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Alternating Apéry-Type Series and Colored Multiple Zeta Values of Level Eight*

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Abstract Apéry-type (inverse) binomial series have appeared prominently in the calculations of Feynman integrals in recent years. In their previous work, the authors showed that a few large classes of the non-alternating Apéry-type (inverse) central binomial series can be evaluated using colored multiple zeta values of level four (i.e., special values of multiple polylogarithms at the fourth roots of unity) by expressing them in terms of iterated integrals. In this sequel, the authors will prove that for several classes of the alternating versions they need to raise the level to eight. Their main idea is to adopt hyperbolic trigonometric 1-forms to replace the ordinary trigonometric ones used in the non-alternating setting.

Keywords Apéry-type series, Colored multiple zeta values, Binomial coefficients, Iterated integrals
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1 Introduction

Significant progress has been made on the calculations of the ε-expansion of multiloop Feynman diagrams in the past quarter of a century. It turns out that special values of multiple polylogarithms have played indispensable roles in these computations. Much experimental work emerged around the beginning of this century (see [4–7]), in which a special class of series emerges. These infinite sums are often called Apéry-type series (or Apéry-like sums) because the simplest cases were used by Apéry in his celebrated proof of irrationality of $\zeta(2)$ and $\zeta(3)$ in 1979 when he discovered the following identities

$$\zeta(2) = 3\sum_{n\geq 1} \frac{1}{n^2\binom{2n}{n}}, \quad \zeta(3) = \frac{5}{2}\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Motivated by Apéry's proof, Leshchiner [10] generalized the above to the following identities

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involving the alternating Riemann zetas $\zeta(\overline{n})$ which are defined by

$$\zeta(\overline{n}) := \sum_{k=1}^{\infty} \frac{(-1)^k}{k^n} = (2^{1-n}-1)\zeta(n), \quad \forall n \geq 2. \tag{1.1}$$

Let $\delta_{j,1}$ denote the Kronecker symbol. For all $j \in \mathbb{N}$, put $A_j = 1 - \frac{\delta_{j,1}}{4}$, $B_j = 1 + \frac{\delta_{j,1}}{4}$.

Theorem 1.1 (see [10]) *For any $k \in \mathbb{N}$, we have*

$$\begin{aligned}
\zeta(\overline{2k}) &= \sum_{n\geq 1} \frac{2}{n^2\binom{2n}{n}} \sum_{j=1}^{k} A_j \frac{(-1)^{k-j}\zeta_{n-1}(2_{k-j})}{n^{2j-2}}, \\
\zeta(2k+1) &= \sum_{n\geq 1} \frac{2(-1)^{n-1}}{n^3\binom{2n}{n}} \sum_{j=1}^{k} B_j \frac{(-1)^{k-j}\zeta_{n-1}(2_{k-j})}{n^{2j-2}}, \\
\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^{2k-1}} &= \sum_{n\geq 0} \frac{\binom{2n}{n}}{4^{2n}} \sum_{j=1}^{k} A_j \frac{(-1)^{k-j} t_n(2_{k-j})}{n^{2j-1}}, \\
\sum_{n\geq 0} \frac{1}{(2n+1)^{2k}} &= \sum_{n\geq 0} \frac{(-1)^n\binom{2n}{n}}{4^{2n}} \sum_{j=1}^{k} B_j \frac{(-1)^{k-j} t_n(2_{k-j})}{(2n+1)^{2j}},
\end{aligned}$$

where 2_p represents the string of 2's with p repetitions, B_j's are Bernoulli numbers, and ζ_{n-1} and t_n are defined by (1.2) *and* (1.6)*, respectively.*

We will call the Apéry-type series a central binomial series (resp. inverse central binomial series) if the central binomial coefficient $\binom{2n}{n}$ appears on the numerator (resp. denominator). It is remarkable that both types appear in Theorem 1.1 and both appear in the evaluations of Feynman integrals (see [4–6] for inverse binomial series and [7] for binomial series). Moreover, odd-indexed variations of both types appeared implicitly, too. See [6, (1.1)] and [14, Remark 4.2] for the former and [7, (A.25)] and [14, Eq. (1.3)] for the latter.

In our previous work [13–15] we also studied both types and considered their even-odd-indexed variations. Our main result is that many such series can be expressed as $\mathbb{Q}$-linear combinations of the real and/or the imaginary part of colored multiple zeta values of level 4. We further considered similar series with $\binom{2n}{n}$ replaced by its square, in which case a possible extra factor of $\frac{1}{\pi}$ may appear in the evaluations of these binomial series. In this paper, we will focus primarily on the alternating versions.

We begin with some basic notations. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A finite sequence $\boldsymbol{k} := (k_1, \cdots, k_r) \in \mathbb{N}^r$ is called a composition. We put

$$|\boldsymbol{k}| := k_1 + \cdots + k_r, \quad \mathrm{dep}(\boldsymbol{k}) := r,$$

and call them the weight and the depth of $\boldsymbol{k}$, respectively. If $k_1 > 1$ then $\boldsymbol{k}$ is called admissible. For any $m, n, d \in \mathbb{N}$ with $m \geq n + d$ and any compositions $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$, set

$$\zeta_m(\boldsymbol{s})_n := \sum_{m\geq n_1 > n_2 > \cdots > n_d > n} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}},$$

$$\zeta^{\star}_m(\boldsymbol{s})_n := \sum_{m\geq n_1\geq\cdots\geq n_d>n} \frac{1}{n_1^{s_1}\cdots n_d^{s_d}},$$

and respectively define the multiple harmonic (star) sums by

$$\zeta_n(\boldsymbol{s}) := \zeta_n(\boldsymbol{s})_0, \quad \zeta^{\star}_n(\boldsymbol{s}) := \zeta^{\star}_n(\boldsymbol{s})_0. \tag{1.2}$$

Then the multiple zeta values (MZVs for short) and the multiple zeta star values (MZSVs for short) are defined by

$$\zeta(\boldsymbol{s}) := \lim_{n\to\infty} \zeta_n(\boldsymbol{s}), \quad \zeta^{\star}(\boldsymbol{s}) := \lim_{n\to\infty} \zeta^{\star}_n(\boldsymbol{s}),$$

respectively, and their n-tails are defined by

$$\zeta(\boldsymbol{s})_n := \lim_{m\to\infty} \zeta_m(\boldsymbol{s})_n, \quad \zeta^{\star}(\boldsymbol{s})_n := \lim_{m\to\infty} \zeta^{\star}_m(\boldsymbol{s})_n,$$

respectively. We call $|\boldsymbol{s}| := s_1+\cdots+s_d$ the weight and d the depth of the corresponding values. Kontsvitch observed that $\zeta(\boldsymbol{s})$ can be expressed using Chen's iterated integrals (see [16, Section 2.1] for a brief summary of this theory):

$$\zeta(\boldsymbol{s}) = \int_0^1 \mathbf{w}(\boldsymbol{s}), \quad \mathbf{w}(\boldsymbol{s}) := \mathtt{a}^{s_1-1}\mathtt{x}_1\cdots\mathtt{a}^{s_d-1}\mathtt{x}_1. \tag{1.3}$$

Here we have put $\mathtt{a} = \mathrm{d}t/t$ and $\mathtt{x}_\xi = \mathrm{d}t/(\xi - t)$ for any Nth roots of unity ξ. We will call $\mathbf{w}(\boldsymbol{s})$ the associated word of $\boldsymbol{s}$.

In general, let $\boldsymbol{s} = (s_1,\cdots,s_d) \in \mathbb{N}^d$ and $\boldsymbol{\eta} = (\eta_1,\cdots,\eta_d)$, where $\eta_1,\cdots,\eta_d$ are Nth roots of unity. We can define the colored MZVs (CMZVs for short) of level N by

$$\mathrm{Li}_{\boldsymbol{s}}(\boldsymbol{\eta}) := \sum_{n_1>\cdots>n_d>0} \frac{\eta_1^{n_1}\ldots\eta_d^{n_d}}{n_1^{s_1}\ldots n_d^{s_d}}, \tag{1.4}$$

which converges if $(s_1,\eta_1)\neq(1,1)$ (see [11] and [16, Chapter 15]), in which case we call $(\boldsymbol{s};\boldsymbol{\eta})$ admissible. The level two colored MZVs are often called Euler sums or alternating MZVs. In this case, namely, when $(\eta_1,\cdots,\eta_d)\in\{\pm1\}^r$ and $(s_1,\eta_1)\neq(1,1)$, we set $\zeta(\boldsymbol{s};\boldsymbol{\eta}) = \mathrm{Li}_{\boldsymbol{s}}(\boldsymbol{\eta})$. Further, to save space we put a bar on top of s_j if the corresponding $\eta_j = -1$, which is consistent with the notation in (1.1). For example,

$$\zeta(\overline{2},3,\overline{1},4) = \zeta(2,3,1,4;-1,1,-1,1).$$

Moreover, CMZVs can be expressed by Chen's iterated integrals. In fact, for any complex number z such that $(s_1, z\eta_1)\neq(1,1)$ we have

$$\mathrm{Li}_{\boldsymbol{s}}(z\eta_1,\eta_2,\cdots,\eta_d) = \int_0^z \mathtt{a}^{s_1-1}\mathtt{x}_{\gamma_1}\cdots\mathtt{a}^{s_d-1}\mathtt{x}_{\gamma_d}, \tag{1.5}$$

where $\gamma_j := \prod\limits_{i=1}^{j}\eta_i^{-1} = \frac{1}{\eta_1\cdots\eta_j}$ for all j.

In [1–2] Akhilesh discovered some very important and surprising connections between MZVs and the following Apéry-type series, or multiple Apéry-like sums:

$$\sigma(\boldsymbol{s};x) := \sum_{n_1>n_2>\cdots>n_d} \binom{2n_1}{n_1}^{-1} \frac{(2x)^{2n_1}}{(2n_1)^{s_1}\cdots(2n_d)^{s_d}}.$$

Here we have renormalized the sums to make them consistent with our previous work. His ingenious idea is to study the n-tails (and more generally, double tails) of such series. We reformulate one of his key result as follows to make it more transparent. Set

$$g_s(t) = g_s^+(t) = \begin{cases} \tan t\,\mathrm{d}t, & \text{if } s=1, \\ \mathrm{d}t \circ (\cot t\,\mathrm{d}t)^{s-2} \circ \mathrm{d}t, & \text{if } s \ge 2. \end{cases}$$

Further, we set $\binom{0}{0}=1$,

$$b_n(x) = 4^n \binom{2n}{n}^{-1} x^{2n}, \quad b_n = b_n(1) = 4^n \binom{2n}{n}^{-1}, \quad \forall n \ge 0.$$

Theorem 1.2 (see [2, Thm. 4]) *For all $n \in \mathbb{N}_0$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$, we have*

$$\begin{aligned}\sigma(\boldsymbol{s};\sin y)_n &:= \sum_{n_1>\cdots>n_d>n} \frac{b_{n_1}(\sin y)}{(2n_1)^{s_1}\cdots(2n_d)^{s_d}} \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\int_0^y g_{s_1}(t)\circ\cdots\circ g_{s_d}(t)\circ b_n(\sin t)\mathrm{d}t.\end{aligned}$$

Here $y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ if $n_1=1$, and $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ if $n_1\ge 2$.

In this paper, motivated by a series of conjectures by Sun [12], and Leshchiner's and Akhilesh's results above, we shall investigate the following alternating Apéry-type series. For any $n\in\mathbb{N}$, $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_d)\in\{\pm1\}^d$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$ and a complex variable x, define

$$\sigma(\boldsymbol{s};\boldsymbol{\eta};x)_n := \sum_{n_1>\cdots>n_d>n} \frac{b_{n_1}(x)\eta_1^{n_1}\cdots\eta_d^{n_d}}{(2n_1)^{s_1}\cdots(2n_d)^{s_d}},$$

which is called the n-tail of the alternating Apéry-type series denoted by

$$\sigma(\boldsymbol{s};\boldsymbol{\eta};x) := \sigma(\boldsymbol{s};\boldsymbol{\eta};x)_0.$$

To save space we will put a bar on top of s_j if the corresponding $\eta_j=-1$. For example,

$$\sigma(\overline{3},4,\overline{1};x) = \sum_{n_1>n_2>n_3} \frac{b_{n_1}(x)(-1)^{n_1+n_3}}{(2n_1)^3(2n_2)^4(2n_3)}.$$

From Leschiner's result we see that studying Apéry-type series naturally leads to the following multiple t-harmonic sums (and their star version). For any $m,n,d\in\mathbb{N}$ and compositions $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$, we define

$$t_m(\boldsymbol{s})_n := \sum_{m\ge n_1>n_2>\cdots>n_d>n} \frac{1}{(2n_1-1)^{s_1}\cdots(2n_d-1)^{s_d}},$$

$$t_m^\star(\boldsymbol{s})_n := \sum_{m\ge n_1\ge\cdots\ge n_d>n} \frac{1}{(2n_1-1)^{s_1}\cdots(2n_d-1)^{s_d}}$$

and the multiple t-harmonic (star) sums

$$t_n(\boldsymbol{s}) := t_n(\boldsymbol{s})_0, \quad t_n^\star(\boldsymbol{s}) := t_n^\star(\boldsymbol{s})_0, \tag{1.6}$$

respectively. Correspondingly, the multiple t-(star) values are the infinite sums

$$t(\boldsymbol{s}) := \lim_{n\to\infty} t_n(\boldsymbol{s}), \quad t^\star(\boldsymbol{s}) := \lim_{n\to\infty} t_n^\star(\boldsymbol{s}).$$

We will use heavily the hyperbolic trigonometric functions throughout this paper. To fix notation, we put

$$\operatorname{sh} x = -\mathrm{i}\sin(\mathrm{i}x) = \frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}, \quad \operatorname{ch} x = \cos(\mathrm{i}x) = \frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}, \quad \operatorname{th} x = -\mathrm{i}\tan(\mathrm{i}x) = \frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}},$$
$$\operatorname{cth} x = \mathrm{i}\cot(\mathrm{i}x) = \frac{\mathrm{e}^x+\mathrm{e}^{-x}}{\mathrm{e}^x-\mathrm{e}^{-x}}, \quad \operatorname{sech} x = \sec(\mathrm{i}x) = \frac{2}{\mathrm{e}^x+\mathrm{e}^{-x}}, \quad \operatorname{csch} x = \mathrm{i}\csc(\mathrm{i}x) = \frac{2}{\mathrm{e}^x-\mathrm{e}^{-x}}.$$

We will extend Chen's iterated integrals by combining 1-forms and functions as follows. For any $r\in\mathbb{N}$, 1-forms $f_1(t)\mathrm{d}t,\cdots,f_{r+1}(t)\mathrm{d}t$ and functions $F_1(t),\cdots,F_r(t)$, define recursively

$$\begin{aligned}
&\int_0^1 (f_1(t)\mathrm{d}t+F_1(t))\circ\cdots\circ(f_r(t)\mathrm{d}t+F_r(t))\circ f_{r+1}(t)\mathrm{d}t \\
:=&\int_0^1 (f_1(t)\mathrm{d}t+F_1(t))\circ\cdots\circ(f_{r-1}(t)\mathrm{d}t+F_{r-1}(t))\circ f_r(t)\mathrm{d}t\circ f_{r+1}(t)\mathrm{d}t \\
&+\int_0^1 (f_1(t)\mathrm{d}t+F_1(t))\circ\cdots\circ(f_{r-1}(t)\mathrm{d}t+F_{r-1}(t))\circ(F_r(t)f_{r+1}(t))\mathrm{d}t.
\end{aligned}$$

2 Alternating Apéry-Type Inverse Central Binomial Series

Define the hyperbolic counterpart of $g_s(t)$ by

$$\widetilde{g}_s(t) = \widetilde{g}_s^{+}(t) = \begin{cases} \operatorname{th} t\,\mathrm{d}t, & \text{if } s=1, \\ \mathrm{d}t\circ(\operatorname{cth} t\,\mathrm{d}t)^{s-2}\circ\mathrm{d}t, & \text{if } s\ge 2. \end{cases}$$

Throughout the paper we put $\nu := \sqrt{2}+1$.

Theorem 2.1 *Set $\psi = \operatorname{sh}^{-1} 1 = \log\nu$. For all $n\in\mathbb{N}_0$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$, we have*

$$\sigma(\boldsymbol{s};\mathrm{i}\operatorname{sh} y)_n = (-1)^d \frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \widetilde{g}_{s_1}(t)\circ\cdots\circ\widetilde{g}_{s_d}(t)\circ b_n(\mathrm{i}\operatorname{sh} y)\mathrm{d}t.$$

Here $y\in(-\psi,\psi)$ if $s_1=1$, and $y\in[-\psi,\psi]$ if $s_1\ge 2$.

Proof Applying the substitution $y\to\mathrm{i}y$ in Theorem 1.2, we obtain immediately

$$\sigma(\boldsymbol{s};\mathrm{i}\operatorname{sh} y)_n := \sum_{n_1>\cdots>n_d>n} \frac{b_{n_1}(\mathrm{i}\operatorname{sh} y)}{(2n_1)^{s_1}\cdots(2n_d)^{s_d}}$$

$$= -\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y g_{s_1}(\mathrm{i}t)\circ\cdots\circ g_{s_d}(\mathrm{i}t)\circ b_n(\mathrm{i}\,\mathrm{sh}\,t)\mathrm{d}(\mathrm{i}t),$$

where we have used the fact that $\sin \mathrm{i}t = \mathrm{i}\,\mathrm{sh}\,t$ and $\cos \mathrm{i}t = \mathrm{ch}\,t$. Note that

$$g_s(\mathrm{i}t) = \begin{cases}\tan(\mathrm{i}t)\mathrm{d}(\mathrm{i}t), & \text{if } s=1,\\ \mathrm{d}(\mathrm{i}t)\circ(\cot(\mathrm{i}t)\mathrm{d}(\mathrm{i}t))^{s-2}\circ\mathrm{d}(\mathrm{i}t), & \text{if } s\ge 2,\end{cases}$$
$$= \begin{cases}-\,\mathrm{th}\,t\,\mathrm{d}t, & \text{if } s=1,\\ -\mathrm{d}t\circ(\mathrm{cth}\,t\,\mathrm{d}t)^{s-2}\circ\mathrm{d}t, & \text{if } s\ge 2.\end{cases}$$

The theorem follows immediately.

Similar to the original form in [2, Theorem 4], setting $\mathbf{w}(\boldsymbol{s}) = \mathtt{x}_{\varepsilon_1}\cdots\mathtt{x}_{\varepsilon_w}$ (see (1.3)) yields

$$\sigma(\boldsymbol{s};\mathrm{i}\,\mathrm{sh}\,y)_n = (-1)^d\,\mathrm{th}^{\varepsilon_1}\,y\int_0^y \mathrm{th}^{\varepsilon_1+\varepsilon_2-1}\,t\mathrm{d}t\cdots\mathrm{th}^{\varepsilon_{w-1}+\varepsilon_w-1}\,t\mathrm{d}t b_n(\mathrm{i}\,\mathrm{sh}\,t)\mathrm{d}t.$$

Corollary 2.1 *For all $p\in\mathbb{N}$, we have*

$$\sigma(1,2_p;\mathrm{i}\,\mathrm{sh}\,y) = \sigma(\overline{1},2_p;\mathrm{sh}\,y) = (-1)^{p+1}\,\mathrm{th}\,y\int_0^y(\mathrm{d}t)^{2p+1} = \frac{(-1)^{p+1}}{(2p+1)!}y^{2p+1}\,\mathrm{th}\,y,$$
$$\sigma(2_p;\mathrm{i}\,\mathrm{sh}\,y) = \sigma(\overline{2},2_{p-1};\mathrm{sh}\,y) = (-1)^p\int_0^y(\mathrm{d}t)^{2p} = \frac{(-1)^p}{(2p)!}y^{2p}.$$

Remark 2.1 Taking $p=0$ and $p=1$ in the first equation of Corollary 2.1, we can recover the two equations (C.8) and (C.16) in [6]. Similarly, taking $p=0$ and $p=1$ in the second equation above, we can obtain (C.9) and (C.17) in [6].

Example 2.1 For $j=1,2,3,4$, we have

$$\mathrm{sh}^{-1}\Big(\frac{\sqrt{j}}{2}\Big) = \log\Big(\frac{\sqrt{j}+\sqrt{j+4}}{2}\Big).$$

Thus for all $p\ge 0$, we get

$$\begin{aligned}\sigma\Big(1,2_p;\frac{\sqrt{j}}{2}\mathrm{i}\Big) &= \sigma\Big(\overline{1},2_p;\frac{\sqrt{j}}{2}\Big)\\ &= \frac{(-1)^{p+1}\sqrt{j}}{\sqrt{j+4}}\int_0^{\mathrm{sh}^{-1}(\frac{\sqrt{j}}{2})}(\mathrm{d}t)^{2p+1}\\ &= \frac{(-1)^{p+1}\sqrt{j}}{(2p+1)!\sqrt{j+4}}\log^{2p+1}\Big(\frac{\sqrt{j}+\sqrt{j+4}}{2}\Big).\end{aligned}$$

Similarly, for all $p\ge 1$,

$$\sigma\Big(2_p;\frac{\sqrt{j}}{2}\mathrm{i}\Big) = \sigma\Big(\overline{2},2_{p-1};\frac{\sqrt{j}}{2}\Big) = (-1)^p\int_0^{\mathrm{sh}^{-1}(\frac{\sqrt{j}}{2})}(\mathrm{d}t)^{2p} = \frac{(-1)^p}{(2p)!}\log^{2p}\Big(\frac{\sqrt{j}+\sqrt{j+4}}{2}\Big).$$

Remark 2.2 Kalmykov and his collaborators encountered and studied similar sums when computing Feynman integrals in [6, 8–9]. By the stuffle relations it is easy to see that those sums can be expressed as $\mathbb{Q}$-linear combinations of σ's and their odd-indexed variations, but not vice versa in general.

We now turn to the general alternating case. Define

$$\omega_0 := \frac{\mathrm{d}t}{t}, \quad \omega_{\pm 1} := \frac{\mathrm{d}t}{\sqrt{1 \mp t^2}}, \quad \omega_{\pm 2} := \frac{t\mathrm{d}t}{1 \mp t^2}, \quad \omega_{\pm 4} := \frac{t\mathrm{d}t}{\sqrt{1 - t^4}}.$$

Theorem 2.2 *For any $d \in \mathbb{N}$, $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$ and $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_d) \in \{\pm 1\}^d$, put $\boldsymbol{\eta}_j = \prod_{k=1}^{j} \eta_k$ and $\mathrm{sgn}(\boldsymbol{\eta}) = \prod_{k=1}^{d} \boldsymbol{\eta}_k$. Set $\mathrm{sih}_1\, y = \sin y$ and $\mathrm{sih}_{-1}\, y = \mathrm{sh}\, y$. Put $b_n^{\pm}(t) = \binom{2n}{n}^{-1} (\pm 4t^2)^n$. Then we have*

$$\sigma(\boldsymbol{s}; \boldsymbol{\eta}; \mathrm{sih}_{\eta_1} y)_n = \mathrm{sgn}(\boldsymbol{\eta}) \frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\mathrm{sih}_{\eta_1} y} \Gamma_{s_1}^{\boldsymbol{\eta}_0, \boldsymbol{\eta}_1}(t) \circ \cdots \circ \Gamma_{s_d}^{\boldsymbol{\eta}_{d-1}, \boldsymbol{\eta}_d}(t) \circ b_n^{\mathrm{sgn}(\boldsymbol{\eta}_d)}(t) \omega_{\boldsymbol{\eta}_d}, \tag{2.1}$$

$$\sigma(\boldsymbol{s}; \boldsymbol{\eta}; x)_n = \mathrm{sgn}(\boldsymbol{\eta}) \sqrt{1 - \eta_1 x^2} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x} \Gamma_{s_1}^{\boldsymbol{\eta}_0, \boldsymbol{\eta}_1}(t) \circ \cdots \circ \Gamma_{s_d}^{\boldsymbol{\eta}_{d-1}, \boldsymbol{\eta}_d}(t) \circ b_n^{\mathrm{sgn}(\boldsymbol{\eta}_d)}(t) \omega_{\boldsymbol{\eta}_d}, \tag{2.2}$$

where $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_1$ and

$$\Gamma_s^{a,b}(t) = \begin{cases} \omega_{3b-a}, & \text{if } s = 1, \\ \omega_a \omega_0^{s-2} \omega_b, & \text{if } s \geq 2. \end{cases}$$

Here if $\eta_1 = 1$ then $\sigma(\boldsymbol{s}; \boldsymbol{\eta}; 1)_n$ is the limit $x \to 1^-$ on the right-hand side of (2.2).

Remark 2.3 Note that in general $\mathrm{sgn}(\boldsymbol{\eta}) = \eta_1^d \cdots \eta_{d-1}^2 \eta_d \neq \prod_{k=1}^{d} \eta_k$.

Proof (2.2) is an easy consequence of (2.1) which we will prove by induction on d. When $d = 1$, from Theorem 1.2 and using the change of variables $t \to \sin^{-1} t$ we see that

$$\sigma(1; \sin y)_n = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \tan t \mathrm{d}t b_n^{+}(\sin t) \mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\sin y} \omega_2 b_n^{+}(t) \omega_1,$$

$$\sigma(s; \sin y)_n = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \mathrm{d}t \Big(\frac{\mathrm{d}t}{\tan t}\Big)^{s-2} \mathrm{d}t b_n^{+}(\sin t) \mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\sin y} \omega_1 \omega_0^{s-2} \omega_1 b_n^{+}(t) \omega_1.$$

Similarly, from Theorem 2.1 and using the change of variables $t \to \mathrm{sh}^{-1} t$ we get

$$\sigma(\overline{1}; \mathrm{sh}\, y)_n = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \mathrm{th}\, t \mathrm{d}t b_n^{-}(\mathrm{sh}\, t) \mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\mathrm{sh}\, y} \omega_{-2} b_n^{-}(t) \omega_{-1},$$

$$\sigma(\overline{s}; \mathrm{sh}\, y)_n = -\frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \mathrm{d}t \Big(\frac{\mathrm{d}t}{\mathrm{th}\, t}\Big)^{s-2} \mathrm{d}t b_n^{-}(\mathrm{sh}\, t) \mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\mathrm{sh}\, y} \omega_{-1} \omega_0^{s-2} \omega_{-1} b_n^{-}(t) \omega_{-1}.$$

Hence the depth $d = 1$ case is proved.

Assume the theorem holds when the depth is $d - 1$ for some $d \geq 2$. Write $s = s_d$, $\boldsymbol{s}' = (s_1, \cdots, s_{d-1})$ and $\boldsymbol{\eta}' = (\eta_1, \cdots, \eta_{d-1})$. By the definition

$$\begin{aligned} \sigma(\boldsymbol{s}; \boldsymbol{\eta}; \mathrm{sih}_{\eta_1} y)_n &= \sum_{k>n} \frac{\sigma(\boldsymbol{s}'; \boldsymbol{\eta}'; \mathrm{sih}_{\eta_1} y)_k \eta_d^k}{(2k)^s} \\ &= \mathrm{sgn}(\boldsymbol{\eta}') \frac{\mathrm{d}}{\mathrm{d}y} \int_0^{\mathrm{sih}_{\eta_1} y} \Gamma_{s_1}^{\boldsymbol{\eta}_0, \boldsymbol{\eta}_1}(t) \circ \cdots \circ \Gamma_{s_d}^{\boldsymbol{\eta}_{d-2}, \boldsymbol{\eta}_{d-1}}(t) \circ \sum_{k>n} \frac{\eta_d^k b_k^{\mathrm{sgn}(\boldsymbol{\eta}_{d-1})}(t)}{(2k)^s} \omega_{\boldsymbol{\eta}_{d-1}} \end{aligned}$$

by the inductive assumption. But

$$\sum_{k>n} \frac{\eta_d^k b_k^{\mathrm{sgn}(\boldsymbol{\eta}_{d-1})}(t)}{(2k)^s} = \sum_{k>n} \binom{2n}{n}^{-1} \frac{(\boldsymbol{\eta}_d 4t^2)^k}{(2k)^s} = \sigma(s;\boldsymbol{\eta}_d;t)_n.$$

If $\boldsymbol{\eta}_d = 1$ then setting $\tau = \sin^{-1} t$ and applying the change of variables $u = \sin^{-1} z$ in the iterated integral in Theorem 1.2, we have

$$\begin{aligned}
\sigma(s;\boldsymbol{\eta}_d;t)_n = \sigma(s;\sin\tau)_n &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau g_s(u)\circ b_n^+(\sin u)\mathrm{d}u \\
&= \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \Gamma_s^{1,1}(z)\circ b_n^+(z)\omega_1(z) \\
&= \begin{cases} \dfrac{t}{\sqrt{1-t^2}}\displaystyle\int_0^t b_n^+(z)\omega_1(z), & \text{if } s=1, \\ \displaystyle\int_0^t \omega_0(z)^{s-2}\omega_1(z)\circ b_n^+(z)\omega_1(z), & \text{if } s\ge 2. \end{cases}
\end{aligned} \tag{2.3}$$

Noticing that

$$\frac{t}{\sqrt{1-t^2}}\omega_{\boldsymbol{\eta}_{d-1}} = \begin{cases} \dfrac{t\mathrm{d}t}{1-t^2} = \omega_2, & \text{if } \boldsymbol{\eta}_{d-1}=1, \\ \dfrac{t\mathrm{d}t}{\sqrt{1-t^4}} = \omega_4, & \text{if } \boldsymbol{\eta}_{d-1}=-1, \end{cases}$$

we deduce that

$$\begin{aligned}
\sum_{k>n} \frac{\eta_d^k b_k^{\mathrm{sgn}(\boldsymbol{\eta}_{d-1})}(t)}{(2k)^s}\omega_{\boldsymbol{\eta}_{d-1}} &= \sigma(s;\boldsymbol{\eta}_d;t)_n\omega_{\boldsymbol{\eta}_{d-1}} \\
&= \begin{cases} \omega_{3-\boldsymbol{\eta}_{d-1}} b_n^+(t)\omega_1, & \text{if } s=1, \\ \omega_{\boldsymbol{\eta}_{d-1}}\omega_0^{s-2}\omega_1 b_n^+(t)\omega_1, & \text{if } s\ge 2 \end{cases} \\
&= \boldsymbol{\eta}_d \Gamma_{s_d}^{\boldsymbol{\eta}_{d-1},\boldsymbol{\eta}_d}(t)\circ b_n^{\mathrm{sgn}(\boldsymbol{\eta}_d)}(t).
\end{aligned} \tag{2.4}$$

This completes the induction proof for the case $\boldsymbol{\eta}_d = 1$. If $\boldsymbol{\eta}_d = -1$ then we only need to modify the above proof slightly by replacing all the trigonometric functions by their hyperbolic counterpart, replacing b_n^+ by b_n^-, replacing ω_j by ω_{-j} for $j = 1,2,4$, and keeping an extra negative sign in the front of (2.3)–(2.4). Thus

$$\sum_{k>n} \frac{(-1)^k f_k^{\mathrm{sgn}(\boldsymbol{\eta}_{d-1})}(t)}{(2k)^s}\omega_{\boldsymbol{\eta}_{d-1}} = \begin{cases} -\omega_{-3-\boldsymbol{\eta}_{d-1}} b_n^-(t)\omega_{-1}, & \text{if } s=1, \\ -\omega_{\boldsymbol{\eta}_{d-1}}\omega_0^{s-2}\omega_{-1} b_n^-(t)\omega_{-1}, & \text{if } s\ge 2. \end{cases}$$

Combining the two cases we see that for $\eta_d = \pm 1$,

$$\begin{aligned}
\sum_{k>n} \frac{\eta_d^k f_k^{\mathrm{sgn}(\boldsymbol{\eta}_{d-1})}(t)}{(2k)^s}\omega_{\boldsymbol{\eta}_{d-1}} &= \begin{cases} \boldsymbol{\eta}_d\omega_{3\boldsymbol{\eta}_d-\boldsymbol{\eta}_{d-1}} b_n^{\mathrm{sgn}(\boldsymbol{\eta}_d)}(t)\omega_{\boldsymbol{\eta}_d}, & \text{if } s=1, \\ \boldsymbol{\eta}_d\omega_{\boldsymbol{\eta}_{d-1}}\omega_0^{s-2}\omega_{\boldsymbol{\eta}_d} b_n^{\mathrm{sgn}(\boldsymbol{\eta}_d)}(t)\omega_{\boldsymbol{\eta}_d}, & \text{if } s\ge 2 \end{cases} \\
&= \boldsymbol{\eta}_d \Gamma_{s_d}^{\boldsymbol{\eta}_{d-1},\boldsymbol{\eta}_d}(t)\circ b_n^{\mathrm{sgn}(\boldsymbol{\eta}_d)}(t).
\end{aligned}$$

This concludes the proof of the theorem.

Theorem 2.3 *Let $d \in \mathbb{N}$, $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$ and $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_d) \in \{\pm 1\}^d$. If $(s_1, \eta_1) \neq (1, 1)$ then*

$$\sigma(\boldsymbol{s}; \boldsymbol{\eta}; 1) = \sum_{n_1 > \cdots > n_d > 0} \frac{b_{n_1} \eta_1^{n_1} \cdots \eta_d^{n_d}}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}} \in \mathsf{CMZV}^8_{|\boldsymbol{s}|} \otimes \mathbb{Q}[\mathrm{i}, \sqrt{2}],$$

where $\mathsf{CMZV}^8_{|\boldsymbol{s}|}$ is the $\mathbb{Q}$-span of all CMZVs of weight $|\boldsymbol{s}|$ and level 8.

Proof This follows immediately from Theorem 2.2 by using the change of variables

$$t \to \frac{\sqrt{2}t}{\sqrt{1+t^4}}. \tag{2.5}$$

Indeed, let $\mu_j = \exp\left(\frac{(2j-1)\pi \mathrm{i}}{4}\right)$ $(j = 1, \cdots, 4)$ be the four 8th roots of unity satisfying $x^4 + 1 = 0$. Under (2.5) we get

$$\omega_0 := \frac{\mathrm{d}t}{t} \to \frac{(1-t^4)\mathrm{d}t}{t(1+t^4)} = \mathtt{a} + \frac{1}{2} \sum_{j=1}^4 \mathtt{x}_{\mu_j}, \tag{2.6}$$

$$\omega_1 := \frac{\mathrm{d}t}{\sqrt{1-t^2}} \to \frac{\sqrt{2}(1+t^2)\mathrm{d}t}{1+t^4} = \frac{\sqrt{2}}{4} \sum_{j=1}^4 (\mu_j + \mu_j^3) \mathtt{x}_{\mu_j}, \tag{2.7}$$

$$\omega_{-1} := \frac{\mathrm{d}t}{\sqrt{1+t^2}} \to \frac{\sqrt{2}(1-t^2)\mathrm{d}t}{1+t^4} = \frac{\sqrt{2}}{4} \sum_{j=1}^4 (\mu_j - \mu_j^3) \mathtt{x}_{\mu_j}, \tag{2.8}$$

$$\omega_2 := \frac{t\mathrm{d}t}{1-t^2} \to \frac{2(1+t^2)\mathrm{d}t}{(1-t^2)(1+t^4)} = \mathtt{x}_1 + \mathtt{x}_{-1} - \frac{1}{2} \sum_{j=1}^4 \mathtt{x}_{\mu_j}, \tag{2.9}$$

$$\omega_{-2} := \frac{t\mathrm{d}t}{1+t^2} \to \frac{2(1-t^2)\mathrm{d}t}{(1+t^2)(1+t^4)} = \frac{1}{2} \sum_{j=1}^4 \mathtt{x}_{\mu_j} - \mathtt{x}_{\mathrm{i}} - \mathtt{x}_{-\mathrm{i}}, \tag{2.10}$$

$$\omega_{-4} := \omega_4 = \frac{t\mathrm{d}t}{\sqrt{1-t^4}} \to \frac{2t\mathrm{d}t}{1+t^4} = \frac{1}{2} \sum_{j=1}^4 \mu_j^2 \mathtt{x}_{\mu_j}. \tag{2.11}$$

This completes the proof of the theorem.

Example 2.2 By the ideas used in the proof of Theorem 2.3, we can compute

$$\begin{aligned}
\sigma(\bar{1}, \bar{1}; 1) &= -\frac{1}{\sqrt{2}} \int_0^1 \omega_4 \omega_1 = -\frac{1}{8} \int_0^1 \Big(\sum_{j=1}^4 \mu_j^2 \mathtt{x}_{\mu_j} \Big) \Big(\sum_{k=1}^4 (\mu_k + \mu_k^3) \mathtt{x}_{\mu_k} \Big) \\
&= -\frac{1}{8} \sum_{j,k=1}^4 \mu_j^2 (\mu_k + \mu_k^3) \int_0^1 \mathtt{x}_{\mu_j} \mathtt{x}_{\mu_k} \\
&= -\frac{1}{8} \sum_{j,k=1}^4 \mu_j^2 (\mu_k + \mu_k^3) \operatorname{Li}_{1,1} \Big(\mu_j^{-1}, \frac{\mu_j}{\mu_k} \Big) \\
&\approx -0.5346431875726234
\end{aligned} \tag{2.12}$$

by using GP-Pari or Au's Mathematica package (see [3]). Similarly,

$$\sigma(\bar{2}, \bar{1}; 1) = -\int_0^1 \omega_{-1} \omega_4 \omega_1 = -\frac{1}{16} \int_0^1 \Big(\sum_{j=1}^4 (\mu_j - \mu_j^3) \mathtt{x}_{\mu_j} \Big) \Big(\sum_{l=1}^4 \mu_l^2 \mathtt{x}_{\mu_l} \Big) \Big(\sum_{k=1}^4 (\mu_k + \mu_k^3) \mathtt{x}_{\mu_k} \Big)$$

$$= -\frac{1}{16}\sum_{j,k,l=1}^{4}(\mu_j+\mu_j^3)\mu_l^2(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1,1}\left(\mu_j^{-1},\frac{\mu_j}{\mu_l},\frac{\mu_l}{\mu_k}\right)\approx 0.0851511799,$$

$$\sigma(\overline{2},1;1)=\int_0^1\omega_{-1}\omega_{-2}\omega_{-1}=\frac{1}{8}\int_0^1\sum_{j=1}^{4}(\mu_j-\mu_j^3)\mathrm{x}_{\mu_j}\Big(\frac{1}{2}\sum_{l=1}^{4}\mathrm{x}_{\mu_l}-\mathrm{x}_{\mathrm{i}}-\mathrm{x}_{-\mathrm{i}}\Big)\sum_{k=1}^{4}(\mu_k-\mu_k^3)\mathrm{x}_{\mu_k}$$

$$=\frac{1}{16}\sum_{j,k,l=1}^{4}(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1,1}\left(\mu_j^{-1},\frac{\mu_j}{\mu_l},\frac{\mu_l}{\mu_k}\right)$$

$$-\frac{1}{8}\sum_{j,k=1}^{4}\sum_{\varepsilon=\pm\mathrm{i}}(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1,1}\left(\mu_j^{-1},\frac{\mu_j}{\varepsilon},\frac{\varepsilon}{\mu_k}\right)$$

$$\approx 0.045805888486699.$$

We have checked these numerically by computing the series σ directly.

Corollary 2.2 *Let $d\in\mathbb{N}$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$ and $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_d)\in\{\pm1\}^d$. For every real algebraic point x with $|x|\le 1$ the value*

$$\sigma(\boldsymbol{s};\boldsymbol{\eta};x),$$

if it exists, can be expressed as a $\mathbb{Q}[\mathrm{i},\sqrt{2},\sqrt{1+\eta_1x^2}]$-linear combination of the multiple polylogarithms evaluated at algebraic points.

Proof Suppose $x\ne 0$. By Theorem 2.2, up to a factor of $\sqrt{1+\eta_1x^2}$ in front $\sigma(\boldsymbol{s};\boldsymbol{\eta};x)$ can be expressed as

$$\int_0^x[\omega_0,\omega_{\pm1},\omega_{\pm2},\omega_4]_{|\boldsymbol{s}|}, \tag{2.13}$$

where $[\omega_0,\omega_{\pm1},\omega_{\pm2},\omega_4]_{|\boldsymbol{s}|}$ is an iteration of 1-forms of length $|\boldsymbol{s}|$. Therefore, after applying the change of variables (2.5) we see that (2.13) is transformed to a $\mathbb{Q}[\mathrm{i},\sqrt{2}]$-linear combination of iterated integrals of the form

$$\int_0^{t(x)}[\mathrm{x}_0,\mathrm{x}_\mu:\mu^8=1]_{|\boldsymbol{s}|},\quad\text{where}\quad t(x)=\frac{x^2-\sqrt{1-x^4}}{x^2}.$$

Note $t(x)\to x$ under the change of variables (2.5). The corollary follows from (1.5) immediately.

Example 2.3 For $j=1,2,3,4$, we have the explicit evaluations

$$\sigma\Big(\overline{1};\frac{\sqrt{j}}{2}\Big)=-\sqrt{\frac{j}{4+j}}\int_0^{\frac{\sqrt{j}}{2}}\omega_{-1}=-\sqrt{\frac{j}{4+j}}\log\Big(\frac{\sqrt{j}+\sqrt{4+j}}{2}\Big).$$

Using the idea of (2.12) we see that for all $j=1,2,3,4$, we have

$$\sigma\Big(\overline{1},\overline{1};\frac{\sqrt{j}}{2}\Big)=-\sqrt{\frac{j}{4+j}}\int_0^{\frac{\sqrt{j}}{2}}\omega_4\omega_{-1}=-\frac{\sqrt{2j}}{8\sqrt{4+j}}\sum_{j,k=1}^{4}\mu_j^2(\mu_k+\mu_k^3)\int_0\mathrm{x}_{\mu_j}\mathrm{x}_{\mu_k}$$

$$= -\frac{\sqrt{2j}}{8\sqrt{4+j}} \sum_{j,k=1}^{4} \mu_j^2(\mu_k+\mu_k^3)\,\mathrm{Li}_{1,1}\Big(c(j)\mu_j^{-1}, \frac{\mu_j}{\mu_k}\Big),$$

where $c(j) = \sqrt{\frac{(4-\sqrt{16-j^2})}{j}}$ such that $c(j) \to \frac{\sqrt{j}}{2}$ under (2.5). We can similarly express $\sigma\big(1,\bar{1},\bar{1};\frac{\sqrt{j}}{2}\big)$. By Maple computation we find the numerical evaluations

$$\begin{aligned}
\sigma\Big(1,\bar{1},\bar{1};\frac{1}{2}\Big) &= -\frac{\sqrt{3}}{3}\int_0^{\frac{1}{2}} \omega_4^2\omega_1 \approx -0.001257459248252,\\
\sigma\Big(1,\bar{1},\bar{1};\frac{\sqrt{2}}{2}\Big) &= -\int_0^{\frac{\sqrt{2}}{2}} \omega_4^2\omega_1 \approx -0.013713567545998,\\
\sigma\Big(1,\bar{1},\bar{1};\frac{\sqrt{3}}{2}\Big) &= -\sqrt{3}\int_0^{\frac{\sqrt{3}}{2}} \omega_4^2\omega_1 \approx -0.08102265305753797,\\
\sigma\Big(\bar{1},\bar{1};\frac{1}{2}\Big) &= -\frac{1}{\sqrt{5}}\int_0^{\frac{1}{2}} \omega_4\omega_1 \approx -0.019408779689355473,\\
\sigma\Big(\bar{1},\bar{1};\frac{\sqrt{2}}{2}\Big) &= -\frac{1}{\sqrt{3}}\int_0^{\frac{\sqrt{2}}{2}} \omega_4\omega_1 \approx -0.07667150401885149,\\
\sigma\Big(\bar{1},\bar{1};\frac{\sqrt{3}}{2}) &= -\frac{\sqrt{3}}{\sqrt{7}}\int_0^{\frac{\sqrt{3}}{2}} \omega_4\omega_1 \approx -0.1845412608250132,\\
\sigma(\bar{1},\bar{1};1) &= -\frac{1}{\sqrt{2}}\int_0^{1} \omega_4\omega_1 \approx -0.5346431875726234.
\end{aligned} \tag{2.14}$$

The above have been verified numerically by directly computing the series and integrals separately. For the last equation also see Example 2.2.

3 Two Odd Variations of Alternating Apéry-Type Inverse Central Binomial Series

In this section, we consider variations of the alternating Apéry-type inverse binomial series studied in Section 2 by restricting the summation indices to odd numbers only. Recall that we have the following results. Define the 1-forms

$$h_s(t) := \begin{cases} 2\csc 2t\mathrm{d}t, & \text{if } s=1,\\ \csc t\mathrm{d}t \circ (\cot t\mathrm{d}t)^{s-2} \circ \csc t\mathrm{d}t, & \text{if } s \ge 2. \end{cases} \tag{3.1}$$

Theorem 3.1 (see [14, Theorem 2.3]) *For all $n \in \mathbb{N}_0$, $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$ we have the tail*

$$\begin{aligned}
&\tau^\star(\boldsymbol{s}; \sin y)_n\\
:= &\sum_{n_1 \ge \cdots \ge n_d \ge n} \frac{b_{n_1}(\sin y)}{(2n_1+1)^{s_1}\cdots(2n_d+1)^{s_d}} = \frac{\mathrm{d}}{\mathrm{d}y}\int_0^y h_{s_1}\circ\cdots h_{s_d}\circ b_n(\sin t)\mathrm{d}t.
\end{aligned} \tag{3.2}$$

Define the hyperbolic counterpart of $h_s(t)$ by

$$\widetilde{h}_s(t) = \widetilde{h}_s^+(t) = \begin{cases} 2\,\mathrm{csch}\, 2t\mathrm{d}t, & \text{if } s=1,\\ \mathrm{csch}\, t\mathrm{d}t \circ (\mathrm{cth}\, t\mathrm{d}t)^{s-2} \circ \mathrm{csch}\, t\mathrm{d}t, & \text{if } s \ge 2. \end{cases}$$

Theorem 3.2 *Set $\psi = \operatorname{sh}^{-1} 1 = \log \nu$. For all $n \in \mathbb{N}_0$, $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$, we have*

$$\tau^\star(\boldsymbol{s}; \mathrm{i}\operatorname{sh} y)_n = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \widetilde{h}_{s_1} \circ \cdots \circ \widetilde{h}_{s_d} \circ b_n(\mathrm{i}\operatorname{sh} t)\mathrm{d}t.$$

Here $y \in (-\psi, \psi)$ if $n_1 = 1$, and $y \in [-\psi, \psi]$ if $n_1 \geq 2$.

Proof Applying the substitution $y \to \mathrm{i}y$ in Theorem 3.1, we obtain the theorem immediately since $\csc \mathrm{i}t = -\mathrm{i}\operatorname{csch} t$ and $\operatorname{sech} \mathrm{i}t = \operatorname{sech} t$.

We now turn to another odd variation. Define the 1-forms

$$\kappa_s(t) = \begin{cases} \sin t\mathrm{d}t \csc t\mathrm{d}t + \tan t\mathrm{d}t, & \text{if } s = 1, \\ \sin t\mathrm{d}t(\cot t\mathrm{d}t + 1)(\cot t\mathrm{d}t)^{s-2} \csc t\mathrm{d}t, & \text{if } s \geq 2. \end{cases} \tag{3.3}$$

Theorem 3.3 (see [14, Theorem 3.1]) *For all $n \in \mathbb{N}_0$ and $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$, the tail*

$$\chi(\boldsymbol{s}; \sin y)_n := \sum_{n_1 > \cdots > n_d > n} \frac{b_{n_1}(\sin y)}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}} = \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \kappa_{s_1} \circ \cdots \circ \kappa_{s_d} \circ b_n(\sin t)\mathrm{d}t.$$

In the above $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ if $s_1 > 1$, and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $s_1 = 1$.

Define the hyperbolic counterpart of $\kappa_s(t)$ by

$$\widetilde{\kappa}_s(t) = \widetilde{\kappa}_s^+(t) = \begin{cases} -\operatorname{sh} t\mathrm{d}t \operatorname{csch} t\mathrm{d}t - \operatorname{th} t\mathrm{d}t, & \text{if } s = 1, \\ -\operatorname{sh} t\mathrm{d}t(\operatorname{cth} t\mathrm{d}t + 1)(\operatorname{cth} t\mathrm{d}t)^{s-2} \operatorname{csch} t\mathrm{d}t, & \text{if } s \geq 2. \end{cases}$$

Theorem 3.4 *Set $\psi = \operatorname{sh}^{-1} 1 = \log \nu$. For all $n \in \mathbb{N}_0$, $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$, we have*

$$\chi(\boldsymbol{s}; \mathrm{i}\operatorname{sh} y)_n := \sum_{n_1 > \cdots > n_d > n} \frac{b_{n_1}(\mathrm{i}\operatorname{sh} y)}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}} = (-1)^d \frac{\mathrm{d}}{\mathrm{d}y} \int_0^y \widetilde{\kappa}_{s_1} \circ \cdots \circ \widetilde{\kappa}_{s_d} \circ b_n(\mathrm{i}\operatorname{sh} t)\mathrm{d}t.$$

Here $y \in (-\psi, \psi)$ if $n_1 = 1$, and $y \in [-\psi, \psi]$ if $n_1 \geq 2$.

Proof Applying the substitution $y \to \mathrm{i}y$ in Theorem 3.3 we can prove the theorem easily since $\csc \mathrm{i}t = -\mathrm{i}\operatorname{csch} t$ and $\operatorname{sech} \mathrm{i}t = \operatorname{sech} t$.

4 Alternating Apéry-Type Central Binomial Series

We studied Apéry-type inverse binomial series in the previous sections. It is natural to see if the same idea works for the binomial series, too. In [15] we successfully carried out this investigation and proved that results similar to those in Theorem 1.2 and its odd-indexed versions, Theorem 3.1 and Theorem 3.3, still hold. We will generalize some of these to the alternating case in this section.

Put $a_n^+(x) = a_n(x) := \frac{\binom{2n}{n} x^{2n}}{4^n}$ and $a_n^+ = a_n := \frac{\binom{2n}{n}}{4^n}$. Define

$$f_{\pm 1}(t) := 1, \quad f_{\pm 2}(t) := \frac{t}{\sqrt{1 \mp t^2}}, \quad f_{\pm 3}(t) = \frac{1}{t}, \quad f_{\pm 20}(t) := \frac{1}{t\sqrt{1 \mp t^2}}, \quad f_5(t) = t.$$

Recall that the 1-forms $\omega_{\pm k}$ are defined by (2.6)–(2.11). Then for all the subscripts $k = 1, 2, 3, 4, 5, 20$, we have

$$f_{\pm k}(t)\omega_{\pm 1} = \omega_{\pm k}, \qquad \text{where } \omega_{\pm 20} = \omega_0 \pm \omega_{\pm 2}.$$

By [15], for all $n \geq 0$, $s \geq 1$, and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have

$$\begin{aligned}
\sum_{m>n} \frac{a_m(\sin y)}{(2m)^s} &= \int_0^y (\cot t\mathrm{d}t)^{s-1}(1 - \csc t\mathrm{d}t \circ \sec t) a_n(\sin t) \tan t\mathrm{d}t, \\
\sum_{m>n} \frac{a_m(\sin y)}{(2m+1)^s} &= \csc y \int_0^y (\cot t\mathrm{d}t)^{s-1}(1 - \mathrm{d}t \circ \csc t \sec t) a_n(\sin t) \sin t \tan t\mathrm{d}t, \\
\sum_{m\geq n} \frac{a_m(\sin y)}{(2m+1)^s} &= \csc y \int_0^y (\cot t\mathrm{d}t)^{s-1}(\csc t - \mathrm{d}t \circ \sec t) a_n(\sin t) \tan t\mathrm{d}t, \\
\sum_{m>n} \frac{a_m(\sin y)}{2m-1} &= \cos y \int_0^y a_n(\sin t) \tan t \sec t\mathrm{d}t, \\
\sum_{m>n} \frac{a_m(\sin y)}{(2m-1)^s} &= \sin y \int_0^y (\cot t\mathrm{d}t)^{s-2}(\cot^2 t\mathrm{d}t) a_n(\sin t) \tan t \sec t\mathrm{d}t, \quad s \geq 2.
\end{aligned}$$

With substitution $\sin y = x$, we get

$$\sum_{m>n} \frac{a_m(x)}{(2m)^s} = \int_0^x \omega_0^{s-1}\Big(1 - \omega_3 \circ \frac{1}{\sqrt{1-t^2}}\Big) a_n(t)\omega_2, \tag{4.1}$$

$$\begin{aligned}
\sum_{m\geq n} \frac{a_m(x)}{(2m+1)^s} &= \frac{1}{x}\int_0^x \omega_0^{s-1}\Big(\frac{1}{t} - \omega_1 \circ \frac{1}{\sqrt{1-t^2}}\Big) a_n(t)\omega_2, \\
\sum_{m>n} \frac{a_m(x)}{2m-1} &= \sqrt{1-x^2}\int_0^x a_n(t)\frac{t\mathrm{d}t}{(1-t^2)^{\frac{3}{2}}}, \\
\sum_{m>n} \frac{a_m(x)}{(2m-1)^s} &= x\int_0^x \omega_0^{s-2}\frac{\sqrt{1-t^2}\mathrm{d}t}{t^2} a_n(t)\frac{t\mathrm{d}t}{(1-t^2)^{\frac{3}{2}}}, \quad s \geq 2.
\end{aligned}$$

The above iteration formulas form the foundation of all the results in [15]. Set $a_m^-(x) = (-1)^m a_m(x)$. Applying substitution $x \to \mathrm{i}x$ in the above we obtain, for all $s \geq 1$,

$$\sum_{m>n} \frac{a_m^-(x)}{(2m)^s} = \int_0^x \omega_0^{s-1}\Big(\omega_{-3} \circ \frac{1}{\sqrt{1+t^2}} - 1\Big) a_n^-(t)\omega_{-2}, \tag{4.2}$$

$$\sum_{m\geq n} \frac{a_m^-(x)}{(2m+1)^s} = \frac{1}{x}\int_0^x \omega_0^{s-1}\Big(\frac{1}{t} + \omega_{-1} \circ \frac{1}{\sqrt{1+t^2}}\Big) a_n^-(t)\omega_{-2}, \tag{4.3}$$

$$\sum_{m>n} \frac{a_m^-(x)}{2m-1} = -\sqrt{1+x^2}\int_0^x a_n^-(t)\frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}, \tag{4.4}$$

$$\sum_{m>n} \frac{a_m^-(x)}{(2m-1)^s} = x\int_0^x \omega_0^{s-2}\frac{\sqrt{1+t^2}\mathrm{d}t}{t^2} a_n^-(t)\frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}, \quad s \geq 2. \tag{4.5}$$

By repeatedly applying (4.2)–(4.5), we can find many results analogous to those in Section 2. See [15] for the approach we used to study the non-alternating version.

Fix a primitive 8th root of unity $\mu = \frac{(1+\mathrm{i})}{\sqrt{2}}$. Using the idea in the proof for the non-alternating case we can prove the following theorem.

Theorem 4.1 *Let $d \in \mathbb{N}$, $\boldsymbol{s} = (s_1, \cdots, s_d) \in \mathbb{N}^d$ and $\eta = \pm 1$. Let $l_j(n) = 2n$ or $l_j(n) = 2n \pm 1$ for all $1 \le j \le d$. Then*

$$\sum_{n_1 \succ n_2 \succ \cdots \succ n_d \succ 0} \frac{a_{n_1} \eta^{n_1}}{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^8_{\le |\boldsymbol{s}|} \otimes \mathbb{Q}[\mathrm{i}, \sqrt{2}], \tag{4.6}$$

where "$\succ$" can be either "$\ge$" or "$>$", provided the series is defined.

Proof If $\eta = 1$ then the Apéry-like sum (4.6) is in $\mathsf{CMZV}^4_{|\boldsymbol{s}|}$ by [15, Theorem 5.3]. If $\eta = -1$, by the same proof for [13, Theorem 5.3] we can show that this sum can be expressed as an iterated integral involving only the following 1-forms:

First block: $\operatorname{csch} t \mathrm{d}t,\ \operatorname{cth} t\, \mathrm{d}t,\ \mathrm{d}t$ (see [13, (5.16)]),

Mid blocks: $\operatorname{th} t \mathrm{d}t,\ \operatorname{sech} t \operatorname{csch} t\, \mathrm{d}t$ (see [13, (5.21)]),

and no additional 1-forms can appear inside the end block. Under the change of variables $t \to \operatorname{sh}^{-1} y$ we have

$$\mathrm{d}t \to \omega_{-1}, \quad \operatorname{cth} t \mathrm{d}t \to \omega_0, \quad \operatorname{th} t \mathrm{d}t \to \omega_{-2}, \quad \operatorname{csch} t \mathrm{d}t \to \omega_{-3}, \quad \operatorname{sech} t \operatorname{csch} t \to \omega_{-20}.$$

Under the change of variables $t \to \frac{\mathrm{i}(1-t^2)}{1+t^2}$, we get

$$\omega_0 = \mathtt{a} = \frac{\mathrm{d}t}{t} \to \mathtt{y}, \quad \omega_{-1} = \frac{\mathrm{d}t}{\sqrt{1+t^2}} \to \mathtt{d}_{\mathrm{i},-\mathrm{i}}, \quad \omega_{-2} = \frac{t\mathrm{d}t}{1+t^2} \to -\mathtt{z}, \tag{4.7}$$

$$\omega_{-3} = \frac{\mathrm{d}t}{t\sqrt{1+t^2}} \to \mathtt{d}_{-1,1}, \quad \omega_{-20} = \frac{\mathrm{d}t}{t(1+t^2)} \to \mathtt{y} + \mathtt{z} = -\mathtt{a} - \mathtt{x}_{-1} - \mathtt{x}_1, \tag{4.8}$$

where $\mathtt{y} = \mathtt{x}_{-\mathrm{i}} + \mathtt{x}_{\mathrm{i}} - \mathtt{x}_{-1} - \mathtt{x}_1$, $\mathtt{z} = -\mathtt{a} - \mathtt{x}_{-\mathrm{i}} - \mathtt{x}_{\mathrm{i}}$ and $\mathtt{d}_{\xi,\xi'} = \mathtt{x}_\xi - \mathtt{x}_{\xi'}$ for any two roots of unity ξ and ξ'. We see that all the ω's transform according to (4.7)–(4.8). Hence the sum (4.6) can be expressed by $\mathbb{Q}[\mathrm{i}, \sqrt{2}]$-linear combination of convergent iterated integrals of the form

$$\int_1^\mu \mathtt{x}_{\gamma_1} \cdots \mathtt{x}_{\gamma_k}, \quad \gamma_j \in \{0, \mu^e : 0 \le e \le 7\}.$$

Now we can apply Chen's theory of iterated integrals (see [16, Lemma 2.1.2(ii)]) to get

$$\begin{aligned} \int_1^\mu \mathtt{x}_{\gamma_1} \cdots \mathtt{x}_{\gamma_k} &= \sum_{\ell=0}^{k} \Big(\operatorname{Reg} \int_0^\mu \mathtt{x}_{\gamma_1} \cdots \mathtt{x}_{\gamma_\ell} \Big) \Big(\operatorname{Reg} \int_1^0 \mathtt{x}_{\gamma_{\ell+1}} \cdots \mathtt{x}_{\gamma_{k-1}} \mathtt{x}_{\gamma_k} \Big) \\ &= (-1)^{k-\ell} \sum_{\ell=0}^{k} \Big(\operatorname{Reg} \int_0^1 \mathtt{x}_{\frac{\gamma_1}{\mu}} \cdots \mathtt{x}_{\frac{\gamma_\ell}{\mu}} \Big) \Big(\operatorname{Reg} \int_0^1 \mathtt{x}_{\gamma_k} \mathtt{x}_{\gamma_{k-1}} \cdots \mathtt{x}_{\gamma_{\ell+1}} \Big), \end{aligned}$$

where Reg means one should take regularized values of the possible divergent integrals (see [16, 13.3.1] for detailed treatment of this mechanism). These regularized values are all polynomials of T whose coefficients are all given by CMZVs of level 8. Therefore the theorem follows by specializing at $T = 0$. This completes the proof of the theorem.

Remark 4.1 When computing concrete examples, sometimes one can avoid to use the full regularization mechanism. Instead, one can often use shuffle products and/or substitutions to combine all divergent integrals into convergent ones. See Examples 4.5 and 5.5 for applications of these ideas.

We now present a few enlightening examples.

Example 4.1 By (4.2), for all $s \geq 1$ we have

$$\begin{aligned}\sum_{n>0} \frac{(-1)^n a_n(x)}{(2n)^s} &= \int_0^x \omega_0^{s-1}\Big(\omega_{-3} \circ \frac{1}{\sqrt{1+t^2}} - 1\Big)\omega_{-2} \\ &= \int_0^x \omega_0^{s-1}\Big(\frac{\mathrm{d}t}{t\sqrt{1+t^2}} \circ \frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} - \frac{t\mathrm{d}t}{1+t^2}\Big) \\ &= \int_0^x \omega_0^{s-1}\Big(\frac{\mathrm{d}t}{t\sqrt{1+t^2}} \cdot \Big[\frac{-1}{\sqrt{1+u^2}}\Big]_0^t - \frac{t\mathrm{d}t}{1+t^2}\Big) \\ &= \int_0^x \omega_0^{s-1}(\omega_{-3} - \omega_{-20} - \omega_{-2}) = \int_0^1 \omega_0^{s-1}(\omega_{-3} - \omega_0).\end{aligned}$$

Applying the change of variables $t \to \frac{\mathrm{i}(1-t^2)}{1+t^2}$ and using (4.7) we see that

$$\sum_{n>0} \frac{(-1)^n a_n(x)}{(2n)^s} = \int_1^{\lambda(x)} \mathtt{y}^{s-1}\mathtt{c}, \tag{4.9}$$

where $\lambda(x) = \sqrt{\frac{1+\mathrm{i}x}{1-\mathrm{i}x}}$ and $\mathtt{c} = 2\mathtt{x}_{-1} - \mathtt{x}_{-\mathrm{i}} - \mathtt{x}_{\mathrm{i}}$. We can take $x = \frac{\sqrt{j}}{2}$ $(1 \leq j \leq 4)$ in (4.9) to get

$$\sum_{n>0} \frac{(-j)^n\binom{2n}{n}}{16^n(2n)} = \log \frac{t^2+1}{(t+1)^2}\Big|_1^{\lambda(\frac{\sqrt{j}}{2})} = \log\Big(\frac{4}{j}(\sqrt{j+4}-2)\Big).$$

Setting $\mathtt{y}_\mu = \mathtt{x}_{-\frac{\mathrm{i}}{\mu}} + \mathtt{x}_{\frac{\mathrm{i}}{\mu}} - \mathtt{x}_{-\frac{1}{\mu}} - \mathtt{x}_{\frac{1}{\mu}}$ and $\mathtt{c}_\mu = 2\mathtt{x}_{-\frac{1}{\mu}} - \mathtt{x}_{-\frac{\mathrm{i}}{\mu}} - \mathtt{x}_{\frac{\mathrm{i}}{\mu}}$, and noticing that $\lambda(1) = \mu$, we obtain

$$\begin{aligned}\sum_{n>0} \frac{(-1)^n a_n}{(2n)^2} &= \int_1^0 \mathtt{yc} + \int_0^\mu \mathtt{y} \int_1^0 \mathtt{c} + \int_0^\mu \mathtt{yc} = \int_0^1 \mathtt{cy} - \int_0^1 \mathtt{y}_\mu \int_0^1 \mathtt{c} + \int_0^1 \mathtt{y}_\mu \mathtt{c}_\mu \\ &= \frac{\pi^2}{8} - \frac{1}{2}(\log^2 2 - 2\log 2 \log \nu + 2\log^2 \nu + 4\,\mathrm{Li}_2(\nu^{-1})) \\ &\approx -0.1074917339.\end{aligned} \tag{4.10}$$

Example 4.2 For all $s \geq 1$, using (4.3) we have

$$\begin{aligned}\sum_{n\geq 0} \frac{(-1)^n a_n(x)}{(2n+1)^s} &= \frac{1}{x}\int_0^x \omega_0^{s-1}\Big(\frac{1}{t} + \omega_{-1} \circ \frac{1}{\sqrt{1+t^2}}\Big)\omega_{-2} \\ &= \frac{1}{x}\int_0^x \omega_0^{s-1}\Big(\frac{\mathrm{d}t}{1+t^2} + \omega_{-1} \circ \frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}\Big) \\ &= \frac{1}{x}\int_0^x \omega_0^{s-1}\Big(\frac{\mathrm{d}t}{1+t^2} - \omega_{-1} \cdot \Big[\frac{1}{\sqrt{1+u^2}}\Big]_0^t\Big)\end{aligned}$$

$$=\frac{1}{x}\int_0^x \omega_0^{s-1}\omega_{-1}=\frac{1}{x}\int_1^{\lambda(x)} \mathtt{y}^{s-1}\mathtt{d}_{\mathrm{i},-\mathrm{i}}$$

by the change of variables $t\to\frac{\mathrm{i}(1-t^2)}{1+t^2}$. Of course, when $x=\frac{\sqrt{j}}{2}$ $(1\le j\le 4)$ and $s=1$ one can integrate without change of variables to get

$$\sum_{n>0}\frac{(-j)^n\binom{2n}{n}}{16^n(2n+1)}=\frac{2}{\sqrt{j}}\int_0^{\frac{\sqrt{j}}{2}}\omega_{-1}=\frac{2}{\sqrt{j}}\log(u+\sqrt{1+u^2})\Big|_0^{\frac{\sqrt{j}}{2}}=\frac{2}{\sqrt{j}}\log\Big(\frac{\sqrt{j}+\sqrt{4+j}}{2}\Big).$$

Similarly to Example 4.1, when $x=1$ and $s=2$ we can replace $\mathtt{c}$ by $\mathtt{d}_{\mathrm{i},-\mathrm{i}}$ in (4.10) to get

$$\begin{aligned}\sum_{n>0}\frac{(-1)^n a_n}{(2n+1)^2}&=\int_0^1 \mathtt{d}_{\mathrm{i},-\mathrm{i}}\mathtt{y}-\int_0^1\mathtt{y}_\mu\int_0^1\mathtt{d}_{\mathrm{i},-\mathrm{i}}+\int_0^1\mathtt{y}_\mu\mathtt{d}_{\frac{\mathrm{i}}{\mu},-\frac{\mathrm{i}}{\mu}}\\&=\frac{5\pi^2}{24}+\log 2\log\nu-\log^2\nu-2\,\mathrm{Li}_2(\nu^{-1})\approx 0.9552018.\end{aligned}$$

Example 4.3 For any $s\ge 1$, (4.2)–(4.3) and the computation in Example 4.1 yield that

$$\begin{aligned}\sum_{n\ge m>0}\frac{(-1)^n a_n}{(2n+1)^s(2m)}&=\int_0^1\omega_0^{s-1}\Big(\frac{1}{t}+\omega_{-1}\circ\frac{1}{\sqrt{1+t^2}}\Big)\sum_{m>0}\frac{a_m^-(t)}{2m}\omega_{-2}\\&=\int_0^1\omega_0^{s-1}\Big(\frac{1}{t}+\omega_{-1}\circ\frac{1}{\sqrt{1+t^2}}\Big)\omega_{-2}(\omega_{-3}-\omega_0)\\&=\int_0^1\omega_0^{s-1}\Big(\frac{\mathrm{d}t}{1+t^2}+\omega_{-1}\circ\frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}\Big)(\omega_{-3}-\omega_0).\end{aligned}$$

Since

$$\int_{t_2}^{t_1}\frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}=\frac{1}{\sqrt{1+t_2^2}}-\frac{1}{\sqrt{1+t_1^2}},$$

we have

$$\begin{aligned}\sum_{n\ge m>0}\frac{(-1)^n a_n}{(2n+1)^s(2m)}&=\int_0^1\omega_0^{s-1}\omega_{-1}\Big[\frac{1}{\sqrt{1+t^2}}(\omega_{-3}-\omega_0)\Big]\\&=\int_0^1\omega_0^{s-1}\omega_{-1}(\omega_{-20}-\omega_{-3})=\int_1^\mu\mathtt{y}^{s-1}\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}\end{aligned}$$

by the change of variables $t\to\frac{\mathrm{i}(1-t^2)}{1+t^2}$, where $\mathtt{e}=\mathtt{x}_0+2\mathtt{x}_{-1}$. If $s=1$ then we get

$$\sum_{n\ge m>0}\frac{(-1)^n a_n}{(2n+1)(2m)}=\int_1^\mu\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}=\lim_{\varepsilon\to 0}\Big(\int_1^\varepsilon\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}+\int_\varepsilon^\mu\mathtt{d}_{-\mathrm{i},\mathrm{i}}\int_1^\varepsilon\mathtt{e}+\int_\varepsilon^\mu\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}\Big)$$

$$=\lim_{\varepsilon\to 0}\Big(\int_\varepsilon^1\mathtt{e}\mathtt{d}_{-\mathrm{i},\mathrm{i}}-\int_\varepsilon^\mu\mathtt{d}_{-\mathrm{i},\mathrm{i}}\Big(\int_\varepsilon^\mu\mathtt{e}+\int_\mu^1\mathtt{e}\Big)+\int_\varepsilon^\mu\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}\Big)$$

$$=\int_0^1\mathtt{e}\mathtt{d}_{-\mathrm{i},\mathrm{i}}-\int_0^\mu\mathtt{d}_{-\mathrm{i},\mathrm{i}}\int_\mu^1\mathtt{e}-\int_0^\mu\mathtt{e}\mathtt{d}_{-\mathrm{i},\mathrm{i}}=\frac{5\pi^2}{48}-\log 2\log\nu-\mathrm{Li}_2(\nu^{-1})\approx -0.0503718221$$

by using Au's mathematica package (see [3]). When $s=2$ we obtain

$$\sum_{n\ge m>0}\frac{(-1)^n a_n}{(2n+1)^2(2m)}=\lim_{\varepsilon\to 0}\Big(\int_1^\varepsilon\mathtt{y}\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}+\int_\varepsilon^\mu\mathtt{y}\int_1^\varepsilon\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}+\int_\varepsilon^\mu\mathtt{y}\mathtt{d}_{-\mathrm{i},\mathrm{i}}\int_1^\varepsilon\mathtt{e}+\int_\varepsilon^\mu\mathtt{y}\mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{e}\Big)$$

$$= \lim_{\varepsilon\to 0}\Big(-\int_\varepsilon^1 \texttt{ed}_{-\mathrm{i},\mathrm{i}}\texttt{y} + \int_\varepsilon^\mu \texttt{y}\int_\varepsilon^1 \texttt{ed}_{-\mathrm{i},\mathrm{i}} - \int_\varepsilon^\mu \texttt{yd}_{-\mathrm{i},\mathrm{i}}\Big(\int_\varepsilon^\mu \texttt{e} + \int_\mu^1 \texttt{e}\Big) + \int_\varepsilon^\mu \texttt{yd}_{-\mathrm{i},\mathrm{i}}\texttt{e}\Big)$$
$$= -\int_0^1 \texttt{ed}_{-\mathrm{i},\mathrm{i}}\texttt{y} + \int_0^\mu \texttt{y}\int_0^1 \texttt{ed}_{-\mathrm{i},\mathrm{i}} - \int_0^\mu \texttt{yd}_{-\mathrm{i},\mathrm{i}}\int_\mu^1 \texttt{e} - \int_0^\mu \texttt{yed}_{-\mathrm{i},\mathrm{i}} - \int_0^\mu \texttt{eyd}_{-\mathrm{i},\mathrm{i}}$$
$$= \frac{9}{8}\zeta(3) - \frac{8}{3}\sqrt{2}L(3,\chi_8) - \frac{\pi^2}{48}\log 2 - \frac{\log^3 2}{12} - \frac{3\log^2 2}{2}\log\nu + 2\log 2\log^2\nu - \frac{2}{3}\log^3\nu$$
$$+ (3\log 2 - 2\log\nu)\,\mathrm{Li}_2(\nu^{-1}) + 4\,\mathrm{Li}_3\Big(\frac{1}{\sqrt{2}}\Big) - 2\,\mathrm{Li}_2\,3(\nu^{-1}) \approx -0.02023197786,$$

where $L(3,\chi_8)$ is the Dirichlet L-function with the primitive character χ_8 modulo 8 satisfying $\chi_8(3) = \chi_8(5) = -1$ and $\chi_8(1) = \chi_8(7) = 1$.

Example 4.4 With the same idea of iteration, (4.2) and (4.4) imply that

$$\sum_{n>m>0} \frac{(-1)^n a_n}{(2n-1)(2m)} = -\sqrt{2}\int_0^1 \frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}\sum_{m>0}\frac{a_m^-(t)}{2m}$$
$$= \sqrt{2}\int_0^1 \Big[\frac{1}{\sqrt{1+t^2}}\Big]_t^1 (\omega_{-3} - \omega_0)$$
$$= \int_0^1 (\omega_{-3} - \omega_0) - \sqrt{2}\int_0^1 \frac{1}{\sqrt{1+t^2}}(\omega_{-3} - \omega_0)$$
$$= \int_0^1 (\omega_{-3} - \omega_0) - \sqrt{2}\int_0^1 \omega_{-20} - \omega_{-3}$$
$$= \nu\int_0^1 (\omega_{-3} - \omega_0) + \sqrt{2}\int_0^1 \omega_{-2}$$
$$= \nu(\log 2 - \log(\sqrt{2}+1)) + \frac{\sqrt{2}}{2}\log 2$$
$$= \Big(1 + \frac{3\sqrt{2}}{2}\Big)\log 2 - \nu\log\nu \approx 0.035710328462762$$

by Example 4.1. Note that the weight at the end drops by 1 as is the general case when $l_1(n) = 2n-1$.

Example 4.5 As a last example in this section, we consider a sum not covered by Theorem 4.1 since the 1-form

$$\omega_6 := \frac{\mathrm{d}t}{t\sqrt{1-t^4}}$$

appears. Using (4.1) and (4.4) we get

$$\sum_{n>m>0} \frac{(-1)^{n+m} a_n}{(2n-1)(2m)} = -\sqrt{2}\int_0^1 \frac{t\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}\sum_{m>0}\frac{a_m(t)}{2m}$$
$$= \sqrt{2}\int_0^1 \Big[\frac{1}{\sqrt{1+t^2}}\Big]_t^1 \Big(1 - \omega_3 \circ \frac{1}{\sqrt{1-t^2}}\Big)\omega_2$$
$$= \int_0^1 (\omega_3 + \omega_2 - \omega_{20}) - \sqrt{2}\int_0^1 \frac{1}{\sqrt{1+t^2}}(\omega_3 + \omega_2 - \omega_{20})$$
$$= \int_0^1 (2\texttt{x}_{-1} - \texttt{x}_{\mathrm{i}} - \texttt{x}_{-\mathrm{i}}) + \sqrt{2}\int_0^1 \frac{1}{\sqrt{1+t^2}}(\texttt{a} - \omega_3) \quad \Big(\text{by } t \to \frac{1-t^2}{1+t^2} \text{ in the first integral}\Big)$$

$$= \log 2 + \sqrt{2}\int_0^1 (\omega_{-3} - \omega_6).$$

To compute the last integral we use the regularization trick as follows:

$$\int_0^1 (\omega_{-3} - \omega_6) = \lim_{\varepsilon\to 0}\Big(\int_\varepsilon^1 \omega_{-3} - \int_\varepsilon^1 \omega_6\Big) = \lim_{\varepsilon\to 0}\Big(\int_\varepsilon^1 \omega_{-3} - \frac{1}{2}\int_{\varepsilon^2}^1 \omega_3\Big)$$

by the change of variables $t \to \sqrt{t}$ in the second integral. Hence by the change of variables $t \to \frac{\mathrm{i}(1-t^2)}{1+t^2}$ in the first integral and $t \to \frac{1-t^2}{1+t^2}$ in the second integral, we see that

$$\int_0^1 (\omega_{-3} - \omega_6) = \lim_{\varepsilon\to 0}\Big(\int_{\lambda(\varepsilon)}^{\lambda(1)} \mathtt{d}_{-1,1} - \frac{1}{2}\int_{\lambda(\varepsilon^2)}^{0} \mathtt{d}_{-1,1}\Big).$$

Since $\lambda(-\mathrm{i}) = \mu$ we have

$$\begin{aligned}\int_0^1 \Big(\omega_{-3} - \frac{\mathrm{d}t}{t\sqrt{1-t^4}}\Big) &= \lim_{\varepsilon\to 0}\Big(\log\frac{t-1}{t+1}\Big|_{\lambda(-\mathrm{i}\varepsilon)}^{\lambda(-\mathrm{i})} - \frac{1}{2}\log\frac{t-1}{t+1}\Big|_{\lambda(\varepsilon^2)}^{0}\Big)\\ &= \log\frac{\mu-1}{\mu+1} + \lim_{\varepsilon\to 0}\Big(\log\frac{\sqrt{1+\mathrm{i}\varepsilon}+\sqrt{1-\mathrm{i}\varepsilon}}{\sqrt{1+\mathrm{i}\varepsilon}-\sqrt{1-\mathrm{i}\varepsilon}} + \frac{1}{2}\log\frac{\sqrt{1+\varepsilon^2}-\sqrt{1-\varepsilon^2}}{\sqrt{1+\varepsilon^2}+\sqrt{1-\varepsilon^2}}\Big)\\ &= \log\frac{\mu-1}{\mu+1} + \log(-\sqrt{2}\mathrm{i}) = \log\sqrt{2} + \log\frac{\mathrm{i}(1-\mu)}{\mu+1} = \log\sqrt{2} - \log\nu.\end{aligned}$$

Thus

$$\sum_{n>m>0}\frac{(-1)^{n+m}a_n}{(2n-1)(2m)} = \Big(1+\frac{\sqrt{2}}{2}\Big)\log 2 - \sqrt{2}\log\nu \approx -0.063174227986,$$

which is clearly a value in $\mathsf{CMZV}_1^8 \otimes \mathbb{Q}[\mathrm{i},\sqrt{2}]$.

5 Alternating Apéry-Type Inverse Central Binomial Series with Summation Indices of Mixed Parities

We now return to the inverse binomial series and consider their alternating versions in this section. Keeping the same notation as before, we write $b_n^+(x) := b_n(x)$ and

$$b_n^-(x) = b_n(\mathrm{i}x) = (-1)^n 4^n \binom{2n}{n}^{-1} x^{2n}, \quad b_n^- = b_n^-(1) = (-1)^n 4^n \binom{2n}{n}^{-1}, \quad \forall n \ge 0.$$

Combining [14, (4.3)–(4.8)] with Theorems 2.1, 3.2 and 3.4 yield that

$$\sum_{n_1>n}\frac{b_{n_1}^\pm(x)}{2n_1} = \pm f_{\pm 2}(x)\int_0^x b_n^\pm(t)\omega_{\pm 1}, \tag{5.1}$$

$$\sum_{n_1>n}\frac{b_{n_1}^\pm(x)}{(2n_1)^s} = \pm f_1(x)\int_0^x \omega_0^{s-2}\omega_{\pm 1}b_n^\pm(t)\omega_{\pm 1}, \quad \forall s \ge 2, \tag{5.2}$$

$$\sum_{n_1\ge n}\frac{b_{n_1}^\pm(x)}{2n_1+1} = f_{\pm 20}(x)\int_0^x b_n^\pm(t)\omega_{\pm 1}, \tag{5.3}$$

$$\sum_{n_1\ge n}\frac{b_{n_1}^\pm(x)}{(2n_1+1)^s} = f_3(x)\int_0^x \omega_0^{s-2}\omega_{\pm 3}b_n^\pm(t)\omega_{\pm 1}, \quad \forall s \ge 2, \tag{5.4}$$

$$\sum_{n_1>n} \frac{b^{\pm}_{n_1}(x)}{2n_1-1} = \pm f_5(x)\int_0^x \omega_{\pm 3} b^{\pm}_n(t)\omega_{\pm 1} \pm f_{\pm 2}(x)\int_0^x b^{\pm}_n(t)\omega_{\pm 1}, \tag{5.5}$$

$$\sum_{n_1>n} \frac{b^{\pm}_{n_1}(x)}{(2n_1-1)^s} = \pm f_5(x)\int_0^x (\omega_0+1)\omega_0^{s-2}\omega_{\pm 3} b^{\pm}_n(t)\omega_{\pm 1}, \quad \forall s\geq 2. \tag{5.6}$$

By applying (5.1)–(5.6) iteratively it is possible to express every alternating Apéry-type inverse binomial series with summation indices of mixed parities by an iterated integral involving only 1-forms $\omega_{\pm j}$ $(0\leq j\leq 6)$, where $\omega_{\pm 5} = \frac{t\mathrm{d}t}{\sqrt{1\mp t^2}}$ and $\omega_{\pm 6} = \frac{\mathrm{d}t}{t\sqrt{1-t^4}}$. Unfortunately, $\omega_{\pm 3}$, $\omega_{\pm 5}$ and $\omega_{\pm 6}$ transform badly under the change of variables (2.5). Using the idea in the proof for the non-alternating case we can slightly extend Theorem 2.3 to the following more general form.

Theorem 5.1 *Let $d\in\mathbb{N}$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$ and $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_d)\in\{\pm 1\}^d$. Let $l_j(n)=2n$ or $l_j(n)=2n+1$ for all $1\leq j\leq d$ so that $s_j=1$ and $\eta_j=\operatorname{sgn}(j-1.5)$ if $l_j(n)=2n+1$. If $(s_1,\eta_1)\neq(1,1)$ then*

$$\sum_{n_1>\cdots>n_d>0} \frac{b_{n_1}\eta_1^{n_1}\cdots\eta_d^{n_d}}{l_1(n_1)^{s_1}\cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^8_{|\boldsymbol{s}|}\otimes\mathbb{Q}[\mathrm{i},\sqrt{2}].$$

Proof We only need to consider the appearance of $l_j(n)=2n+1$ raised to the first power on the denominator. If it only appear at the beginning then we need to have $\eta_1=-1$ to guarantee convergence, in which case the theorem is clear since (5.3) only involves the 1-form $\omega_{\pm 1}$. If $l_j(n)=2n+1$ appears for some $j\geq 2$ then (5.3) may produce $\omega_{\pm 20}=\omega_0\pm\omega_{\pm 2}$ after combining $f_{\pm 20}$ with the 1-form $\omega_{\pm 1}$ produced by the proceeding iteration since $\eta_j=1$ guarantees the same sign pattern. The rest of the proof follows from the same reasoning as used in that of Theorem 2.3 and thus is left to the interested reader.

Remark 5.1 If $l_j(n)=2n+1$ appears for some $j\geq 2$ but $\eta_j=-1$ then the 1-form ω_6 is produced. We do not know how to handle this in general with our approach yet.

The following result as well as its proof is analogous to Theorem 4.1.

Theorem 5.2 *Let $d\in\mathbb{N}$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$ and $\eta=\pm 1$. Let $l_j(n)=2n$ or $l_j(n)=2n\pm 1$ for all $1\leq j\leq d$. If $(s_1,\eta_1)\neq(1,1)$ then*

$$\sum_{n_1\succ n_2\succ\cdots\succ n_d\succ 0} \frac{b_{n_1}\eta^{n_1}}{l_1(n_1)^{s_1}\cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^8_{\leq|\boldsymbol{s}|}\otimes\mathbb{Q}[\mathrm{i},\sqrt{2}], \tag{5.7}$$

where "$\succ$" can be either "$\geq$" or "$>$", provided the series is defined.

Proof The proof is almost exactly the same as that of Theorem 4.1 so we omit the details here. The key idea is to follow the proof of [14, Theorem 4.3] step by step but use the substitution $t\to\frac{\mathrm{i}(1-t^2)}{1+t^2}$ (instead of $t\to\frac{1-t^2}{1+t^2}$ in the original proof) at the end.

We now consider several alternating Apéry-type inverse binomial series with summation indices having mixed parity.

Example 5.1 Using (5.1) and (5.3) successively, we get

$$\begin{aligned}
&\sum_{n\geq m>0}\frac{(-1)^n b_n}{(2n+1)(2m)}=\sum_{n\geq m>0}\frac{b_n^-(1)}{(2n+1)(2m)}\\
&=\frac{1}{\sqrt{2}}\int_0^1\sum_{m>0}\frac{b_m^-(t)}{(2m)}\omega_{-1}=\frac{-1}{\sqrt{2}}\int_0^1\omega_{-2}\omega_{-1}\\
&=\frac{1}{4}\int_0^1\Big(\mathtt{x}_{\mathrm{i}}+\mathtt{x}_{-\mathrm{i}}-\frac{1}{2}\sum_{l=1}^4\mathtt{x}_{\mu_l}\Big)\sum_{k=1}^4(\mu_k-\mu_k^3)\mathtt{x}_{\mu_k}\\
&=\frac{1}{8}\sum_{k=1}^4\sum_{\varepsilon=\pm\mathrm{i}}(\mu_k-\mu_k^3)\int_0^1\mathtt{x}_\varepsilon\mathtt{x}_{\mu_k}-\frac{1}{4}\sum_{k,l=1}^4(\mu_k-\mu_k^3)\int_0^1\mathtt{x}_{\mu_l}\mathtt{x}_{\mu_k}\\
&=\frac{1}{4}\sum_{k,l=1}^4(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1}\Big(\mu_l^{-1},\frac{\mu_l}{\mu_k}\Big)-\frac{1}{8}\sum_{k=1}^4\sum_{\varepsilon=\pm\mathrm{i}}(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1}\Big(\varepsilon^{-1},\frac{\varepsilon}{\mu_k}\Big)\\
&=\frac{3\sqrt{2}}{32}(8\log^2\nu+16\,\mathrm{Li}_2(\sqrt{2}-1)-8\log 2\log\nu-\pi^2)\approx -0.14078648719
\end{aligned}$$

by GP-Pari or Au's mathematica package (see [3]). We have also verified this by numerically computing the series directly and by numerically evaluating the integral with mathematica:

$$\frac{1}{\sqrt{2}}\int_0^1\omega_{-2}\omega_{-1}=\frac{1}{\sqrt{2}}\int_0^1\frac{t\ln(t+\sqrt{1+t^2})}{1+t^2}\mathrm{d}t\approx -0.14078648719.$$

Example 5.2 In depth 3, using (5.1)–(5.3) successively, we find that

$$\begin{aligned}
&\sum_{n>m\geq k>0}\frac{(-1)^n b_n}{(2n)(2m+1)(2k)^2}=\sum_{n>m\geq k>0}\frac{b_n^-(1)}{(2n)(2m+1)(2k)^2}\\
&=\frac{-1}{\sqrt{2}}\int_0^1\sum_{m\geq k>0}\frac{b_m^-(t)}{(2m+1)(2k)^2}\omega_{-1}=\frac{-1}{\sqrt{2}}\int_0^1\omega_{-20}\sum_{k>0}\frac{b_k^-(t)}{(2k)^2}\omega_{-1}=\frac{1}{\sqrt{2}}\int_0^1\omega_{-20}\omega_{-1}^3\\
&=\frac{1}{8}\int_0^1(\mathtt{a}+\mathtt{x}_i+\mathtt{x}_{-i})\Big(\sum_{k=1}^4(\mu_k-\mu_k^3)\mathtt{x}_{\mu_k}\Big)^3\\
&=\frac{1}{32}\sum_{j,k,l=1}^4(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)(\mu_l-\mu_l^3)\,\mathrm{Li}_{2,1,1}\Big(\mu_j^{-1},\frac{\mu_j}{\mu_k},\frac{\mu_k}{\mu_l}\Big)\\
&\quad+\frac{1}{32}\sum_{j,k,l=1}^4\sum_{\varepsilon=\pm\mathrm{i}}(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)(\mu_l-\mu_l^3)\,\mathrm{Li}_{1,1,1,1}\Big(\varepsilon^{-1},\frac{\varepsilon}{\mu_j},\frac{\mu_j}{\mu_k},\frac{\mu_k}{\mu_l}\Big)\\
&=\frac{7\sqrt{2}}{64}G^2+\frac{3739}{8192}\pi^4+\text{other terms in }\mathsf{CMZV}_4^8\approx 0.0202649114985
\end{aligned}$$

by GP-Pari or Au's mathematica package (see [3]). We have also checked this by numerically computing the series directly and by numerically evaluating the integral with mathematica:

$$\frac{1}{\sqrt{2}}\int_0^1\omega_{-20}\omega_{-1}^3=\frac{1}{6\sqrt{2}}\int_0^1\frac{\log^3(t+\sqrt{1+t^2})}{t(1+t^2)}\mathrm{d}t\approx 0.0202649114985.$$

Example 5.3 Similarly, using (5.1)–(5.3) successively, we can get

$$
\begin{aligned}
&\sum_{n\geq m>k>0} \frac{(-1)^{n+k} b_n}{(2n+1)(2m)^2(2k)} = \sum_{n\geq m>k>0} \frac{b_n^-(1)(-1)^k}{(2n+1)(2m)^2(2k)} \\
=&\frac{1}{\sqrt{2}}\int_0^1 \sum_{m>k>0} \frac{b_m^-(t)(-1)^k}{(2m)^2(2k)}\omega_{-1} = \frac{-1}{\sqrt{2}}\int_0^1 \omega_{-1}^2 \sum_{k>0}\frac{b_k(t)}{(2k)}\omega_{-1} = \frac{-1}{\sqrt{2}}\int_0^1 \omega_{-1}^2\omega_4\omega_1 \\
=&\frac{-1}{64}\int_0^1 \sum_{h=1}^4 (\mu_h-\mu_h^3)\mathrm{x}_{\mu_h}\sum_{j=1}^4(\mu_j-\mu_j^3)\mathrm{x}_{\mu_j}\sum_{l=1}^4\mu_l^2\mathrm{x}_{\mu_l}\sum_{k=1}^4(\mu_k+\mu_k^3)\mathrm{x}_{\mu_k} \\
=&\frac{-1}{64}\sum_{h,j,k,l=1}^4 \mu_l^2(\mu_h-\mu_h^3)(\mu_j-\mu_j^3)(\mu_k+\mu_k^3)\,\mathrm{Li}_{1,1,1,1}\left(\mu_h^{-1},\frac{\mu_h}{\mu_j},\frac{\mu_j}{\mu_l},\frac{\mu_l}{\mu_k}\right) \\
=&\frac{\sqrt{2}}{128}\Big(3\pi^3\log(1+\sqrt{2})-\pi^3\log 2-128\Im\,\mathrm{Li}_3\left(\frac{1+\mathrm{i}}{2}\right)\log(1+\sqrt{2}) \\
&+\pi(4\log^2 2\log(1+\sqrt{2})-8\log 2\log^2(1+\sqrt{2})+3\zeta(3))\Big)\approx -0.00777369894.
\end{aligned}
$$

We have also checked this by numerically computing the series directly and by numerically evaluating the integral with Mathematica:

$$
\begin{aligned}
\frac{-1}{\sqrt{2}}\int_0^1 \omega_{-1}^2\omega_4\omega_1 &= \frac{-1}{2\sqrt{2}}\int_0^1 \frac{t(\log\nu-\log(t+\sqrt{1+t^2}))^2\log(t+\sqrt{1+t^2})}{\sqrt{1-t^4}}\mathrm{d}t \\
&\approx -0.00777369894.
\end{aligned}
$$

Example 5.4 This example converges very slowly. Using (5.1)–(5.3) successively, we see that

$$
\begin{aligned}
&\sum_{n\geq m>k>0} \frac{(-1)^n b_n}{(2n+1)(2m)^2(2k)} = \sum_{n\geq m>k>0} \frac{b_n^-(1)}{(2n+1)(2m)^2(2k)} \\
=&\frac{1}{\sqrt{2}}\int_0^1 \sum_{m>k>0}\frac{b_m^-(t)}{(2m)^2(2k)}\omega_{-1} = \frac{-1}{\sqrt{2}}\int_0^1 \omega_{-1}^2\sum_{k>0}\frac{b_k^-(t)}{(2k)}\omega_{-1} = \frac{1}{\sqrt{2}}\int_0^1 \omega_{-1}^2\omega_{-2}\omega_{-1} \\
=&\frac{1}{32}\int_0^1 \sum_{h=1}^4(\mu_h-\mu_h^3)\mathrm{x}_{\mu_h}\sum_{j=1}^4(\mu_j-\mu_j^3)\mathrm{x}_{\mu_j}\left(\frac{1}{2}\sum_{l=1}^4\mathrm{x}_{\mu_l}-\mathrm{x}_{\mathrm{i}}-\mathrm{x}_{-\mathrm{i}}\right)\sum_{k=1}^4(\mu_k-\mu_k^3)\mathrm{x}_{\mu_k} \\
=&\frac{1}{64}\sum_{h,j,k,l=1}^4(\mu_h-\mu_h^3)(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)\int_0^1 \mathrm{x}_{\mu_h}\mathrm{x}_{\mu_j}\mathrm{x}_{\mu_l}\mathrm{x}_{\mu_k} \\
&-\frac{1}{32}\sum_{h,j,k=1}^4\sum_{\varepsilon=\pm\mathrm{i}}(\mu_h-\mu_h^3)(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)\int_0^1 \mathrm{x}_{\mu_h}\mathrm{x}_{\mu_j}\mathrm{x}_{\varepsilon}\mathrm{x}_{\mu_k} \\
=&\frac{1}{64}\sum_{h,j,k,l=1}^4(\mu_h-\mu_h^3)(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1,1,1}\left(\mu_h^{-1},\frac{\mu_h}{\mu_j},\frac{\mu_j}{\mu_l},\frac{\mu_l}{\mu_k}\right) \\
&-\frac{1}{32}\sum_{h,j,k=1}^4\sum_{\varepsilon=\pm\mathrm{i}}(\mu_h-\mu_h^3)(\mu_j-\mu_j^3)(\mu_k-\mu_k^3)\,\mathrm{Li}_{1,1,1,1}\left(\mu_h^{-1},\frac{\mu_h}{\mu_j},\frac{\mu_j}{\varepsilon},\frac{\varepsilon}{\mu_k}\right) \\
\approx& 0.00585622967.
\end{aligned}
$$

We have also checked this by numerically computing the series directly and by numerically evaluating the integral with mathematica:

$$\begin{aligned}\frac{1}{\sqrt{2}}\int_0^1 \omega_{-1}^2\omega_{-2}\omega_{-1} &= \frac{1}{2\sqrt{2}}\int_0^1 \frac{t(\log\nu-\log(t+\sqrt{1+t^2}))^2\log(t+\sqrt{1+t^2})}{1+t^2}\mathrm{d}t\\ &\approx 0.00585622967.\end{aligned}$$

Example 5.5 Using (5.1) and (5.5) successively, we find that

$$\begin{aligned}\sum_{n>m>0}\frac{(-1)^n b_n}{(2n-1)(2m)} &= -\int_0^1 \omega_{-3}\sum_{m>0}\frac{b_m^-(t)}{2m}\omega_{-1} - \frac{1}{\sqrt{2}}\int_0^1\sum_{m>0}\frac{b_m^-(t)}{2m}\omega_{-1}\\ &= \int_0^1 \omega_{-3}\omega_{-2}\omega_{-1} + \frac{1}{\sqrt{2}}\int_0^1 \omega_{-2}\omega_{-1}.\end{aligned}$$

Under the change of variables $t\to\frac{\mathrm{i}(1-t^2)}{1+t^2}$ for the first integral (see (4.7)) and (2.5) for the second, we get

$$\sum_{n>m>0}\frac{(-1)^n b_n}{(2n-1)(2m)} = \int_1^\mu \mathtt{d}_{-1,1}\mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} + \frac{1}{4}\int_0^1\Big(\frac{1}{2}\sum_{j=1}^4 \mathtt{x}_{\mu_j} - \mathtt{x}_{\mathrm{i}} - \mathtt{x}_{-\mathrm{i}}\Big)\sum_{k=1}^4(\mu_k-\mu_k^3)\mathtt{x}_{\mu_k}.$$

For some very small $\varepsilon>0$, by Chen's theory of iterated integrals (see [16, Lemma 2.1.2(ii)]) we may compute the first integral as

$$\begin{aligned}\int_1^\mu \mathtt{d}_{-1,1}\mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} &= \int_1^\varepsilon \mathtt{d}_{-1,1}\mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_\varepsilon^\mu \mathtt{d}_{-1,1}\int_1^\varepsilon \mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_\varepsilon^\mu \mathtt{d}_{-1,1}\mathtt{z}\int_1^\varepsilon \mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_\varepsilon^\mu \mathtt{d}_{-1,1}\mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}}\\ &= -\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{z}\mathtt{d}_{-1,1} + \int_\varepsilon^\mu \mathtt{d}_{-1,1}\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{z} - \int_\varepsilon^\mu \mathtt{d}_{-1,1}\mathtt{z}\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_\varepsilon^\mu \mathtt{d}_{-1,1}\mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}}.\end{aligned}$$

Note that by the shuffle relation

$$\begin{aligned}&\int_\varepsilon^\mu \mathtt{d}_{-1,1}\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{z} - \int_\varepsilon^\mu \mathtt{d}_{-1,1}\mathtt{z}\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\\ &= \int_\varepsilon^\mu \mathtt{d}_{-1,1}\Big(\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\int_\varepsilon^1 \mathtt{z} - \int_\varepsilon^1 \mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}}\Big) - \Big(\int_\varepsilon^\mu \mathtt{d}_{-1,1}\int_\varepsilon^\mu \mathtt{z} - \int_\varepsilon^\mu \mathtt{z}\mathtt{d}_{-1,1}\Big)\int_\varepsilon^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}.\end{aligned}$$

Thus taking $\varepsilon\to 0$, we have

$$\begin{aligned}\sum_{n>m>0}\frac{(-1)^n b_n}{(2n-1)(2m)} &= -\int_0^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{z}\mathtt{d}_{-1,1} - \int_0^\mu \mathtt{d}_{-1,1}\int_0^1 \mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_0^\mu \mathtt{d}_{-1,1}\int_0^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\int_\mu^1 \mathtt{z}\\ &\quad + \int_0^\mu \mathtt{z}\mathtt{d}_{-1,1}\int_0^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_0^\mu \mathtt{d}_{-1,1}\mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} + \frac{1}{4}\int_0^1\Big(\frac{1}{2}\sum_{j=1}^4 \mathtt{x}_{\mu_j} - \mathtt{x}_{\mathrm{i}} - \mathtt{x}_{-\mathrm{i}}\Big)\sum_{k=1}^4(\mu_k-\mu_k^3)\mathtt{x}_{\mu_k}\\ &= -\int_0^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\mathtt{z}\mathtt{d}_{-1,1} - \int_0^1 \mathtt{d}_{\frac{-1}{\mu},\frac{1}{\mu}}\int_0^1 \mathtt{z}\mathtt{d}_{-\mathrm{i},\mathrm{i}} + \int_0^1 \mathtt{d}_{\frac{-1}{\mu},\frac{1}{\mu}}\int_0^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}}\int_\mu^1 \mathtt{z}\\ &\quad - \int_0^1 (\mathtt{a}+\mathtt{x}_{-\frac{\mathrm{i}}{\mu}}+\mathtt{x}_{\frac{\mathrm{i}}{\mu}})\mathtt{d}_{\frac{-1}{\mu},\frac{1}{\mu}}\int_0^1 \mathtt{d}_{-\mathrm{i},\mathrm{i}} - \int_0^1 \mathtt{d}_{\frac{-1}{\mu},\frac{1}{\mu}}(\mathtt{a}+\mathtt{x}_{\frac{-\mathrm{i}}{\mu}}+\mathtt{x}_{\frac{\mathrm{i}}{\mu}})\mathtt{d}_{\frac{-\mathrm{i}}{\mu},\frac{\mathrm{i}}{\mu}}\\ &\quad + \frac{1}{4}\Big(\frac{1}{2}\sum_{j=1}^4 \mathtt{x}_{\mu_j} - \mathtt{x}_{\mathrm{i}} - \mathtt{x}_{-\mathrm{i}}\Big)\sum_{k=1}^4(\mu_k-\mu_k^3)\mathtt{x}_{\mu_k}.\end{aligned}$$

By using Au's mathematica package (see [3]), we find the evaluation

$$\sum_{n>m>0} \frac{(-1)^n b_n}{(2n-1)(2m)}$$
$$= \frac{\pi^2}{32}(3\sqrt{2}+4\log 2-6\log\nu)+\frac{1}{6}(16\sqrt{2}L(3,\chi_8)+\log^3\nu)+\frac{\log 2}{4}(3\sqrt{2}-2\log\nu)\log\nu$$
$$-\frac{3\sqrt{2}}{4}(\log^2\nu+\mathrm{Li}_2(\nu^{-1}))-4\,\mathrm{Li}_3(\nu^{-1})-\log\nu\,\mathrm{Li}_2(\nu^{-1})-\frac{3}{8}\zeta(3)\approx 0.20569096448.$$

In view of the theorems and the examples obtained so far, we would like to conclude the paper with the following question.

Question 5.1 Let $d\in\mathbb{N}$, $\boldsymbol{s}=(s_1,\cdots,s_d)\in\mathbb{N}^d$ and $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_d)\in\{\pm1\}^d$. Let $l_j(n)=2n$ or $l_j(n)=2n\pm1$ for all $1\le j\le d$. Is it true that

$$\sum_{n_1\succ n_2\succ\cdots\succ n_d\succ 0} \frac{a_{n_1}\eta_1^{n_1}\cdots\eta_d^{n_d}}{l_1(n_1)^{s_1}\cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^8_{\le|\boldsymbol{s}|}\otimes\mathbb{Q}[\mathrm{i},\sqrt{2}],$$
$$\sum_{n_1\succ n_2\succ\cdots\succ n_d\succ 0} \frac{b_{n_1}\eta_1^{n_1}\cdots\eta_d^{n_d}}{l_1(n_1)^{s_1}\cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^8_{\le|\boldsymbol{s}|+1}\otimes\mathbb{Q}[\mathrm{i},\sqrt{2}],$$

if the sums converge? Here "$\succ$" can be either "$\ge$" or "$>$".

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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