\ &\{q_t^k\}_{t \in [0,T]} \in L^2_{\mathcal{P}^\star}(\Omega \times [0, T]; \mathbb{R}^n), \quad k = 1, \cdots, d, \\ &\{z_t\}_{t \in [0,T]} \in L^2_{\mathcal{P}^\star}(Q, [0, T]). \end{aligned} \tag{3.45}$$

We can prove that, under the assumptions (A1)–(A4), for any admissible pair, there exists a unique solution of the adjoint equation (3.42).

Theorem 3.1 *Let assumptions* (A1)–(A4) *hold. For the given admissible pair* $(\overline{X}, \overline{u})$*, the adjoint equation* (3.42) *admits a unique solution* $(p_t^\infty, q_t^\infty, z_t^\infty)_{t \geq 0}$ *satisfying the Definition* 3.1.

In order to prove Theorem 3.1, we will first introduce a group of the following BSDE named truncated adjoint equation and a group of SDE named affine equation. Then, we present the duality relation between these two equations.

For the given admissible pair (X, u), terminal time T, the truncated adjoint equation, describing the unknown $\mathbb{R}^n \times \mathbb{R}^{n \times d} \times (\mathbb{R}^{N \times N})^n$-valued adapted processes $p_t^{T,\nu}$, $q_t^{T,\nu} = (q_t^{1,T,\nu}, \cdots, q_t^{d,T,\nu})$, and $z_t^{T,\nu} = (z_t^{(1),T,\nu}, \cdots, z_t^{(n),T,\nu})$, is the following finite horizon regime-switching BSDE on interval $[0, T]$ with the given terminal time $T > 0$ and terminal $p_T^{T,\nu} = \nu \in L^2(\Omega, \mathscr{F}_T; \mathbb{R}^n)$:

$$\begin{cases} -\mathrm{d}p_t^{T,\nu} = \Big[C_t^{\mathrm{T}} p_t^{T,\nu} + \sum_{k=1}^{d} D_t^{k\,\mathrm{T}} q_t^{k,T,\nu} - E_t\Big]\mathrm{d}t - \sum_{k=1}^{d} q_t^{k,T,\nu} \mathrm{d}B_t^k \\ \qquad\qquad -z_t^{T,\nu} \cdot \mathrm{d}Q_t, \quad t \in [0, T], \\ p_T^{T,\nu} = \nu. \end{cases} \tag{3.46}$$

The existence and uniqueness of the solution by (3.46) can be get similarly by the result of Orrieri et al. [15, Theorem 6.2], Briand et al. [6, Theorem 4.1], Pardoux [16, Theorems

2.1–2.2], Huang et al. [12, Theorem 5.13]. The proof is technically and lengthy, we omit it and only give the conclusion.

Lemma 3.4 *Let assumptions* (A1)–(A4) *hold. For the given admissible pair* (X,u), *terminal time* $T>0$ *and terminal* $p_T^{T,\nu}=\nu\in L^2(\Omega,\mathscr{F}_T;\mathbb{R}^n)$, *the truncated adjoint equation* (3.46) *admits a unique progressively measurable solutions* $(p_t^{T,\nu},q_t^{T,\nu},z_t^{T,\nu})_{t\in[0,T]}$ *such that*:

(1) *For all* t *in* $[0,T]$,

$$\begin{aligned} p_t^{T,\nu} &= \nu+\int_t^T\Big[C_s^{\mathrm{T}}p_s^{T,\nu}+\sum_{k=1}^d {D_s^k}^{\mathrm{T}}q_s^{k,T,\nu}-E_s\Big]\mathrm{d}s\\ &\quad -\sum_{k=1}^d\int_t^T q_s^{k,T,\nu}\mathrm{d}B_s^k-\int_t^T z_t^{T,\nu}\cdot\mathrm{d}Q_t,\quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{3.47}$$

(2) $\{p_t^{T,\nu}\}_{t\in[0,T]}$ *has continuous trajectories.*

(3)

$$\begin{aligned} &\{p_t^{T,\nu}\}_{t\in[0,T]}\in L^\infty([0,T];L^2(\Omega;\mathbb{R}^n)),\\ &\{q_t^{k,T,\nu}\}_{t\in[0,T]}\in L^2_{\mathcal{P}^\star}(\Omega\times[0,T];\mathbb{R}^n),\quad k=1,\cdots,d,\\ &\{z_t^{T,\nu}\}_{t\in[0,T]}\in L^2_{\mathcal{P}^\star}(Q,[0,T]). \end{aligned} \tag{3.48}$$

The affine equation is a forward SDE:

$$\begin{cases} \mathrm{d}\mathcal{Y}_s^{t,\eta,\gamma,\rho}=C_s\mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}s+\displaystyle\sum_{k=1}^d D_s^k\mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}B_s^k+\gamma_s\mathrm{d}s+\sum_{k=1}^d\rho_s^k\mathrm{d}B_s^k,\quad s\ge t,\\ \mathcal{Y}_t^{t,\eta,\gamma,\rho}=\eta. \end{cases} \tag{3.49}$$

It is a forward SDE with initial condition $\eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)$ and general forcing term $(\gamma_t,\rho_t^1,\cdots,\rho_t^d)$ with γ_t and ρ_t^i, $i=1,\cdots,d$ in $L^2([0,T];L^2(\Omega;\mathbb{R}^n))$. When $\gamma\equiv0$ (resp. $\rho\equiv0$, $\gamma=\rho\equiv0$), we write $\mathcal{Y}_s^{t,\eta,\gamma,\rho}$ as $\mathcal{Y}_s^{t,\eta,\rho}$ (resp. $\mathcal{Y}_s^{t,\eta,\gamma}$, $\mathcal{Y}_s^{t,\eta}$) for short.

Specifically, when we take $t=0$, $\eta=0$, $\gamma_s=B_u(X_s,u_s,\alpha_s)v_s$ and $\rho_s^k=\sigma_u^k(X_s,u_s,\alpha_s)v_s$, the affine equation (3.49) is variational equation (3.5).

Let assumptions (A1)–(A4) hold. By the same method we derive that, like the proof of Theorem 2.1, the affine equation admits a unique adapted solution and we have the following estimate

$$\mathbb{E}|\mathcal{Y}_s^{t,\eta,\gamma,\rho}|^2\le \mathrm{e}^{-2\beta(s-t)}\mathbb{E}|\eta|^2+K\int_t^s\mathrm{e}^{-2\beta(s-r)}\mathbb{E}[|\gamma_r|^2+|\rho_r^1|^2+\cdots+|\rho_r^d|^2]\mathrm{d}r. \tag{3.50}$$

We conclude this section with the duality relation between the truncated adjoint equation (3.46) and the affine equation (3.49).

Lemma 3.5 *The truncated adjoint equation* (3.46) *and the affine equation* (3.49) *have the following duality relation*:

$$\begin{aligned} &\mathbb{E}\int_t^T\langle p_s^{T,\nu},\gamma_s\rangle\mathrm{d}s+\sum_{k=1}^d\mathbb{E}\int_t^T\langle q_s^{k,T,\nu},\rho_s^k\rangle\mathrm{d}s+\mathbb{E}\langle p_t^{T,\nu},\eta\rangle\\ &=\mathbb{E}\int_t^T\langle\mathcal{Y}_s^{t,\eta,\gamma,\rho},-E_s\rangle\mathrm{d}s+\mathbb{E}\langle\nu,\mathcal{Y}_T^{t,\eta,\gamma,\rho}\rangle, \end{aligned} \tag{3.51}$$

where $0\le t\le T$, $\nu\in L^2(\Omega,\mathscr{F}_T;\mathbb{R}^n)$, $\eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)$, $\gamma,\rho^k\in L^2([0,T];L^2(\Omega;\mathbb{R}^n))$.

Proof Firstly, we have

$$\begin{aligned}&\mathbb{E}\int_t^T \mathrm{d}\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, p_s^{T,\nu}\rangle = \mathbb{E}\langle \mathcal{Y}_T^{t,\eta,\gamma,\rho}, p_T^{T,\nu}\rangle - \mathbb{E}\langle \mathcal{Y}_t^{t,\eta,\gamma,\rho}, p_t^{T,\nu}\rangle \\ =\ & \mathbb{E}\langle \nu, \mathcal{Y}_T^{t,\eta,\gamma,\rho}\rangle - \mathbb{E}\langle p_t^{T,\nu}, \eta\rangle. \end{aligned} \tag{3.52}$$

By using Itô's formula on $[t,T]$ to $\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, p_s^{T,\nu}\rangle$, we obtain that

$$\begin{aligned}
&\mathbb{E}\int_t^T \mathrm{d}\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, p_s^{T,\nu}\rangle \\
=\ & \mathbb{E}\int_t^T [\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, \mathrm{d}p_s^{T,\nu}\rangle + \langle p_s^{T,\nu}, \mathrm{d}\mathcal{Y}_s^{t,\eta,\gamma,\rho}\rangle + \langle \mathrm{d}p_s^{T,\nu}, \mathrm{d}\mathcal{Y}_s^{t,\eta,\gamma,\rho}\rangle] \\
=\ & \mathbb{E}\int_t^T \Big[\Big\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, -\Big[C_s^{\mathrm{T}} p_s^{T,\nu} + \sum_{k=1}^d {D_s^k}^{\mathrm{T}} q_s^{k,T,\nu} - E_s\Big]\mathrm{d}s + \sum_{k=1}^d q_s^{k,T,\nu}\mathrm{d}B_s + z_s^{T,\nu}\cdot \mathrm{d}Q_s\Big\rangle \\
&+ \Big\langle p_s^{T,\nu}, C_s\mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}s + \sum_{k=1}^d D_s^k \mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}B_s^k + \gamma_s\mathrm{d}s + \sum_{k=1}^d \rho_s^k \mathrm{d}B_s^k\Big\rangle \\
&+ \Big\langle -\Big[C_s^{\mathrm{T}} p_s^{T,\nu} + \sum_{k=1}^d {D_s^k}^{\mathrm{T}} q_s^{k,T,\nu} - E_s\Big]\mathrm{d}s + \sum_{k=1}^d q_s^{k,T,\nu}\mathrm{d}B_s + z_t^{T,\nu}\cdot \mathrm{d}Q_t, \\
&\quad C_s\mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}s + \sum_{k=1}^d D_s^k \mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}B_s^k + \gamma_s\mathrm{d}s + \sum_{k=1}^d \rho_s^k \mathrm{d}B_s^k\Big\rangle\Big] \\
=\ & \mathbb{E}\int_t^T \Big[\Big\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, -\Big[C_s^{\mathrm{T}} p_s^{T,\nu} + \sum_{k=1}^d {D_s^k}^{\mathrm{T}} q_s^{k,T,\nu} - E_s\Big]\mathrm{d}s\Big\rangle \\
&+ \langle p_s^{T,\nu}, C_s\mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}s + \gamma_s\mathrm{d}s\rangle + \Big\langle \sum_{k=1}^d q_s^{k,T,\nu}\mathrm{d}B_s, \sum_{k=1}^d D_s^k \mathcal{Y}_s^{t,\eta,\gamma,\rho}\mathrm{d}B_s^k + \sum_{k=1}^d \rho_s^k \mathrm{d}B_s^k\Big\rangle\Big] \\
=\ & \mathbb{E}\int_t^T \Big[\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, E_s\rangle\mathrm{d}s + \langle p_s^{T,\nu}, \gamma_s\rangle\mathrm{d}s + \sum_{k=1}^d\sum_{j=1}^d \langle q_s^{k,T,\nu}, \rho_s^j\rangle \mathrm{d}B_s^k\mathrm{d}B_s^j\Big] \\
=\ & \mathbb{E}\int_t^T \Big[\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, E_s\rangle\mathrm{d}s + \langle p_s^{T,\nu}, \gamma_s\rangle\mathrm{d}s + \sum_{k=1}^d \langle q_s^{k,T,\nu}, \rho_s^k\rangle \mathrm{d}s\Big],
\end{aligned}$$

which implies the required result.

Now, we prove Theorem 3.1.

Proof of Theorem 3.1 (1) Existence. First, let us take $\nu = \gamma = \rho \equiv 0$, $\eta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$ in duality relation (3.51), then we have

$$\mathbb{E}\langle p_t^T, \eta\rangle = \mathbb{E}\int_t^T \langle \mathcal{Y}_s^{t,\eta}, -E_s\rangle \mathrm{d}s. \tag{3.53}$$

By previous estimates, we can obtain that

$$\sup_{s\geq 0}\mathbb{E}|E_s|^2 \leq \sup_{s\geq 0}\mathbb{E}K(1+|X_s|^2+|u_s|^2) < \infty, \tag{3.54}$$

as well as

$$\mathbb{E}\int_t^{\infty}|\mathcal{Y}_s^{t,\eta}|^2\mathrm{d}s\le\int_t^{\infty}\mathrm{e}^{-2\beta(s-t)}\mathbb{E}|\eta|^2\mathrm{d}s<\infty.\tag{3.55}$$

Thus

$$\lim_{T\to\infty}\mathbb{E}\int_t^{T}\langle\mathcal{Y}_s^{t,\eta},-E_s\rangle\mathrm{d}s=\mathbb{E}\int_t^{\infty}\langle\mathcal{Y}_s^{t,\eta},-E_s\rangle\mathrm{d}s.\tag{3.56}$$

Define $\Theta{:}L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)\to\mathbb{R}$ by

$$\Theta(\eta):=\mathbb{E}\int_t^{\infty}\langle\mathcal{Y}_s^{t,\eta},-E_s\rangle\mathrm{d}s.\tag{3.57}$$

Since $\mathcal{Y}_s^{t,\eta}$ satisfies (3.49) with $\gamma=\rho\equiv 0$, then Θ is a bounded linear operator. By Riesz representation theorem, there exists $P_t\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)$ such that

$$\mathbb{E}\langle P_t,\eta\rangle=\mathbb{E}\int_t^{\infty}\langle\mathcal{Y}_s^{t,\eta},-E_s\rangle\mathrm{d}s=\lim_{T\to\infty}\mathbb{E}\langle p_t^T,\eta\rangle,\quad \eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n).\tag{3.58}$$

This means that p_t^T weakly converges to P_t in $L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)$. Thanks to (3.55) and (3.58), we can obtain that

$$\mathbb{E}|P_t|^2\le\beta^{-1}\sup_{s\ge 0}(\mathbb{E}|E_s|^2)^{\frac{1}{2}},\quad t>0.\tag{3.59}$$

Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathscr{F}$-stopping times which strictly increase monotonically to the infinity. For all $n\in\mathbb{N}$, let $(\widetilde{p}_t^{\tau_n},\widetilde{q}_t^{\,1,\tau_n},\cdots,\widetilde{q}_t^{\,d,\tau_n},\widetilde{z}_t^{\tau_n})_{t\in[0,\tau_n]}$ be the solution of the following truncated adjoint equation

$$\begin{cases}-\mathrm{d}\widetilde{p}_t^{\tau_n}=\Big[C_t^{\mathrm{T}}\widetilde{p}_t^{\tau_n}+\displaystyle\sum_{k=1}^{d}D_t^{k\,\mathrm{T}}\widetilde{q}_t^{\,k,\tau_n}-E_t\Big]\mathrm{d}t-\sum_{k=1}^{d}\widetilde{q}_t^{\,k,\tau_n}\mathrm{d}B_t\\ \qquad\qquad -\widetilde{z}_t\cdot\mathrm{d}Q_t,\quad t\in[0,\tau_n],\\ \widetilde{p}_{\tau_n}^{\tau_n}=P_{\tau_n}.\end{cases}\tag{3.60}$$

We claim that, for all $n,m\in\mathbb{N}$ such that $0\le n\le m$ and for all $t\in[0,\tau_n]$, we have that

$$\begin{aligned}&\widetilde{p}_t^{\tau_n}=\widetilde{p}_t^{\tau_m},\quad \mathbb{P}\text{-a.s.},\\ &\widetilde{q}^{\,\tau_n}=\widetilde{q}^{\,\tau_m},\quad \mathbb{P}\otimes\mathrm{d}t\text{-a.s.}\quad\text{in }\Omega\times[0,\tau_m],\\ &\widetilde{z}^{\,\tau_n}=\widetilde{z}^{\,\tau_m},\quad \mathbb{P}\otimes\mathrm{d}t\text{-a.s.}\quad\text{in }\Omega\times[0,\tau_m].\end{aligned}\tag{3.61}$$

Recalling duality relation (3.51), it follows that

$$\mathbb{E}\langle\widetilde{p}_t^{\tau_n},\eta\rangle=\mathbb{E}\int_t^{\tau_n}\langle\mathcal{Y}_s^{t,\eta},-E_s\rangle\mathrm{d}s+\mathbb{E}\langle P_{\tau_n},\mathcal{Y}_{\tau_n}^{t,\eta}\rangle,\quad \eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n).\tag{3.62}$$

By (3.58), we can obtain that

$$\mathbb{E}\langle P_{\tau_n},\mathcal{Y}_{\tau_n}^{t,\eta}\rangle=\mathbb{E}\int_{\tau_n}^{\infty}\langle\mathcal{Y}_s^{\tau_n,\mathcal{Y}_{\tau_n}^{t,\eta}},-E_s\rangle\mathrm{d}s,\quad \eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n).\tag{3.63}$$

Thus

$$\mathbb{E}\langle \widetilde{p}_t^{\tau_n}, \eta\rangle = \mathbb{E}\int_t^{\tau_n} \langle \mathcal{Y}_s^{t,\eta}, -E_s\rangle \mathrm{d}s + \mathbb{E}\int_{\tau_n}^{\infty} \langle \mathcal{Y}_s^{\tau_n,\mathcal{Y}_{\tau_n}^{t,\eta}}, -E_s\rangle \mathrm{d}s. \tag{3.64}$$

Recalling the existence and uniqueness of the solution of affine equation (3.49), it follows that

$$\mathcal{Y}_s^{\tau_n,\mathcal{Y}_{\tau_n}^{t,\eta}} = \mathcal{Y}_s^{t,\eta}, \quad s \in [\tau_n, \infty],\ \mathbb{P}\text{-a.s.} \tag{3.65}$$

Moreover, we can obtain

$$\mathbb{E}\langle \widetilde{p}_t^{\tau_n}, \eta\rangle = \mathbb{E}\int_t^{\infty} \langle \mathcal{Y}_s^{t,\eta}, -E_s\rangle \mathrm{d}s = \mathbb{E}\langle P_t, \eta\rangle, \quad t \in [0,\tau_n],\ \eta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n). \tag{3.66}$$

By the same method we have

$$\mathbb{E}\langle \widetilde{p}_t^{\tau_m}, \eta\rangle = \mathbb{E}\langle P_t, \eta\rangle, \quad t \in [0,\tau_m],\ \eta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n). \tag{3.67}$$

Since $\mathbb{E}\langle P_t, \eta\rangle$ does not dependent on n or m, we have $\widetilde{p}_t^{\tau_n} = \widetilde{p}_t^{\tau_m}, \mathbb{P}$-a.s. for all t in $[0, \tau_n]$ and $\widetilde{p}_{\tau_n}^{\tau_n} = \widetilde{p}_{\tau_n}^{\tau_m}, \mathbb{P}$-a.s. in particular. By the existence and uniqueness of the solution of truncated adjoint equation (3.46) our claim (3.61) is proved.

Since (3.55), it follows that

$$\sup_{t\in[0,\tau_n]} (\mathbb{E}|\widetilde{p}_t^{\tau_n}|^2)^{\frac{1}{2}} \le \beta^{-1} \sup_{s\ge 0}(\mathbb{E}|E_s|^2)^{\frac{1}{2}}. \tag{3.68}$$

Now we first define $(p_t^{\infty}, q_t^{1,\infty}, \cdots, q_t^{d,\infty}, z_t^{\infty})_{t\ge 0}$ by

$$p_t^{\infty} = \sum_{n=1}^{\infty} \widetilde{p}_t^{\tau_n} I_{[\tau_{n-1},\tau_n)}(t), \quad q_t^{k,\infty} = \sum_{n=1}^{\infty} \widetilde{p}_t^{k,\tau_n} I_{[\tau_{n-1},\tau_n)}(t), \quad z_t^{\infty} = \sum_{n=1}^{\infty} \widetilde{z}_t^{\tau_n} I_{[\tau_{n-1},\tau_n)}(t). \tag{3.69}$$

We claim that $(p_t^{\infty}, q_t^{1,\infty}, \cdots, q_t^{d,\infty}, z_t^{\infty})_{t\ge 0}$ is the desired solution. Indeed it satisfies the desired integrability and adaptedness conditions. Now we prove it satisfies (3.44). Fixed $0 \le t \le T$, since $\{\tau_k\}_{k\in\mathbb{N}}$ is a strictly increase monotonically stopping time sequence, there exist $n_t^{\omega}, n_T^{\omega} \in \mathbb{N}$ such that

$$\tau_{n_t^{\omega}} \le t < \tau_{n_t^{\omega}+1}, \quad \tau_{n_T^{\omega}} \le T < \tau_{n_T^{\omega}+1} \tag{3.70}$$

hold for a.e.-$\omega \in \Omega$, which implies

$$p_t^{\infty} = \widetilde{p}_t^{\tau_{n_t^{\omega}+1}}, \quad p_T^{\infty} = \widetilde{p}_t^{\tau_{n_T^{\omega}+1}}. \tag{3.71}$$

Then we have

$$\begin{aligned} p_t^{\infty} - p_T^{\infty} &= \widetilde{p}_t^{\tau_{n_t^{\omega}+1}} - \widetilde{p}_t^{\tau_{n_T^{\omega}+1}} \\ &= (\widetilde{p}_t^{\tau_{n_t^{\omega}+1}} - \widetilde{p}_{\tau_{n_t^{\omega}+1}}^{\tau_{n_t^{\omega}+2}}) + (\widetilde{p}_{\tau_{n_T^{\omega}}}^{\tau_{n_T^{\omega}+1}} - \widetilde{p}_T^{\tau_{n_T^{\omega}+1}}) + \sum_{n=n_t^{\omega}+1}^{n_T^{\omega}-1} (\widetilde{p}_{\tau_n}^{\tau_{n+1}} - \widetilde{p}_{\tau_{n+1}}^{\tau_{n+2}}). \end{aligned} \tag{3.72}$$

Recalling claim (3.61), it leads to

$$\widetilde{p}_t^{\tau_n} = \widetilde{p}_t^{\tau_{n+1}}, \quad t \le \tau_n,\ \mathbb{P}\text{-a.s.} \tag{3.73}$$

In particular,

$$\widetilde{p}_{\tau_n}^{\tau_n} = \widetilde{p}_{\tau_n}^{\tau_{n+1}}, \quad \mathbb{P}\text{-a.s.}$$

Then we can obtain

$$\begin{aligned}
&p_t^\infty - p_T^\infty \\
&= (\widetilde{p}_t^{\tau_{n_t^\omega+1}} - \widetilde{p}_{\tau_{n_t^\omega+1}}^{\tau_{n_t^\omega+1}}) + (\widetilde{p}_{\tau_{n_T^\omega}}^{\tau_{n_T^\omega+1}} - \widetilde{p}_T^{\tau_{n_T^\omega+1}}) + \sum_{n=n_t^\omega+1}^{n_T^\omega-1} (\widetilde{p}_{\tau_n}^{\tau_{n+1}} - \widetilde{p}_{\tau_{n+1}}^{\tau_{n+1}}) \\
&= \int_t^{\tau_{n_t^\omega+1}} \Big[\Big(C_s^{\mathrm{T}} \widetilde{p}_s^{\tau_{n_t^\omega+1}} + \sum_{k=1}^d D_s^{k\,\mathrm{T}} \widetilde{q}_s^{\,k,\tau_{n_t^\omega+1}} - E_s\Big)\mathrm{d}s - \sum_{k=1}^d \widetilde{q}_s^{\,k,\tau_{n_t^\omega+1}} \mathrm{d}B_s - \widetilde{z}_s^{\tau_{n_t^\omega+1}} \cdot \mathrm{d}Q_s\Big] \\
&\quad + \int_{\tau_{n_T^\omega}}^T \Big[\Big(C_s^{\mathrm{T}} \widetilde{p}_s^{\tau_{n_T^\omega+1}} + \sum_{k=1}^d D_s^{k\,\mathrm{T}} \widetilde{q}_s^{\,k,\tau_{n_T^\omega+1}} - E_s\Big)\mathrm{d}s - \sum_{k=1}^d \widetilde{q}_s^{\,k,\tau_{n_T^\omega+1}} \mathrm{d}B_s - \widetilde{z}_s^{\tau_{n_T^\omega+1}} \cdot \mathrm{d}Q_s\Big] \\
&\quad + \sum_{n=n_t^\omega+1}^{n_T^\omega-1} \int_{\tau_n}^{\tau_{n+1}} \Big[\Big(C_s^{\mathrm{T}} \widetilde{p}_s^{\tau_{n+1}} + \sum_{k=1}^d D_s^{k\,\mathrm{T}} \widetilde{q}_s^{\tau_{n+1}} - E_s\Big)\mathrm{d}t - \sum_{k=1}^d \widetilde{q}_s^{\tau_{n+1}} \mathrm{d}B_s - \widetilde{z}_s^{\tau_{n+1}} \cdot \mathrm{d}Q_s\Big] \\
&= \int_t^{\tau_{n_t^\omega+1}} \Big[\Big(C_s^{\mathrm{T}} p_s^\infty + \sum_{k=1}^d D_s^{k\,\mathrm{T}} q_s^{k,\infty} - E_s\Big)\mathrm{d}s - \sum_{k=1}^d q_s^{k,\infty} \mathrm{d}B_s - z_s^\infty \cdot \mathrm{d}Q_s\Big] \\
&\quad + \int_{\tau_{n_T^\omega}}^T \Big[\Big(C_s^{\mathrm{T}} p_s^\infty + \sum_{k=1}^d D_s^{k\,\mathrm{T}} q_s^{k,\infty} - E_s\Big)\mathrm{d}s - \sum_{k=1}^d q_s^{k,\infty} \mathrm{d}B_s - z_s^\infty \cdot \mathrm{d}Q_s\Big] \\
&\quad + \sum_{n=n_t^\omega+1}^{n_T^\omega-1} \int_{\tau_n}^{\tau_{n+1}} \Big[\Big(C_s^{\mathrm{T}} p_s^\infty + \sum_{k=1}^d D_s^{k\,\mathrm{T}} q_s^{k,\infty} - E_s\Big)\mathrm{d}t - \sum_{k=1}^d q_s^{k,\infty} \mathrm{d}B_s - z_s^\infty \cdot \mathrm{d}Q_s\Big] \\
&= \int_t^T \Big[\Big(C_s^{\mathrm{T}} p_s^\infty + \sum_{k=1}^d D_s^{k\,\mathrm{T}} q_s^{k,\infty} - E_s\Big)\mathrm{d}t - \sum_{k=1}^d q_s^{k,\infty} \mathrm{d}B_s - z_s^\infty \cdot \mathrm{d}Q_s\Big].
\end{aligned}$$

This completes the proof of existence.

(2) Uniqueness. Let $(p_t^\infty, q_t^{1,\infty}, \cdots, q_t^{d,\infty}, z_t^\infty)_{t\geq 0}$ and $({p'}_t^\infty, {q'}_t^{1,\infty}, \cdots, {q'}_t^{d,\infty}, {z'}_t^\infty)_{t\geq 0}$ be the solution of (3.42). Then for all $T>0$, $(p_t^\infty, q_t^{1,\infty}, \cdots, q_t^{d,\infty}, z_t^\infty)_{t\in[0,T]}$ and $({p'}_t^\infty, {q'}_t^{1,\infty}, \cdots, {q'}_t^{d,\infty}, {z'}_t^\infty)_{t\in[0,T]}$ are the solutions of truncated adjoint equation (3.46) with terminal conditions $\nu = p_T^\infty$ and $\nu' = {p'}_T^\infty$, respectively.

We fix $T_0 > 0$ now, and let $\rho \in L^2(\Omega \times [0,\infty); \mathbb{R}^n)$ has support in the finite interval $[0, T_0]$ on $\mathbb{R}^+$ ($\rho_s \equiv 0, s > T_0$). Recalling duality relation (3.51), it follows that

$$\mathbb{E}\int_t^T \langle \mathcal{Y}_s^{t,\eta,\rho}, -E_s\rangle \mathrm{d}s + \mathbb{E}\langle p_T^\infty, \mathcal{Y}_T^{t,\eta,\rho}\rangle = \sum_{k=1}^d \mathbb{E}\int_t^T \langle q_s^\infty, \rho_s^k\rangle \mathrm{d}s + \mathbb{E}\langle p_t^\infty, \eta\rangle, \quad 0 \leq t \leq T \tag{3.74}$$

as well as

$$\begin{aligned}
&\mathbb{E}\int_t^T \langle \mathcal{Y}_s^{t,\eta,\rho}, -E_s\rangle \mathrm{d}s + \mathbb{E}\langle {p'}_T^\infty, \mathcal{Y}_T^{t,\eta,\rho}\rangle \\
&= \sum_{k=1}^d \mathbb{E}\int_t^T \langle {q'}_s^\infty, \rho_s^k\rangle \mathrm{d}s + \mathbb{E}\langle {p'}_t^\infty, \eta\rangle, \quad 0 \leq t \leq T.
\end{aligned} \tag{3.75}$$

Define $(\widehat{p}_t^\infty, \widehat{q}_t^{1,\infty}, \cdots, \widehat{q}_t^{d,\infty}, \widehat{z}_t^\infty)_{t\geq=0} = (p_t^\infty - {p'}_t^\infty, q_t^{1,\infty} - {q'}_t^{1,\infty}, \cdots, q_t^{d,\infty} - {q'}_t^{d,\infty}, z_t^\infty - {z'}_t^\infty)_{t\geq 0}$, then we have

$$\mathbb{E}\langle \widehat{p}_T^\infty, \mathcal{Y}_T^{t,\eta,\rho}\rangle = \sum_{k=1}^d \mathbb{E}\int_t^T \langle \widehat{q}_s^\infty, \rho_s^k\rangle \mathrm{d}s + \mathbb{E}\langle \widehat{p}_t^\infty, \eta\rangle, \quad 0 \leq t \leq T. \tag{3.76}$$

Since $\sup\limits_{s\geq 0} \mathbb{E}|p_s|^2 < \infty$ and recalling (3.50), it follows that

$$\begin{aligned}&\sum_{k=1}^d \mathbb{E}\int_t^{T_0} \langle \widehat{q}_s^\infty, \rho_s^k\rangle \mathrm{d}s + \mathbb{E}\langle \widehat{p}_t^\infty, \eta\rangle \\ &= \lim_{T\to\infty} \Big[\sum_{k=1}^d \mathbb{E}\int_t^T \langle \widehat{q}_s^\infty, \rho_s^k\rangle \mathrm{d}s + \mathbb{E}\langle \widehat{p}_t^\infty, \eta\rangle\Big] \\ &= \lim_{T\to\infty} \mathbb{E}\langle \widehat{p}_T^\infty, \mathcal{Y}_T^{t,\eta,\rho}\rangle = 0. \end{aligned}\tag{3.77}$$

Due to the arbitrariness of t, T_0, ρ and η, we have

$$\widehat{p}_t^\infty = 0, \quad \mathbb{P}\text{-a.s.}$$

This completes the proof of uniqueness.

4 Necessary Stochastic Maximum Principle

In this section, we give two versions of the SMP which are the necessary conditions of the optimal controls. The first is based on the well-posedness result for the infinite horizon BSDE. The second is written in terms of the family of truncated backward equations introduced in the previous section. The Hamiltonian associated to the system is

$$H(x,u,e,p,q) := \langle b(x,u,e), p\rangle + \sum_{k=1}^d \langle \sigma^k(x,u,e), q^k\rangle - f(x,u,e). \tag{4.1}$$

Theorem 4.1 (Necessary SMP infinite horizon case) *Let* $(\overline{X}, \overline{u})$ *be the optimal pairs,* $(\overline{p}_t^\infty, \overline{q}_t^\infty, \overline{z}_t^\infty)$ *be the solution of the adjoint equation* (3.42)*. Then under* (A1)–(A4) *the following variational inequality holds*:

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle H_u(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{u}_t - u_t\rangle \mathrm{d}t \geq 0, \quad \forall u(\cdot) \in U_{ad}. \tag{4.2}$$

Proof Let $v(\cdot) = u(\cdot) - \overline{u}(\cdot)$, and $t = 0$, $\eta = 0$, $\nu = \overline{p}_T^\infty$, $\gamma = b_u(\overline{X}, \overline{u}, \alpha)v$, $\rho = \sigma_u(\overline{X}, \overline{u}, \alpha)v$, then affine equation (3.49) becomes variational equation (3.5). It follows from the duality relation (3.51) that

$$\begin{aligned}&\mathbb{E}\int_0^T \langle \overline{p}_s^\infty, b_u(\overline{X}_s, \overline{u}_s, \alpha_s)v_s\rangle \mathrm{d}s + \sum_{k=1}^d \mathbb{E}\int_0^T \langle \overline{q}_s^{k,\infty}, \sigma_u^k(\overline{X}_s, \overline{u}_s, \alpha_s)v_s\rangle \mathrm{d}s \\ &= \mathbb{E}\int_0^T \langle Y_s, -f_x(X_s, u_s, \alpha_s)\rangle \mathrm{d}s + \mathbb{E}\langle \overline{p}_T^\infty, Y_T\rangle. \end{aligned}\tag{4.3}$$

By Lemma 3.3, we have

$$
\begin{aligned}
0 &\le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [\langle f_x(\overline{X}_t,\overline{u}_t,\alpha_t),Y_t\rangle + \langle f_u(\overline{X}_t,\overline{u}_t,\alpha_t),v_t\rangle]\mathrm{d}t \\
&= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\langle \overline{p}_T^\infty, Y_T\rangle - \int_0^T (\langle \overline{p}_t^\infty, b_u(\overline{X}_t,\overline{u}_t,\alpha_t)v_t\rangle \\
&\quad + \sum_{k=1}^d \langle \overline{q}_t^{k,\infty}, \sigma_u^k(\overline{X}_t,\overline{u}_t,\alpha_t)v_t\rangle - \langle f_u(\overline{X}_t,\overline{u}_t,\alpha_t),v_t\rangle)\mathrm{d}t\Big] \\
&\le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\langle \overline{p}_T^\infty, Y_T\rangle + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle -H_u(\overline{X}_t,\overline{u}_t,\alpha_t,\overline{p}_t^\infty,\overline{q}_t^\infty), u_t-\overline{u}_t\rangle\mathrm{d}t. \qquad (4.4)
\end{aligned}
$$

Recalling that $\sup\limits_{t\ge 0}\mathbb{E}|Y_t|^2<\infty$ and $\sup\limits_{t\ge 0}\mathbb{E}|\overline{p}_t^\infty|^2<\infty$, then we can conclude (4.2).

Theorem 4.2 (Necessary SMP) *Let assumptions* (A1)–(A4) *hold. Let* $(\overline{X},\overline{u})$ *be the optimal pair,* $(\overline{p}_t^T,\overline{q}_t^T,\overline{z}_t^T)$ *be the solution of truncated adjoint equation* (3.46) *on* $[0,T]$ *with terminal conditions* $\nu=0$. *Then the following variational inequality holds*:

$$
\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle H_u(\overline{X}_t,\overline{u}_t,\alpha_t,\overline{p}_t^T,\overline{q}_t^T),\overline{u}_t-u_t\rangle\mathrm{d}t \ge 0, \quad \forall u(\cdot)\in U_{ad}. \qquad (4.5)
$$

Proof Let $v(\cdot)=u(\cdot)-\overline{u}(\cdot)$, and $t=0$, $\eta=0$, $\gamma=b_u(\overline{X},\overline{u},\alpha)v$, $\rho=\sigma_u(\overline{X},\overline{u},\alpha)v$, then affine equation (3.49) becomes variational equation (3.5). Thanks to duality relation (3.51), it follows that

$$
\begin{aligned}
&\mathbb{E}\int_0^T \langle \overline{p}_s^T, b_u(\overline{X}_s,\overline{u}_s,\alpha_s)v_s\rangle\mathrm{d}s + \sum_{k=1}^d \mathbb{E}\int_0^T \langle \overline{q}_s^{k,T}, \sigma_u^k(\overline{X}_s,\overline{u}_s,\alpha_s)v_s\rangle\mathrm{d}s \\
&= \mathbb{E}\int_0^T \langle Y_s, -f_x(X_s,u_s,\alpha_s)\rangle\mathrm{d}s. \qquad (4.6)
\end{aligned}
$$

By Lemma 3.3, we have

$$
\begin{aligned}
0 &\le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [\langle f_x(\overline{X}_t,\overline{u}_t,\alpha_t),Y_t\rangle + \langle f_u(\overline{X}_t,\overline{u}_t,\alpha_t),v_t\rangle]\mathrm{d}t \\
&= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T -\Big(\langle \overline{p}_t^T, b_u(\overline{X}_t,\overline{u}_t,\alpha_t)v_t\rangle \\
&\quad + \sum_{k=1}^d \langle \overline{q}_t^{k,T}, \sigma_u^k(\overline{X}_t,\overline{u}_t,\alpha_t)v_t\rangle - \langle f_u(\overline{X}_t,\overline{u}_t,\alpha_t),v_t\rangle\Big)\mathrm{d}t \\
&\le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle -H_u(\overline{X}_t,\overline{u}_t,\alpha_t,\overline{p}_t^T,\overline{q}_t^T), u_t-\overline{u}_t\rangle\mathrm{d}t. \qquad (4.7)
\end{aligned}
$$

Then we can conclude (4.5).

5 Sufficient Stochastic Maximum Principle

In this section, we give a version of the SMP which is the sufficient condition of finding optimal control under some additional concavity assumption on the Hamiltonian function H.

Theorem 5.1 (Sufficient SMP) *Let assumptions* (A1)–(A4) *hold. Let* $\overline{u}(\cdot)$ *be an admissible control,* $\overline{X}(\cdot)$ *be the related state process,* $(\overline{p}_t^\infty, \overline{q}_t^\infty, \overline{z}_t^\infty)$ *be the solution of related adjoint equation* (3.42). *Furthermore, for all* (t, e) *in* $\mathbb{R}^+ \times I$, *let mapping* $\mathcal{H}(x, u) = H(x, u, e, \overline{p}_t^\infty, \overline{q}_t^\infty)$ *be a concave function* $\mathbb{P} \otimes \mathrm{d}t$ *-a.s. in* $\Omega \times [0, \infty)$, *and the following minimality condition holds*:

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle H_u(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{u}_t - u_t\rangle \mathrm{d}t \geq 0, \quad \forall u(\cdot) \in U_{ad}. \tag{5.1}$$

Then $\overline{u}(\cdot)$ *is the optimal control.*

Proof We only need to prove for all $u(\cdot) \in U_{ad}$, $J(\overline{u}(\cdot)) - J(u(\cdot)) \leq 0$. Let $u(\cdot)$ be an arbitrary admissible control, $X(\cdot)$ be the related state process, $(p_t^\infty, q_t^\infty, z_t^\infty)$ be the solution of related adjoint equation (3.42).

$$\begin{aligned}
&J(\overline{u}(\cdot)) - J(u(\cdot)) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [f(\overline{X}_t, \overline{u}_t, \alpha_t) - f(X_t, u_t, \alpha_t)]\mathrm{d}t \\
&= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \{-[H(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty)) - H(X_t, u_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty)] \\
&\quad + [\langle b(\overline{X}_t, \overline{u}_t, \alpha_t) - b(X_t, u_t, \alpha_t), \overline{p}_t^\infty\rangle] + [\langle \sigma(\overline{X}_t, \overline{u}_t, \alpha_t) - \sigma(X_t, u_t, \alpha_t), \overline{p}_t^\infty\rangle]\}\mathrm{d}t \\
&=: \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [-\mathrm{I} + \mathrm{II} + \mathrm{III}]\mathrm{d}t.
\end{aligned} \tag{5.2}$$

Thanks to the concavity of $\mathcal{H}$, it follows that

$$\mathrm{I} \geq \langle H_x(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{X}_t - X_t\rangle + \langle H_u(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{u}_t - u_t\rangle. \tag{5.3}$$

Then we obtain

$$\begin{aligned}
&J(\overline{u}(\cdot)) - J(u(\cdot)) \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [\langle -H_x(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{X}_t - X_t\rangle + \mathrm{II} + \mathrm{III}]\mathrm{d}t \\
&\quad + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle -H_u(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{u}_t - u_t\rangle \mathrm{d}t.
\end{aligned} \tag{5.4}$$

Recalling the estimate of X, $\overline{X}$ and $\overline{p}_t^\infty$, we have

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\langle \overline{p}_T^\infty, \overline{X}_T - X_T\rangle = 0, \tag{5.5}$$

which implies

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \mathrm{d}\langle \overline{p}_t^\infty, \overline{X}_t - X_t\rangle = 0. \tag{5.6}$$

Finally, we apply Itô's formula to $\langle \overline{p}_t^\infty, \overline{X}_t - X_t\rangle$,

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [\langle -H_x(\overline{X}_t, \overline{u}_t, \alpha_t, \overline{p}_t^\infty, \overline{q}_t^\infty), \overline{X}_t - X_t\rangle + \mathrm{II} + \mathrm{III}]\mathrm{d}t \\
&= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \mathrm{d}\langle \overline{p}_t^\infty, \overline{X}_t - X_t\rangle = 0.
\end{aligned} \tag{5.7}$$

This completes the proof of Theorem 5.1.

6 Application

We illustrate the application of the maximum principle by virtue of the following example. The state equation is

$$\begin{cases} \mathrm{d}x_t = (A(\alpha_t)x_t - u_t)\mathrm{d}t - C(\alpha_t)x_t\mathrm{d}B_t, \\ x_0 = \xi, \end{cases} \tag{6.1}$$

where u_t takes values in $U = \left[0, \frac{1}{2}\pi\right]$, $\xi, C(e) > 0, A(e) + 3C^2(e) < 0$, for every $e \in I$. The objective of optimal control problem is to minimize the following cost functional

$$J(u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^T - \sin(u_t)\mathrm{d}t\Big]. \tag{6.2}$$

We first give the Hamiltonian function

$$H(x, u, e, p, q) = \langle (A(e)x - u), p\rangle - \langle C(e)x, q\rangle + \sin(u). \tag{6.3}$$

Then, we have

$$H_u = -p + \cos(u), \tag{6.4}$$

$$H_x = A(e)p - C(e)q. \tag{6.5}$$

And the adjoint equation is

$$\mathrm{d}p_t = (-A(\alpha_t)p_t + C(\alpha_t)q_t)\mathrm{d}t + q_t\mathrm{d}B_t + z_t\mathrm{d}Q_t. \tag{6.6}$$

It is reasonable to guess that $q_t = z_t \equiv 0$. Then, we can reformulate the adjoint equation as

$$\mathrm{d}p_t = (-A(\alpha_t)p_t)\mathrm{d}t, \tag{6.7}$$

which can be explicitly solved as

$$p_t = p_0\mathrm{e}^{-\int_0^t A(\alpha_s)\mathrm{d}s} \tag{6.8}$$

for some constant p_0. By the maximum condition, we obtain a candidate of optimal control

$$\overline{u}_t = \arccos(p_t). \tag{6.9}$$

The next goal is to determine p_0. Substituting $\overline{u}_t$ back into (6.1), we have

$$\begin{cases} \mathrm{d}\overline{x}_t = (A(\alpha_t)\overline{x}_t - \arccos(p_t))\mathrm{d}t - C(\alpha_t)\overline{x}_t\mathrm{d}B_t, \\ \overline{x}_0 = \xi. \end{cases} \tag{6.10}$$

Solving this linear SDE yields that

$$\overline{x}_t = \xi Z_t^{-1} - Z_t^{-1}\int_0^t Z_s \arccos(p_s)\mathrm{d}s, \tag{6.11}$$

where the integrating factor Z_t is of the form as

$$Z_t = \exp\Big(\int_0^t \Big(\frac{1}{2}C(\alpha_s)^2 - A(\alpha_s)\Big)\mathrm{d}s - \int_0^t C(\alpha_s)\mathrm{d}B_s\Big). \tag{6.12}$$

It follows from (6.11) that

$$\mathbb{E}[x_t p_t] = \mathbb{E}\Big[\xi Z_t^{-1} p_t - Z_t^{-1} p_t \int_0^t Z_s \arccos(p_s) \mathrm{d}s\Big]. \tag{6.13}$$

By (6.8) and the Definition 3.1, we can get $p_0 \equiv 0$ which implies $p_t \equiv 0$. Moreover, by the sufficient SMP, we can verify that the candidate $\overline{u}_t = \arccos(p_t) \equiv \frac{\pi}{2}$ is indeed an optimal control.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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