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Exact-Approximate Controllability of the Abstract Thermoelasticity of Type I

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Abstract In this paper, the author considers a general control problem about the system of thermoelasticity of type I. By introducing some unique continuation property of the corresponding adjoint system and a suitable observability inequality for an elastic equation, using compact decoupling technique and variational approach, the exact-approximate controllability of the abstract thermoelasticity of type I is obtained. Finally, the author applies her abstract result to the exact-approximate controllability of the linear system of thermoelasticity.

Keywords Thermoelasticity, Exact-approximate controllability, Compact decoupling
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1 Introduction

The abstract thermoelasticity of type I we considered is as follows

$$\begin{cases} u_{tt} + A_1 u + B\theta = Ff, & t \in (0, T], \\ \theta_t + A_2\theta - B^* u_t = 0, & t \in (0, T], \\ u(0) = u_0, \quad u_t(0) = u_1, \quad \theta(0) = \theta_0, \end{cases} \tag{1.1}$$

where u denotes the displacement, θ is the temperature and f is the control. Moreover, (u_0, u_1, θ_0) belongs to some Hilbert space and A_1, A_2, B, B^*, F will be showed later.

Due to the regularizing effect of the heat equation that the temperature $\theta(\cdot)$ cannot satisfy exactly controllable property. Then the aim of this paper is to study the exact-approximate controllability of system (1.1), i.e., for any given time $T > 0$, try to find a control $f(\cdot)$ such that the displacement $u(\cdot)$ is exactly controllable and the temperature $\theta(\cdot)$ is approximately controllable, respectively (see a precise definition in Section 2).

For the concrete system of (1.1), Zuazua [14] combined decoupling result and multiplier techniques to obtain exact-approximate controllability with a control supported in a neighborhood of the boundary of a domain. Moreover, in [4], for thermoelastic plates, Teresa and

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Zuazua [14] proved similar results hold. Other related results can be found in [2, 5, 9] and the works cited therein.

The control problems about various abstract systems have been widely discussed by many authors, see [3, 10–13] and references cited therein. Especially, Zhang in [12] considered the exact controllability of an abstract semilinear control system by making some interesting assumptions.

However, for the above abstract system, as far as I know, there are no papers concerning its control problems. Only Ait Ben Hassi et al. in [1] and Henry et al. [6] analyzed its compact decoupling by different ways. In this note, with Arzela-Ascoli lemma (see [8]), we deduce a little stronger compact decoupling result (see Corollary 3.1). Then, motivated by [1, 12, 14], we study the controllability of system (1.1) with control in the elastic component. Using variational techniques, combining compact result Corollary 3.1, with some assumptions, we derive system (1.1) is exact-approximately controllable. This result generalizes the similar result in [14].

The rest of the paper is organized as follows. In Section 2, we show the main result of this paper. Some preliminaries are presented in Section 3. Section 4 is devoted to proving our main result. Finally, we will apply the abstract result to a concrete system in Section 5.

2 Statement of the Main Result

Throughout this paper, let $|\cdot|_H$ and $\langle\cdot,\cdot\rangle_H$ denote the norm and inner product of a Hilbert space H, respectively. To begin, we make the following assumptions.

(A1) H_1 and H_2 are two Hilbert spaces. Operators $A_1 : D(A_1) \subset H_1 \to H_1$ and $A_2 : D(A_2) \subset H_2 \to H_2$ are self-adjoint positive, and $B : D(B) \subset H_2 \to H_1$ is a closed operator with adjoint B^* such that $D(A_2^{\frac{1}{2}}) \subset D(B)$, $D(A_1^{\frac{1}{2}}) \subset D(B^*)$, and the operator $A_2^{-1}B^*A_1^{\frac{1}{2}}$ can be extended to a bounded linear operator from H_1 to H_2. Moreover, $A_1^{-\frac{1}{2}}BA_2^{-1}$ is a compact operator from H_2 to H_1.

(A2) Similar to [12], assume the embeddings $V \triangleq D(A_1)^{\frac{1}{2}} \hookrightarrow H_1$ and $Y \triangleq D(A_2)^{\frac{1}{2}} \hookrightarrow H_2$ are compact. Next, we identify H_1 and H_1', and H_2 and H_2', respectively. Further, suppose that $V \subset H_1 \equiv H_1' \subset V'$ and $Y \subset H_2 \equiv H_2' \subset Y'$ be two Gelfand triples, where H_1', H_2', V' and Y' are the dual spaces of H_1, H_2, V and Y, respectively, i.e., embeddings $V \triangleq D(A_1)^{\frac{1}{2}} \hookrightarrow H_1$ and $Y \triangleq D(A_2)^{\frac{1}{2}} \hookrightarrow H_2$ are both continuous and dense, and the duality pairing $\langle\cdot,\cdot\rangle_{V,V'}, \langle\cdot,\cdot\rangle_{Y,Y'}$ and the inner product $\langle\cdot,\cdot\rangle_{H_1}, \langle\cdot,\cdot\rangle_{H_2}$ are compatible in the sense that

$$\langle v, a_1\rangle_{V,V'} = \langle v, a_1\rangle_{H_1}, \quad \forall v \in V,\ a_1 \in H_1, \tag{2.1}$$

$$\langle y, a_2\rangle_{Y,Y'} = \langle y, a_2\rangle_{H_2}, \quad \forall y \in Y,\ a_2 \in H_2. \tag{2.2}$$

(A3) Set

$$X = V \times H_1 \times H_2, \tag{2.3}$$

and X is a Hilbert space with the following norm

$$|(b_1, b_2, b_3)|_X = \sqrt{|A_1^{\frac{1}{2}}b_1|_{H_1}^2 + |b_2|_{H_1}^2 + |b_3|_{H_2}^2}, \quad \forall (b_1, b_2, b_3) \in X. \tag{2.4}$$

Let U be another Hilbert space and $F \in L(U, X)$. Next, assume X and U are the state space and controllability space of system (1.1), respectively.

(A4) For any fixed time T, there exists a constant C such that

$$|\vartheta^0|_V^2 + |\vartheta^1|_{H_1}^2 \le C \int_0^T |F^* \vartheta_t|_U^2 \mathrm{d}t, \quad \forall (\vartheta^0, \vartheta^1) \in V \times H_1, \tag{2.5}$$

where ϑ satisfies equation

$$\begin{cases} \vartheta_{tt} + A_1 \vartheta - B A_2^{-1} B^* \vartheta_t = 0, & t \in [0, T), \\ \vartheta(T) = \vartheta^0, \quad \vartheta_t(T) = \vartheta^1. \end{cases} \tag{2.6}$$

And define F^* to be the adjoint operator of F.

(A5) Denote

$$\overline{X} \triangleq H_1' \times V' \times H_2' = H_1 \times V' \times H_2. \tag{2.7}$$

Let $T > 0$ be given. Assume that, for any $(\varphi^0, \varphi^1, \psi^0) \in \overline{X}$, we have

$$F^* \varphi = 0, \quad \forall t \in (0, T) \Rightarrow (\varphi, \psi) \equiv (0, 0), \quad \forall t \in (0, T), \tag{2.8}$$

where (φ, ψ) satisfies the equation as follows:

$$\begin{cases} \varphi_{tt} + A_1 \varphi + B \psi_t = 0, & t \in [0, T), \\ -\psi_t + A_2 \psi + B^* \varphi = 0, & t \in [0, T), \\ \varphi(T) = \varphi^0, \quad \varphi_t(T) = \varphi^1, \quad \psi(T) = \psi^0. \end{cases} \tag{2.9}$$

Now, we introduce the following definition.

Definition 2.1 *Fix a state space X and a control time T. If for any initial data $(u_0, u_1, \theta_0) \in X$, any final data $(z_0, z_1, \zeta_0) \in X$ and any $\varepsilon > 0$, there exists a control $f(\cdot) \in L^2(0, T; U)$ such that the mild solution of system* (1.1) *satisfies*

$$\begin{cases} u(T) = z_0, \quad u_t(T) = z_1, \\ |\theta(T) - \zeta_0|_{H_2} \le \varepsilon. \end{cases} \tag{2.10}$$

We will say the system (1.1) *is exact-approximately controllable.*

Then, we have the following main result.

Theorem 2.1 *Suppose* (A1)–(A5) *hold, then the system* (1.1) *is exact-approximately controllable.*

3 Some Preliminaries

In this section, we present some preliminary results, which will play a key role of the proof of the main result.

Proposition 3.1 *Let* (A1)–(A5) *hold and assume* $T > 0$. *For any bounded set* K *of* H_2, *if the final data* $(\varphi^0, \varphi^1, \psi^0) \in \overline{X}$ *of system* (2.9) *satisfies* $|(\varphi^0, \varphi^1 + B\psi^0)|_{H_1 \times V'} \geq 1,\ \psi^0 \in K$, *then there exists a constant* $\delta = \delta(K)$ *such that*

$$\delta \leq \int_0^T |F^*\varphi|_U^2 \mathrm{d}t. \tag{3.1}$$

Here, (φ, ψ) *is the mild solution of system* (2.9).

Put

$$w(t) = -\int_t^T \varphi(s)\mathrm{d}s + \chi, \tag{3.2}$$

where χ satisfies

$$-A_1\chi = \varphi^1 + B\psi^0. \tag{3.3}$$

Therefore, by (2.9) and (3.2), we obtain that

$$\begin{cases} w_{tt} + A_1 w + B\psi = 0, & t \in [0, T), \\ -\psi_t + A_2\psi + B^* w_t = 0, & t \in [0, T), \\ w(T) = \chi, \quad w_t(T) = \varphi^0, \quad \psi(T) = \psi^0. \end{cases} \tag{3.4}$$

Then, Proposition 3.1 is equivalent to the following one.

Proposition 3.2 *Suppose* (A1)–(A5) *hold. For any bounded set* K *of* H_2, *if* $(\chi, \varphi^0, \psi^0) \in X$ *satisfies* $|(\chi, \varphi^0)|_{V \times H_1} \geq 1,\ \psi^0 \in K$, *then there exists a constant* $\delta = \delta(K)$ *such that*

$$\delta \leq \int_0^T |F^* w_t|_U^2 \mathrm{d}t, \tag{3.5}$$

where (w, ψ) *satisfies* (3.4).

Next, to prove Proposition 3.2, first we introduce the decoupled system of (3.4),

$$\begin{cases} \overline{w}_{tt} + A_1\overline{w} - BA_2^{-1}B^*\overline{w}_t = 0, & t \in [0, T), \\ -\overline{\psi}_t + A_2\overline{\psi} + B^*\overline{w}_t = 0, & t \in [0, T), \\ \overline{w}(T) = \chi, \quad \overline{w}_t(T) = \varphi^0, \quad \overline{\psi}(T) = \psi^0. \end{cases} \tag{3.6}$$

With (3.6), we know $\overline{w}$ satisfies the following equation

$$\begin{cases} \overline{w}_{tt} + A_1\overline{w} - BA_2^{-1}B^*\overline{w}_t = 0, \quad t \in [0, T), \\ \overline{w}(T) = \chi, \quad \overline{w}_t(T) = \varphi^0. \end{cases} \tag{3.7}$$

Therefore, by (A4), we get

$$|\chi|_V^2 + |\varphi^0|_{H_1}^2 \leq C \int_0^T |F^*\overline{w}_t|_U^2 \mathrm{d}t. \tag{3.8}$$

Set

$$(\widetilde{w}, \widetilde{\psi}) = (w, \psi) - (\overline{w}, \overline{\psi}). \tag{3.9}$$

Then, taking into account (3.4) and (3.6), we obtain that

$$\begin{cases} \widetilde{w}_{tt} + A_1\widetilde{w} = -BA_2^{-1}B^*\overline{w}_t - B\psi, & t \in [0,T), \\ -\widetilde{\psi}_t + A_2\widetilde{\psi} + B^*\widetilde{w}_t = 0, & t \in [0,T), \\ \widetilde{w}(T) = 0, \quad \widetilde{w}_t(T) = 0, \quad \widetilde{\psi}(T) = 0. \end{cases} \tag{3.10}$$

Further, combining (3.8) and (3.9), it follows

$$|\chi|_V^2 + |\varphi^0|_{H_1}^2 \le C\int_0^T (|F^*w_t|_U^2 + |F^*\widetilde{w}_t|_U^2)\mathrm{d}t. \tag{3.11}$$

Next, we recall some known results.

In [6], one could get the following lemma.

Lemma 3.1 *The corresponding operators of systems* (3.4) *and* (3.6) *generate contractive* C_0*-semigroups* (*denoted by* $\{S(t)\}_{t\ge 0}$ *and* $\{S_d(t)\}_{t\ge 0}$*, respectively*) *on the Hilbert space* X*.*

On the other hand, the C_0-semigroups $\{S(t)\}_{t\ge 0}$ and $\{S_d(t)\}_{t\ge 0}$ have property as follows (see [1]).

Lemma 3.2 *For any* $t \ge 0$*, operator* $S(t) - S_d(t) : X \to X$ *is compact.*

Similar to [7], we also need the following infinite dimensional version of Arzela-Ascoli lemma (see [8]).

Lemma 3.3 *Let* $T > 0$*. Let* $\mathcal{W} \subset C([0,T];X)$ *such that for each* $t \in [0,T]$*, the set* $\{h(t) \mid h \in \mathcal{W}\}$ *is relatively compact in* X*. Moreover,* $\mathcal{W}$ *is uniformly bounded and equicontinuous, i.e.,*

$$\sup_{h\in\mathcal{W},\ t\in[0,T]} |h(t)|_X < \infty,$$

and for any $\varepsilon > 0$*, there exists a* $\delta = \delta(\varepsilon) > 0$*, such that*

$$|h(t_1) - h(t_2)|_X < \varepsilon, \quad \forall t_1, t_2 \in [0,T],\ |t_1 - t_2| < \delta,\ h \in \mathcal{W}.$$

Then, there exist a sequence $h_k \in \mathcal{W}$ *and* $h_0 \in C([0,T];X)$*, such that*

$$\lim_{k\to\infty} |h_k(\cdot) - h_0(\cdot)|_{C([0,T];X)} = 0.$$

Corollary 3.1 *For any given* $T > 0$*, operator* $S(\cdot) - S_d(\cdot) : X \to C([0,T];X)$ *is compact.*

Proof With [1, Theorem 2.4], we know that the map $t \mapsto S(t) - S_d(t)$ is norm continuous on $[0,\infty)$. Thus, combining Lemmas 3.2–3.3, we get Corollary 3.1.

We start to prove Proposition 3.2. We will follow the strategy in [14] to argue by contradiction.

Proof of Proposition 3.2 First, suppose Proposition 3.2 does not hold. Thus, there exists a bounded set K of H_2 and a sequence of initial data $(\chi_j, \varphi_j^0, \psi_j^0) \in X$ such that

$$\int_0^T |F^*w_{j,t}|_U^2 \mathrm{d}t \to 0 \quad \text{as } j \to \infty, \tag{3.12}$$

where $(\chi_j, \varphi_j^0, \psi_j^0)$ satisfies

$$|(\chi_j, \varphi_j^0)|_{V\times H_1} \geq 1, \quad \psi_j^0 \in K. \tag{3.13}$$

Therefore, by (3.11)–(3.13), it is easy to deduce that

$$\liminf_{j\to\infty} \int_0^T |F^* \widetilde{w}_{j,t}|_U^2 \mathrm{d}t > 0. \tag{3.14}$$

Next, we put

$$(\widehat{\chi}_j, \widehat{\varphi}_j^0, \widehat{\psi}_j^0) = \frac{(\chi_j, \varphi_j^0, \psi_j^0)}{|F^* \widetilde{w}_{j,t}|_{L^2(0,T;U)}}. \tag{3.15}$$

Moreover, assume that $(\widehat{w}_j, \widehat{\varphi}_j)$ and $(\widehat{\widetilde{w}}_j, \widehat{\widetilde{\psi}}_j)$ are the corresponding solutions of systems (3.4) and (3.10), respectively.

Hence, we have

$$\int_0^T |F^* \widehat{\widetilde{w}}_{j,t}|_U^2 \mathrm{d}t = 1, \quad \forall j \geq 1 \tag{3.16}$$

and

$$\int_0^T |F^* \widehat{w}_{j,t}|_U^2 \mathrm{d}t \to 0 \quad \text{ as } j \to \infty. \tag{3.17}$$

Combining (3.11) and (3.16)–(3.17), it follows

$$|\widehat{\chi}_j|_V^2 + |\widehat{\varphi}_j^0|_{H_1}^2 \leq C. \tag{3.18}$$

Note $\psi_j^0 \in K$, then we get $|\widehat{\psi}_j^0|_{H_2} \leq C$. And with (3.18), one could obtain (for convenience, the extracted subsequence is denoted by the original sequence)

$$\begin{cases} (\widehat{\chi}_j, \widehat{\varphi}_j^0) \rightharpoonup (\widehat{\chi}, \widehat{\varphi}^0) & \text{in } V \times H_1 \text{ as } j \to \infty, \\ \widehat{\psi}_j^0 \rightharpoonup \widehat{\psi}^0 & \text{in } H_2 \text{ as } j \to \infty \end{cases} \tag{3.19}$$

and

$$\begin{cases} \widehat{w}_{j,t} \rightharpoonup \widehat{w}_t & \text{in } L^2(0,T;H_1) \text{ as } j \to \infty, \\ \widehat{\widetilde{w}}_{j,t} \rightharpoonup \widehat{\widetilde{w}}_t & \text{in } L^2(0,T;H_1) \text{ as } j \to \infty, \end{cases} \tag{3.20}$$

where $(\widehat{\varphi}, \widehat{\psi})$, $(\widehat{w}, \widehat{\psi})$ and $(\widehat{\widetilde{w}}, \widehat{\widetilde{\psi}})$ are the solutions of systems (2.9), (3.4) and (3.10) corresponding to the limit initial data, respectively.

Moreover, with Corollary 3.1, we know $(\widehat{\widetilde{w}}_{j,t})$ is relatively compact in $C([0,T];H_1)$, thus

$$\widehat{\widetilde{w}}_{j,t} \to \widehat{\widetilde{w}}_t \quad \text{in } L^2(0,T;H_1) \text{ as } j \to \infty. \tag{3.21}$$

Next, by (3.16) and (3.21), we conclude

$$\int_0^T |F^* \widehat{\widetilde{w}}_t|_U^2 \mathrm{d}t = 1. \tag{3.22}$$

Combining (3.17) and (3.20), we arrive at

$$F^* \widehat{\varphi} = F^* \widehat{w}_t = 0, \quad \forall t \in (0,T). \tag{3.23}$$

Now, taking into account (A5) and (3.23), it holds that

$$(\widehat{\varphi}, \widehat{\psi}) \equiv (0,0), \quad \forall t \in (0,T). \tag{3.24}$$

Furthermore, we get

$$(\widehat{\varphi}^0, \widehat{\varphi}^1, \widehat{\psi}^0) \equiv (0,0,0). \tag{3.25}$$

Finally, recalling systems (2.9), (3.4), (3.6) and combining (3.24)–(3.25), we deduce

$$\widehat{\widehat{w}} \equiv 0, \quad \forall t \in (0,T), \tag{3.26}$$

and this contradicts (3.22). Then, we know Proposition 3.2 holds.

Lemma 3.4 *Assume* (A1)–(A5) *hold. For any given* $(z_0, z_1, \zeta_0) \in X$ *(see* (2.3) *for* X*) and any* $\varepsilon > 0$*, set*

$$\begin{aligned} J(\xi^0, \xi^1, \eta^0) = \frac{1}{2}\int_0^T |F^*\xi|_U^2 \mathrm{d}t - \langle z_1, \xi^0\rangle_{H_1} + \langle z_0, \xi^1\rangle_{V,V'} \\ + \varepsilon|\eta^0|_{H_2} - \langle \zeta_0 - B^* z_0, \eta_j^0\rangle_{H_2}, \quad \forall (\xi^0, \xi^1, \eta^0) \in \overline{X}, \end{aligned} \tag{3.27}$$

where (ξ, η) *is the mild solution of* (2.9) *corresponding to the initial data* (ξ^0, ξ^1, η^0) *and* $\overline{X}$ *is given in* (2.7)*. Then, the function* $J(\cdot,\cdot,\cdot)$ *admits a unique minimizer* $(\overline{\xi}^0, \overline{\xi}^1, \overline{\eta}^0)$ *in* $\overline{X}$*.*

Proof Noting (3.27), we know J is uniformly convex and continuous in $\overline{X}$. Now, we will deduce the function $J : \overline{X} \to \mathbb{R}$ is coercive, i.e., for any $\varepsilon > 0$, there exists

$$\liminf_{|(\xi^0,\xi^1,\eta^0)|_{\overline{X}} \to \infty} \frac{J(\xi^0, \xi^1, \eta^0)}{|(\xi^0, \xi^1, \eta^0)|_{\overline{X}}} \geq \varepsilon. \tag{3.28}$$

Obviously, it suffices to show

$$\liminf_{|(\xi^0,\xi^1+B\eta^0,\eta^0)|_{\overline{X}} \to \infty} \frac{J(\xi^0, \xi^1, \eta^0)}{|(\xi^0, \xi^1 + B\eta^0, \eta^0)|_{\overline{X}}} \geq \varepsilon. \tag{3.29}$$

Similar to [14], we introduce a sequence $(\xi_j^0, \xi_j^1, \eta_j^0) \in \overline{X}$ such that

$$M_j = |(\xi_j^0, \xi_j^1 + B\eta_j^0, \eta_j^0)|_{\overline{X}} \to \infty \quad \text{as } j \to \infty. \tag{3.30}$$

Let us denote

$$(\widehat{\xi}_j^0, \widehat{\xi}_j^1, \widehat{\eta}_j^0) = \frac{(\xi_j^0, \xi_j^1, \eta_j^0)}{M_j}. \tag{3.31}$$

Then

$$(\widehat{\xi}_j, \widehat{\eta}_j) = \frac{(\xi_j, \eta_j)}{M_j}$$

is the corresponding solution of system (2.9). Hence, we have

$$\begin{aligned} \frac{J_j}{M_j} = \frac{J(\xi_j^0, \xi_j^1, \eta_j^0)}{M_j} = \frac{M_j}{2}\int_0^T |F^*\widehat{\xi}_j|_U^2 \mathrm{d}t - \langle z_1, \widehat{\xi}_j^0\rangle_{H_1} + \langle z_0, \widehat{\xi}_j^1\rangle_{V,V'} \\ + \varepsilon|\widehat{\eta}_j^0|_{H_2} - \langle \zeta_0 - B^* z_0, \widehat{\eta}_j^0\rangle_{H_2}. \end{aligned} \tag{3.32}$$

In what follows, we will consider two cases.

(i)

$$\liminf_{j\to\infty}\int_0^T |F^*\widehat{\xi}_j|_U^2\mathrm{d}t > 0; \tag{3.33}$$

(ii)

$$\liminf_{j\to\infty}\int_0^T |F^*\widehat{\xi}_j|_U^2\mathrm{d}t = 0. \tag{3.34}$$

By (3.33), one obviously gets that

$$\liminf_{j\to\infty}\frac{J_j}{M_j} = \infty. \tag{3.35}$$

On the other hand, by (3.34), we could obtain a sequence $(\widehat{\xi}_j)$ (the original sequence denote the subsequence for convenience) such that

$$\int_0^T |F^*\widehat{\xi}_j|_U^2\mathrm{d}t \to 0 \quad \text{as } j\to\infty. \tag{3.36}$$

Moreover, from (3.31), we know $(\widehat{\xi}_j^0, \widehat{\xi}_j^1, \widehat{\eta}_j^0)$ is bounded in $\overline{X}$. Therefore, there exists a sequence such that $(\widehat{\xi}_j^0, \widehat{\xi}_j^1, \widehat{\eta}_j^0) \rightharpoonup (\widehat{\xi}^0, \widehat{\xi}^1, \widehat{\eta}^0)$ (the extracted subsequence is denoted by the original sequence for convenience) in $\overline{X}$ as $j\to\infty$. Also, we denote $(\widehat{\xi}, \widehat{\eta})$ to be the mild solution of system (2.9) with final data $(\widehat{\xi}^0, \widehat{\xi}^1, \widehat{\eta}^0)$.

Thus, combining (3.36), we deduce

$$F^*\widehat{\xi} \equiv 0, \quad \forall t\in(0,T). \tag{3.37}$$

Next, we use (A5) again to get $(\widehat{\xi}^0, \widehat{\xi}^1, \widehat{\eta}^0) \equiv (0,0,0)$. Finally, it yields

$$(\widehat{\xi}_j^0, \widehat{\xi}_j^1, \widehat{\eta}_j^0) \rightharpoonup (0,0,0) \quad \text{in } \overline{X} \text{ as } j\to\infty. \tag{3.38}$$

Applying (3.32) and (3.38), we have

$$\liminf_{j\to\infty}\frac{J_j}{M_j} = \liminf_{j\to\infty}\Big[\frac{M_j}{2}\int_0^T |F^*\widehat{\xi}_j|_U^2\mathrm{d}t + \varepsilon|\widehat{\eta}_j^0|_{H_2}\Big]. \tag{3.39}$$

Now, from (3.31), one could conclude

$$|(\widehat{\xi}_j^0, \widehat{\xi}_j^1 + B\widehat{\eta}_j^0, \widehat{\eta}_j^0)|_{\overline{X}} = 1 \tag{3.40}$$

for all $j>0$. Therefore, we see

$$|\widehat{\eta}_j^0|_{H_2} \le 1, \quad \forall j>0. \tag{3.41}$$

And it is easy to get

$$\liminf_{j\to\infty}|\widehat{\eta}_j^0|_{H_2} \le 1. \tag{3.42}$$

However, we claim that

$$\liminf_{j\to\infty}|\widehat{\eta}_j^0|_{H_2} = 1. \tag{3.43}$$

Let us check (3.43). First of all, we assume

$$\liminf_{j\to\infty} |\widehat{\eta}_j^0|_{H_2} < 1. \tag{3.44}$$

By (3.40) and (3.44), it holds

$$\liminf_{j\to\infty} |(\widehat{\xi}_j^0, \widehat{\xi}_j^1 + B\widehat{\eta}_j^0)|_{H_1\times V'} > 0. \tag{3.45}$$

Further, combining (3.41), (3.45) and using Proposition 3.1, we arrive at

$$\liminf_{j\to\infty} \int_0^T |F^*\widehat{\xi}_j|_U^2 \mathrm{d}t > 0, \tag{3.46}$$

which contradicts (3.36). Hence, we obtain (3.43).

Next, by (3.39) and (3.43), we could still get (3.35).

Finally, we know that the function $J : \overline{X} \to \mathbb{R}$ is coercive. Thus, this completes the proof of Lemma 3.4.

4 Proof of Theorem 2.1

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1 To begin with, without loss of generality, we suppose $(u_0, u_1, \theta_0) = (0, 0, 0)$.

Thanks to Lemma 3.4, we know the function J admits a unique minimizer $(\overline{\xi}^0, \overline{\xi}^1, \overline{\eta}^0)$ in $\overline{X}$. Note $J : \overline{X} \to \mathbb{R}$ is Fréchet differentiable, then at the minimizer $(\overline{\xi}^0, \overline{\xi}^1, \overline{\eta}^0)$ we have

$$\Big|\int_0^T \langle FF^*\overline{\xi}, \varrho\rangle_{H_1} \mathrm{d}t - \langle z_1, \varrho^0\rangle_{H_1} + \langle z_0, \varrho^1\rangle_{V,V'} - \langle \zeta_0 - B^*z_0, \iota^0\rangle_{H_2}\Big| \le \varepsilon|\iota^0|_{H_2} \tag{4.1}$$

for any $(\varrho^0, \varrho^1, \iota^0) \in \overline{X}$ and any $\varepsilon > 0$, where (ϱ, ι) satisfies (2.9) with final data $(\varrho^0, \varrho^1, \iota^0)$.

Moreover, for (1.1), putting $f = F^*\overline{\xi}$, we deduce

$$\begin{aligned}\int_0^T \langle FF^*\overline{\xi}, \varrho\rangle_{H_1} \mathrm{d}t = {}& \langle u_t(T), \varrho^0\rangle_{H_1} - \langle u(T), \varrho^1\rangle_{V,V'} \\ & + \langle \theta(T) - B^*u(T), \iota^0\rangle_{H_2}.\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we see

$$\begin{aligned}&|\langle u_t(T) - z_1, \varrho^0\rangle_{H_1} - \langle u(T) - z_0, \varrho^1\rangle_{V,V'} \\ &+ \langle \theta(T) - B^*u(T) - \zeta_0 + B^*z_0, \iota^0\rangle_{H_2}| \le \varepsilon|\iota^0|_{H_2}\end{aligned} \tag{4.3}$$

for any $\varepsilon > 0$ and any $(\varrho^0, \varrho^1, \iota^0) \in \overline{X}$.

In what follows, by (4.3), one obtains

$$\begin{cases} u(T) = z_0, \quad u_t(T) = z_1, \\ |\theta(T) - \zeta_0|_{H_2} \le \varepsilon. \end{cases} \tag{4.4}$$

Then the proof is finished.

5 Example

Finally, we apply our abstract result to the thermoelasticity system in [14]. Now, we will consider the system in one dimension.

Set $\Omega = (0,1)$ and define

$$H_1 = H_2 = L^2(0,1), \quad U = L^2((0,T)\times(0,1)).$$

Furthermore, let

$$\begin{cases} D(A_1) = D(A_2) = H^2(0,1)\cap H_0^1(0,1), \\ A_1 = A_2 = -\partial_x^2, \end{cases} \qquad \begin{cases} D(B) = H_0^1(0,1), \\ B = \partial_x. \end{cases} \tag{5.1}$$

Then, we get the adjoint operator B^* of B is given by $B^* = -\partial_x,\ D(B^*) = H_0^1(0,1)$.

In this case, system (1.1) could be written as follows:

$$\begin{cases} u_{tt} - u_{xx} + \theta_x = f1_\omega & \text{in } (0,T)\times(0,1), \\ \theta_t - \theta_{xx} + u_{tx} = 0 & \text{in } (0,T)\times(0,1), \\ u(t,0) = u(t,1) = \theta(t,0) = \theta(t,1) = 0 & \text{in } (0,T), \\ u(0) = u_0, \quad u_t(0) = u_1, \quad \theta(0) = \theta_0 & \text{in } (0,1), \end{cases} \tag{5.2}$$

where $\omega = (l_1, l_2)$ and $0 < l_1, l_2 < 1$. Also, we assume $f \in L^2(0,T;L^2(0,1))$ and the initial data (u_0, u_1, θ_0) belongs to the Hilbert space

$$X \triangleq H_0^1(0,1)\times L^2(0,1)\times L^2(0,1).$$

Clearly, the adjoint system of (5.2) is

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + \psi_{tx} = 0 & \text{in } (0,T)\times(0,1), \\ -\psi_t - \psi_{xx} - \varphi_x = 0 & \text{in } (0,T)\times(0,1), \\ \varphi(t,0) = \varphi(t,1) = \psi(t,0) = \psi(t,1) = 0 & \text{in } (0,T), \\ \varphi(T) = \varphi^0, \quad \varphi_t(T) = \varphi^1, \quad \psi(T) = \psi^0 & \text{in } (0,1), \end{cases} \tag{5.3}$$

where $(\varphi^0, \varphi^1, \psi^0)$ is taken in the following space

$$\overline{X} \triangleq L^2(0,1)\times H^{-1}(0,1)\times L^2(0,1).$$

Denote

$$w = -\int_t^T \varphi(s)\mathrm{d}s + \chi \tag{5.4}$$

with χ satisfying

$$\begin{cases} -\chi_{xx} = -\varphi^1 - \psi_x^0 & \text{in } (0,1), \\ \chi(0) = \chi(1) = 0. \end{cases} \tag{5.5}$$

Therefore, we have

$$\begin{cases} w_{tt} - w_{xx} + \psi_x = 0 & \text{in } (0,T)\times(0,1), \\ -\psi_t - \psi_{xx} - w_{tx} = 0 & \text{in } (0,T)\times(0,1), \\ w(t,0) = w(t,1) = \psi(t,0) = \psi(t,1) = 0 & \text{in } (0,T), \\ w(T) = \chi, \quad w_t(T) = \varphi^0, \quad \psi(T) = \psi^0 & \text{in } (0,1) \end{cases} \tag{5.6}$$

and $(\chi, \varphi^0, \psi^0) \in X$. Consequently, we get the adjoint system of (5.6) is as follows:

$$\begin{cases} \overline{w}_{tt} - \overline{w}_{xx} - P\overline{w}_t = 0 & \text{in } (0,T)\times(0,1), \\ -\overline{\psi}_t - \overline{\psi}_{xx} - \overline{w}_{tx} = 0 & \text{in } (0,T)\times(0,1), \\ \overline{w}(t,0) = \overline{w}(t,1) = \overline{\psi}(t,0) = \overline{\psi}(t,1) = 0 & \text{in } (0,T), \\ \overline{w}(T) = \chi, \quad \overline{w}_t(T) = \varphi^0, \quad \overline{\psi}(T) = \psi^0 & \text{in } (0,1), \end{cases} \tag{5.7}$$

where

$$Pv = v - \int_0^1 v(x)\mathrm{d}x, \quad \forall v \in L^2(0,1).$$

Next, we recall the known result in [14].

Lemma 5.1 *Assume $T > 2\max(l_1, 1-l_2)$, then there exists a constant $C > 0$ such that*

$$|\chi|^2_{H_0^1(0,1)} + |\varphi^0|^2_{L^2(0,1)} \le C \int_0^T \int_{l_1}^{l_2} |\overline{w}_t|^2 \mathrm{d}x\mathrm{d}t, \tag{5.8}$$

where $\overline{w}$ satisfies

$$\begin{cases} \overline{w}_{tt} - \overline{w}_{xx} - P\overline{w}_t = 0 & \textit{in } (0,T)\times(0,1), \\ \overline{w}(T) = \chi, \quad \overline{w}_t(T) = \varphi^0 & \textit{in } (0,1). \end{cases} \tag{5.9}$$

In this case, it is easy to get the assumption (A4) holds.

On the other hand, we apply the result in [14] again.

Lemma 5.2 *Suppose that $T > 2\max(l_1, 1-l_2)$ and (φ, ψ) is mild solution of* (5.3). *If*

$$\varphi = 0 \quad \textit{in } (0,T)\times(l_1, l_2),$$

then we have

$$(\varphi, \psi) \equiv (0,0) \quad \textit{in } (0,T)\times(0,1).$$

Obviously, the assumption (A5) also holds.

Now, with Theorem 2.1, one obtains the following corollary.

Corollary 5.1 *Assume $T > 2\max(l_1, 1-l_2)$, then* (5.2) *is exact-approximately controllable.*

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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