Chin. Ann. Math. Ser. B
46(4), 2025, 583–610
DOI: 10.1007/s11401-025-0030-3

**Chinese Annals of Mathematics, Series B**


# Stochastic Linear Quadratic Optimal Control Problems for Stochastic Evolution Equations with Unbounded Control Operator*

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**Abstract** The author studies a stochastic linear quadratic (SLQ for short) optimal control problem for systems governed by stochastic evolution equations, where the control operator in the drift term may be unbounded. Under the condition that the cost functional is uniformly convex, the well-posedness of the operator-valued Riccati equation is proved. Based on that, the optimal feedback control of the control problem is given.

**Keywords** Stochastic evolution equation, Stochastic linear quadratic control problem, Optimal feedback control, Unbounded control operator
**2020 MR Subject Classification** 93E20, 49N10, 93B52

## 1 Introduction

We begin with some notations. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W(\cdot) = \{W(t)\}_{t\geq 0}$ is defined, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(\cdot)$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$.

Let $T > 0$. For any $t \in [0, T)$ and Banach space $\mathbb{H}$, let

$$
\begin{aligned}
&L^2_{\mathcal{F}_t}(\Omega; \mathbb{H}) = \{\xi : \Omega \to \mathbb{H} \mid \xi \text{ is } \mathcal{F}_t\text{-measurable}, \ \mathbb{E}|\xi|^2_{\mathbb{H}} < \infty\},\\
&L^2_{\mathbb{F}}(t, T; \mathbb{H}) = \Big\{\varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted}, \ \ \mathbb{E}\int_t^T |\varphi(s)|^2_{\mathbb{H}} \mathrm{d}s < \infty\Big\},\\
&C_{\mathbb{F}}([t, T]; L^2(\Omega; \mathbb{H})) = \{\varphi : [t, T]\times\Omega \to \mathbb{H} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted},\\
&\qquad\qquad \varphi : [t, T] \to L^2_{\mathcal{F}_T}(\Omega; \mathbb{H}) \ \text{ is continuous}\},\\
&L^2_{\mathbb{F}}(\Omega; L^1(t, T; \mathbb{H})) = \Big\{\varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted}, \ \mathbb{E}\Big(\int_t^T |\varphi(s)|_{\mathbb{H}} \mathrm{d}s\Big)^2 < \infty\Big\}.
\end{aligned}
$$

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Banach spaces. Denote by $\mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ (resp. $\mathcal{L}(\mathbb{H}_1)$) the set of all bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$ (resp. $\mathbb{H}_1$). If $\mathbb{H}$ is a Hilbert space, then we set

$$
\mathbb{S}(\mathbb{H}) \triangleq \{F \in \mathcal{L}(\mathbb{H}) \mid F \text{ is self-adjoint}\}
$$

Manuscript received September 16, 2022. Revised May 28, 2023.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11971334, 12025105).

and

$$\overline{\mathbb{S}_+}(\mathbb{H}) \triangleq \{F \in \mathbb{S}(\mathbb{H}) \mid \langle F\xi, \xi \rangle \geq 0, \forall \xi \in \mathbb{H}\}.$$

Here and in what follows, for simplicity of notations, when there is no confusion, we shall use $\langle \cdot, \cdot \rangle$ for inner products in possibly different Hilbert spaces.

For any interval $[t_1, t_2] \subset [0, +\infty)$, denote by $C([t_1, t_2]; \mathbb{S}(\mathbb{H}))$ the set of all continuous mappings from $[t_1, t_2]$ to $\mathbb{S}(\mathbb{H})$, which is a Banach space equipped with the norm

$$|F|_{C([t_1,t_2];\mathbb{S}(\mathbb{H}))} \triangleq \sup_{t \in [t_1,t_2]} |F(t)|_{\mathcal{L}(\mathbb{H})}.$$

Denote by $C_{\mathcal{S}}([t_1, t_2]; \mathbb{S}(\mathbb{H}))$ the set of all strongly continuous mappings $F : [t_1, t_2] \to \mathbb{S}(\mathbb{H})$, that is, $F(\cdot)\xi$ is continuous on $[t_1, t_2]$ for each $\xi \in \mathbb{H}$. Let $\{F_n\}_{n=1}^{\infty} \subset C_{\mathcal{S}}([t_1, t_2]; \mathbb{S}(\mathbb{H}))$. We say that $\{F_n\}_{n=1}^{\infty}$ converges strongly to $F \in C_{\mathcal{S}}([t_1, t_2]; \mathbb{S}(\mathbb{H}))$ if

$$\lim_{n\to\infty} F_n(\cdot)\xi = F(\cdot)\xi, \quad \forall \xi \in \mathbb{H}.$$

In this case, we write

$$\lim_{n\to\infty} F_n = F \quad \text{in } C_{\mathcal{S}}([t_1, t_2]; \mathbb{S}(\mathbb{H})).$$

If $F \in C_{\mathcal{S}}([t_1, t_2]; \mathbb{S}(\mathbb{H}))$, then, by the uniform boundedness theorem, the quantity

$$|F|_{C_{\mathcal{S}}([t_1,t_2];\mathbb{S}(\mathbb{H}))} \triangleq \sup_{t \in [t_1,t_2]} |F(t)|_{\mathcal{L}(\mathbb{H})}$$

is finite, and $C_{\mathcal{S}}([t_1, t_2]; \mathbb{S}(\mathbb{H}))$ is a Banach space with this norm (see [2]).

We next introduce the Banach space $C_\gamma([t_1, t_2]; \mathbb{H})$ (see [2]) of continuous mappings on $[t_1, t_2)$ into a space $\mathbb{H}$, which is equipped with the norm

$$|f|_{C_\gamma([t_1,t_2];\mathbb{H})} \triangleq \sup_{s \in [t_1,t_2]} (t_2 - s)^\gamma |f(s)|_{\mathbb{H}} < \infty.$$

The space accounts for possible singularities at the time $t_2$ of the order $\gamma$.

Denote by $C_{\gamma,\mathcal{S}}([t_1, t_2]; \mathcal{L}(\mathbb{H}))$ the set of all strongly continuous operators in $\mathcal{L}(\mathbb{H})$ with the norm

$$|F|_{C_{\gamma,\mathcal{S}}([t_1,t_2];\mathcal{L}(\mathbb{H}))} \triangleq \sup_{s \in [t_1,t_2]} (t_2 - s)^\gamma |F(s)|_{\mathcal{L}(\mathbb{H})}.$$

The norm is finite (see [2]).

Let

$$L^{2,S}(t_1, t_2; \mathcal{L}(\mathbb{H})) \triangleq \{F : (t_1, t_2) \to \mathcal{L}(\mathbb{H}) \mid \ F\eta \in L^2(t_1, t_2; \mathbb{H}) \\ \forall \eta \in \mathbb{H}, \ |F|_{\mathcal{L}(\mathbb{H})} \in L^2(t_1, t_2)\}.$$

Now we turn to our control problem. Let $H$ and $U$ be two separable Hilbert spaces. Let $A$ be the generator of a $C_0$-semigroup $\{e^{\tau A}\}_{\tau\geq 0}$ on $H$. Denote by $H_{-1}$ the completion of $H$ with respect to the norm $|x|_{H_{-1}} \triangleq |(\beta I - A)^{-1}x|_H$, where $\beta \in \rho(A)$ (the resolvent of $A$) is fixed. Let $B \in \mathcal{L}(U, H_{-1})$ satisfy the following condition. There exists a number $\gamma \in \left(0, \frac{1}{2}\right)$ and a constant $c > 0$, such that the control $u$ to the state map kernel $e^{A\tau}B$ satisfies the singular estimate

$$|e^{A\tau}Bu|_H \leq \frac{c}{\tau^\gamma}|u|_U \tag{1.1}$$

for every $u \in U$ and $0 < \tau < 1$.

We consider the following controlled linear stochastic evolution equation (SEE for short):

$$\begin{cases} dx = (Ax + A_1x + Bu)ds + (Cx + Du)dW(s) \quad \text{in } (t, T], \\ x(t) = \eta \in H, \end{cases} \tag{1.2}$$

where $A_1(\cdot) \in L^1(0, T; \mathcal{L}(H))$, $C(\cdot) \in L^2(0, T; \mathcal{L}(H))$, $D(\cdot) \in L^\infty(0, T; \mathcal{L}(H, U))$ and $u(\cdot) \in \mathcal{U}[t, T] \triangleq L^2_{\mathbb{F}}(t, T; U)$.

The control system (1.2) covers many systems of stochastic partial differential equations with boundary controls. For these systems, the control operators are unbounded. To guarantee the well-posedness of such systems, people introduce the notion of admissible control operator (see [14]). The estimate (1.1) can be used to guarantee that $B$ is an admissible control operator. Then, it follows from the well-posedness of control systems with admissible control operators that, for any $\eta \in H$ and $u(\cdot) \in \mathcal{U}[t, T]$, the control system (1.2) admits a unique solution $x(\cdot) \in C_{\mathbb{F}}([t, T]; L^2(\Omega; H))$ (see [17]).

Consider the following cost functional:

$$\mathcal{J}(t, \eta; u(\cdot)) \triangleq \mathbb{E}\langle Gx(T), x(T)\rangle + \mathbb{E}\int_t^T (\langle Qx(s), x(s)\rangle + \langle Ru(s), u(s)\rangle)ds, \tag{1.3}$$

where

$$G \in \mathbb{S}(H), \quad Q(\cdot) \in L^1(0, T; \mathbb{S}(H)), \quad R(\cdot) \in L^\infty(0, T; \mathbb{S}(U)).$$

**Problem (SLQ)** For any given initial pair $(t, \eta) \in [0, T) \times H$, find a control $\overline{u}(\cdot) \in \mathcal{U}[t, T]$ such that

$$V(t, \eta) \triangleq \mathcal{J}(t, \eta; \overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)). \tag{1.4}$$

Any $\overline{u}(\cdot) \in \mathcal{U}[t, T]$ satisfying (1.4) is called an optimal control of Problem (SLQ) for the initial pair $(t, \eta)$, and the corresponding $\overline{x}(\cdot) \equiv x(\cdot; t, \eta, \overline{u}(\cdot))$ is called an optimal state process; the pair $(\overline{x}(\cdot), \overline{u}(\cdot))$ is called an optimal pair. The function $V(\cdot, \cdot)$ is called a value function of Problem (SLQ).

When $H = \mathbb{R}^n$ $(n \in \mathbb{N})$, Problem (SLQ) is extensively studied. Early works focus on the following case (see [1, 7]):

$$G \geq 0, \quad R(s) \geq \delta I, \quad Q(s) \geq 0 \quad \text{a.e. } s \in [0, T] \tag{1.5}$$

for some $\delta > 0$. Unfortunately, the condition (1.5) is not satisfied by several important examples (see [21]). A breakthrough is done in [3], in which the authors found that Problem (SLQ) might still be solvable when $R(s)$ is not positive definite for a.e. $s \in [0, T]$. This new phenomenon inspires many further researches (see [4–5, 11, 18–19] and the rich references therein).

When $H$ is an infinite dimensional space, the system (1.2) is used to describe a lot of random phenomena appearing in physics, chemistry, biology, and so on (see [12, 17]). Thus, there are many works addressing the optimal control problems for SEEs. In particular, we refer the readers to [2, 6, 8–10, 16] and the rich references therein for Problem (SLQ) for controlled SEEs. In those works, the condition (1.5) was assumed. As we said before, this is not satisfied for several important examples of Problem (SLQ). The condition (1.5) is dropped in [15] when $B$ is a bounded linear operator. The main purpose of this paper is to generalize the result in [15] to the case that $B$ is an admissible control operator. Compared with the works in [15], the unboundedness of the control operator $B$ leads to many technical difficulties. To overcome them, we borrow some ideas from [10, 20]. More details can be found in Section 3.

The unbounded operator $B$ leads to some substantial technical difficulties. For example, the system (1.2) admits a unique solution $x(\cdot) \in C_{\mathbb{F}}([t, T]; L^2(\Omega; H))$, but $B \in \mathcal{L}(U, H_{-1})$. Hence, we use the smoothing effect of the operator semigroup generated by the operator $A$ to deal with the unboundedness. In this paper, we prove that the existence of the strongly regular solution to the Riccati equation is equivalence to the uniform convexity of the cost functional. Under the strongly regular solution to the Riccati equation, we obtain the existence of the optimal feedback control for Problem (SLQ).

An outline of this paper is as follows. In Section 2, we present some preliminary results and the main result of the paper. Section 3 is devoted to the proof of the main result and giving an example of the uniform convexity of the cost functional.

## 2 Preliminaries and the Main Result

In this section, we provide the main result of this paper. To begin with, let us first introduce the notion of the optimal feedback operator.

**Definition 2.1** *We call $\overline{\Theta}(\cdot) \in L^{2,S}(t, T; \mathcal{L}(U, H))$ an optimal feedback operator of Problem (SLQ) on $[t, T]$ if*

$$\mathcal{J}(t, \eta; \overline{\Theta}(\cdot)\overline{x}(\cdot)) \leq \mathcal{J}(t, \eta; u(\cdot)), \quad \forall \eta \in H, \ u(\cdot) \in \mathcal{U}[t, T], \tag{2.1}$$

*where $\overline{x}(\cdot)$ is the mild solution to (1.2) with $u(\cdot) = \overline{\Theta}(\cdot)\overline{x}(\cdot)$.*

To study the optimal feedback operator of Problem (SLQ), we introduce the Riccati equation associated with Problem (SLQ) below:

$$\begin{cases}\dot{P}+P(A+A_1)+(A+A_1)^*P+C^*PC+Q-L^*K^{-1}L=0 \quad \text{in } [t,T),\\ P(T)=G,\end{cases} \tag{2.2}$$

where

$$L(\cdot)=B(\cdot)^*P(\cdot)+D(\cdot)^*P(\cdot)C(\cdot), \quad K(\cdot)=R(\cdot)+D(\cdot)^*P(\cdot)D(\cdot).$$

**Definition 2.2** *We call $P(\cdot)\in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$ a mild solution to (2.2) if for any $\eta\in H$ and $s\in[t,T]$,*

$$\begin{aligned}P(s)\eta=\mathrm{e}^{(T-s)A^*}G\mathrm{e}^{(T-s)A}\eta+\int_s^T \mathrm{e}^{(\tau-s)A^*}(PA_1+A_1^*P\\ +C^*PC+Q-L^*K^{-1}L)\mathrm{e}^{(\tau-s)A}\eta\mathrm{d}\tau.\end{aligned} \tag{2.3}$$

**Definition 2.3** *A mild solution $P(\cdot)$ of (2.2) is strongly regular if*

$$K(s)\geq\lambda I, \quad a.e.\, s\in[t,T]$$

*for some $\lambda>0$.*

To deal with (2.2), we need to concern the operator-valued equations:

$$\begin{cases}\dot{P}+P(A+A_1+B\Theta)+(A+A_1+B\Theta)^*P\\ +(C+D\Theta)^*P(C+D\Theta)+\Theta^*R\Theta+Q=0 \quad \text{in } [t,T),\\ P(T)=G,\end{cases} \tag{2.4}$$

where $\Theta(\cdot)\in L^{2,S}(t,T;\mathcal{L}(H,U))$. We also introduce the definition of the mild solution to (2.4).

**Definition 2.4** *We call $P(\cdot)\in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$ a mild solution to (2.4) if for any $s\in[t,T]$,*

$$\begin{aligned}P(s)\eta=\mathrm{e}^{(T-s)A^*}G\mathrm{e}^{(T-s)A}\eta+\int_s^T \mathrm{e}^{(\tau-s)A^*}[P(A_1+B\Theta)+(A_1+B\Theta)^*P\\ +(C+D\Theta)^*P(C+D\Theta)+\Theta^*R\Theta+Q]\mathrm{e}^{(\tau-s)A}\eta\mathrm{d}\tau, \quad \forall\eta\in H.\end{aligned} \tag{2.5}$$

We need the following lemmas.

**Lemma 2.1** (see [15, Lemma 3.6]) *Let $\widehat{A}(\cdot)\in L^1(0,T;\mathcal{L}(H))$, $\widehat{C}(\cdot)\in L^2(0,T;\mathcal{L}(H))$, $\widehat{G}\in\mathbb{S}(H)$, $\widehat{Q}(\cdot)\in L^1(0,T;\mathbb{S}(H))$ and $P(\cdot)$ be the solution to the following equation:*

$$\begin{cases}\dot{P}+P(A+\widehat{A})+(A+\widehat{A})^*P+\widehat{C}^*P\widehat{C}+\widehat{Q}=0 \quad in\ [t,T),\\ P(T)=\widehat{G}.\end{cases} \tag{2.6}$$

*Then (2.6) admits a unique solution $P(\cdot)\in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$. Moreover,*

$$\widehat{G}\geq0, \quad \widehat{Q}(s)\geq0, \quad a.e.\ s\in[t,T], \tag{2.7}$$

*then $P(\cdot)\in C_{\mathcal{S}}([t,T];\ \overline{\mathbb{S}_+}(H))$.*

**Lemma 2.2** (see [13, Proposition 6.5.3]) (1) *The map* $L_t \equiv \int_t^\tau \mathrm{e}^{(\tau-s)A}B\mathrm{d}s$ *is continuous from* $C_\gamma([t,T];U)$ *to* $C([t,T];H)$ *for* $\gamma < \frac{1}{2}$;

(2) *The adjoint map* $L_t^* \equiv \int_t^T B^*\mathrm{e}^{(\tau-t)A^*}\mathrm{d}\tau$ *is continuous from* $C_\gamma([t,T];H)$ *to* $C([t,T];U)$ *for* $\gamma < \frac{1}{2}$.

The main result of this paper is stated as follows.

**Theorem 2.1** *The following statements are equivalent*:

(1) *The map* $u(\cdot) \mapsto \mathcal{J}(0,0;u(\cdot))$ *is uniformly convex, i.e., there exists a constant* $\lambda > 0$ *such that*

$$\mathcal{J}(0,0;u(\cdot)) \geq \lambda\mathbb{E}\int_0^T |u(s)|_U^2\mathrm{d}s, \quad \forall u(\cdot) \in \mathcal{U}[0,T]. \tag{2.8}$$

(2) *The Riccati equation* (2.2) *admits a strongly regular solution* $P(\cdot) \in C_{\mathcal{S}}([0,T];\mathbb{S}(H))$.

*In such case, the unique optimal control* $\overline{u}(\cdot)$ *of Problem* (*SLQ*) *is*

$$\overline{u}(\cdot) = \overline{\Theta}(\cdot)\overline{x}(\cdot), \tag{2.9}$$

*where*

$$\overline{\Theta} = -K(\cdot)^{-1}L(\cdot). \tag{2.10}$$

## 3 Problem (SLQ)

In this section, we study the optimal feedback of Problem (SLQ) with that the weighted operator $R(\cdot) \geq \lambda I$ may not hold. Since the operator $B$ in the stochastic control system (1.2) is unbounded, we use the smoothing effect of the operator semigroup of $A$ to overcome the difficulty caused by the unboundedness.

We now define four operators as follows:

$$\Psi_t : H \to \mathcal{X}[t,T] \triangleq L^2_{\mathbb{F}}(t,T;H), \quad \Psi_t\eta = x(\cdot;t,\eta,0), \quad \forall \eta \in H,$$

where $x(\cdot;t,\eta,0)$ is the solution to (1.2) with $u \equiv 0$;

$$\Xi_t : \mathcal{U}[t,T] \to \mathcal{X}[t,T], \quad \Xi_t u = x(\cdot;t,0,u), \quad \forall u \in \mathcal{U}[t,T],$$

where $x(\cdot;t,0,u)$ is the solution to (1.2) with $\eta = 0$;

$$\widehat{\Psi}_t : H \to L^2_{\mathcal{F}_T}(\Omega;H), \quad \widehat{\Psi}_t\eta = x(T;t,\eta,0), \quad \forall \eta \in H$$

and

$$\widehat{\Xi}_t : \mathcal{U}[t,T] \to L^2_{\mathcal{F}_T}(\Omega;H), \quad \widehat{\Xi}_t u = x(T;t,0,u), \quad \forall \eta \in H.$$

By the inequality (1.1), the system (1.2) is well-posed. We can obtain that the four operators are bounded linear operators. Therefore, the state process $x(\cdot)$ and its terminal value $x(T)$ can be written as

$$x(\cdot) = (\Psi_t \eta)(\cdot) + (\Xi_t u)(\cdot), \quad x(T) = \widehat{\Psi}_t \eta + \widehat{\Xi}_t u$$

for any $t \in [0, T)$ and $(\eta, u(\cdot)) \in H \times \mathcal{U}[t, T]$.

The following lemma shows that how to handle the unbounded operator $B$ by using the smoothing effect of the operator semigroup generated by $A$.

**Lemma 3.1** *Let $\widehat{C} = C + D\Theta$. Assume that $\widehat{C}(\cdot)$ and $\Theta(\cdot)$ are given bounded operator-valued function for every $s \in [t, T)$ satisfying the following conditions:*

$$|\widehat{C}(s)x|_H \le \frac{r|x|_H}{(T-s)^\gamma}, \quad |\Theta(s)x|_H \le \frac{r|x|_H}{(T-s)^\gamma}, \quad \forall x \in H \tag{3.1}$$

*for some suitably chosen $r > 0$. Then, for (2.4), there exists a unique mild solution $P(\cdot) \in C_{\mathcal{S}}([t, T]; \mathbb{S}(H))$ such that*

$$|B^* P(s)x|_H \le \frac{\mathcal{C}|x|_H}{(T-s)^\gamma}, \quad |PB(s)u|_H \le \frac{\mathcal{C}|u|_U}{(T-s)^\gamma}, \quad \forall x \in H, \ \forall u \in U. \tag{3.2}$$

**Proof** We first show that there exists a unique local-in-time solution

$$P(\cdot) \in C_{\mathcal{S}}([T_0, T]; \mathbb{S}(H))$$

to (2.4). To prove the existence of a solution $P$, we use a fixed point argument on the map $\Gamma$ given by

$$\Gamma(f, g, h)(s) \triangleq \begin{pmatrix} \Gamma_1(f, g, h)(s) \\ \Gamma_2(f, g, h)(s) \\ \Gamma_3(f, g, h)(s) \end{pmatrix}$$

for $s \in [t, T]$ on the space $X \equiv C_{\mathcal{S}}([t, T]; \mathcal{L}(H)) \times C_{\gamma,\mathcal{S}}([t, T]; \mathcal{L}(H, U)) \times C_{\gamma,\mathcal{S}}([t, T]; \mathcal{L}(U, H))$ equipped with the norm

$$|(f, g, h)|_X = |f|_{C_{\mathcal{S}}([t,T];\mathcal{L}(H))} + |g|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))} + |h|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))},$$

where

$$\begin{aligned}
\Gamma_1(f, g, h)(s) \triangleq & \int_s^T \mathrm{e}^{(\tau - s)A^*} (f A_1 + A_1^* f + h\Theta + \Theta^* g + \widehat{C}^* f \widehat{C} + \Theta^* R \Theta + Q) \mathrm{e}^{(\tau - s)A} \mathrm{d}\tau \\
& + \mathrm{e}^{(T-s)A^*} G \mathrm{e}^{(T-s)A}, \\
\Gamma_2(f, g, h)(s) \triangleq & \ B^* \int_s^T \mathrm{e}^{(\tau - s)A^*} (f A_1 + A_1^* f + h\Theta + \Theta^* g + \widehat{C}^* f \widehat{C} + \Theta^* R \Theta + Q) \mathrm{e}^{(\tau - s)A} \mathrm{d}\tau \\
& + B^* \mathrm{e}^{(T-s)A^*} G \mathrm{e}^{(T-s)A},
\end{aligned}$$

$$\Gamma_3(f,g,h)(s) \triangleq \int_s^T \mathrm{e}^{(\tau-s)A^*}(fA_1 + A_1^* f + h\Theta + \Theta^* g + \widehat{C}^* f \widehat{C} + \Theta^* R \Theta + Q)\mathrm{e}^{(\tau-s)A} B \mathrm{d}\tau + \mathrm{e}^{(T-s)A^*} G \mathrm{e}^{(T-s)A} B.$$

In order to deal with the unboundedness of the control operator $B$, we wish to find a fixed point of the system of three equations defined by three variables (operators) which are $f = P, g = B^*P$ and $h = PB$.

First, we prove that $\Gamma$ maps $X$ into $X$ by working component by component. Let $(f,g,h) \in X$. Let $\varepsilon$ be small enough and $\eta \in H$. We obtain

$$\begin{aligned}
&\Gamma_1(f,g,h)(s)\eta - \Gamma_1(f,g,h)(s+\varepsilon)\eta \\
&= \int_s^T \mathrm{e}^{(\tau-s)A^*}(fA_1 + h\Theta + A_1^* f + \Theta^* g + \widehat{C}^* f \widehat{C} + \Theta^* R\Theta + Q)\mathrm{e}^{(\tau-s)A}\eta \mathrm{d}\tau \\
&\quad - \int_{s+\varepsilon}^T \mathrm{e}^{(\tau-s-\varepsilon)A^*}(fA_1 + h\Theta + A_1^* f + \Theta^* g + \widehat{C}^* f \widehat{C} + Q + \Theta^* R\Theta)\mathrm{e}^{(\tau-s-\varepsilon)A}\eta \mathrm{d}\tau \\
&\quad + \mathrm{e}^{(T-s)A^*} G \mathrm{e}^{(T-s)A}\eta - \mathrm{e}^{(T-s-\varepsilon)A^*} G \mathrm{e}^{(T-s-\varepsilon)A}\eta \\
&= \int_{s+\varepsilon}^T \mathrm{e}^{(\tau-s-\varepsilon)A^*}[\mathrm{e}^{\varepsilon A^*}(fA_1 + h\Theta + A_1^* f + \Theta^* g + \widehat{C}^* f \widehat{C} + Q + \Theta^* R\Theta)\mathrm{e}^{(\tau-s)A}\eta \\
&\quad - (fA_1 + h\Theta + A_1^* f + \Theta^* g + \widehat{C}^* f \widehat{C} + Q + \Theta^* R\Theta)\mathrm{e}^{(T-s-\varepsilon)A}\eta]\mathrm{d}\tau \\
&\quad + \int_s^{s+\varepsilon} \mathrm{e}^{(\tau-s)A^*}(fA_1 + h\Theta + A_1^* f + \Theta^* g + \widehat{C}^* f \widehat{C} + \Theta^* R\Theta + Q)\mathrm{e}^{(\tau-s)A}\eta \mathrm{d}\tau \\
&\quad + \mathrm{e}^{(T-s-\varepsilon)A^*}(\mathrm{e}^{\varepsilon A^*} G \mathrm{e}^{(T-s)A}\eta - G\mathrm{e}^{(T-s-\varepsilon)A}\eta).
\end{aligned}$$

Therefore, by setting

$$\Pi = fA_1 + h\Theta + A_1^* f + \Theta^* g + \widehat{C}^* f \widehat{C} + \Theta^* R\Theta + Q,$$

we see that

$$\begin{aligned}
&|\Gamma_1(f,g,h)(s)\eta - \Gamma_1(f,g,h)(s+\varepsilon)\eta|_H \\
&\leq \Big| \int_{s+\varepsilon}^T \mathrm{e}^{(\tau-s-\varepsilon)A^*}[\mathrm{e}^{\varepsilon A^*}\Pi \mathrm{e}^{(\tau-s)A}\eta - \Pi \mathrm{e}^{(\tau-s-\varepsilon)A}\eta]\mathrm{d}\tau \Big|_H + \Big| \int_s^{s+\varepsilon} \mathrm{e}^{(\tau-s)A^*}\Pi \mathrm{e}^{(\tau-s)A}\eta \mathrm{d}\tau \Big|_H \\
&\quad + |\mathrm{e}^{(T-s-\varepsilon)A^*}(\mathrm{e}^{\varepsilon A^*} G \mathrm{e}^{(T-s)A}\eta - G\mathrm{e}^{(T-s-\varepsilon)A}\eta)|_H. \qquad (3.3)
\end{aligned}$$

Since

$$|\mathrm{e}^{(\tau-s-\varepsilon)A^*}[\mathrm{e}^{\varepsilon A^*}\Pi \mathrm{e}^{(\tau-s)A}\eta - \Pi \mathrm{e}^{(\tau-s-\varepsilon)A}\eta]|_H \leq C(t)|\Pi \mathrm{e}^{(\tau-s)A}\eta|_H,$$

by Lebesgue's dominated convergence theorem and strong continuity of the operator semigroup $\{\mathrm{e}^{sA}\}_{s\geq 0}$,

$$\lim_{\varepsilon\to 0} \Big| \int_{s+\varepsilon}^T \mathrm{e}^{(\tau-s-\varepsilon)A^*}[\mathrm{e}^{\varepsilon A^*}\Pi \mathrm{e}^{(\tau-s)A}\eta - \Pi \mathrm{e}^{(\tau-s-\varepsilon)A}\eta]\mathrm{d}\tau \Big|_H = 0.$$

The second integrand in (3.3) approaches to 0 as $\varepsilon \to 0$ by using the absolute continuity of the integral. Further, the third integrand in (3.3) approaches to 0 as $\varepsilon \to 0$, which follows from the strong continuity of $\{e^{sA}\}_{s\geq 0}$. Therefore, for (3.3), when $\varepsilon \to 0$, we have

$$|\Gamma_1(f,g,h)(s)\eta - \Gamma_1(f,g,h)(s+\varepsilon)\eta|_H \to 0.$$

This means that $\Gamma_1(f,g,h) \in C_{\mathcal{S}}([t,T];\mathcal{L}(H))$.

Similarly, we deduce that

$$\begin{aligned}
&|\Gamma_2(f,g,h)(s)\eta - \Gamma_2(f,g,h)(s+\varepsilon)\eta|_U \\
\leq &\Big|B^*\int_{s+\varepsilon}^T e^{(\tau-s-\varepsilon)A^*}[e^{\varepsilon A^*}\Pi e^{(\tau-s)A}\eta - \Pi e^{(\tau-s-\varepsilon)A}\eta]d\tau\Big|_U \\
&+\Big|B^*\int_s^{s+\varepsilon} e^{(\tau-s)A^*}\Pi e^{(\tau-s)A}\eta d\tau\Big|_U \\
&+|B^*e^{(T-s-\varepsilon)A^*}(e^{\varepsilon A^*}Ge^{(T-s)A}\eta - Ge^{(T-s-\varepsilon)A}\eta)|_U
\end{aligned}$$

and

$$\begin{aligned}
&|\Gamma_3(f,g,h)(s)u - \Gamma_3(f,g,h)(s+\varepsilon)u|_H \\
\leq &\Big|\int_{s+\varepsilon}^T e^{(\tau-s-\varepsilon)A^*}[e^{\varepsilon A^*}\Pi e^{(\tau-s)A}Bu - \Pi e^{(\tau-s-\varepsilon)A}Bu]d\tau\Big|_H \\
&+\Big|\int_s^{s+\varepsilon} e^{(\tau-s)A^*}\Pi e^{(\tau-s)A}Bud\tau\Big|_U \\
&+|e^{(T-s-\varepsilon)A^*}(e^{\varepsilon A^*}Ge^{(T-s)A}Bu - Ge^{(T-s-\varepsilon)A}Bu)|_H.
\end{aligned}$$

When $\varepsilon \to 0$, we conclude that

$$|\Gamma_2(f,g,h)(s)\eta - \Gamma_2(f,g,h)(s+\varepsilon)\eta|_U \to 0$$

and

$$|\Gamma_3(f,g,h)(s)u - \Gamma_3(f,g,h)(s+\varepsilon)u|_H \to 0.$$

Therefore, the right continuity holds. The left continuity can be proved similarly.

We denote the ball $B(0,r;X) \triangleq \{y \in X \mid |y|_X \leq r\}$. Let $(f,g,h)$ and $(f_0,g_0,h_0)$ be in $B(0,r;X)$. Our next goal is to determine $r$ and $t$ so that $\Gamma : B(0,r;X) \to B(0,r;X)$, and $\Gamma$ is a contraction on $B(0,r;X)$ for suitably chosen $r$ and $t$. To meet these two conditions, we firstly estimate the norms of $\Gamma_1, \Gamma_2$ and $\Gamma_3$, and thus, the norm of $\Gamma$. Then we estimate $|\Gamma(f,g,h) - \Gamma(f_0,g_0,h_0)|_X$. Finally, we combine the conditions on $r$. Now, we demonstrate the process.

For $s \in [t,T]$, we obtain

$$|\Gamma_1(f,g,h)|_{C_{\mathcal{S}}([t,T];\mathcal{L}(H))}$$

$$
\begin{aligned}
&\le \sup_{t\le s\le T}\int_s^T \mathcal{C}M^2\mathrm{e}^{2\alpha(T-s)}\Big[\frac{2r^2}{(T-\tau)^\gamma}+\frac{r^3}{(T-\tau)^{2\gamma}}+\frac{r^2}{(T-\tau)^{2\gamma}}|R|_{\mathcal{L}(U)}\\
&\quad +2r|A_1|_{\mathcal{L}(H)}+|Q|_{\mathcal{L}(H)}\Big]\mathrm{d}\tau+\mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}|G|_{\mathcal{L}(H)}\\
&\le \mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}\Big\{\frac{r^3(T-t)^{1-2\gamma}}{1-2\gamma}+r^2\Big[\frac{2(T-t)^{1-\gamma}}{1-\gamma}+\frac{(T-t)^{1-2\gamma}}{1-2\gamma}||R|_{\mathcal{L}(U)}|_{L^\infty(t,T)}\Big]\\
&\quad +||Q|_{\mathcal{L}(H)}|_{L^1(t,T)}+2r||A_1|_{\mathcal{L}(H)}|_{L^1(t,T)}\Big\}+\mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}\\
&\le \mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}+3\mathcal{C}_t(r^3+r^2+r+1)M^2\mathrm{e}^{2\alpha(T-t)},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{C}_t=\max\Big\{&\mathcal{C}||A_1|_{\mathcal{L}(H)}|_{L^1(t,T)},\ \mathcal{C}||Q|_{\mathcal{L}(H)}|_{L^1(t,T)},\ \mathcal{C}\frac{(T-t)^{1-\gamma}}{1-\gamma},\\
&\mathcal{C}\frac{(T-t)^{1-2\gamma}}{1-2\gamma},\ \mathcal{C}\frac{(T-t)^{1-2\gamma}}{1-2\gamma}||R|_{\mathcal{L}(U)}|_{L^\infty(t,T)}\Big\}.
\end{aligned}
$$

We next estimate $|\Gamma_2(f,g,h)|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))}$ as follows:

$$
\begin{aligned}
&|\Gamma_2(f,g,h)|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))}\\
&=\sup_{s\in[t,T]}(T-s)^\gamma\Big|B^*\mathrm{e}^{(T-s)A^*}G\mathrm{e}^{(T-s)A}+B^*\int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau\Big|_{\mathcal{L}(H,U)}\\
&\le \sup_{s\in[t,T]}(T-s)^\gamma\Big[\frac{1}{(T-s)^\gamma}|G|_{\mathcal{L}(H)}M^2\mathrm{e}^{2\alpha(T-s)}+\int_s^T\frac{1}{(\tau-s)^\gamma}|\Pi|_{\mathcal{L}(H)}M^2\mathrm{e}^{2\alpha(\tau-s)}\mathrm{d}\tau\Big]\\
&\le \mathcal{C}(T)M^2\mathrm{e}^{2\alpha(T-t)}\Big\{r^3\frac{(T-t)^{1-2\gamma}}{1-2\gamma}+r^2\Big[\frac{2(T-t)^{1-\gamma}}{1-\gamma}+\frac{(T-t)^{1-2\gamma}}{1-2\gamma}||R|_{\mathcal{L}(U)}|_{L^\infty(t,T)}\Big]\\
&\quad +2r||A_1|_{\mathcal{L}(H)}|_{L^1(t,T)}+||Q|_{\mathcal{L}(H)}|_{L^1(t,T)}\Big\}+\mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}\\
&\le \mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}+3\widetilde{\mathcal{C}}_tM^2\mathrm{e}^{2\alpha(T-t)}(r^3+r^2+r+1),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{\mathcal{C}}_t=\max\Big\{&\mathcal{C}_t,\ \mathcal{C}(T)||A_1|_{\mathcal{L}(H)}|_{L^1(t,T)},\ \mathcal{C}(T)||Q|_{\mathcal{L}(H)}|_{L^1(t,T)},\ \mathcal{C}(T)\frac{(T-t)^{1-\gamma}}{1-\gamma},\\
&\mathcal{C}(T)\frac{(T-t)^{1-2\gamma}}{1-2\gamma},\ \mathcal{C}(T)\frac{(T-t)^{1-2\gamma}}{1-2\gamma}||R|_{\mathcal{L}(U)}|_{L^\infty(t,T)}\Big\}.
\end{aligned}
$$

Finally, we estimate $|\Gamma_3(f,g,h)|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))}$ as following:

$$
\begin{aligned}
&|\Gamma_3(f,g,h)|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))}\\
&=\sup_{s\in[t,T]}(T-s)^\gamma\Big|\mathrm{e}^{(T-s)A^*}G\mathrm{e}^{(T-s)A}B+\int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi\mathrm{e}^{(\tau-s)A}B\mathrm{d}\tau\Big|_{\mathcal{L}(U,H)}\\
&\le \sup_{s\in[t,T]}(T-s)^\gamma\Big[\frac{1}{(T-s)^\gamma}|G|_{\mathcal{L}(H)}M^2\mathrm{e}^{2\alpha(T-s)}+\int_s^T\frac{1}{(\tau-s)^\gamma}|\Pi|_{\mathcal{L}(H)}M^2\mathrm{e}^{2\alpha(\tau-s)}\mathrm{d}\tau\Big]\\
&\le \mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)}+3\widetilde{\mathcal{C}}_tM^2\mathrm{e}^{2\alpha(T-t)}(r^3+r^2+r+1),
\end{aligned}
$$

where $\widetilde{\mathcal{C}}_t$ has been set as previously mentioned.

Let $t = T_0$ such that $T - T_0$ is sufficiently small. We impose the condition

$$3\mathcal{C}M^2\mathrm{e}^{2\alpha(T-t)} + 9\widetilde{\mathcal{C}}_t M^2\mathrm{e}^{2\alpha(T-t)}(r^3 + r^2 + r + 1) < r.$$

Let $r = 12\mathcal{C}M^2\mathrm{e}^{2\alpha T}$ and so that

$$\widetilde{\mathcal{C}}_t < \frac{\mathcal{C}}{r^3 + r^2 + r + 1}. \tag{3.4}$$

This guarantees that $\Gamma$ acts from $B(0, r; X)$ into $B(0, r; X)$ for our choices of $r$ and $t$.

For $s \in [t, T]$, we deduce that

$$\begin{aligned}
&|\Gamma_1(f, g, h)(s) - \Gamma_1(f_0, g_0, h_0)(s)|_{C_{\mathcal{S}}([t,T];\mathcal{L}(H))} \\
&= \sup_{s\in[t,T]} \Big| \int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau - \int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi_0\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau\Big|_{\mathcal{L}(H)} \\
&\leq M^2\mathrm{e}^{2\alpha T} \sup_{s\in[t,T]} \Big\{ \Big[2||A_1|_{\mathcal{L}(H)}|_{L^1(s,T)} + \frac{r^2(T-s)^{1-2\gamma}}{1-2\gamma}\Big]|f - f_0|_{C_{\mathcal{S}}([t,T];\mathcal{L}(H))} \\
&\quad + \frac{r(T-s)^{1-\gamma}}{1-\gamma}|g - g_0|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))} + \frac{r(T-s)^{1-\gamma}}{1-\gamma}|h - h_0|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))}\Big\} \\
&\leq M^2\mathrm{e}^{2\alpha T}(\widetilde{\mathcal{C}}_t + \widetilde{\mathcal{C}}_t r + \widetilde{\mathcal{C}}_t r^2)|(f - f_0, g - g_0, h - h_0)|_X.
\end{aligned}$$

Combining $\widetilde{\mathcal{C}}_t < \frac{\mathcal{C}}{r^3+r^2+r+1}$ with $r = 12\mathcal{C}M^2\mathrm{e}^{2\alpha T}$, we conclude that

$$M^2\mathrm{e}^{2\alpha T}(\widetilde{\mathcal{C}}_t + \widetilde{\mathcal{C}}_t r^2 + \widetilde{\mathcal{C}}_t r) < \frac{1}{12}.$$

Next,

$$\begin{aligned}
&|\Gamma_2(f, g, h)(t) - \Gamma_2(f_0, g_0, h_0)(t)|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))} \\
&= \sup_{s\in[t,T]} (T-s)^\gamma \Big| B^* \int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau - B^* \int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi_0\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau\Big|_{\mathcal{L}(H,U)} \\
&\leq \sup_{s\in[t,T]} (T-s)^\gamma \Big| \int_s^T \frac{1}{(\tau-s)^\gamma}(\Pi - \Pi_0)\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau\Big|_{\mathcal{L}(H,U)} \\
&\leq \sup_{s\in[t,T]} \mathcal{C}(T)M^2\mathrm{e}^{2\alpha T}\Big\{ \Big[2||A_1|_{\mathcal{L}(H)}|_{L^1(s,T)} + \frac{r^2(T-s)^{1-2\gamma}}{1-2\gamma}\Big]|f - f_0|_{C_{\mathcal{S}}([t,T];\mathcal{L}(H))} \\
&\quad + \frac{r(T-s)^{1-\gamma}}{1-\gamma}|g - g_0|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))} + \frac{r(T-s)^{1-\gamma}}{1-\gamma}|h - h_0|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))}\Big\} \\
&\leq M^2\mathrm{e}^{2\alpha T}(\widetilde{\mathcal{C}}_t + \widetilde{\mathcal{C}}_t r + \widetilde{\mathcal{C}}_t r^2)|(f - f_0, g - g_0, h - h_0)|_X.
\end{aligned}$$

Using $\widetilde{\mathcal{C}}_t < \frac{\mathcal{C}}{r^3+r^2+r+1}$ and $r = 12\mathcal{C}M^2\mathrm{e}^{2\alpha T}$ again, we have

$$M^2\mathrm{e}^{2\alpha T}(\widetilde{\mathcal{C}}_t + \widetilde{\mathcal{C}}_t r^2 + \widetilde{\mathcal{C}}_t r) < \frac{1}{12}.$$

Furthermore,

$$|\Gamma_3(f, g, h)(t) - \Gamma_3(f_0, g_0, h_0)(t)|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))}$$

$$
\begin{aligned}
&= \sup_{s\in[t,T]} (T-s)^{\gamma}\Big|\int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi \mathrm{e}^{(\tau-s)A}B\mathrm{d}\tau - \int_s^T \mathrm{e}^{(\tau-s)A^*}\Pi_0 \mathrm{e}^{(\tau-s)A}B\mathrm{d}\tau\Big|_{\mathcal{L}(U,H)} \\
&\le \sup_{s\in[t,T]} (T-s)^{\gamma}\Big|\int_s^T \mathrm{e}^{(\tau-s)A^*}(\Pi-\Pi_0)\frac{1}{(\tau-s)^{\gamma}}\mathrm{d}\tau\Big|_{\mathcal{L}(U,H)} \\
&\le \sup_{s\in[t,T]} M^2\mathrm{e}^{2\alpha T}\mathcal{C}(T)\Big\{\Big[2||A_1|_{\mathcal{L}(H)}|_{L^1(s,T)} + \frac{r^2(T-s)^{1-2\gamma}}{1-2\gamma}\Big]|f-f_0|_{C_{\mathcal{S}}([t,T];\mathcal{L}(H))} \\
&\quad + \frac{r(T-s)^{1-\gamma}}{1-\gamma}|g-g_0|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(U,H))} + \frac{r(T-s)^{1-\gamma}}{1-\gamma}|h-h_0|_{C_{\gamma,\mathcal{S}}([t,T];\mathcal{L}(H,U))}\Big\} \\
&\le M^2\mathrm{e}^{2\alpha T}(\widetilde{\mathcal{C}}_t + \widetilde{\mathcal{C}}_t r + \widetilde{\mathcal{C}}_t r^2)|(f-f_0, g-g_0, h-h_0)|_X.
\end{aligned}
$$

Combining $\widetilde{\mathcal{C}}_t < \frac{\mathcal{C}}{r^3+r^2+r+1}$ with $r = 12\mathcal{C}M^2\mathrm{e}^{2\alpha T}$, we obtain

$$
M^2\mathrm{e}^{2\alpha T}(\widetilde{\mathcal{C}}_t + \widetilde{\mathcal{C}}_t r^2 + \widetilde{\mathcal{C}}_t r) < \frac{1}{12}.
$$

That guarantees $\Gamma$ is a contraction from $B(0,r;X)$ into $B(0,r;X)$ for our choices of $r$ and $t$. Hence $\Gamma(f,g,h)$ has a unique fixed point $(f,g,h)\in X$. Estimate (3.2) follows from $B^*P$ in $C_\gamma([t,T];U)$ and $PB\in C_\gamma([t,T];H)$. Since $PB$ admits a unique solution in $C_\gamma([t,T];H)$, it has $(PB)^* = B^*P^*$. Therefore, if $P\in C_{\mathcal{S}}([t,T];\mathcal{L}(H))$ satisfying (2.4), then $P^*\in C_{\mathcal{S}}([t,T];\mathcal{L}(H))$ does. Hence, we have $P=P^*$. That is, $P\in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$.

We next prove that there exists a global solution to (2.4). Let $T_{\max}\ge T_0$. We extend the solution from $[T_{\max},T]$ to any time interval $[t,T]$. Since

$$
\begin{aligned}
\langle P(s)\eta,\eta\rangle &\le \mathcal{J}(s,\eta;0) = \mathbb{E}\Big(\int_s^T \langle Qx(\tau),x(\tau)\rangle\mathrm{d}\tau + \langle Gx(T),x(T)\rangle\Big) \\
&\le \mathcal{C}M^2T\mathrm{e}^{2\alpha T}|\eta|_H^2 + \mathcal{C}M^2\mathrm{e}^{2\alpha T}|\eta|_H^2 \\
&\le \mathcal{C}_T|\eta|_H^2,
\end{aligned}
$$

it implies that $|P(s)|_{\mathcal{L}(H)}\le\mathcal{C}_T$ for all $s\in[T_{\max},T]$. We use the bound to reiterate the above proof on a new interval $[T_1,T_{\max}]$ with $G=P(T_{\max})$. This bound yields that the choice of $\mathcal{C}$ in (3.4) is global and all the estimates are uniform and that $r$ and the time step $T_{\max}-T_1$ are the same. Therefore, the solution can be extended by repeated iteration on the equal time steps to any initial time $t\ge 0$.

Finally, we claim that there exists a unique solution $P\in C_{\mathcal{S}}([t,T];\mathcal{L}(H))$ to (2.4) satisfying $B^*P\in C_\gamma([t,T];U)$. Assume that there is another solution $\widetilde{P}(s)$ to (2.4), then

$$
\inf_{u(\cdot)\in\mathcal{U}[s,T]}\mathcal{J}(s,\eta;u) = \langle P(s)\eta,\eta\rangle = \langle\widetilde{P}(s)\eta,\eta\rangle,\quad \forall\eta\in H.
$$

Hence, for any $\xi,\zeta\in H$, we have

$$
\begin{aligned}
0 &= \langle (P(s)-\widetilde{P}(s))(\xi+\zeta),(\xi+\zeta)\rangle \\
&= \langle (P(s)-\widetilde{P}(s))\xi,\xi\rangle + \langle (P(s)-\widetilde{P}(s))\xi,\zeta\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\langle (P(s)-\widetilde{P}(s))\zeta,\xi\rangle+\langle (P(s)-\widetilde{P}(s))\zeta,\zeta\rangle\\
&=2\langle (P(s)-\widetilde{P}(s))\xi,\zeta\rangle
\end{aligned}
$$

by self-adjointness of $P$ and $\widetilde{P}$. Thus, $P(s)=\widetilde{P}(s)$.

We next give a result about the differentiability of $P$.

**Lemma 3.2** *Let $P$ be a mild solution to* (2.4). *Then for any $\eta,\xi\in D(A)$, $\langle P(\cdot)\eta,\xi\rangle$ is differentiable in $[t,T]$ and*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\langle P\eta,\xi\rangle &= -\langle P\eta,(A+A_1+B\Theta)\xi\rangle-\langle P(A+A_1+B\Theta)\eta,\xi\rangle\\
&\quad -\langle P(C+D\Theta)\eta,(C+D\Theta)\xi\rangle-\langle R\Theta\eta,\Theta\xi\rangle-\langle Q\eta,\xi\rangle. \qquad (3.5)
\end{aligned}
$$

**Proof** We can obtain that the operators $B^*P$ and $PB$ are bounded on $U$ and $H$ from Lemma 3.1. Hence, for any $\eta,\xi\in H$, using (2.5), we can obtain

$$
\begin{aligned}
&\langle P(s)\eta,\xi\rangle\\
&=\langle G\mathrm{e}^{(T-s)A}\eta,\mathrm{e}^{(T-s)A}\xi\rangle+\int_s^T\langle[P(A_1+B\Theta)+(A_1+B\Theta)^*P\\
&\quad+(C+D\Theta)^*P(C+D\Theta)+\Theta^*R\Theta+Q]\mathrm{e}^{(\tau-s)A}\eta,\mathrm{e}^{(\tau-s)A}\xi\rangle\mathrm{d}\tau\\
&=\langle G\mathrm{e}^{(T-s)A}\eta,\mathrm{e}^{(T-s)A}\xi\rangle+\int_s^T[\langle P(A_1+B\Theta)\mathrm{e}^{(\tau-s)A}\eta,\mathrm{e}^{(\tau-s)A}\xi\rangle\\
&\quad+\langle P\mathrm{e}^{(\tau-s)A}\eta,(A_1+B\Theta)\mathrm{e}^{(\tau-s)A}\xi\rangle+\langle Q\mathrm{e}^{(\tau-s)A}\eta,\mathrm{e}^{(\tau-s)A}\xi\rangle\\
&\quad+\langle P(C+D\Theta)\mathrm{e}^{(\tau-s)A}\eta,(C+D\Theta)\mathrm{e}^{(\tau-s)A}\xi\rangle+\langle R\Theta\mathrm{e}^{(\tau-s)A}\eta,\Theta\mathrm{e}^{(\tau-s)A}\xi\rangle]\mathrm{d}\tau. \qquad (3.6)
\end{aligned}
$$

Taking $\eta,\xi\in D(A)$ and using (3.6), we can get that $\langle P(s)\eta,\xi\rangle$ is differentiable with respect to $s$. Further, we conclude (3.5).

**Lemma 3.3** *Let $\Theta(\cdot)\in L^{2,S}(t,T;\mathcal{L}(H,U))$. Assume that $P(\cdot)\in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$ is the mild solution to* (2.4). *Then*

$$
\mathcal{J}(t,\eta;\Theta(\cdot)x(\cdot)+u(\cdot))=\langle P(t)\eta,\eta\rangle+\mathbb{E}\int_t^T\{2\langle (L+K\Theta)x,u\rangle+\langle Ku,u\rangle\}\mathrm{d}s. \qquad (3.7)
$$

**Proof** Taking $(t,\eta)\in[0,T)\times H$ and $u(\cdot)\in\mathcal{U}[t,T]$, let $x(\cdot)$ satisfy the following equation:

$$
\begin{cases}
\mathrm{d}x=[(A+A_1+B\Theta)x+Bu]\mathrm{d}s+[(C+D\Theta)x+Du]\mathrm{d}W(s) \quad \text{in } [t,T],\\
x(t)=\eta.
\end{cases} \qquad (3.8)
$$

(3.8) admits a unique solution $x(\cdot)\in C_{\mathbb{F}}([t,T];L^2(\Omega;H))$ for the admissibility of $B$. Let $\mathbb{R}(\lambda)\triangleq\lambda I(\lambda I-A)^{-1}$ for $\lambda\in\rho(A)$ and $x_\lambda(\cdot)=\mathbb{R}(\lambda)x(\cdot)$. Then $x_\lambda(\cdot)$ is the mild solution to

$$
\begin{cases}
\mathrm{d}x_\lambda=\{Ax_\lambda+\mathbb{R}(\lambda)[(A_1+B\Theta)x+Bu]\}\mathrm{d}s+\mathbb{R}(\lambda)[(C+D\Theta)x\\
\qquad\qquad +Du]\mathrm{d}W(s) \quad \text{in } [t,T],\\
x_\lambda(t)=\mathbb{R}(\lambda)\eta.
\end{cases} \qquad (3.9)
$$

Applying Ito's formula to $s \mapsto \langle P(s)x(s), x(s)\rangle$ and Lemma 3.2, we conclude that

$$
\begin{aligned}
&\mathbb{E}\Big[\langle Gx_\lambda(T), x_\lambda(T)\rangle + \int_t^T (\langle Qx_\lambda, x_\lambda\rangle + \langle R(\Theta x_\lambda + u), \Theta x_\lambda + u\rangle)\mathrm{d}s\Big] \\
&= \langle P(t)\mathbb{R}(\lambda)\eta, \mathbb{R}(\lambda)\eta\rangle + \mathbb{E}\int_t^T \{-\langle P(t)x_\lambda, (A + A_1 + B\Theta)x_\lambda\rangle \\
&\quad - \langle P(s)(A + A_1 + B\Theta)x_\lambda, x_\lambda\rangle - \langle P(C + D\Theta)x_\lambda, (C + D\Theta)x_\lambda\rangle - \langle R\Theta x_\lambda, \Theta x_\lambda\rangle \\
&\quad - \langle Qx_\lambda, x_\lambda\rangle + \langle PAx_\lambda, x_\lambda\rangle + \langle P\mathbb{R}(\lambda)[(A_1 + B\Theta)x + Bu], x_\lambda\rangle + \langle Px_\lambda, Ax_\lambda\rangle \\
&\quad + \langle Qx_\lambda, x_\lambda\rangle + \langle P\mathbb{R}(\lambda)[(C + D\Theta)x + Du], \mathbb{R}(\lambda)[(C + D\Theta)x + Du]\rangle \\
&\quad + \langle Px_\lambda, \mathbb{R}(\lambda)[(A + B\Theta)x + Bu]\rangle + \langle R(\Theta x_\lambda + u), \Theta x_\lambda + u\rangle\}\mathrm{d}s \\
&= \langle P(t)\eta, \eta\rangle + \int_t^T [2\langle (L + K\Theta)x, u\rangle + \langle Ku, u\rangle]\mathrm{d}s + F(\lambda),
\end{aligned} \tag{3.10}
$$

where

$$
\begin{aligned}
F(\lambda) = {}& \langle P(t)\mathbb{R}(\lambda)\eta, \mathbb{R}(\lambda)\eta\rangle - \langle P(t)\eta, \eta\rangle + \mathbb{E}\int_t^T \{\langle P\mathbb{R}(\lambda)(A_1 + B\Theta)x, x_\lambda\rangle \\
& - \langle P(A_1 + B\Theta)x_\lambda, x_\lambda\rangle + \langle Px_\lambda, \mathbb{R}(\lambda)(A + B\Theta)x\rangle - \langle P(s)(A_1 + B\Theta)x_\lambda, x_\lambda\rangle \\
& + \langle P\mathbb{R}(\lambda)(C + D\Theta)x, \mathbb{R}(\lambda)(C + D\Theta)x\rangle - \langle P(C + D\Theta)x_\lambda, (C + D\Theta)x_\lambda\rangle \\
& + \langle P\mathbb{R}(\lambda)Bu, x_\lambda\rangle - \langle PBu, x\rangle + \langle Px_\lambda, \mathbb{R}(\lambda)Bu\rangle + \langle P\mathbb{R}(\lambda)Du, \mathbb{R}(\lambda)Du\rangle \\
& - \langle PDu, Du\rangle + \langle P\mathbb{R}(\lambda)(C + D\Theta)x, \mathbb{R}(\lambda)Du\rangle - \langle P(C + D\Theta)x, Du\rangle \\
& - \langle Px, Bu\rangle + \langle P\mathbb{R}(\lambda)Du, \mathbb{R}(\lambda)(C + D\Theta)x\rangle - \langle PDu, (C + D\Theta)x\rangle\}\mathrm{d}s.
\end{aligned}
$$

Since

$$
\lim_{\lambda \to \infty} \mathbb{R}(\lambda)\zeta = \zeta \quad \text{in } H \tag{3.11}
$$

for any $\zeta \in H$, we obtain

$$
\lim_{\lambda \to \infty} \langle P(t)\mathbb{R}(\lambda)\eta, \mathbb{R}(\lambda)\eta\rangle = \langle P(t)\eta, \eta\rangle
$$

and

$$
\lim_{\lambda \to \infty} x_\lambda = x \quad \text{in } C_{\mathbb{F}}([t, T]; L^2(\Omega; H)). \tag{3.12}
$$

By (3.11)–(3.12), it yields that

$$
\begin{aligned}
&\lim_{\lambda \to \infty} [\langle P(s)\mathbb{R}(\lambda)(A_1(s) + B(s)\Theta(s))x(s), x_\lambda(s)\rangle \\
&\qquad - \langle P(s)(A_1(s) + B(s)\Theta(s))x_\lambda(s), x_\lambda(s)\rangle] = 0, \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.13}
$$

for a.e. $s \in [t, T]$. Since $B \in \mathcal{L}(U, H_{-1})$, we obtain $\mathbb{R}(\lambda)B \in \mathcal{L}(U, H)$. Therefore, from Lemma 3.2, we get that

$$
|\langle P(s)\mathbb{R}(\lambda)(A_1(s) + B(s)\Theta(s))x(s), x_\lambda(s)\rangle| - \langle P(s)(A_1(s) + B(s)\Theta(s))x(s), x_\lambda(s)\rangle
$$

$$\leq \mathcal{C}[|P(s)|_{\mathcal{L}(H)}|A_1(s)|_{\mathcal{L}(H)} + |P(s)|_{\mathcal{L}(H)}|(\lambda I - A)^{-1}B(s)|_{\mathcal{L}(U;H)}|\Theta|_{\mathcal{L}(U;H)} + |P(s)B(s)|_{\mathcal{L}(U;H)}|\Theta(s)|_{\mathcal{L}(U;H)}]|x(s)|_H^2.$$

By (3.13) and Lebesgue's dominated convergence theorem, it holds that

$$\lim_{\lambda\to\infty}\mathbb{E}\int_t^T [\langle P(s)\mathbb{R}(\lambda)(A_1(s)+B(s)\Theta(s))x(s), x_\lambda(s)\rangle - \langle P(s)(A_1(s)+B(s)\Theta(s))x_\lambda(s), x_\lambda(s)\rangle]\mathrm{d}s = 0.$$

Using a same argument, it has

$$\lim_{\lambda\to+\infty} F(\lambda) = 0.$$

Letting $\lambda$ tend to $+\infty$ in both sides of (3.10), we obtain (3.7).

**Lemma 3.4** *Let $\Theta(\cdot)\in L^{2,S}((t,T);\mathcal{L}(H;U))$, the following inequality holds*

$$\mathbb{E}\int_t^T |u(s)-\Theta(s)x(s)|_U^2\mathrm{d}s \geq c_0\mathbb{E}\int_t^T |u(s)|_U^2\mathrm{d}s,$$

*where $c_0$ is a constant.*

**Proof** Define a bounded linear operator as follows

$$\Upsilon : \mathcal{U}[t,T]\to\mathcal{U}[t,T],$$

where $\Upsilon u = u - \Theta x$. Therefore, the operator $\Upsilon$ is bijective and its inverse $\Upsilon^{-1}$ is given by $\Upsilon^{-1}u = u + \Theta\widehat{x}$, where $\widehat{x}$ is the solution to

$$\begin{cases} \mathrm{d}\widehat{x}(s) = [(A+A_1+B\Theta)\widehat{x}+Bu]\mathrm{d}s + [(C+D\Theta)\widehat{x}+Du]\mathrm{d}W(s) \quad \text{in } (t,T], \\ \widehat{x}(t) = 0. \end{cases} \tag{3.14}$$

Because of the admissibility of control operator $B$, the (3.14) is well-posed. Applying the bounded inverse theorem, the operator $\Upsilon^{-1}$ is bounded with the norm $|\Upsilon^{-1}|_{\mathcal{L}(\mathcal{U}[t,T])} > 0$. Therefore, we have

$$\mathbb{E}\int_t^T |u(s)|_U^2\mathrm{d}s = \mathbb{E}\int_t^T |(\Upsilon^{-1}\Upsilon u)(s)|_U^2\mathrm{d}s \leq |\Upsilon^{-1}|_{\mathcal{L}(\mathcal{U}[t,T])}\mathbb{E}\int_t^T |(\Upsilon u)(s)|_U^2\mathrm{d}s = |\Upsilon^{-1}|_{\mathcal{L}(\mathcal{U}[t,T])}\mathbb{E}\int_t^T |u(s)-\Theta(s)x(s)|_U^2\mathrm{d}s.$$

When choosing $c_0 = |\Upsilon^{-1}|^{-1}_{\mathcal{L}(\mathcal{U}[t,T])}$, we derive that

$$\mathbb{E}\int_t^T |u(s)-\Theta(s)x(s)|_U^2\mathrm{d}s \geq c_0\mathbb{E}\int_t^T |u(s)|_U^2\mathrm{d}s.$$

This completes the proof.

**Proposition 3.1** *Suppose that the map $u(\cdot)\mapsto \mathcal{J}(0,0;u(\cdot))$ is uniformly convex. Then Problem (SLQ) admits a unique optimal control, and there exists a constant $\alpha\in\mathbb{R}$ such that*

$$V(t,\eta)\geq \alpha|\eta|^2,\quad \forall (t,\eta)\in[0,T]\times H. \tag{3.15}$$

**Proof** Since $u(\cdot)\mapsto \mathcal{J}(0,0;u(\cdot))$ is uniformly convex, then for some constant $\lambda>0$ and for any $u(\cdot)\in\mathcal{U}[0,T]$, we get that

$$\mathcal{J}(0,0;u(\cdot))\geq \lambda\mathbb{E}\int_0^T |u(s)|_U^2\mathrm{d}s.$$

For any $t\in[0,T)$ and $u(\cdot)\in\mathcal{U}[t,T]$, define the zero-extension of $u(\cdot)$ as follows:

$$v(s)\triangleq\begin{cases}0, & s\in[0,t),\\ u(s), & s\in[t,T).\end{cases} \tag{3.16}$$

It holds that $v(\cdot)\in\mathcal{U}[0,T]$, because the initial state is 0, the solution $x(\cdot)$ of

$$\begin{cases}\mathrm{d}x(s)=[(A+A_1)x+Bv]\mathrm{d}s+(Cx+Dv)\mathrm{d}W(s) \quad \text{in } [0,T),\\ x(0)=0\end{cases} \tag{3.17}$$

satisfies $x(s)=0,\ s\in[0,T]$, and that (3.17) admits a unique mild solution

$$x(\cdot)\in C_{\mathbb{F}}([0,T];L^2(\Omega;H))$$

by the admissibility of the unbounded operator $B$. Thus, we have

$$\mathcal{J}(t,0;u(\cdot))=\mathcal{J}(0,0;v(\cdot))\geq\lambda\mathbb{E}\int_0^T|v(s)|_U^2\mathrm{d}s=\lambda\mathbb{E}\int_t^T|u(s)|_U^2\mathrm{d}s.$$

Therefore, for any given $(t,x)\in[0,T)\times H$, the map $u(\cdot)\mapsto\mathcal{J}(t,0;u(\cdot))$ is uniformly convex.

Let $\overline{u}(\cdot)\in\mathcal{U}[t,T]$ be an optimal control. Then, we can rewrite the cost functional as follows:

$$\begin{aligned}
&\mathcal{J}(t,\eta;u(\cdot))\\
&=\mathbb{E}\int_0^T[\langle G(\widehat{\Psi}_t\eta+\widehat{\Xi}_tu),\widehat{\Psi}_t\eta+\widehat{\Xi}_tu\rangle+\langle Q(\Psi_t\eta+\Xi_tu),\Psi_t\eta+\Xi_tu\rangle+\langle Ru,u\rangle]\mathrm{d}s\\
&=\mathbb{E}\int_0^T[\langle(\widehat{\Xi}_t^*G\widehat{\Xi}_t+\Xi_t^*Q\Xi_t+R)u,u\rangle+2\langle(\widehat{\Xi}_t^*G\widehat{\Psi}_t+\Xi_t^*Q\Psi_t)\eta,u\rangle\\
&\quad+\langle(\widehat{\Psi}_t^*G\widehat{\Psi}_t+\Psi_t^*Q\Psi_t)\eta,\eta\rangle]\mathrm{d}s\\
&=\mathcal{J}(t,\eta;0)+\mathcal{J}(t,0;u(\cdot))+2\int_t^T\langle(\widehat{\Xi}_t^*G\widehat{\Psi}_t+\Xi_t^*Q\Psi_t)\eta,u\rangle\mathrm{d}s\\
&\geq\lambda\mathbb{E}\int_t^T|\overline{u}(s)|_U^2\mathrm{d}s+\mathcal{J}(t,\eta;0)-\frac{\lambda}{2}\mathbb{E}\int_t^T|\overline{u}(s)|_U^2\mathrm{d}s-\frac{1}{2\lambda}\mathbb{E}\int_t^T|(\widehat{\Xi}_t^*G\widehat{\Psi}_t+\Xi_t^*Q\Psi_t)\eta|_H^2\mathrm{d}s\\
&\geq\frac{\lambda}{2}\mathbb{E}\int_t^T|\overline{u}(s)|_U^2\mathrm{d}s+\mathcal{J}(t,\eta;0)-\frac{\lambda}{2}\mathbb{E}\int_t^T|\overline{u}(s)|_U^2\mathrm{d}s\\
&\quad-\frac{1}{2\lambda}\mathbb{E}\int_t^T|(\widehat{\Xi}_t^*G\widehat{\Psi}_t+\Xi_t^*Q\Psi_t)\eta|_H^2\mathrm{d}s,
\end{aligned}\tag{3.18}$$

where the operators $\Psi_t, \Xi_t, \widehat{\Psi}_t, \widehat{\Xi}_t$ are defined at the beginning of Section 3. (3.18) implies that the map $u(\cdot) \mapsto \mathcal{J}(t,\eta;u(\cdot))$ is coercivity and that is continuous and convex. Therefore, $u(\cdot) \mapsto \mathcal{J}(t,0;u(\cdot))$ has a unique minimizer. Further, by (3.18), it implies that

$$V(t,\eta) \geq \mathcal{J}(t,\eta;0) - \frac{1}{2\lambda}\mathbb{E}\int_t^T |(\widehat{\Xi}_t^* G\widehat{\Psi}_t + \Xi_t^* Q\Psi_t)\eta|_H^2 \mathrm{d}s. \tag{3.19}$$

Because the functions on the right-hand side of (3.19) are quadratic in $x$ and continuous in $t$, we derive (3.15).

**Proposition 3.2** *Let* (2.8) *hold. Then for any* $\Theta(\cdot) \in L^{2,S}(t,T;\mathcal{L}(U,H))$, *the solution* $P(\cdot) \in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$ *to* (2.4) *satisfies*

$$K(s) \geq \lambda I, \quad a.e.\ s \in [t,T], \quad P(s) \geq \alpha I, \quad \forall s \in [t,T], \tag{3.20}$$

*where* $\alpha \in \mathbb{R}$ *is the constant appearing in* (3.15).

**Proof** For any $u(\cdot) \in \mathcal{U}[0,T]$, by the admissibility of $B$, let $x(\cdot)$ be the solution of

$$\begin{cases} \mathrm{d}x(s) = [(A + A_1 + B\Theta)x + Bu]\mathrm{d}s + [(C + D\Theta)x + Du]\mathrm{d}W(s) \quad \text{in } [0,T], \\ x(0) = 0. \end{cases} \tag{3.21}$$

Let $P(\cdot)$ be the solution to (2.4). Since (2.8) holds, we derive that

$$\mathcal{J}(0,0;\Theta(\cdot)x(\cdot) + u(\cdot)) \geq \lambda\mathbb{E}\int_0^T |\Theta(s)x(s) + u(s)|^2 \mathrm{d}s. \tag{3.22}$$

Because of Lemma 3.1, we know that the operator $B^*P \in \mathcal{L}(H)$ appearing in $L$ is bounded. Further, using Lemma 3.3, we deduce that

$$\mathcal{J}(0,0;\Theta(\cdot)x(\cdot) + u(\cdot)) = \mathbb{E}\int_0^T [2\langle (L + K\Theta)x, u\rangle + \langle Ku, u\rangle]\mathrm{d}s. \tag{3.23}$$

Hence, for any $u(\cdot) \in \mathcal{U}[0,T]$, we get that

$$\mathbb{E}\int_0^T \{2\langle [L + (K - \lambda I)\Theta]x, u\rangle + \langle (K - \lambda I)u, u\rangle\}\mathrm{d}s = \lambda\mathbb{E}\int_0^T |\Theta(s)x(s)|^2 \mathrm{d}s \geq 0. \tag{3.24}$$

We first prove that $K - \lambda I \geq 0$ for a.e. $s \in [t,T]$. In fact, if there exists a constant $\alpha > 0$ and a measurable set $\mathcal{T} \in [t,T]$ such that

$$K - \lambda I < -\alpha I \quad \text{for a.e. } s \in \mathcal{T}, \tag{3.25}$$

where the Lebesgue measure $m(\mathcal{T}) > 0$. Let $N > 0$ such that $\frac{1}{N} \leq m(\mathcal{T})$. Let $\{\mathcal{T}_n\}_{n=1}^\infty$ be a sequence of measurable subsets of $\mathcal{T}$ such that $m(\mathcal{T}_n) = \frac{1}{N+n}$. We assume $\zeta \in U$ and $u_n = n\chi_{\mathcal{T}_n}\zeta$ for $n = 1, 2, \cdots$. Let $x_n$ be the solution to (3.21) with $u = u_n$. Under the singular estimate (1.1), we obtained that

$$|x_n|_{C_{\mathbb{F}}([t,T];L^2(\Omega;H))} \leq c,$$

where $c$ is a constant independent of $n$. It follows from (3.25) that

$$\varliminf_{n\to\infty}\frac{1}{n^2}\int_t^T[\langle(K-\lambda I)u_n,u_n\rangle+2\langle[(K-\lambda I)\Theta+L]x_n,u_n\rangle]\mathrm{d}s\le-\alpha|\zeta|_U^2. \tag{3.26}$$

From (3.24), it holds that

$$\varliminf_{n\to\infty}\frac{1}{n^2}\int_t^T[\langle(K-\lambda I)u_n,u_n\rangle+2\langle[(K-\lambda I)\Theta+L]x_n,u_n\rangle]\mathrm{d}s\ge 0, \tag{3.27}$$

which contradicts (3.26). Thus, we have

$$K-\lambda I\ge 0\quad\text{for a.e. }\ s\in[t,T]. \tag{3.28}$$

Therefore, we get the first inequality in (3.20).

We next prove the second inequality in (3.20). For any $(t,\eta)\in[0,T)\times H$ and $u(\cdot)\in\mathcal{U}[t,T]$, we assume that $x(\cdot)$ is the solution of

$$\begin{cases}\mathrm{d}x(s)=[(A+A_1+B\Theta)x+Bu]\mathrm{d}s+[(C+D\Theta)x+Du]\mathrm{d}W(s)\quad\text{in }(t,T],\\ x(t)=\eta.\end{cases}$$

Since $B^*P\in\mathcal{L}(H)$ appearing in $L$ is bounded and Lemma 3.3, we have

$$\begin{aligned}&\mathcal{J}(t,\eta;\Theta(\cdot)x(\cdot)+u(\cdot))\\&=\langle P(t)\eta,\eta\rangle+\mathbb{E}\int_t^T\{2\langle(L+K\Theta)x,u\rangle+\langle Ku,u\rangle\}\mathrm{d}s.\end{aligned} \tag{3.29}$$

According to Proposition 3.1, we get that

$$\mathcal{J}(t,\eta;\Theta(\cdot)x(\cdot)+u(\cdot))\ge V(t,\eta)\ge\alpha|x|^2. \tag{3.30}$$

Setting $u(\cdot)=0$ in (3.30) and using (3.29), we deduce that

$$\langle P(t)\eta,\eta\rangle\ge\alpha|x|^2,\quad\forall(t,\eta)\in[0,T]\times H.$$

Thus, it implies the second inequality in (3.20).

In the following, we prove the main result. When the cost functional is uniformly convex, we apply some iteration scheme with (2.4) to obtain the solvability of Riccati equation (2.2) and further get the strongly regular solution of (2.2).

Furthermore, we derive the optimal feedback control of Problem (SLQ).

**Proof of Theorem 2.1** Without loss of generality, we assume that $t=0$.

(1) ⇒ (2). We define the initial variable $P_0$. Assume that $P_0$ is the solution to

$$\begin{cases}\dot{P}_0+P_0(A+A_1)+(A+A_1)^*P_0+C^*P_0C+Q=0\quad\text{in }[0,T],\\ P_0(T)=G.\end{cases} \tag{3.31}$$

Using the same argument in Lemma 3.1, we can easily obtain $B^*P_0 \in \mathcal{L}(H,U)$ and $P_0B \in \mathcal{L}(U,H)$. By Proposition 3.2 and taking $\Theta = 0$ in (3.31), we can get that

$$R(s) + D(s)^*P_0(s)D(s) \geq \lambda I, \quad P_0(s) \geq \alpha I, \quad \text{a.e. } s \in [0,T]. \tag{3.32}$$

We next set up the following iteration scheme as follows:

$$\begin{cases} \dot{P}_{i+1} + P_{i+1}(A + A_1 + B\Theta_i) + (A + A_1 + B\Theta_i)^*P_{i+1} \\ +C_i^*P_{i+1}C_i + \Theta_i^*R\Theta_i + Q = 0, \\ P_{i+1}(T) = G, \end{cases}$$

where

$$\begin{aligned} K_i &= R + D^*P_iD, \quad L_i = B^*P_i + D^*P_iC, \\ \Theta_i &= -K_i^{-1}L_i, \qquad C_i = C + D\Theta_i \end{aligned}$$

for $i = 0, 1, 2, \cdots$. Applying Lemma 3.1 and setting $P = P_{i+1}$ and $\Theta = \Theta_i$ in (2.4), we see that

$$|B^*P_i(\tau)x|_U \leq \frac{\mathcal{C}}{(T-\tau)^\gamma}|x|_H, \quad |P(\tau)B_iu|_H \leq \frac{\mathcal{C}}{(T-\tau)^\gamma}|u|_U, \\ \forall x \in H, \ \tau \in [t,T), \ i = 0, 1, 2, \cdots.$$

By (3.32), we can derive that $\Theta_0 = -K_0^{-1}L_0 \in L^{2,S}(0,T;\mathcal{L}(H;U))$. Noticing Proposition 3.2, we obtain

$$K_1 \geq \lambda I, \quad P_1(s) \geq \alpha I \quad \text{a.e. } s \in [0,T] \tag{3.33}$$

by replacing $P$ and $\Theta$ in (2.4) with $P_1$ and $\Theta_0$, respectively. Similarly, we derive that

$$K_{i+1} \geq \lambda I, \quad P_{i+1}(s) \geq \alpha I \quad \text{a.e. } s \in [0,T], \quad i = 0, 1, 2, \cdots. \tag{3.34}$$

We denote

$$\Delta_i \triangleq P_i - P_{i+1}, \quad \Lambda_i \triangleq \Theta_{i-1} - \Theta_i, \quad i \geq 1.$$

Therefore, for $i \geq 1$, we conclude that

$$\begin{aligned} &- \Delta_i(s)\eta = P_{i+1}(s)\eta - P_i(s)\eta \\ =& \int_s^T e^{(\tau-s)A^*}[(P_i(s) - P_{i+1}(s))(A_1 + B\Theta_i) + (A_1 + B\Theta_i)^*(P_i(s) - P_{i+1}(s)) \\ &+ C_{i-1}^*P_iC_{i-1} + \Theta_{i-1}^*R\Theta_{i-1} - C_i^*P_{i+1}C_i - \Theta_i^*R\Theta_i]e^{(\tau-s)A}\eta d\tau \\ =& \int_s^T e^{(\tau-s)A^*}[\Delta_i(A_1 + B\Theta_i) + (A_1 + B\Theta_i)^*\Delta_i + P_i(B\Theta_{i-1} - B\Theta_i) \\ &+ (B\Theta_{i-1} - B\Theta_i)^*P_i + C_i^*\Delta_iC_i + C_{i-1}^*P_iC_{i-1} \\ &+ \Theta_{i-1}^*R\Theta_{i-1} - C_i^*P_iC_i - \Theta_i^*R\Theta_i]e^{(\tau-s)A}\eta d\tau. \end{aligned} \tag{3.35}$$

By calculating, we deduce that

$$
\begin{aligned}
& C_{i-1}^* P_i C_{i-1} - C_i^* P_i C_i \\
= & (C + D\Theta_{i-1})^* P_i (C + D\Theta_{i-1}) - (C + D\Theta_i)^* P_i (C + D\Theta_i) \\
= & \Theta_{i-1}^* D^* P_i D\Theta_{i-1} - \Theta_i^* D^* P_i D\Theta_i + \Theta_{i-1}^* D^* P_i C \\
& + C^* P_i D\Theta_{i-1} - \Theta_i^* D^* P_i C - C^* P_i D\Theta_i \\
= & (\Theta_{i-1} - \Theta_i)^* D^* P_i D(\Theta_{i-1} - \Theta_i) + (C + D\Theta_i)^* P_i D(\Theta_{i-1} - \Theta_i) \\
& + (\Theta_{i-1} - \Theta_i)^* D^* P_i (C + D\Theta) \\
= & \Lambda_i^* D^* P_i D\Lambda_i + C_i^* P_i D\Lambda_i + \Lambda_i^* D^* P_i C_i.
\end{aligned} \tag{3.36}
$$

Likewise, we see that

$$
B^* P_i + D^* P_i C_i + R\Theta_i = B^* P_i + D^* P_i C + (R + D^* P_i D)\Theta_i = 0, \tag{3.37}
$$

$$
\Theta_{i-1}^* R\Theta_{i-1} - \Theta_i^* R\Theta_i = \Lambda_i^* R\Lambda_i + \Lambda_i^* R\Theta_i + \Theta_i^* R\Lambda_i. \tag{3.38}
$$

From (3.35)–(3.38), we can obtain that

$$
\begin{aligned}
& -\Delta_i(s) - \int_s^T \mathrm{e}^{(\tau-s)A^*} [\Delta_i(A_1 + B\Theta_i) + (A_1 + B\Theta_i)^* \Delta_i + C_i^* \Delta_i C_i] \mathrm{e}^{(\tau-s)A} \mathrm{d}\tau \\
= & \int_s^T \mathrm{e}^{(\tau-s)A^*} (P_i B\Lambda_i + \Lambda_i^* B^* P_i + \Lambda_i^* D^* P_i D\Lambda_i + C_i^* P_i D\Lambda_i + \Lambda_i^* D^* P_i C_i \\
& + \Lambda_i^* R\Lambda_i + \Lambda_i^* R\Theta_i + \Theta_i^* R\Lambda_i) \mathrm{e}^{(\tau-s)A} \mathrm{d}\tau \\
= & \int_s^T \mathrm{e}^{(\tau-s)A^*} [\Lambda_i^* K_i \Lambda_i + (P_i B + C_i^* P_i D + \Theta_i^* R)\Lambda_i \\
& + \Lambda_i^* (B^* P_i + D^* P_i C_i + R\Theta_i)] \mathrm{e}^{(\tau-s)A} \mathrm{d}\tau \\
= & \int_s^T \mathrm{e}^{(\tau-s)A^*} [\Lambda_i^* K_i \Lambda_i - \Delta_{i-1} B\Lambda_i + (P_{i-1} B + C_i^* P_i D + \Theta_i^* R)\Lambda_i] \mathrm{e}^{(\tau-s)A} \mathrm{d}\tau.
\end{aligned} \tag{3.39}
$$

Taking $\widehat{G} = 0, \widehat{A} = A_1 + B\Theta_i, \widehat{C} = C_i$ and $\widehat{Q} = \Lambda_i^* K_i \Lambda_i$ in (2.6), by (3.39), we can derive that $\Delta_i(\cdot)$ is a solution to (2.6). Applying Lemma 2.1, we obtain $\Delta_i(\cdot) \geq 0$, that is, $P_i(\cdot) - P_{i+1}(\cdot) \geq 0$ for $i \geq 1$. By (3.33), we get that for any $s \in [0, T]$,

$$
P_1(s) \geq P_i(s) \geq P_{i+1}(s) \geq \alpha I, \quad \forall i \geq 1.
$$

Hence, we get that the sequence $\{P_i\}_{i=1}^\infty$ is uniformly bounded. Further, there exist constants $\widetilde{c} > 0$ and $c > 0$ such that

$$
\begin{aligned}
& |P_i(s)|_{\mathcal{L}(H)} \leq \widetilde{c}, \quad |K_i(s)|_{\mathcal{L}(H)} \leq \widetilde{c}, \\
& |\Theta_i(s)|_{\mathcal{L}(H,U)} \leq \frac{c}{(T-s)^\gamma}, \\
& |C_i(s)|_{\mathcal{L}(H)} \leq |C(s)|_{\mathcal{L}(H)} + |D(s)|_{\mathcal{L}(H)} \frac{c}{(T-s)^\gamma}.
\end{aligned} \tag{3.40}
$$

We next show the convergence of the sequence $\{P_i, B^*P_i, P_iB\}_{i=1}^{\infty}$. Observe that

$$\Lambda_i = K_i^{-1}D^*\Delta_{i-1}DK_{i-1}^{-1}(B^*P_i + D^*P_iC) - K_{i-1}^{-1}(B^*\Delta_{i-1} + D^*\Delta_{i-1}C). \tag{3.41}$$

Therefore, it follows from (3.40)–(3.41) that

$$\begin{aligned}
&|\Lambda_i(s)^*K_i(s)\Lambda_i(s)|_{\mathcal{L}(H)}\\
&= |(\Theta_{i-1}(s)K_i(s)\Lambda_i(s))^*K_i(s)(\Theta_{i-1}(s) - \Theta_i(s))|_{\mathcal{L}(H)}\\
&\le (|\Theta_{i-1}(s)|_{\mathcal{L}(H,U)} + |\Theta_i(s)|_{\mathcal{L}(H,U)})|K_i(s)|_{\mathcal{L}(U)}|\Theta_{i-1}(s) - \Theta_i(s)|_{\mathcal{L}(H,U)}\\
&\le \Big[\frac{2c^2}{(T-s)^{2\gamma}}\widetilde{c} + \frac{2c^2}{(T-s)^{\gamma}} + \frac{2c}{(T-s)^{\gamma}}\widetilde{c}|C(s)|_{\mathcal{L}(H)}\Big]|\Delta_{i-1}|_{\mathcal{L}(H)}\\
&\quad + \frac{2c}{(T-s)^{\gamma}}\widetilde{c}|B^*\Delta_{i-1}(s)|_{\mathcal{L}(H,U)}.
\end{aligned} \tag{3.42}$$

Combining (3.39) with $\Delta_i(T) = 0$, we derive that

$$\begin{aligned}
\Delta_i(s) = \int_s^T &\mathrm{e}^{(\tau-s)A^*}[\Delta_i(A_1 + B\Theta_i) + (A_1 + B\Theta_i)^*\Delta_i + C_i^*\Delta_iC_i + \Lambda_i^*K_i\Lambda_i\\
&- \Delta_{i-1}B\Lambda_i + (P_{i-1}B + C_i^*P_iD + \Theta_i^*R)\Lambda_i]\mathrm{e}^{(\tau-s)A}\mathrm{d}\tau.
\end{aligned} \tag{3.43}$$

It holds from (3.40), (3.42)–(3.43) that

$$\begin{aligned}
&|\Delta_i(s)|_{\mathcal{L}(H)}\\
&\le \int_s^T CM\mathrm{e}^{2\alpha T}\Big\{\Big[2|A_1(\tau)|_{\mathcal{L}(H)} + \Big(|C(\tau)|_{\mathcal{L}(H)} + \frac{c}{(T-\tau)^{\gamma}}\Big)^2\Big]|\Delta_i(\tau)|_{\mathcal{L}(H)}\\
&\quad + \frac{c}{(T-\tau)^{\gamma}}|B^*\Delta_i(\tau)|_{\mathcal{L}(H,U)} + \frac{c}{(T-\tau)^{\gamma}}|\Delta_i(\tau)B|_{\mathcal{L}(U,H)}\\
&\quad + \Big[\frac{2c^3|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}} + c^2\Big(\frac{4\widetilde{c}}{(T-\tau)^{2\gamma}} + \frac{5\widetilde{c}|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}} + |C(\tau)|_{\mathcal{L}(H)}\Big)\\
&\quad + c\Big(\frac{5\widetilde{c}|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}} + |C(\tau)|_{\mathcal{L}(H)}\Big) + \frac{|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}}\Big]|\Delta_{i-1}(\tau)|_{\mathcal{L}(H)}\\
&\quad + \Big[\frac{c^2}{(T-\tau)^{\gamma}} + c\Big(\frac{4\widetilde{c}}{(T-\tau)^{\gamma}} + |C(\tau)|_{\mathcal{L}(H)}\Big)\frac{1}{(T-\tau)^{\gamma}}\Big]|B^*\Delta_{i-1}(\tau)|_{\mathcal{L}(H,U)}\\
&\quad + \frac{2c}{(T-\tau)^{\gamma}}|\Delta_{i-1}B|_{\mathcal{L}(U,H)}\Big\}\mathrm{d}\tau,
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
&|B^*\Delta_i(s)|_{\mathcal{L}(H,U)}\\
&\le \int_s^T \frac{CM\mathrm{e}^{\alpha T}}{(\tau-s)^{\gamma}}\Big\{\Big[2|A_1(\tau)|_{\mathcal{L}(H)} + \Big(|C(\tau)|_{\mathcal{L}(H)} + \frac{c}{(T-\tau)^{\gamma}}\Big)^2\Big]|\Delta_i(\tau)|_{\mathcal{L}(H)}\\
&\quad + \frac{c}{(T-\tau)^{\gamma}}|B^*\Delta_i(\tau)|_{\mathcal{L}(H,U)} + \frac{c}{(T-\tau)^{\gamma}}|\Delta_i(\tau)B|_{\mathcal{L}(U,H)}\\
&\quad + \Big[c^3\Big(\frac{2|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}}\Big) + c^2\Big(\frac{4\widetilde{c}}{(T-\tau)^{2\gamma}} + \frac{5\widetilde{c}|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}} + |C(\tau)|_{\mathcal{L}(H)}\Big)\\
&\quad + c\Big(\frac{5\widetilde{c}|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}} + |C(\tau)|_{\mathcal{L}(H)}\Big) + \frac{|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^{\gamma}}\Big]|\Delta_{i-1}(\tau)|_{\mathcal{L}(H)}
\end{aligned}$$

$$+ \Big[\frac{c^2}{(T-\tau)^\gamma} + c\Big(\frac{4\widetilde{c}}{(T-\tau)^\gamma} + |C(\tau)|_{\mathcal{L}(H)}\Big)\frac{1}{(T-\tau)^\gamma}\Big]|B^*\Delta_{i-1}(\tau)|_{\mathcal{L}(H,U)}$$
$$+ \frac{cr}{(T-\tau)^\gamma}|\Delta_{i-1}B|_{\mathcal{L}(U,H)}\Big\}\mathrm{d}\tau \tag{3.45}$$

and

$$\begin{aligned}
&|\Delta_i(s)B|_{\mathcal{L}(H,U)}\\
\le& \int_s^T CM\mathrm{e}^{\alpha T}\Big\{\Big[2|A_1(\tau)|_{\mathcal{L}(H)} + \Big(|C(\tau)|_{\mathcal{L}(H)} + \frac{c}{(T-\tau)^\gamma}\Big)^2\Big]|\Delta_i(\tau)|_{\mathcal{L}(H)}\\
&+ \frac{c}{(T-\tau)^\gamma}|B^*\Delta_i(\tau)|_{\mathcal{L}(H,U)} + \frac{c}{(T-\tau)^\gamma}|\Delta_i(\tau)B|_{\mathcal{L}(U,H)}\\
&+ \Big[c^3\Big(\frac{2|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^\gamma}\Big) + c^2\Big(\frac{4\widetilde{c}}{(T-\tau)^{2\gamma}} + \frac{5\widetilde{c}|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^\gamma} + |C(\tau)|_{\mathcal{L}(H)}\Big)\\
&+ c\Big(\frac{5\widetilde{c}|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^\gamma} + |C(\tau)|_{\mathcal{L}(H)}\Big) + \frac{|C(\tau)|_{\mathcal{L}(H)}}{(T-\tau)^\gamma}\Big]|\Delta_{i-1}(\tau)|_{\mathcal{L}(H)}\\
&+ \Big[\frac{c^2}{(T-\tau)^\gamma} + c\Big(\frac{4\widetilde{c}}{(T-\tau)^\gamma} + |C(\tau)|_{\mathcal{L}(H)}\Big)\frac{1}{(T-\tau)^\gamma}\Big]|B^*\Delta_{i-1}(\tau)|_{\mathcal{L}(H,U)}\\
&+ \frac{2c}{(T-\tau)^\gamma}|\Delta_{i-1}B|_{\mathcal{L}(U,H)}\Big\}\frac{1}{(\tau-s)^\gamma}\mathrm{d}\tau.
\end{aligned} \tag{3.46}$$

From (3.44)–(3.46), we see that

$$\begin{aligned}
&|\Delta_i(s)|_{\mathcal{L}(H)} + |B^*\Delta_i(s)|_{\mathcal{L}(H,U)} + |\Delta_i(s)B|_{\mathcal{L}(U,H)}\\
\le& \int_s^T \phi(\tau)[|\Delta_i(\tau)|_{\mathcal{L}(H)} + |B^*\Delta_i(\tau)|_{\mathcal{L}(H,U)} + |\Delta_i(\tau)B|_{\mathcal{L}(U,H)}\\
&+ |\Delta_{i-1}(\tau)|_{\mathcal{L}(H)} + |B^*\Delta_{i-1}(\tau)|_{\mathcal{L}(H,U)} + |\Delta_{i-1}(\tau)B|_{\mathcal{L}(U,H)}]\mathrm{d}\tau, \quad \forall s\in[0,T],\ \forall i\ge 1,
\end{aligned}$$

where $\phi(\cdot)$ is a nonnegative integrable function, which is independent of $\Delta(\cdot)$. Further, by Gronwall's inequality, it infers that

$$\begin{aligned}
&|\Delta_i(s)|_{\mathcal{L}(H)} + |B^*\Delta_i(s)|_{\mathcal{L}(H,U)} + |\Delta_i(s)B|_{\mathcal{L}(U,H)}\\
&\le \mathrm{e}^{\int_0^T\phi(\tau)\mathrm{d}\tau}\int_s^T \phi(\tau)(|\Delta_{i-1}(\tau)|_{\mathcal{L}(H)} + |B^*\Delta_{i-1}(\tau)|_{\mathcal{L}(H,U)} + |\Delta_{i-1}(\tau)B|_{\mathcal{L}(U,H)})\mathrm{d}\tau\\
&= c\int_s^T [\phi(\tau)|(\Delta_{i-1}(\tau)|_{\mathcal{L}(H)} + |B^*\Delta_{i-1}(\tau)|_{\mathcal{L}(H,U)} + |\Delta_{i-1}(\tau)B|_{\mathcal{L}(U,H)})]\mathrm{d}\tau,
\end{aligned}$$

where $c = \mathrm{e}^{\int_0^T\phi(\tau)\mathrm{d}\tau}$. Setting $a \triangleq \max_{0\le\tau\le T}|\Delta_0(\tau)|_{\mathcal{L}(H)}$ and by induction, it has

$$|\Delta_i(s)|_{\mathcal{L}(H)} + |B^*\Delta_i(s)|_{\mathcal{L}(H,U)} + |\Delta_i(s)B|_{\mathcal{L}(U,H)} \le a\frac{c^i}{i!}\Big(\int_s^T\phi(\tau)\mathrm{d}\tau\Big)^i, \quad \forall s\in[0,T]. \tag{3.47}$$

Therefore, the inequality (3.47) implies that the sequence $\{P_i, B^*P_i, P_iB\}_{i=1}^\infty$ is uniform convergence in $X$.

We denote $(P, g(s), h(s)) \in X$ the limit of $\{P_i, B^*P_i, P_iB\}_{i=1}^\infty$ with $g(s) = B^*P(s)$, $h(s) = P(s)B$. Then

$$K(s) = \lim_{i\to\infty} K_i(s) \ge \lambda I \quad \text{a.e. } s\in[0,T].$$

Furthermore, when $i \to \infty$, it holds that

$$\begin{aligned}&\Theta_i \to -K^{-1}L \equiv \Theta \quad \text{in } L^{2.S}(0,T;\mathcal{L}(H,U)),\\ &C_i \to C + D\Theta \quad \text{in } L^2(0,T;\mathcal{L}(H)).\end{aligned}$$

Hence, the operator $P(\cdot)$ solves (2.2) in the sense of mild solution.

(2) $\Rightarrow$ (1). Since the operator $B$ is unbounded, we need to estimate $PB \in \mathcal{L}(U,H)$ and $B^*P \in \mathcal{L}(H,U)$ when $P$ is the mild solution to (2.2). Let $P(\cdot) \in C_{\mathcal{S}}([t,T];\mathbb{S}(H))$ be the mild solution to (2.4). For any $v(\cdot) \in \mathcal{U}[t,T]$, the system (1.2) with the control $u(\cdot) = \overline{\Theta}(\cdot)x(\cdot) + v(\cdot)$ becomes

$$\begin{cases}\mathrm{d}x = [(A + A_1 + B\overline{\Theta})x + Bv]\mathrm{d}s + [(C + D\overline{\Theta})x + Dv]\mathrm{d}W(s) \quad \text{in } [t,T],\\ x(t) = \eta.\end{cases} \tag{3.48}$$

By Itô's formula and Lemma 3.2, we have

$$\begin{aligned}&\mathcal{J}(t,\eta;\Theta x(\cdot) + v(\cdot))\\ &= \langle P(t)\eta,\eta\rangle + \mathbb{E}\int_t^T \{\langle[\dot{P} + P(A + A_1 + B\Theta) + (A + A_1 + B\Theta)^*P\\ &\quad + (C + D\Theta)^*P(C + D\Theta) + Q + \Theta^*R\Theta]x, x\rangle\\ &\quad + \langle Kv, v\rangle + 2\langle[(L + K\Theta)]x, v\rangle\}\mathrm{d}s\\ &= \langle P(t)\eta,\eta\rangle + \mathbb{E}\int_t^T \{2\langle[L + K\Theta]x, v\rangle + \langle Kv, v\rangle\}\mathrm{d}s.\end{aligned} \tag{3.49}$$

Because $\Theta$ is an optimal feedback operator, we have

$$\mathcal{J}(t,\eta;\Theta x(\cdot) + v(\cdot)) \geq \mathcal{J}(t,\eta;\Theta x(\cdot)) = \langle P(t)\eta,\eta\rangle.$$

This, together with (3.49), implies that

$$\mathbb{E}\int_t^T [\langle Kv, v\rangle + \langle(K\Theta + L)x, v\rangle]\mathrm{d}s \geq 0. \tag{3.50}$$

We next show that $K(s) \geq 0$ for a.e. $s \in [t,T]$ by contradiction. Otherwise, there exists $\delta > 0$ and a measurable set $\mathcal{T} \in [t,T]$ with Lebesgue measure $m(\mathcal{T}) > 0$ such that

$$K(s) < -\delta I \quad \text{for a.e. } s \in \mathcal{T}. \tag{3.51}$$

Let $N > 0$ such that $\frac{1}{N} \leq m(\mathcal{T})$. Let $\{\mathcal{T}_n\}_{n=1}^{\infty}$ be a sequence of the measurable subsets of $\mathcal{T}$ such that $m(\mathcal{T}_n) = \frac{1}{N+n}$. Let $\zeta \in U$ and $v_n = n\chi_{\mathcal{T}_n}\zeta$ for $n = 1, 2, \cdots$. Denote by $x_n$ the solution of (3.48) with $\eta = 0$ and $v = v_n$. Under the singular estimate (1.1), then we obtained that

$$|x_n|_{C_{\mathbb{F}}([t,T];L^2(\Omega;H))} \leq \mathcal{C},$$

where $\mathcal{C}$ is a constant independent of $n$. This, together with (3.51), implies that

$$\varliminf_{n\to\infty} \frac{1}{n^2} \int_t^T [\langle K v_n, v_n\rangle + \langle (K\Theta + L)x_n, v_n\rangle] \mathrm{d}s \le -\delta|\zeta|_U^2. \tag{3.52}$$

On the other hand, by (3.50), we have

$$\varliminf_{n\to\infty} \frac{1}{n^2} \int_t^T [\langle K v_n, v_n\rangle + \langle (K\Theta + L)x_n, v_n\rangle] \mathrm{d}s \ge 0, \tag{3.53}$$

which contradicts (3.52). Therefore, we see that

$$K(s) \ge 0 \quad \text{for a.e. } s \in [t,T]. \tag{3.54}$$

We use the following iteration scheme

$$\begin{aligned} P_{i+1}(s)\eta = \mathrm{e}^{(T-s)A^*} G \mathrm{e}^{(T-s)A}\eta + \int_s^T \mathrm{e}^{(\tau-s)A^*} [P_{i+1}(A_1 + B\Theta_i) + (A_1 + B\Theta_i)^* P_{i+1} \\ + (C + D\Theta_i)^* P_{i+1}(C + D\Theta_i) + \Theta_i^* R\Theta_i + Q]\mathrm{e}^{(\tau-s)A}\eta \mathrm{d}\tau, \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} \Theta_i &= -K_i^{\dagger} L_i, & K_i &= R + D^* P_i D, \\ L_i &= B^* P_i + D^* P_i C, & P_0 &= \mathrm{e}^{(T-s)A^*} G \mathrm{e}^{(T-s)A}. \end{aligned}$$

Using the result of Lemma 3.1, each iteration $P_i$ is well defined and bounded with

$$|P_i|_{C_{\mathcal{S}([t,T]\mathcal{L}(H))}} \le \mathcal{C}_T, \quad |B^* P(s)x|_H \le \frac{\mathcal{C}|x|_H}{(T-s)^\gamma}, \quad |PB(s)u|_H \le \frac{\mathcal{C}|u|_U}{(T-s)^\gamma},$$

$\forall x \in H$ and $i = 0, 1, 2, \cdots$. Thanks to (3.54), each $K_i(s)^\dagger$ is well defined and bounded on H at each step. Taking estimates, it implies that the sequence $\{P_i, B^* P_i, P_i B\}_{i=1}^\infty$ is convergence in $X$ for $(T-t)$ is sufficiently small, and thus converging to some $(P, p(s), q(s)) \in X$ with $p(s) = B^* P(s)$ and $q(s) = P(s)B$. Passing through the limit in (3.55), we obtain (2.2). Further, when $P(s)$ is the mild solution to (2.2), $B^* P$ and $PB$ are bounded.

Let $P(\cdot)$ be the strongly regular solution of (2.2). Then there exists a $\lambda \ge 0$ such that

$$K(s) \ge \lambda I \quad \text{for a.e. } s \in [0,T]. \tag{3.56}$$

Because $B^* P \in \mathcal{L}(H, U)$ and (3.56) hold, we have $\Theta \equiv -K^{-1} L \in L^{2,S}(0,T;\mathcal{L}(H;U))$. For any $u(\cdot) \in \mathcal{U}[0,T]$, let $x^{(u)}(\cdot)$ be the solution of

$$\begin{cases} \mathrm{d}x^{(u)}(s) = [Ax^{(u)}(s) + Bu(s)]\mathrm{d}s + [Cx^{(u)}(s) + Du(s)]\mathrm{d}W(s) \quad \text{in } [0,T], \\ x^{(u)}(0) = 0. \end{cases} \tag{3.57}$$

Applying Itô's formula to $s \mapsto \langle P(s)x^{(u)}(s), x^{(u)}(s)\rangle$, it has

$$\mathcal{J}(0,0;u(\cdot))$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^T \{\langle[\dot{P} + P(A + A_1) + (A + A_1)^* P + C^* PC + Q]x^{(u)}, x^{(u)}\rangle \\
&\quad + 2\langle(B^* P + D^* PC)x^{(u)}, u\rangle + \langle(R + D^* PD)u, u\rangle\}\mathrm{d}\tau \\
&= \mathbb{E}\int_0^T [\langle\Theta^* K\Theta x^{(u)}, x^{(u)}\rangle - 2\langle K\Theta x^{(u)}, u\rangle + \langle Ku, u\rangle]\mathrm{d}\tau \\
&= \mathbb{E}\int_0^T \langle K(u - \Theta x^{(u)}), u - \Theta x^{(u)}\rangle \mathrm{d}\tau.
\end{aligned}
$$

Due to (3.56) and Lemma 3.4, we deduce that

$$
\begin{aligned}
&\mathcal{J}(0, 0; u(\cdot)) \\
&= \mathbb{E}\int_0^T \langle K(u - \Theta x^{(u)}), u - \Theta x^{(u)}\rangle \mathrm{d}\tau \\
&\geq \lambda c_0 \mathbb{E}\int_0^T |u(\tau)|_U^2 \mathrm{d}\tau, \quad \forall u(\cdot) \in \mathcal{U}[0, T]
\end{aligned}
$$

for some $\gamma = \lambda c_0 > 0$. Hence, the statement (1) holds.

In the end, we introduce an example about the uniform convexity of the cost functional.

Let $O \subset \mathbb{R}^m (m \in \mathbb{N})$ be a bounded domain with the $C^2$ boundary $\partial O$. We consider the following stochastic Schrödinger equation

$$
\begin{cases}
\mathrm{d}x + \mathrm{i}\Delta x \mathrm{d}t = ax\mathrm{d}t + bx\mathrm{d}W(t) + c\widetilde{u}\mathrm{d}W(t) & \text{in } O \times (0, T], \\
x = v & \text{on } \partial O \times [0, T], \\
x(0) = x_0 & \text{in } O,
\end{cases}
\tag{3.58}
$$

where $a, b, c \in L^\infty_{\mathbb{F}}(0, T; W_0^{1,+\infty}(O))$ and $x_0 \in H^{-1}(O)$.

Let $H = H^{-1}(O)$ and $U = L^2(O) \times L^2(\partial O)$. Define an unbounded linear operator on $H$ as follows:

$$
\begin{cases}
D(\mathcal{A}) = H_0^1(O), \\
\langle \mathcal{A}f, g\rangle_{H^{-1}(O), H_0^1(O)} = \displaystyle\int_G \nabla f(\xi) \cdot \overline{\nabla g(\xi)} \mathrm{d}\xi, \quad \forall f, g \in H_0^1(O).
\end{cases}
$$

Define a map $A_D : L^2(\partial O) \to L^2(O)$ as follows:

$$
A_D g = h,
$$

where $h$ is the solution to

$$
\begin{cases}
\Delta h = 0 & \text{in } O, \\
h = g & \text{on } \partial O.
\end{cases}
$$

Define three operators $A_1, C \in \mathcal{L}(L^2_{\mathbb{F}}(0, T; H))$ and $\mathcal{D} \in \mathcal{L}(L^2_{\mathbb{F}}(0, T; U), L^2_{\mathbb{F}}(0, T; H))$ as

$$
A_1\xi = a\xi, \quad C\xi = b\xi, \quad \mathcal{D}u = cu, \quad \forall \xi \in L^2_{\mathbb{F}}(0, T; H),\ \forall u \in L^2_{\mathbb{F}}(0, T; U).
$$

Write $A = \mathrm{i}\mathcal{A}$. The operators $B, D$ are $B = (0, AA_D)$ and $D = (\mathcal{D}, 0)$. The system can be expressed in an abstract form as follows

$$\begin{cases} \mathrm{d}x = (Ax + A_1x + Bu)\mathrm{d}t + (Cx + Du)\mathrm{d}W(t) \quad \text{in } [0, T], \\ x(0) = x_0, \end{cases} \tag{3.59}$$

where $u = \binom{\widetilde{u}}{v}$ with $|u|_{L^2_{\mathbb{F}}(0,T;U)} = |\widetilde{u}|_{L^2_{\mathbb{F}}(0,T;U)} + |v|_{L^2_{\mathbb{F}}(0,T;U)}$. Therefore, The operator $B$ is an admissible control operator with respect to the semigroup $\{S(t)\}_{t\geq 0}$ generated by $A$ (see [14]). That is, there exists a constant $\overset{\approx}{\mathcal{C}} > 0$ such that for any $u \in L^2_{\mathbb{F}}(0, T; U)$,

$$\mathbb{E}\Big|\int_0^T S(T-s)Bu(s)\mathrm{d}s\Big|_H \leq \overset{\approx}{\mathcal{C}}|u|_{L^2_{\mathbb{F}}(0,T;U)}.$$

Further, we have

$$\mathbb{E}\Big|\int_0^T S(T-s)AA_Dv(s)\mathrm{d}s\Big|_H \leq \overset{\approx}{\mathcal{C}}|v|_{L^2_{\mathbb{F}}(0,T;U)}. \tag{3.60}$$

There is a unique mild solution $x(\cdot) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$ to the system (3.59).

We consider the following cost functional

$$\mathcal{J}(t, x_0; u(\cdot)) \triangleq \mathbb{E}\langle Gx(T), x(T)\rangle + \mathbb{E}\int_t^T (\langle Qx, x\rangle + \langle Ru, u\rangle)\mathrm{d}s, \tag{3.61}$$

where $G \in \mathbb{S}(H), Q(\cdot) \in L^1(0, T; \mathbb{S}(H))$ and $R(\cdot) \in L^\infty(0, T; \mathbb{S}(U))$. Assume that $Q(\cdot) \geq 0$ and that $R(\cdot) \geq \lambda I$ may not hold. We next study the uniform convexity of the cost functional. We make some assumptions.

(AS1) Assume that $\mathcal{D} = \sigma I$ for $\sigma > \overset{\approx}{\mathcal{C}}$.

(AS2) There exists a positive constant $\alpha \geq \mathcal{C}_0(|R|_{L^\infty(0,T;\mathcal{L}(U))} + \xi_0)$ with $\xi_0 > 0$ such that $\langle Ru, u\rangle \geq \alpha|v|^2$.

(AS3) There is a $\beta \geq \mathcal{C}_0(|R|_{L^\infty(0,T;\mathcal{L}(U))} + \xi_0)$ with $\xi_0 > 0$ such that for any $\zeta \in H$, $\langle G\zeta, \zeta\rangle \geq \mu_0|\zeta|^2_H$.

Let $\widehat{x}(\cdot) = D(\cdot)u(\cdot) = \mathcal{D}\widetilde{u}(\cdot)$. Further, the system (3.59) becomes

$$\begin{cases} \mathrm{d}x = (Ax + A_1x + AA_Dv)\mathrm{d}s + (Cx + \widehat{x})\mathrm{d}W(s) \quad \text{in } [0, T), \\ x(T) \in L^2_{\mathcal{F}_T}(\Omega; H), \end{cases} \tag{3.62}$$

where $x(T)$ is the value of solution to (3.59) at the time $T$. Thus, the system (3.62) is well-posed. Since the inequality (3.60) holds, the following inequality holds:

$$|x(\cdot), \widehat{x}(\cdot)|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H))\times L^2_{\mathbb{F}}(0,T;H)} \leq \mathcal{C}|x(T)|_{L^2_{\mathcal{F}_T}(\Omega;H)} + \mathcal{C}|v|_{L^2_{\mathbb{F}}(0,T;U)}. \tag{3.63}$$

Further, the inequality (3.63) implies that there exists a positive constant $\mathcal{C}_0$ such that

$$|\widetilde{u}(\cdot)|^2_{L^2_{\mathbb{F}}(0,T;U)} \leq \mathcal{C}_0|x(T)|^2_{L^2_{\mathcal{F}_T}(\Omega;H)} + \mathcal{C}_0|v|^2_{L^2_{\mathbb{F}}(0,T;U)}. \tag{3.64}$$

From the inequality (3.63), the state process $x(\cdot)$ can be controlled by the final state process $x(T)$, which means that we only need to consider the operator $G$ is large enough to supply the negative of the operator $R(\cdot)$, when $R(\cdot) \geq \lambda I$ may not hold in the cost functional (3.61). The map $u(\cdot) \mapsto \mathcal{J}(0,0;u(\cdot))$ is uniformly convex, that is, for some $\lambda > 0$, it holds that

$$\mathcal{J}(0,0;u(\cdot)) \geq \lambda \mathbb{E} \int_0^T |u(s)|_U^2 \mathrm{d}s, \quad \forall u(\cdot) \in \mathcal{U}[t,T]. \tag{3.65}$$

According to (3.64), (AS2)–(AS3), we deduce that

$$\begin{aligned}
&\int_0^T \langle Ru, u\rangle \mathrm{d}t + \langle Gx(T), x(T)\rangle \\
&\geq \alpha |v|^2_{L^2_{\mathbb{F}}(0,T;U)} + \beta |x(T)|^2_{L^2_{\mathcal{F}_T}(\Omega;H)} \\
&\geq (\alpha-\beta)|v|^2_{L^2_{\mathbb{F}}(0,T;U)} + \beta(|v|^2_{L^2_{\mathbb{F}}(0,T;U)} + |x(T)|^2_{L^2_{\mathcal{F}_T}(\Omega;H)}) \\
&\geq (\alpha-\beta)|v|^2_{L^2_{\mathbb{F}}(0,T;U)} + \frac{\beta}{\mathcal{C}_0}|\widetilde{u}|^2_{L^2_{\mathbb{F}}(0,T;U)} \\
&\geq \frac{\beta}{\mathcal{C}_0}(|v|^2_{L^2_{\mathbb{F}}(0,T;U)} + |\widetilde{u}|^2_{L^2_{\mathbb{F}}(0,T;U)}) \\
&\geq (|R|_{L^\infty(0,T;\mathcal{L}(U))} + \xi_0)|u|^2_{L^2_{\mathbb{F}}(0,T;U)},
\end{aligned}$$

where $\beta < \alpha < \big(1 + \frac{1}{\mathcal{C}_0}\big)\beta$. Therefore, taking $\lambda = \xi_0$, the following inequality

$$\begin{aligned}
\mathcal{J}(0,t;u(\cdot)) &= \mathbb{E}\langle Gx(T), x(T)\rangle + \mathbb{E}\int_t^T (\langle Qx, x\rangle + \langle Ru, u\rangle)\mathrm{d}s \\
&\geq \lambda \mathbb{E}\int_t^T |u(s)|_U^2 \mathrm{d}s
\end{aligned}$$

holds for any $Q(\cdot) \geq 0$.

Further, we can derive the following result.

**Theorem 3.1** *Under the assumptions* (AS1)–(AS3), *the map* $u(\cdot) \mapsto \mathcal{J}(t,\eta;u(\cdot))$ *is uniformly convex.*

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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