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A Note on the Convergence Along Tangential Curve Associated with Fractional Schrödinger Propagator and Boussinesq Operator*

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Abstract In this paper, the authors study the almost everywhere pointwise convergence problem along a class of restricted curves in $\mathbb{R}\times\mathbb{R}$ given by $\{(y,t): y\in\Gamma(x,t)\}$ for each $t\in[0,1]$, where $\Gamma(x,t)=\{\gamma(x,t,\theta):\theta\in\Theta\}$ for a given compact set Θ in $\mathbb{R}$ of the fractional Schrödinger propagator and Boussinesq operator. They focus on the relationship between the upper Minkowski dimension of Θ and the optimal s for which

$$\lim_{\substack{y\in\Gamma(x,t)\\(y,t)\to(x,0)}} \mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(y)=f(x), \qquad \lim_{\substack{y\in\Gamma(x,t)\\(y,t)\to(x,0)}} \mathcal{B}_t f(y)=f(x), \quad \text{a.e.},$$

whenever $f\in H^s(\mathbb{R})$.

Keywords Fractional Schrödinger propagator, Boussinesq operator, Pointwise convergence, Tangential curves, Sobolev space
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1 Introduction

1.1 The pointwise convergence along vertical lines

Let us consider the free Schrödinger equation in $\mathbb{R}^n\times\mathbb{R}$, $n\geq 1$,

$$\mathrm{i}\partial_t u+\triangle_x u=0$$

with initial datum f. Then the solution which is defined by Schrödinger propagator can be formally written as

$$u(x,t)=\mathrm{e}^{\mathrm{i}t\triangle}f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(x\cdot\xi+t|\xi|^2)}\widehat{f}(\xi)\mathrm{d}\xi,$$

where

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}x\cdot\xi}f(x)\mathrm{d}x.$$

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The problem which was considered by Carleson [6] is to determine the minimal regularity s for which

$$\lim_{t\to 0} \mathrm{e}^{\mathrm{i}t\triangle} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1.1}$$

whenever $f \in H^s(\mathbb{R}^n)$, where $H^s(\mathbb{R}^n)$ is the L^2 Sobolev space of order s which is defined by

$$\|f\|_{H^s(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

In 1979, Carleson [6] first proved that the almost everywhere convergence (1.1) holds for any $f \in H^{\frac{1}{4}}(\mathbb{R})$ by making use of the stationary phase method. Dahlberg-Kenig [10] proved the condition $s \geq \frac{1}{4}$ given by Carleson is sharp.

For the situation in higher dimensions, many researchers such as Carbery [5] and Cowling [9] studied this problem, and Sjölin [28] and Vega [31] proved independently that (1.1) holds when $s > \frac{1}{2}$ in any dimensions. After that some important positive results have been obtained by many references (see [1–2, 7, 11–12, 14, 17–19, 23–25, 30]). More recently, Bourgain [3] gave counterexamples showing that (1.1) can fail if $s < \frac{n}{2(n+1)}$. Du-Guth-Li [13] and Du-Zhang [15] improved the sufficient condition to the almost sharp range $s > \frac{n}{2(n+1)}$ when $n = 2$ and $n \geq 3$, respectively. Hence, the Carleson problem was essentially solved except the endpoint.

1.2 The pointwise convergence along a wider approach region

A natural generalization of the pointwise convergence problem is to ask almost everywhere convergence along a wider approach region instead of vertical lines. One may consider the Schrödinger propagator $\mathrm{e}^{\mathrm{i}t\Delta} f(x)$ converges to $f(x)$ nontangentially for almost everywhere $x \in \mathbb{R}^n$. That is, for $\alpha > 0$ and $f \in H^s(\mathbb{R}^n)$, for which s such that

$$\lim_{\substack{(y,t)\in\Gamma_\alpha(x) \\ (y,t)\to(x,0)}} \mathrm{e}^{\mathrm{i}t\Delta} f(y) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

where $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$. If $s > \frac{n}{2}$, then by Sobolev imbedding theorem, we find that

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} |\mathrm{e}^{\mathrm{i}t\Delta} f(x)| \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

Thus, (1.2) holds for $s > \frac{n}{2}$. However, Sjögren-Sjölin [27] proved that (1.2) fails for $s \leq \frac{n}{2}$. In fact, in [27], Sjögren-Sjölin proved that there is an $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and a strictly increasing function $\gamma(t)$ with $\gamma(0) = 0$, such that for all $x \in \mathbb{R}^n$,

$$\limsup_{\substack{(y,t)\to(x,0) \\ |x-y|<\gamma(t)}} |\mathrm{e}^{\mathrm{i}t\Delta} f(y)| = +\infty.$$

In this section, we study the almost everywhere pointwise convergence problem along a class of restricted curves in $\mathbb{R} \times \mathbb{R}$ given by $\{(y,t) : y \in \Gamma(x,t)\}$ for each $t \in [0,1]$, where $\Gamma(x,t) = \{\gamma(x,t,\theta) : \theta \in \Theta\}$ for a given compact set Θ in $\mathbb{R}$ of the fractional Schrödinger propagator and Boussinesq operator. Let $\gamma(x,t,\theta)$ be a map from $\mathbb{R} \times [0,1] \times \Theta$ to $\mathbb{R}$, which satisfies the following conditions (A1)–(A3).

(A1) (Bilipschitz condition in x) For fixed $t \in [0,1], \theta \in \Theta$, $\gamma(x,t,\theta)$ has at least C^1 regularity in x, and there exists a constant $C_1 \geq 1$ such that for each $x, x' \in \mathbb{R}$, $t \in [0,1]$, $\theta \in \Theta$,

$$C_1^{-1}|x-x'| \leq |\gamma(x,t,\theta)-\gamma(x',t,\theta)| \leq C_1|x-x'|.$$

(A2) (Hölder condition of order α in t) There exists a constant $C_2 > 0$ and $\alpha \in (0,1)$ such that for each $x \in \mathbb{R},\ t, t' \in [0,1],\ \theta \in \Theta$,

$$|\gamma(x,t,\theta)-\gamma(x,t',\theta)| \leq C_2|t-t'|^{\alpha}.$$

(A3) (Hölder condition of order 1 in θ) There exists a constant $C_3 > 0$ such that for each $x \in \mathbb{R},\ t \in [0,1],\ \theta, \theta' \in \Theta$,

$$|\gamma(x,t,\theta)-\gamma(x,t,\theta')| \leq C_3|\theta-\theta'|.$$

In order to characterize the size of Θ, we introduce the upper Minkowski dimension of Θ which is defined by

$$\beta(\Theta) = \limsup_{\delta\to 0^+} \frac{\log N(\delta)}{-\log\delta},$$

where $N(\delta)$ is the minimum number of closed balls of diameter δ to cover Θ. As a consequence, when Θ is a single point, $\beta(\Theta)=0$; when Θ is a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $\beta(\Theta)=n$.

In recent years, many authors study the relationship between the upper Minkowski dimension of Θ and the optimal s for which

$$\lim_{\substack{y\in\Gamma(x,t)\\(y,t)\to(x,0)}} \mathrm{e}^{\mathrm{i}t\Delta} f(y) = f(x), \quad \text{a.e.},$$

whenever $f \in H^s(\mathbb{R}^n)$.

Recently a lot of works have been done on this type of problems (see [8, 20–21, 26] and the references given there). In [8], this question is considered when $n=1$ for a class of restricted straight lines. Exactly, for $t \in [-1,1]$, let $\Gamma(x,t) = \{x+t\theta : \theta \in \Theta\}$, where Θ is a given compact set in $\mathbb{R}$. In [8], Cho-Lee-Vargas proved that the corresponding non-tangential convergence result holds for $s > \frac{\beta(\Theta)+1}{4}$. Then Shiraki [26] generalized this result to a wide class of operators which includes the fractional Schrödinger propagator. Li-Wang-Yan [21] obtained the corresponding non-tangential convergence result in any dimensions and improved the straight line to more general curves with Lipschitz regularity in time variable. Very recently, Li-Wang [20] gave an answer when the curves satisfy just lower α-Hölder regularity ($0<\alpha<1$) associated with Schrödinger propagator. So it is interesting to study that whether the conclusion of Li-Wang [20] is true for fractional Schrödinger propagator and Boussinesq operator. First we consider the fractional Schrödinger propagator which is defined by

$$\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x\cdot\xi+t|\xi|^a)}\widehat{f}(\xi)\mathrm{d}\xi, \quad a>1.$$

In this paper, we focus on the relationship between the upper Minkowski dimension of Θ and the optimal s for which

$$\lim_{\substack{y\in\Gamma(x,t)\\(y,t)\to(x,0)}} \mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(y) = f(x), \quad \text{a.e.}, \tag{1.3}$$

whenever $f \in H^s(\mathbb{R})$. And we have the following convergence result for fractional Schrödinger propagator along a class of tangential curves.

Theorem 1.1 *Let $a > 1$. Suppose that $\gamma(x,t,\theta)$ satisfies the conditions* (A1)–(A3)*. The convergence result* (1.3) *holds almost everywhere whenever $f \in H^s(\mathbb{R})$ if*

(1) $\alpha \in \left(0, \frac{1}{2a}\right]$ *and* $s > s_0 = \frac{a\alpha\beta(\Theta)}{2} + \frac{1}{2} - \frac{a\alpha}{2}$;

(2) $\alpha \in \left(\frac{1}{2a}, 1\right)$ *and* $s > s_0 = \frac{\beta(\Theta)+1}{4}$.

The fractional Schrödinger propagator is Schrödinger propagator when $a = 2$. Hence Theorem 1.1 improves the previous known result in Li-Wang [20]. We study the fractional Schrödinger propagator whose phase function is more complicated than Schrödinger propagator, which causes the difficulty when we establish the estimate of kernel. Theorem 1.1 is sharp when $\alpha \in \left(\frac{1}{2a}, 1\right)$ (see [26]). Besides, Theorem 1.1 extends the result of [26] to generalized curve.

By a standard argument, Theorem 1.1 follows from the maximal function estimate below.

Theorem 1.2 *Let $a > 1$. Suppose that $\gamma(x,t,\theta)$ satisfies the conditions* (A1)–(A3)*, considering the L^p estimate of fractional Schrödinger maximal function*

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \le C\|f\|_{H^s(\mathbb{R})}, \quad f \in H^s(\mathbb{R}), \tag{1.4}$$

where $B(x_0,R) \subset \mathbb{R}$. Then

(1) *for* $\alpha \in \left(0, \frac{1}{2a}\right]$, (1.4) *holds if* $s > s_0 = \frac{a\alpha\beta(\Theta)}{2} + \frac{1}{2} - \frac{a\alpha}{2}$ *and* $p = 2$;

(2) *for* $\alpha \in \left(\frac{1}{2a}, \frac{1}{a}\right)$, (1.4) *holds if* $s > s_0 = \frac{\beta(\Theta)+1}{4}$ *and* $p = 4a\alpha$;

(3) *for* $\alpha \in \left[\frac{1}{a}, 1\right)$, (1.4) *holds if* $s > s_0 = \frac{\beta(\Theta)+1}{4}$ *and* $p = 4$.

Moreover, the constant C depends only on C_1, C_2, C_3, Θ and $B(x_0,R)$, but does not depend on f.

As a result of Theorem 1.2, we achieve the sharp L^p estimate of fractional Schrödinger maximal function along a class of tangential curves in $\mathbb{R} \times \mathbb{R}$. In fact, we take Θ to be the set only consisting of a single point θ_0, which implies $\beta(\Theta) = 0$. And here we rewrite $\gamma(x,t,\theta_0)$ as $\gamma(x,t)$. Let $\gamma(x,t)$ be a map from $\mathbb{R} \times [0,1]$ to $\mathbb{R}$, which satisfies the following conditions (A1)$'$–(A2)$'$.

(A1)$'$ (Bilipschitz condition in x) For fixed $t \in [0,1]$, $\gamma(x,t)$ has at least C^1 regularity in x, and there exists a constant $C_1 \ge 1$ such that for each x, $x' \in \mathbb{R}$, $t \in [0,1]$,

$$C_1^{-1}|x-x'| \le |\gamma(x,t) - \gamma(x',t)| \le C_1|x-x'|.$$

(A2)$'$ (Hölder condition of order α in t) There exists a constant $C_2 > 0$ and $\alpha \in (0,1)$ such that for each $x \in \mathbb{R}$, $t, t' \in [0,1]$,

$$|\gamma(x,t) - \gamma(x,t')| \le C_2|t-t'|^{\alpha}.$$

Theorem 1.3 *Let $a > 1$. Suppose that $\gamma(x,t)$ satisfies the conditions* (A1)$'$–(A2)$'$ *for arbitrary $x, x' \in B(x_0,r) \subset \mathbb{R}$ and $t, t' \in [0,1]$. Considering the L^p estimate of fractional Schrödinger maximal function*

$$\Big\| \sup_{t\in(0,1)} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t))| \Big\|_{L^p(B(x_0,r))} \le C\|f\|_{H^s(\mathbb{R})}, \tag{1.5}$$

we have

(1) *for* $s > \frac{1}{2} - \frac{a\alpha}{2}$ *and* $\alpha \in \left(0, \frac{1}{2a}\right]$, (1.5) *holds if* $p \le 2$;
(2) *for* $s > \frac{1}{4}$ *and* $\alpha \in \left(\frac{1}{2a}, \frac{1}{a}\right)$, (1.5) *holds if* $p \le 4a\alpha$;
(3) *for* $s > \frac{1}{4}$ *and* $\alpha \in \left[\frac{1}{a}, 1\right)$, (1.5) *holds if* $p \le 4$.

Moreover, the constant C *depends only on* C_1, C_2 *and* $B(x_0, r)$, *but does not depend on* f.

It is clear that Theorem 1.3 improves the previous results of [26] when $\alpha \in \left(\frac{1}{2a}, 1\right)$. Next we will show that the upper bound for p obtained by Theorem 1.3 cannot be improved when $\gamma(x,t)$ are chosen as in Theorem 1.4 below.

Theorem 1.4 *Taking* $\gamma(x,t) = x - t^\alpha$ *and considering the* L^p *estimate of fractional Schrödinger maximal function*

$$\left\| \sup_{t \in (0,1)} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t))| \right\|_{L^p(B(0,1))} \le C \|f\|_{H^s(\mathbb{R})}, \tag{1.6}$$

we have

(1) (1.6) *holds for* $s > \frac{1}{2} - \frac{a\alpha}{2}$ *and* $\alpha \in \left(0, \frac{1}{2a}\right]$ *only if* $p \le 2$;
(2) (1.6) *holds for* $s > \frac{1}{4}$ *and* $\alpha \in \left(\frac{1}{2a}, \frac{1}{a}\right)$ *only if* $p \le 4a\alpha$;
(3) (1.6) *holds for* $s > \frac{1}{4}$ *and* $\alpha \in \left[\frac{1}{a}, 1\right)$ *only if* $p \le 4$.

Next we study the Boussinesq operator (see [4]) which is defined by

$$\mathcal{B}_t f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x\cdot\xi + t|\xi|\sqrt{1+|\xi|^2})} \widehat{f}(\xi) \mathrm{d}\xi.$$

In this paper, we also focus on the relationship between the upper Minkowski dimension of Θ and the optimal s for which

$$\lim_{\substack{y \in \Gamma(x,t) \\ (y,t) \to (x,0)}} \mathcal{B}_t f(y) = f(x), \quad \text{a.e.}, \tag{1.7}$$

whenever $f \in H^s(\mathbb{R})$. And we have the following convergence result for Boussinesq operator along a class of tangential curves.

Theorem 1.5 *Suppose that* $\gamma(x,t,\theta)$ *satisfies the conditions* (A1)–(A3). *The convergence result* (1.7) *holds almost everywhere whenever* $f \in H^s(\mathbb{R})$ *if*

(1) $\alpha \in \left(0, \frac{1}{4}\right]$ *and* $s > s_0 = \alpha\beta(\Theta) + \frac{1}{2} - \alpha$;
(2) $\alpha \in \left(\frac{1}{4}, 1\right)$ *and* $s > s_0 = \frac{\beta(\Theta)+1}{4}$.

By a standard argument, Theorem 1.5 follows from the maximal function estimate below.

Theorem 1.6 *Suppose that* $\gamma(x,t,\theta)$ *satisfies the conditions* (A1)–(A3), *considering the* L^p *estimate of Boussinesq maximal function*

$$\left\| \sup_{t \in (0,1), \theta \in \Theta} |\mathcal{B}_t f(\gamma(x,t,\theta))| \right\|_{L^p(B(x_0,R))} \le C \|f\|_{H^s(\mathbb{R})}, \quad f \in H^s(\mathbb{R}), \tag{1.8}$$

where $B(x_0, R) \subset \mathbb{R}$. *Then*

(1) *for* $\alpha \in \left(0, \frac{1}{4}\right]$, (1.8) *holds if* $s > s_0 = \alpha\beta(\Theta) + \frac{1}{2} - \alpha$ *and* $p = 2$;
(2) *for* $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$, (1.8) *holds if* $s > s_0 = \frac{\beta(\Theta)+1}{4}$ *and* $p = 8\alpha$;
(3) *for* $\alpha \in \left[\frac{1}{2}, 1\right)$, (1.8) *holds if* $s > s_0 = \frac{\beta(\Theta)+1}{4}$ *and* $p = 4$.

Moreover, the constant C depends only on C_1, C_2, C_3, Θ and $B(x_0, R)$, but does not depend on f.

As a result of Theorem 1.6, we achieve the sharp L^p estimate of Boussinesq maximal function along a class of tangential curves in $\mathbb{R} \times \mathbb{R}$.

Theorem 1.7 *Suppose that $\gamma(x,t)$ satisfies the conditions $(\mathrm{A1})'$, $(\mathrm{A2})'$ for arbitrary $x, x' \in B(x_0, r) \subset \mathbb{R}$ and $t, t' \in [0,1]$. Considering the L^p estimate of Boussinesq maximal function*

$$\Big\| \sup_{t\in(0,1)} |\mathcal{B}_t f(\gamma(x,t))| \Big\|_{L^p(B(x_0,r))} \le C\|f\|_{H^s(\mathbb{R})}, \tag{1.9}$$

we have

(1) *for $s > \frac{1}{2} - \alpha$ and $\alpha \in \left(0, \frac{1}{4}\right]$, (1.9) holds if $p \le 2$;*
(2) *for $s > \frac{1}{4}$ and $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$, (1.9) holds if $p \le 8\alpha$;*
(3) *for $s > \frac{1}{4}$ and $\alpha \in \left[\frac{1}{2}, 1\right)$, (1.9) holds if $p \le 4$.*

Moreover, the constant C depends only on C_1, C_2 and $B(x_0, r)$, but does not depend on f.

Finally, we will show that the upper bound for p obtained by Theorem 1.7 cannot be improved when $\gamma(x,t)$ are chosen as in Theorem 1.8 below.

Theorem 1.8 *Taking $\gamma(x,t) = x - t^\alpha$ and considering the L^p estimate of Boussinesq maximal function*

$$\Big\| \sup_{t\in(0,1)} |\mathcal{B}_t f(\gamma(x,t))| \Big\|_{L^p(B(0,1))} \le C\|f\|_{H^s(\mathbb{R})}, \tag{1.10}$$

we have

(1) *(1.10) holds for $s > \frac{1}{2} - \alpha$ and $\alpha \in \left(0, \frac{1}{4}\right]$ only if $p \le 2$;*
(2) *(1.10) holds for $s > \frac{1}{4}$ and $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$ only if $p \le 8\alpha$;*
(3) *(1.10) holds for $s > \frac{1}{4}$ and $\alpha \in \left[\frac{1}{2}, 1\right)$ only if $p \le 4$.*

Finally we give the main idea for the proof of Theorem 1.2 and we prove Theorem 1.2 in Section 2. By Littlewood-Paley decompositon, we study f with $\operatorname{supp}\widehat{f} \subset \{\xi \in \mathbb{R} : |\xi| \sim \lambda\}, \lambda \gg 1$. We decompose Θ into small subsets $\{\Theta_k\}$ such that $\Theta = \cup_k \Theta_k$ with bounded overlap, where each Θ_k is contained in a closed ball with diameter $\lambda^{-\mu}$. In order to prove Theorem 1.2, it is enough to consider

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \le C\lambda^\nu \|f\|_{L^2(\mathbb{R})}, \tag{1.11}$$

where p is chosen as in Theorem 1.2 and $\nu = \max\left\{\frac{1}{2} - \frac{a\alpha}{2}, \frac{1}{4}\right\}$. Moreover, the constant C depends on C_1, C_2, C_3, Θ and $B(x_0, R)$, but does not depend on f and k. We use Hardy-Littlewood-Sobolev's inequality, Van der Corput's lemma and Schur's lemma to prove (1.11).

The main approach for the proof of Theorem 1.4 depends on [8]. Here we choose $\widehat{f}(\xi) = \chi_{B(0,\lambda^{\frac{1}{2}})}(\xi)$ and we give the proof of Theorem 1.4 in Section 2.

The proof of Theorem 1.6 is similar to Theorem 1.2 and the proof of Theorem 1.8 is similar to Theorem 1.4. We give the proofs of Theorems 1.6 and 1.8 in Section 3.

Throughout this paper, we always use C to denote a positive constant, independent of the main parameters involved, but whose value may change at each occurrence. The positive constants with subscripts, such as C_1 and C_2, do not change in different occurrences. For two real functions f and g, we always use $f \lesssim g$ or $g \gtrsim f$ to denote that f is smaller than a positive constant C times g, and we always use $f \sim g$ as shorthand for $f \lesssim g \lesssim f$. We shall use the notation $f \gg g$, which means that there is a sufficiently large constant C, which does not depend on the relevant parameters arising in the context in which the quantities f and g appear, such that $f \geq Cg$. If the function f has compact support, we use supp$f$ to denote the support of f.

2 Proofs of Theorems 1.2 and 1.4

In this section, we prove Theorems 1.2 and 1.4. We will use the following key lemma to prove Theorem 1.2.

Lemma 2.1 *Under the assumption of Theorem* 1.2*, if f is a Schwartz function and* $\mathrm{supp}\widehat{f} \subset A_\lambda = \{\xi \in \mathbb{R} : |\xi| \sim \lambda\}$*. Then for each k,*

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \leq C\lambda^{\nu}\|f\|_{L^2(\mathbb{R})},$$

where p is chosen as in Theorem 1.2 *and* $\nu = \max\big\{\frac{1}{2} - \frac{a\alpha}{2}, \frac{1}{4}\big\}$*. Moreover, the constant C depends on C_1, C_2, C_3, Θ and $B(x_0,R)$, but does not depend on f and k.*

2.1 Proof of Theorem 1.2

Using Littlewood-Paley decomposition, it is enough to demonstrate that for f with supp $\widehat{f} \subset A_\lambda = \{\xi \in \mathbb{R} : |\xi| \sim \lambda\}$, $\lambda \gg 1$,

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \leq C\lambda^{s_0+\varepsilon}\|f\|_{L^2(\mathbb{R})}, \quad \forall\varepsilon > 0, \tag{2.1}$$

where s_0 and p are chosen in Theorem 1.2.

We decompose Θ into small subsets $\{\Theta_k\}$ such that $\Theta = \cup_k\Theta_k$ with bounded overlap, where each Θ_k is contained in a closed ball with diameter $\lambda^{-\mu}$, $\mu = \min\{1, a\alpha\}$. By the definition of $\beta(\Theta)$, we have

$$1 \leq k \leq \lambda^{\mu\beta(\Theta)+\varepsilon}. \tag{2.2}$$

By Lemma 2.1, we get

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \leq C\lambda^{\nu}\|f\|_{L^2(\mathbb{R})},$$

which implies

$$\sup_k \Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \leq C\lambda^{\nu+\frac{(p-1)\varepsilon}{p}}\|f\|_{L^2(\mathbb{R})}, \tag{2.3}$$

where $\nu = \max\big\{\frac{1}{2} - \frac{a\alpha}{2}, \frac{1}{4}\big\}$. We may combine the above inequalities (2.2)–(2.3) to conclude

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))}$$

$$\leq \Big(\sum_k \Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a} f(\gamma(x,t,\theta))| \Big\|^p_{L^p(B(x_0,R))}\Big)^{\frac{1}{p}}$$
$$\leq C\Big(\sum_k \lambda^{p\nu+(p-1)\varepsilon} \|f\|^p_{L^2(\mathbb{R})}\Big)^{\frac{1}{p}}$$
$$\leq C\lambda^{\frac{\mu\beta(\Theta)}{p}+\nu+\varepsilon}\|f\|_{L^2(\mathbb{R})},$$

which implies (2.1). Theorem 1.2 follows from (2.1). In fact, from the above discussion, we get $\mu = \min\{1, a\alpha\}$, $\nu = \max\left\{\frac{1}{2} - \frac{a\alpha}{2}, \frac{1}{4}\right\}$, $s_0 = \frac{\mu\beta(\Theta)}{p} + \nu$.

(1) For $\alpha \in \big(0, \frac{1}{2a}\big]$, then $\mu = a\alpha, \nu = \frac{1}{2} - \frac{a\alpha}{2}$. Since $p = 2$, we get $s_0 = \frac{a\alpha\beta(\Theta)}{2} + \frac{1}{2} - \frac{a\alpha}{2}$;

(2) for $\alpha \in \big(\frac{1}{2a}, \frac{1}{a}\big)$, then $\mu = a\alpha, \nu = \frac{1}{4}$. Since $p = 4a\alpha$, we get $s_0 = \frac{\beta(\Theta)+1}{4}$;

(3) for $\alpha \in \big[\frac{1}{a}, 1\big)$, then $\mu = 1, \nu = \frac{1}{4}$. Since $p = 4$, we get $s_0 = \frac{\beta(\Theta)+1}{4}$,

which implies Theorem 1.2.

2.2 Four lemmas

In order to prove Lemma 2.1, we introduce the following four lemmas first.

Oscillatory integrals have played an important role in harmonic analysis from its outset. We recall the following well-known variant of Van der Corput's lemma.

Lemma 2.2 (Van der Corput's lemma) (see [29]) *For $a < b$, let $F \in C^\infty([a, b])$ be real valued and $\psi \in C^\infty([a, b])$.*

(i) *If $|F'(x)| \geq \lambda > 0$, $\forall x \in [a, b]$ and $F'(x)$ is monotonic on $[a, b]$, then*

$$\Big|\int_a^b \mathrm{e}^{\mathrm{i}F(x)}\psi(x)\,\mathrm{d}x\Big| \leq \frac{C}{\lambda}\Big(|\psi(b)| + \int_a^b |\psi'(x)|\mathrm{d}x\Big),$$

where C does not depend on F, ψ or $[a, b]$.

(ii) *If $|F''(x)| \geq \lambda > 0$, $\forall x \in [a, b]$, then*

$$\Big|\int_a^b \mathrm{e}^{\mathrm{i}F(x)}\psi(x)\,\mathrm{d}x\Big| \leq \frac{C}{\lambda^{\frac{1}{2}}}\Big(|\psi(b)| + \int_a^b |\psi'(x)|\mathrm{d}x\Big),$$

where C does not depend on F, ψ or $[a, b]$.

Lemma 2.3 *Suppose that $\gamma(x, t, \theta)$ satisfies the conditions* (A1)–(A3). *Assume $t(x)$ and $\theta(x)$ are measurable functions defined on $B(x_0, R)$, $t(x) \in (0, 1)$, $\theta(x) \in \Theta_k$. Let $a > 1$, $\rho \in C_c^\infty(\mathbb{R})$, $\lambda \gg 1$ and*

$$K(x, y) = \int_{A_\lambda} \mathrm{e}^{\mathrm{i}\gamma(y,t(y),\theta(y))\cdot\xi - \mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi + \mathrm{i}t(y)|\xi|^a - \mathrm{i}t(x)|\xi|^a} \rho\Big(\frac{\xi}{\lambda}\Big)\mathrm{d}\xi.$$

(1) *For $x, y \in B(x_0, R)$, then*

$$|K(x, y)| \leq C\lambda.$$

(2) *For $x, y \in B(x_0, R)$ and $x \neq y$, if $|t(x) - t(y)| > 5a^{-1}\lambda^{1-a}(C_1R + C_2 + C_3\mathrm{diam}(\Theta))$, then*

$$|K(x, y)| \leq C\lambda^{-N}, \quad \forall N > 0.$$

(3) *For $x, y \in B(x_0, R)$ and $x \neq y$, if $|t(x)-t(y)| \leq 5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))$ and $|x-y| \geq 2C_1C_3\mathrm{diam}(\Theta_k)$, then*

$$|K(x,y)| \leq C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, \lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

Proof of Lemma 2.3 We replace (1)–(3) in Lemma 2.3 by Cases 1–3, respectively. We note that the constant C in Cases 2–3 depends only on C_1, C_2, C_3, R and Θ.

The change of variables $\xi = \lambda\eta$ gives

$$K(x,y) = \lambda\int_{A_1} \mathrm{e}^{\mathrm{i}\lambda[\gamma(y,t(y),\theta(y))\cdot\eta-\gamma(x,t(x),\theta(x))\cdot\eta+t(y)\lambda^{a-1}|\eta|^a-t(x)\lambda^{a-1}|\eta|^a]}\rho(\eta)\mathrm{d}\eta.$$

First, we consider Case 1. We have the following trivial estimate

$$\begin{aligned}|K(x,y)| &= \lambda\Big|\int_{A_1} \mathrm{e}^{\mathrm{i}\lambda[\gamma(y,t(y),\theta(y))\cdot\eta-\gamma(x,t(x),\theta(x))\cdot\eta+t(y)\lambda^{a-1}|\eta|^a-t(x)\lambda^{a-1}|\eta|^a]}\rho(\eta)\mathrm{d}\eta\Big|\\ &\leq \lambda\int_{A_1}|\rho(\eta)|\mathrm{d}\eta\\ &\leq C\lambda.\end{aligned}$$

Next we prove Cases 2–3. Denote

$$\psi(x,y,\eta) := \gamma(y,t(y),\theta(y))\cdot\eta - \gamma(x,t(x),\theta(x))\cdot\eta + \lambda^{a-1}t(y)|\eta|^a - \lambda^{a-1}t(x)|\eta|^a.$$

Then we get

$$\begin{cases}\dfrac{\partial}{\partial\eta}\psi(x,y,\eta) = \gamma(y,t(y),\theta(y)) - \gamma(x,t(x),\theta(x)) + a\lambda^{a-1}\eta^{a-1}(t(y)-t(x));\\ \dfrac{\partial^2}{\partial\eta^2}\psi(x,y,\eta) = a(a-1)\lambda^{a-1}\eta^{a-2}(t(y)-t(x)).\end{cases}$$

Second, we consider Case 2.

(1) On one hand, since $|t(x)-t(y)| > 5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))$, we have

$$\begin{aligned}|a\lambda^{a-1}\eta^{a-1}(t(y)-t(x))| &\geq a\lambda^{a-1}5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))\\ &= 5(C_1R+C_2+C_3\mathrm{diam}(\Theta)).\end{aligned}\tag{2.4}$$

(2) On the other hand, since $\gamma(x,t,\theta)$ satisfies the conditions (A1), (A2) and (A3), we find

$$\begin{aligned}&|\gamma(y,t(y),\theta(y)) - \gamma(x,t(x),\theta(x))|\\ &\leq |\gamma(y,t(y),\theta(y)) - \gamma(x,t(y),\theta(y))| + |\gamma(x,t(y),\theta(y)) - \gamma(x,t(x),\theta(y))|\\ &\quad + |\gamma(x,t(x),\theta(y)) - \gamma(x,t(x),\theta(x))|\\ &\leq C_1|x-y| + C_2|t(x)-t(y)|^\alpha + C_3|\theta(x)-\theta(y)|\\ &\leq 2C_1R + C_2 + C_3\mathrm{diam}(\Theta).\end{aligned}\tag{2.5}$$

(2.4)–(2.5) imply that

$$\Big|\frac{\partial}{\partial\eta}\psi(x,y,\eta)\Big| = |\gamma(y,t(y),\theta(y)) - \gamma(x,t(x),\theta(x)) + a\lambda^{a-1}\eta^{a-1}(t(y)-t(x))|$$

$$\geq |a\lambda^{a-1}\eta^{a-1}(t(y)-t(x))| - |\gamma(y,t(y),\theta(y)) - \gamma(x,t(x),\theta(x))|$$
$$\geq 5(C_1R + C_2 + C_3\mathrm{diam}(\Theta)) - (2C_1R + C_2 + C_3\mathrm{diam}(\Theta))$$
$$\geq 3(C_1R + C_2 + C_3\mathrm{diam}(\Theta)),$$

which implies that

$$|K(x,y)| \leq C\lambda^{-N}, \quad \forall N > 0.$$

Therefore, if $|t(x)-t(y)| > 5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))$, then we obtain Case 2, and the constant C depends only on C_1, C_2, C_3, R and Θ.

Finally, we prove Case 3. Since $\gamma(x,t,\theta)$ satisfies the conditions (A1)–(A3), we find

$$\begin{aligned}
&|\gamma(y,t(y),\theta(y)) - \gamma(x,t(x),\theta(x))| \\
&\geq |\gamma(y,t(y),\theta(y)) - \gamma(x,t(y),\theta(y))| - |\gamma(x,t(y),\theta(y)) - \gamma(x,t(y),\theta(x))| \\
&\quad - |\gamma(x,t(y),\theta(x)) - \gamma(x,t(x),\theta(x))| \\
&\geq C_1^{-1}|x-y| - C_3|\theta(x)-\theta(y)| - C_2|t(x)-t(y)|^{\alpha} \\
&\geq C_1^{-1}|x-y| - C_3\mathrm{diam}(\Theta_k) - C_2|t(x)-t(y)|^{\alpha}. \qquad (2.6)
\end{aligned}$$

We divide Case 3 into three parts.

(1) $|x-y| \geq 2C_1C_3\mathrm{diam}(\Theta_k)$, $|x-y| \geq 100C_1C_2|t(x)-t(y)|^{\alpha}$ and $|x-y| \geq 100C_1a\lambda^{a-1}|t(x)-t(y)|$.

By (2.6), we have

$$\begin{aligned}
\Big|\frac{\partial}{\partial\eta}\psi(x,y,\eta)\Big| &\geq |\gamma(y,t(y),\theta(y)) - \gamma(x,t(x),\theta(x))| - |a\lambda^{a-1}\eta^{a-1}(t(y)-t(x))| \\
&\geq C_1^{-1}|x-y| - C_3\mathrm{diam}(\Theta_k) - C_2|t(x)-t(y)|^{\alpha} - a\lambda^{a-1}|t(y)-t(x)| \\
&\geq \frac{1}{100C_1}|x-y|,
\end{aligned}$$

which implies

$$|K(x,y)| \leq \frac{C\lambda}{[\lambda(100C_1)^{-1}|x-y|]^N}, \quad \forall N > 0.$$

(2) $|x-y| \geq 2C_1C_3\mathrm{diam}(\Theta_k)$, $|x-y| \geq 100C_1C_2|t(x)-t(y)|^{\alpha}$ and $|x-y| < 100C_1a\lambda^{a-1}|t(x)-t(y)|$.

We have

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial\eta^2}\psi(x,y,\eta)\Big| &= |a(a-1)\lambda^{a-1}\eta^{a-2}(t(y)-t(x))| \\
&\geq a(a-1)\lambda^{a-1}\frac{1}{100C_1a\lambda^{a-1}}|x-y| \\
&= \frac{a-1}{100C_1}|x-y|.
\end{aligned}$$

Using Lemma 2.2, we have

$$|K(x,y)| \leq \lambda\Big(\lambda\frac{a-1}{100C_1}|x-y|\Big)^{-\frac{1}{2}} = \Big(\frac{100C_1}{a-1}\Big)^{\frac{1}{2}}\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}.$$

(3) $|x-y| \geq 2C_1C_3\mathrm{diam}(\Theta_k)$, $|x-y| < 100C_1C_2|t(x)-t(y)|^{\alpha}$.

We have

$$\Big|\frac{\partial^2}{\partial\eta^2}\psi(x,y,\eta)\Big| = |a(a-1)\lambda^{a-1}\eta^{a-2}(t(y)-t(x))|$$
$$\geq a(a-1)\lambda^{a-1}\Big(\frac{|x-y|}{100C_1C_2}\Big)^{\frac{1}{\alpha}}$$
$$= a(a-1)\lambda^{a-1}(100C_1C_2)^{-\frac{1}{\alpha}}|x-y|^{\frac{1}{\alpha}}.$$

Using Lemma 2.2, we have

$$|K(x,y)| \leq \lambda(\lambda a(a-1)\lambda^{a-1}(100C_1C_2)^{-\frac{1}{\alpha}}|x-y|^{\frac{1}{\alpha}})^{-\frac{1}{2}}$$
$$= (a(a-1))^{-\frac{1}{2}}(100C_1C_2)^{\frac{1}{2\alpha}}\lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}.$$

By the estimates of three parts, we get

$$|K(x,y)| \leq C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, \lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

Therefore, the proof of Lemma 2.3 is completed.

In order to prove Lemma 2.1, we also need the famous Hardy-Littlewood-Sobolev's inequality and Schur's lemma.

Lemma 2.4 (Hardy-Littlewood-Sobolev's inequality) (see [22]) *The Riesz potential is an operator defined by*

$$I_\alpha f(x) = \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y = \Big(f * \frac{1}{|\cdot|^{n-\alpha}}\Big)(x).$$

Suppose that $0<\alpha<n$, $1\leq p<\frac{n}{\alpha}$ *and* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.

(i) *If* $f\in L^p(\mathbb{R}^n)$ $\big(1<p<\frac{n}{\alpha}\big)$, *then*

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *if* $f\in L^1(\mathbb{R}^n)$, *then for all* $\lambda>0$,

$$|\{x\in\mathbb{R}^n : |I_\alpha f(x)|>\lambda\}| \leq \Big(\frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}\Big)^{\frac{n}{n-\alpha}},$$

where $C=C(\alpha,n,p)$.

Schur's lemma provides sufficient conditions for linear operators to be bounded on $L^p(\mathbb{R}^n)$. So we describe the details as follows.

Lemma 2.5 (Schur's lemma) (see [16]) *Suppose that* $K(x,y)$ *is a locally integral function on a product of two* σ*-finite measure spaces* $(X,\mu)\times(Y,\nu)$, *and let* T *be a linear operator given by*

$$Tf(x) = \int_Y K(x,y)f(y)\mathrm{d}\nu(y),$$

when f *is bounded and compactly supported. Assume*

$$\sup_{x\in X}\int_Y |K(x,y)|\,\mathrm{d}\nu(y) = A < \infty,$$

$$\sup_{y\in Y}\int_X |K(x,y)|\,\mathrm{d}\mu(x)=B<\infty.$$

Then the operator T extends to a bounded operator from $L^p(Y)$ to $L^p(X)$ with norm $A^{1-\frac{1}{p}}B^{\frac{1}{p}}$ for $1\le p\le\infty$.

2.3 Proof of Lemma 2.1

By linearizing the maximal operator, we choose $t(x),\theta(x)$ be measurable functions defined on $B(x_0,R)$, $t(x)\in(0,1)$, $\theta(x)\in\Theta_k$, such that

$$\sup_{t\in(0,1),\theta\in\Theta_k}|\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a}f(\gamma(x,t,\theta))|\le C|\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a}f(\gamma(x,t(x),\theta(x)))|.$$

Denote

$$Tf(x):=\int_{A_\lambda}\mathrm{e}^{\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi+it(x)|\xi|^a}f(\xi)\mathrm{d}\xi.$$

It is enough to show that

$$\|Tf\|_{L^p(B(x_0,R))}\le C\lambda^\nu\|f\|_{L^2(A_\lambda)}$$

holds for all f with $\operatorname{supp} f\subset A_\lambda$, here we use the Plancherel's theorem to replace $\widehat{f}$ by f. It is easy to see that the adjoint operator T^* of T is given by

$$T^*g(\xi)=\int_{B(x_0,R)}\mathrm{e}^{-\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi-\mathrm{i}t(x)|\xi|^a}g(x)\mathrm{d}x.$$

By duality, this is equivalent to demonstrating that

$$\|T^*g\|_{L^2(A_\lambda)}\le C\lambda^\nu\|g\|_{L^{p'}(B(x_0,R))},\quad \frac{1}{p}+\frac{1}{p'}=1$$

holds for all $g\in L^{p'}(B(x_0,R))$.

Taking $\rho\in C_c^\infty(\mathbb{R})$ such that $\rho(x)=1$ if $|x|\le 1$, and $\rho(x)=0$ if $|x|>2$. Then we have

$$\begin{aligned}
&\|T^*g\|^2_{L^2(A_\lambda)}\\
&=\int_{A_\lambda}\rho\Big(\frac{\xi}{\lambda}\Big)\int_{B(x_0,R)}\mathrm{e}^{-\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi-\mathrm{i}t(x)|\xi|^a}g(x)\mathrm{d}x\int_{B(x_0,R)}\mathrm{e}^{\mathrm{i}\gamma(y,t(y),\theta(y))\cdot\xi+\mathrm{i}t(y)|\xi|^a}\overline{g(y)}\mathrm{d}y\mathrm{d}\xi\\
&=\int_{B(x_0,R)}\int_{B(x_0,R)}g(x)\overline{g(y)}\int_{A_\lambda}\mathrm{e}^{\mathrm{i}\gamma(y,t(y),\theta(y))\cdot\xi-\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi+\mathrm{i}t(y)|\xi|^a-\mathrm{i}t(x)|\xi|^a}\rho\Big(\frac{\xi}{\lambda}\Big)\mathrm{d}\xi\mathrm{d}x\mathrm{d}y\\
&=\int_{B(x_0,R)}\int_{B(x_0,R)}g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where

$$K(x,y)=\int_{A_\lambda}\mathrm{e}^{\mathrm{i}\gamma(y,t(y),\theta(y))\cdot\xi-\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi+\mathrm{i}t(y)|\xi|^a-\mathrm{i}t(x)|\xi|^a}\rho\Big(\frac{\xi}{\lambda}\Big)\mathrm{d}\xi.$$

By Lemma 2.3, we have the following three estimates.

Case 1 For $x,y\in B(x_0,R)$,

$$|K(x,y)|\le C\lambda.$$

Case 2 For $x,y\in B(x_0,R)$ and $x\ne y$, if $|t(x)-t(y)|>5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))$, then

$$|K(x,y)|\le C\lambda^{-N},\quad \forall N>0.$$

Case 3 For $x, y \in B(x_0, R)$ and $x \neq y$, if $|t(x)-t(y)| \leq 5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))$ and $|x-y| \geq 2C_1C_3\mathrm{diam}(\Theta_k)$, then

$$|K(x,y)| \leq C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, \lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

To prove the desired estimates, it suffices to break $B(x_0,R)\times B(x_0,R)$ into Ω_1, Ω_2, where

$$\begin{cases}\Omega_1 := \{(x,y) \in B(x_0,R)\times B(x_0,R) : |t(x)-t(y)| > 5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))\};\\ \Omega_2 := \{(x,y) \in B(x_0,R)\times B(x_0,R) : |t(x)-t(y)| \leq 5a^{-1}\lambda^{1-a}(C_1R+C_2+C_3\mathrm{diam}(\Theta))\}.\end{cases}$$

By Case 2, we have

$$\Big|\int\int_{\Omega_1} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big| \leq C\lambda^{-N}\|g\|^2_{L^{p'}(B(x_0,R))}, \quad \forall N>0,$$

where the constant C depends on $C_1, C_2, C_3, B(x_0,R)$ and Θ.

To achieve the estimate on Ω_2, we will consider the following three cases $\alpha \in \big(0, \frac{1}{2a}\big], \alpha \in \big(\frac{1}{2a}, \frac{1}{a}\big)$ and $\alpha \in \big[\frac{1}{a}, 1\big)$, respectively.

(i) $\alpha \in \big(0, \frac{1}{2a}\big]$.

$$\begin{aligned}&\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big|\\ &\leq \int\int_{\{(x,y)\in\Omega_2:|x-y|<2C_1C_3\lambda^{-a\alpha}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\ &\quad+\int\int_{\{(x,y)\in\Omega_2:2C_1C_3\lambda^{-a\alpha}\leq|x-y|<\lambda^{\frac{\alpha(a-1)}{\alpha-1}}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\ &\quad+\int\int_{\{(x,y)\in\Omega_2:|x-y|\geq\lambda^{\frac{\alpha(a-1)}{\alpha-1}}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\ &= \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

By Case 1, Hölder's inequality and the L^2 boundedness of Hardy-Littlewood maximal operator, we get

$$\begin{aligned}\mathrm{I}_1 &= \int\int_{\{(x,y)\in\Omega_2:|x-y|<2C_1C_3\lambda^{-a\alpha}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\ &\leq C\lambda\int\int_{\{(x,y)\in\Omega_2:|x-y|<2C_1C_3\lambda^{-a\alpha}\}} |g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y\\ &\leq C\lambda^{1-a\alpha}\int_{\mathbb{R}} M(|g|\chi_{B(x_0,R)})(y)|\overline{g(y)}|\chi_{B(x_0,R)}(y)\mathrm{d}y\\ &\leq C\lambda^{1-a\alpha}\|M(|g|\chi_{B(x_0,R)})\|_{L^2(\mathbb{R})}\|g\chi_{B(x_0,R)}\|_{L^2(\mathbb{R})}\\ &\leq C\lambda^{1-a\alpha}\|g\chi_{B(x_0,R)}\|^2_{L^2(\mathbb{R})}\\ &= C\lambda^{1-a\alpha}\|g\|^2_{L^2(B(x_0,R))}.\end{aligned} \tag{2.7}$$

By Case 3, Hölder's inequality and Lemma 2.5, we find

$$\mathrm{I}_2 = \int\int_{\{(x,y)\in\Omega_2:2C_1C_3\lambda^{-a\alpha}\leq|x-y|<\lambda^{\frac{\alpha(a-1)}{\alpha-1}}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y$$

$$\begin{aligned}
&\le C\lambda^{1-\frac{a}{2}}\int\int_{\{(x,y)\in\Omega_2:2C_1C_3\lambda^{-a\alpha}\le|x-y|<\lambda^{\frac{\alpha(a-1)}{\alpha-1}}\}}|x-y|^{-\frac{1}{2\alpha}}|g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y\\
&\le C\lambda^{1-\frac{a}{2}}\Big\|\int_{\{x\in\mathbb{R}:2C_1C_3\lambda^{-a\alpha}\le|x-y|<\lambda^{\frac{\alpha(a-1)}{\alpha-1}}\}}|x-y|^{-\frac{1}{2\alpha}}\chi_{B(x_0,R)}(x)|g(x)|\mathrm{d}x\Big\|_{L^2(B(x_0,R))}\\
&\quad\cdot\|g\|_{L^2(B(x_0,R))}\\
&\le C\lambda^{1-\frac{a}{2}}\lambda^{-a\alpha(1-\frac{1}{2\alpha})}\|g\|^2_{L^2(B(x_0,R))}\\
&= C\lambda^{1-a\alpha}\|g\|^2_{L^2(B(x_0,R))}.
\end{aligned}\tag{2.8}$$

By Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned}
\mathrm{I}_3&=\int\int_{\{(x,y)\in\Omega_2:|x-y|\ge\lambda^{\frac{\alpha(a-1)}{\alpha-1}}\}}|g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\
&\le C\lambda^{\frac12}\int_{\mathbb{R}}\int_{\mathbb{R}}|g(x)|\chi_{B(x_0,R)}(x)|g(y)|\chi_{B(x_0,R)}(y)\frac{1}{|x-y|^{\frac12}}\mathrm{d}x\mathrm{d}y\\
&\le C\lambda^{\frac12}\|g\|_{L^{\frac43}(B(x_0,R))}\Big\||g|\chi_{B(x_0,R)}*\frac{1}{|\cdot|^{\frac12}}\Big\|_{L^4(\mathbb{R})}\\
&\le C\lambda^{\frac12}\|g\|^2_{L^{\frac43}(B(x_0,R))}\\
&\le C\lambda^{\frac12}\|g\|^2_{L^2(B(x_0,R))}\\
&\le C\lambda^{1-a\alpha}\|g\|^2_{L^2(B(x_0,R))},
\end{aligned}\tag{2.9}$$

where we used $\alpha\in\big(0,\frac{1}{2a}\big]$ in the last inequality.

Therefore, we get $\mu=a\alpha$.

From the estimates of I_1–I_3, we obtain

$$\|T^*g\|^2_{L^2(A_\lambda)}\le C\lambda^{1-a\alpha}\|g\|^2_{L^2(B(x_0,R))}.$$

Hence we get $\nu=\frac12-\frac{a\alpha}{2}$ and $p'=2$, which implies $p=2$.

(ii) $\alpha\in\big(\frac{1}{2a},\frac1a\big)$.

$$|K(x,y)|\le C\max\Big\{\frac{\lambda^{\frac12}}{|x-y|^{\frac12}},\lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}\Big\}\le C\max\Big\{\frac{\lambda^{\frac12}}{|x-y|^{\frac{1}{2a\alpha}}},\lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

By Case 1 and Hölder's inequality, we find

$$\begin{aligned}
&\Big|\int\int_{\Omega_2}g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big|\\
&\le\int\int_{\{(x,y)\in\Omega_2:|x-y|<\lambda^{-a\alpha}\}}|g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\
&\quad+\int\int_{\{(x,y)\in\Omega_2:|x-y|\ge\lambda^{-a\alpha}\}}|g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\
&\le C\lambda\int\int_{\{(x,y)\in\Omega_2:|x-y|<\lambda^{-a\alpha}\}}|g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y\\
&\quad+C\lambda^{\frac12}\int\int_{\{(x,y)\in\Omega_2:|x-y|\ge\lambda^{-a\alpha}\}}|g(x)\overline{g(y)}|\frac{1}{|x-y|^{\frac{1}{2a\alpha}}}\mathrm{d}x\mathrm{d}y\\
&\le C\lambda^{\frac12}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{B(x_0,R)}(x)\,|g(x)|\,\chi_{B(x_0,R)}(y)|\overline{g(y)}|\frac{1}{|x-y|^{\frac{1}{2a\alpha}}}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$\leq C\lambda^{\frac{1}{2}}\|g\chi_{B(x_0,R)}\|_{L^{p_1}(\mathbb{R})}\Big\||g|\chi_{B(x_0,R)} * \frac{1}{|\cdot|^{\frac{1}{2a\alpha}}}\Big\|_{L^{p_1'}(\mathbb{R})}. \tag{2.10}$$

Therefore, we obtain $\mu = a\alpha$. By Lemma 2.4, we have

$$\begin{aligned}\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big| &\leq C\lambda^{\frac{1}{2}}\|g\chi_{B(x_0,R)}\|_{L^{p_1}(\mathbb{R})}\Big\||g|\chi_{B(x_0,R)} * \frac{1}{|\cdot|^{\frac{1}{2a\alpha}}}\Big\|_{L^{p_1'}(\mathbb{R})}\\ &\leq C\lambda^{\frac{1}{2}}\|g\|^2_{L^{\frac{4a\alpha}{4a\alpha-1}}(B(x_0,R))}. \end{aligned}\tag{2.11}$$

In fact, when we use Hardy-Littlewood-Sobolev's inequality, it needs the condition

$$\frac{1}{p_1'} = \frac{1}{p_1} - \Big(1 - \frac{1}{2a\alpha}\Big),$$

which implies

$$p_1 = \frac{4a\alpha}{4a\alpha - 1}.$$

From (2.11), we obtain

$$\|T^*g\|^2_{L^2(A_\lambda)} \leq C\lambda^{\frac{1}{2}}\|g\|^2_{L^{\frac{4a\alpha}{4a\alpha-1}}(B(x_0,R))}.$$

Hence we get $\nu = \frac{1}{4}$ and $p' = \frac{4a\alpha}{4a\alpha-1}$, which implies $p = 4a\alpha$.

(iii) $\alpha \in \big[\frac{1}{a}, 1\big)$.

$$|K(x,y)| \leq C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, \lambda^{1-\frac{a}{2}}|x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

By Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned}&\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big|\\ &\leq \int\int_{\{(x,y)\in\Omega_2:|x-y|<\lambda^{-1}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\ &\quad + \int\int_{\{(x,y)\in\Omega_2:|x-y|\geq\lambda^{-1}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y\\ &\leq C\lambda\int\int_{\{(x,y)\in\Omega_2:|x-y|<\lambda^{-1}\}} |g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y\\ &\quad + C\lambda^{\frac{1}{2}}\int\int_{\{(x,y)\in\Omega_2:|x-y|\geq\lambda^{-1}\}} |g(x)\overline{g(y)}|\frac{1}{|x-y|^{\frac{1}{2}}}\mathrm{d}x\mathrm{d}y\\ &\leq C\lambda^{\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{B(x_0,R)}(x)|g(x)|\chi_{B(x_0,R)}(y)|\overline{g(y)}|\frac{1}{|x-y|^{\frac{1}{2}}}\mathrm{d}x\mathrm{d}y\\ &\leq C\lambda^{\frac{1}{2}}\|g\|_{L^{\frac{4}{3}}(B(x_0,R))}\Big\||g|\chi_{B(x_0,R)} * \frac{1}{|\cdot|^{\frac{1}{2}}}\Big\|_{L^4(\mathbb{R})}\\ &\leq C\lambda^{\frac{1}{2}}\|g\|^2_{L^{\frac{4}{3}}(B(x_0,R))}, \end{aligned}\tag{2.12}$$

which implies $\mu = 1$. From (2.12), we obtain

$$\|T^*g\|^2_{L^2(A_\lambda)} \leq C\lambda^{\frac{1}{2}}\|g\|^2_{L^{\frac{4}{3}}(B(x_0,R))}.$$

Hence we get $\nu = \frac{1}{4}$ and $p' = \frac{4}{3}$, which implies $p = 4$.

From the discussions of (i)–(iii), we have $\mu = \min\{1, a\alpha\}$ and $\nu = \max\big\{\frac{1}{2} - \frac{a\alpha}{2}, \frac{1}{4}\big\}$. Thus we have shown the cases of (i)–(iii), and the proof is completed.

2.4 Proof of Theorem 1.4

Adopting the arguments in [8], we choose

$$\widehat{f}(\xi)=\chi_{B(0,\lambda^{\frac{1}{2}})}(\xi).$$

As a consequence, we have

$$\|f\|_{H^s(\mathbb{R})}\le C\lambda^{\frac{1}{4}+\frac{s}{2}}. \tag{2.13}$$

The change of variables $\xi=\lambda^{\frac{1}{2}}\eta$ gives

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a}f(\gamma(x,t))| &= \Big|\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}(x-t^\alpha)\cdot\xi+\mathrm{i}t|\xi|^a}\chi_{B(0,\lambda^{\frac{1}{2}})}(\xi)\mathrm{d}\xi\Big| \\ &= \frac{\lambda^{\frac{1}{2}}}{2\pi}\Big|\int_{B(0,1)}\mathrm{e}^{\mathrm{i}\lambda^{\frac{1}{2}}(x-t^\alpha)\cdot\eta+\mathrm{i}t\lambda^{\frac{a}{2}}|\eta|^a}\mathrm{d}\eta\Big|.\end{aligned}$$

If $t\in\big(0,\frac{1}{200}\lambda^{-\frac{a}{2}}\big)$ and $x\in S=\bigcup\limits_{t\in(0,\frac{1}{200}\lambda^{-\frac{a}{2}})}\big\{y:|y-t^\alpha|\le\frac{1}{200}\lambda^{-\frac{1}{2}}\big\}$, then

$$|\lambda^{\frac{1}{2}}(x-t^\alpha)\cdot\eta+t\lambda^{\frac{a}{2}}|\eta|^a|\le\frac{1}{100}$$

and

$$|\mathrm{e}^{\mathrm{i}t(\sqrt{-\Delta})^a}f(\gamma(x,t))|\ge C\lambda^{\frac{1}{2}}. \tag{2.14}$$

(1) When $\alpha\in\big(0,\frac{1}{2a}\big]$, we get $|S|\sim\lambda^{-\frac{a\alpha}{2}}$ and it follows from (1.6) and (2.13)–(2.14) that

$$\lambda^{\frac{1}{2}-\frac{a\alpha}{2p}}\lesssim\lambda^{\frac{1}{4}+\frac{s}{2}}.$$

We obtain that p cannot be larger than 2 when s is sufficiently close to $\frac{1}{2}-\frac{a\alpha}{2}$, since λ can be sufficiently large.

(2) When $\alpha\in\big(\frac{1}{2a},\frac{1}{a}\big)$, we get $|S|\sim\lambda^{-\frac{a\alpha}{2}}$ and it follows from (1.6) and (2.13)–(2.14) that

$$\lambda^{\frac{1}{2}-\frac{a\alpha}{2p}}\lesssim\lambda^{\frac{1}{4}+\frac{s}{2}}.$$

We obtain that p cannot be larger than $4a\alpha$ when s is sufficiently close to $\frac{1}{4}$, since λ can be sufficiently large.

(3) When $\alpha\in\big[\frac{1}{a},1\big)$, we get $|S|\sim\lambda^{-\frac{1}{2}}$ and it follows from (1.6) and (2.13)–(2.14) that

$$\lambda^{\frac{1}{2}-\frac{1}{2p}}\lesssim\lambda^{\frac{1}{4}+\frac{s}{2}}.$$

Apparently, p cannot be larger than 4 when s is sufficiently close to $\frac{1}{4}$, since λ can be sufficiently large.

3 Proofs of Theorems 1.6 and 1.8

In this section, we prove Theorems 1.6 and 1.8. Next we use the following lemma to prove Theorem 1.6.

Lemma 3.1 *Under the assumption of Theorems* 1.6*, if f is a Schwartz function and* $\operatorname{supp}\widehat{f} \subset A_\lambda = \{\xi \in \mathbb{R} : |\xi| \sim \lambda\}$*. Then for each k,*

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathcal{B}_t f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \le C\lambda^{\nu}\|f\|_{L^2(\mathbb{R})},$$

where p is chosen as in Theorem 1.6 *and* $\nu = \max\left\{\frac{1}{2}-\alpha, \frac{1}{4}\right\}$*. Moreover, the constant C depends on C_1, C_2, C_3, Θ and $B(x_0,R)$, but does not depend on f and k.*

3.1 Proof of Theorem 1.6

The proof of Theorem 1.6 is similar to Theorem 1.2. Here we give a simple proof. Using Littlewood-Paley decomposition, we only need to show that for f with $\operatorname{supp}\widehat{f} \subset A_\lambda = \{\xi \in \mathbb{R} : |\xi| \sim \lambda\}$, $\lambda \gg 1$,

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta} |\mathcal{B}_t f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \le C\lambda^{s_0+\varepsilon}\|f\|_{L^2(\mathbb{R})}, \quad \forall\varepsilon > 0, \tag{3.1}$$

where s_0 and p are chosen in Theorem 1.6.

We also decompose Θ into small subsets $\{\Theta_k\}$ such that $\Theta = \cup_k\Theta_k$ with bounded overlap, where each Θ_k is contained in a closed ball with diameter $\lambda^{-\mu}$, $\mu = \min\{1, 2\alpha\}$. Using the definition of $\beta(\Theta)$, we have

$$1 \le k \le \lambda^{\mu\beta(\Theta)+\varepsilon}. \tag{3.2}$$

By Lemma 3.1, we get

$$\sup_k \Big\| \sup_{t\in(0,1),\theta\in\Theta_k} |\mathcal{B}_t f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \le C\lambda^{\nu+\frac{(p-1)\varepsilon}{p}}\|f\|_{L^2(\mathbb{R})}, \tag{3.3}$$

where $\nu = \max\left\{\frac{1}{2} - \alpha, \frac{1}{4}\right\}$. By (3.2)–(3.3), we obtain

$$\Big\| \sup_{t\in(0,1),\theta\in\Theta} |\mathcal{B}_t f(\gamma(x,t,\theta))| \Big\|_{L^p(B(x_0,R))} \le C\lambda^{\frac{\mu\beta(\Theta)}{p}+\nu+\varepsilon}\|f\|_{L^2(\mathbb{R})},$$

which implies (3.1). Therefore, the proof of Theorem 1.6 is completed.

3.2 Proof of Lemma 3.1

The proof of Lemma 3.1 is similar to Lemma 2.1. By linearizing the maximal operator, we choose $t(x), \theta(x)$ be measurable functions defined on $B(x_0,R), t(x) \in (0,1), \theta(x) \in \Theta_k$, such that

$$\sup_{t\in(0,1),\theta\in\Theta_k} |\mathcal{B}_t f(\gamma(x,t,\theta))| \le C|\mathcal{B}_{t(x)} f(\gamma(x,t(x),\theta(x)))|.$$

Set

$$Tf(x) := \int_{A_\lambda} \mathrm{e}^{\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi+\mathrm{i}t(x)|\xi|\sqrt{1+\xi^2}} f(\xi)\mathrm{d}\xi.$$

It is sufficient to demonstrate that

$$\|Tf\|_{L^p(B(x_0,R))} \le C\lambda^{\nu}\|f\|_{L^2(A_\lambda)}$$

holds for all f with $\operatorname{supp} f \subset A_\lambda$, here we use the Plancherel's theorem to replace $\widehat{f}$ by f. It is easy to see that the adjoint operator T^* of T is given by

$$T^*g(\xi) = \int_{B(x_0,R)} \mathrm{e}^{-\mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi - \mathrm{i}t(x)|\xi|\sqrt{1+\xi^2}} g(x)\mathrm{d}x.$$

By duality, this is equivalent to showing that

$$\|T^*g\|_{L^2(A_\lambda)} \le C\lambda^\nu \|g\|_{L^{p'}(B(x_0,R))}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

holds for all $g \in L^{p'}(B(x_0,R))$.

Taking $\rho \in C_c^\infty(\mathbb{R})$ such that $\rho(x) = 1$ if $|x| \le 1$, and $\rho(x) = 0$ if $|x| > 2$. Then we have

$$\|T^*g\|^2_{L^2(A_\lambda)} = \int_{B(x_0,R)} \int_{B(x_0,R)} g(x)\overline{g(y)} K(x,y)\mathrm{d}x\mathrm{d}y,$$

where

$$K(x,y) = \int_{A_\lambda} \mathrm{e}^{\mathrm{i}\gamma(y,t(y),\theta(y))\cdot\xi - \mathrm{i}\gamma(x,t(x),\theta(x))\cdot\xi + \mathrm{i}t(y)|\xi|\sqrt{1+\xi^2} - \mathrm{i}t(x)|\xi|\sqrt{1+\xi^2}} \rho\Big(\frac{\xi}{\lambda}\Big)\mathrm{d}\xi.$$

We have the following kernel estimate.

Case 1 For $x, y \in B(x_0,R)$,

$$|K(x,y)| \le C\lambda.$$

Case 2 For $x, y \in B(x_0,R)$ and $x \ne y$, if $|t(x) - t(y)| > 5(C_1R + C_2 + C_3\mathrm{diam}(\Theta))\lambda^{-1}$, then

$$|K(x,y)| \le C\lambda^{-N}, \quad \forall N > 0.$$

Case 3 For $x, y \in B(x_0,R)$ and $x \ne y$, if $|t(x) - t(y)| \le 5(C_1R + C_2 + C_3\mathrm{diam}(\Theta))\lambda^{-1}$ and $|x - y| \ge 2C_1C_3\mathrm{diam}(\Theta_k)$, then

$$|K(x,y)| \le C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, |x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

We also note that the constant C in Cases 2–3 depends only on C_1, C_2, C_3, R and Θ. By rescaling, we find

$$K(x,y) = \lambda \int_{A_1} \mathrm{e}^{\mathrm{i}\lambda[\gamma(y,t(y),\theta(y))\cdot\eta - \gamma(x,t(x),\theta(x))\cdot\eta + t(y)|\eta|\sqrt{1+\lambda^2\eta^2} - t(x)|\eta|\sqrt{1+\lambda^2\eta^2}]} \rho(\eta)\mathrm{d}\eta.$$

By the proof of Lemma 2.3, we can prove Cases 1–3 here.

To prove the desired estimates, it suffices to break $B(x_0,R) \times B(x_0,R)$ into Ω_1, Ω_2, where

$$\begin{cases} \Omega_1 := \{(x,y) \in B(x_0,R) \times B(x_0,R) : |t(x) - t(y)| > 5(C_1R + C_2 + C_3\mathrm{diam}(\Theta))\lambda^{-1}\}; \\ \Omega_2 := \{(x,y) \in B(x_0,R) \times B(x_0,R) : |t(x) - t(y)| \le 5(C_1R + C_2 + C_3\mathrm{diam}(\Theta))\lambda^{-1}\}. \end{cases}$$

By Case 2, we have

$$\Big|\int\int_{\Omega_1} g(x)\overline{g(y)} K(x,y)\mathrm{d}x\mathrm{d}y\Big| \le C\lambda^{-N} \|g\|^2_{L^{p'}(B(x_0,R))}, \quad \forall N > 0,$$

where the constant C depends on $C_1, C_2, C_3, B(x_0,R)$ and Θ.

To achieve the estimate on Ω_2, we will consider the following three cases $\alpha \in \left(0, \frac{1}{4}\right], \alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$ and $\alpha \in \left[\frac{1}{2}, 1\right)$, respectively.

(i) $\alpha \in \left(0, \frac{1}{4}\right]$.

$$\begin{aligned}
&\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big| \\
\le &\int\int_{\{(x,y)\in\Omega_2:|x-y|<2C_1C_3\lambda^{-2\alpha}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y \\
&+\int\int_{\{(x,y)\in\Omega_2:2C_1C_3\lambda^{-2\alpha}\le|x-y|<\lambda^{\frac{\alpha}{\alpha-1}}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y \\
&+\int\int_{\{(x,y)\in\Omega_2:|x-y|\ge\lambda^{\frac{\alpha}{\alpha-1}}\}} |g(x)\overline{g(y)}K(x,y)|\mathrm{d}x\mathrm{d}y \\
= &\,\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}$$

By Case 1, Hölder's inequality and the L^2 boundedness of Hardy-Littlewood maximal operator, we get

$$\begin{aligned}
\mathrm{I}_1 &\le C\lambda\int\int_{\{(x,y)\in\Omega_2:|x-y|<2C_1C_3\lambda^{-2\alpha}\}} |g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{1-2\alpha}\int_{\mathbb{R}} M(|g|\chi_{B(x_0,R)})(y)|\overline{g(y)}|\chi_{B(x_0,R)}(y)\mathrm{d}y \\
&\le C\lambda^{1-2\alpha}\|g\|^2_{L^2(B(x_0,R))}. \qquad (3.4)
\end{aligned}$$

By Case 3, Hölder's inequality and Lemma 2.5, we find

$$\begin{aligned}
\mathrm{I}_2 &\le C\int\int_{\{(x,y)\in\Omega_2:2C_1C_3\lambda^{-2\alpha}\le|x-y|<\lambda^{\frac{\alpha}{\alpha-1}}\}} |x-y|^{-\frac{1}{2\alpha}}|g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{-2\alpha(1-\frac{1}{2\alpha})}\|g\|^2_{L^2(B(x_0,R))} \\
&= C\lambda^{1-2\alpha}\|g\|^2_{L^2(B(x_0,R))}. \qquad (3.5)
\end{aligned}$$

By Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned}
\mathrm{I}_3 &\le C\lambda^{\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}} |g(x)|\chi_{B(x_0,R)}(x)|g(y)|\chi_{B(x_0,R)}(y)\frac{1}{|x-y|^{\frac{1}{2}}}\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{\frac{1}{2}}\|g\|_{L^{\frac{4}{3}}(B(x_0,R))}\Big\||g|\chi_{B(x_0,R)}*\frac{1}{|\cdot|^{\frac{1}{2}}}\Big\|_{L^4(\mathbb{R})} \\
&\le C\lambda^{\frac{1}{2}}\|g\|^2_{L^{\frac{4}{3}}(B(x_0,R))} \\
&\le C\lambda^{1-2\alpha}\|g\|^2_{L^2(B(x_0,R))}, \qquad (3.6)
\end{aligned}$$

where we used $\alpha \in (0, \frac{1}{4}]$ in the last inequality. Therefore, we get $\mu = 2\alpha$.

From the estimates of I_1–I_3, we obtain

$$\|T^*g\|^2_{L^2(A_\lambda)} \le C\lambda^{1-2\alpha}\|g\|^2_{L^2(B(x_0,R))}.$$

Hence we get $\nu = \frac{1}{2} - \alpha$ and $p' = 2$, which implies $p = 2$.

(ii) $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

$$|K(x,y)| \le C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{2}}}, |x-y|^{-\frac{1}{2\alpha}}\Big\} \le C\max\Big\{\frac{\lambda^{\frac{1}{2}}}{|x-y|^{\frac{1}{4\alpha}}}, |x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big| \\
&\le C\lambda\int\int_{\{(x,y)\in\Omega_2:|x-y|<\lambda^{-2\alpha}\}}|g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y \\
&\quad + C\lambda^{\frac12}\int\int_{\{(x,y)\in\Omega_2:|x-y|\ge\lambda^{-2\alpha}\}}|g(x)\overline{g(y)}|\frac{1}{|x-y|^{\frac{1}{4\alpha}}}\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{\frac12}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{B(x_0,R)}(x)|g(x)|\chi_{B(x_0,R)}(y)|\overline{g(y)}|\frac{1}{|x-y|^{\frac{1}{4\alpha}}}\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{\frac12}\|g\chi_{B(x_0,R)}\|_{L^{p_1}(\mathbb{R})}\Big\||g|\chi_{B(x_0,R)}*\frac{1}{|\cdot|^{\frac{1}{4\alpha}}}\Big\|_{L^{p_1'}(\mathbb{R})}. \qquad (3.7)
\end{aligned}$$

Therefore, we obtain $\mu=2\alpha$. By Lemma 2.4, we have

$$\begin{aligned}
\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big| &\le C\lambda^{\frac12}\|g\chi_{B(x_0,R)}\|_{L^{p_1}(\mathbb{R})}\Big\||g|\chi_{B(x_0,R)}*\frac{1}{|\cdot|^{\frac{1}{4\alpha}}}\Big\|_{L^{p_1'}(\mathbb{R})} \\
&\le C\lambda^{\frac12}\|g\|^2_{L^{\frac{8\alpha}{8\alpha-1}}(B(x_0,R))}. \qquad (3.8)
\end{aligned}$$

In fact, when we use Hardy-Littlewood-Sobolev's inequality, it needs the condition $\frac{1}{p_1'}=\frac{1}{p_1}-\big(1-\frac{1}{4\alpha}\big)$, which implies $p_1=\frac{8\alpha}{8\alpha-1}$. From (3.8), we obtain

$$\|T^*g\|^2_{L^2(A_\lambda)}\le C\lambda^{\frac12}\|g\|^2_{L^{\frac{8\alpha}{8\alpha-1}}(B(x_0,R))}.$$

Hence we get $\nu=\frac14$ and $p'=\frac{8\alpha}{8\alpha-1}$, which implies $p=8\alpha$.

(iii) $\alpha\in\big[\frac12,1\big)$.

$$|K(x,y)|\le C\max\Big\{\frac{\lambda^{\frac12}}{|x-y|^{\frac12}},|x-y|^{-\frac{1}{2\alpha}}\Big\}.$$

By Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned}
&\Big|\int\int_{\Omega_2} g(x)\overline{g(y)}K(x,y)\mathrm{d}x\mathrm{d}y\Big| \\
&\le C\lambda\int\int_{\{(x,y)\in\Omega_2:|x-y|<\lambda^{-1}\}}|g(x)\overline{g(y)}|\mathrm{d}x\mathrm{d}y \\
&\quad + C\lambda^{\frac12}\int\int_{\{(x,y)\in\Omega_2:|x-y|\ge\lambda^{-1}\}}|g(x)\overline{g(y)}|\frac{1}{|x-y|^{\frac12}}\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{\frac12}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{B(x_0,R)}(x)|g(x)|\chi_{B(x_0,R)}(y)|\overline{g(y)}|\frac{1}{|x-y|^{\frac12}}\mathrm{d}x\mathrm{d}y \\
&\le C\lambda^{\frac12}\|g\|_{L^{\frac43}(B(x_0,R))}\Big\||g|\chi_{B(x_0,R)}*\frac{1}{|\cdot|^{\frac12}}\Big\|_{L^4(\mathbb{R})} \\
&\le C\lambda^{\frac12}\|g\|^2_{L^{\frac43}(B(x_0,R))}, \qquad (3.9)
\end{aligned}$$

which implies $\mu=1$. From (3.9), we find

$$\|T^*g\|^2_{L^2(A_\lambda)}\le C\lambda^{\frac12}\|g\|^2_{L^{\frac43}(B(x_0,R))}.$$

Hence we get $\nu=\frac14$ and $p'=\frac43$, which implies $p=4$.

From the discussions of (i)–(iii), we have $\mu=\min\{1,2\alpha\}$ and $\nu=\max\big\{\frac12-\alpha,\frac14\big\}$.

3.3 Proof of Theorem 1.8

By the proof of Theorem 1.4, we also take $\widehat{f}(\xi) = \chi_{B(0,\lambda^{\frac{1}{2}})}(\xi)$. Then $\|f\|_{H^s(\mathbb{R})} \leq C\lambda^{\frac{1}{4}+\frac{s}{2}}$. By rescaling, we find

$$|\mathcal{B}_t f(\gamma(x,t))| = \frac{\lambda^{\frac{1}{2}}}{2\pi}\Big|\int_{B(0,1)} \mathrm{e}^{\mathrm{i}\lambda^{\frac{1}{2}}(x-t^\alpha)\cdot\eta+\mathrm{i}t\lambda^{\frac{1}{2}}|\eta|\sqrt{1+\lambda\eta^2}}\mathrm{d}\eta\Big|.$$

If $t \in \big(0, \frac{1}{200}\lambda^{-1}\big)$ and $x \in S = \bigcup_{t\in(0,\frac{1}{200}\lambda^{-1})} \big\{y : |y - t^\alpha| \leq \frac{1}{200}\lambda^{-\frac{1}{2}}\big\}$, then $\big|\lambda^{\frac{1}{2}}(x-t^\alpha)\cdot\eta + t\lambda^{\frac{1}{2}}|\eta|\sqrt{1+\lambda\eta^2}\big| \leq \frac{1}{100}$ and $\big|\mathcal{B}_t f(\gamma(x,t))\big| \geq C\lambda^{\frac{1}{2}}$.

(1) When $\alpha \in \big(0, \frac{1}{4}\big]$, we get $|S| \sim \lambda^{-\alpha}$.

(2) When $\alpha \in \big(\frac{1}{4}, \frac{1}{2}\big)$, we get $|S| \sim \lambda^{-\alpha}$.

(3) When $\alpha \in \big[\frac{1}{2}, 1\big)$, we get $|S| \sim \lambda^{-\frac{1}{2}}$.

Combining the proof of Theorem 1.4 with the estimates of $|S|$, we can prove Theorem 1.8.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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