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Compact Intertwining Relation for Composition Operators and Volterra Operators Between Mixed-norm Spaces and Zygmund Spaces*

Zixing YUAN[1] Cezhong TONG[2]

Abstract The authors study the compact intertwining relation for the composition operators and integral operators between mixed-norm spaces and Zygmund spaces.

Keywords Composition operator, Volterra operator, Mixed-norm space, Zygmund space, Compact intertwining relation
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1 Introduction

Let $H(\mathbb{D})$ denote the class of all holomorphic functions on the complex unit disk $\mathbb{D}$, and $S(\mathbb{D})$ denote the collection of all the holomorphic self mappings of $\mathbb{D}$. For $0 < p, q < \infty$ and $-1 < \gamma < \infty$, recall that the mixed-norm space $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_{p,q,\gamma}} = \Big(\int_0^1 M_q^p(f,r)(1-r)^\gamma r\mathrm{d}r\Big)^{\frac{1}{p}} < \infty,$$

where

$$M_q(f,r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^q \mathrm{d}\theta\Big)^{\frac{1}{q}}.$$

For $0 < \alpha < \infty$, we denote by $\mathcal{Z}^\alpha$ the Zygmund type space of those functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)| < \infty.$$

Zygmund type space is a Banach space with the norm defined as follows:

$$\|f\|_{\mathcal{Z}^\alpha} := |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)|.$$

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[1]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail: yuan980127@163.com
[2]Institute of Mathematics, Hebei University of Technology, Tianjin 300401, China.
E-mail: ctong@hebut.edu.cn cezhongtong@hotmail.com
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The little Zygmund type space, denoted by $\mathcal{Z}_0^\alpha$, is the closed subspace of $\mathcal{Z}^\alpha$ consisting of those functions $f \in \mathcal{Z}^\alpha$ with

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f''(z)| = 0.$$

When $\alpha = 1$, we get the classical Zygmund spaces $\mathcal{Z}$ and $\mathcal{Z}_0$. It is known that

$$|f'(z) - f'(0)| \le C\|f\|_{\mathcal{Z}} \log \frac{1}{1-|z|^2}. \tag{1.1}$$

For $0 < \beta < \infty$, the weighted Bloch space $\mathcal{B}^\beta$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\beta} := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}^\beta}$ is a complete semi-norm on $\mathcal{B}^\beta$, which is Möbius invariant. The weighted Bloch space is a Banach space under the norm

$$\|f\| := |f(0)| + \|f\|_{\mathcal{B}^\beta}.$$

Let $\mathcal{B}_0^\beta$ denote the subspace of $\mathcal{B}^\beta$ consisting of those $f \in \mathcal{B}^\beta$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |f'(z)| = 0.$$

For $\delta \ge 0$, recall the space of weighted bounded analytic functions on $\mathbb{D}$ is

$$H^{\infty,\delta} = \Big\{ f \in H(\mathbb{D}) : \|f\|_\infty = \sup_{z\in\mathbb{D}}(1-|z|^2)^\delta |f(z)| < \infty \Big\}.$$

We write non-weighted bounded analytic functions space $H^{\infty,0}$ as H^∞. And let $H_0^{\infty,\delta}$ be the subspace of $H^{\infty,\delta}$ consisting of $f \in H^{\infty,\delta}$ with

$$\lim_{|z|\to 1}(1-|z|^2)^\delta |f(z)| = 0.$$

It is well-known that, for $\delta > 0$, $H^{\infty,\delta} = \mathcal{B}^{1+\delta}$ and $H_0^{\infty,\delta} = \mathcal{B}_0^{1+\delta}$ (see [12, Proposition 7]).

For $f \in H(\mathbb{D})$, every $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ by

$$C_\varphi f = f \circ \varphi.$$

The boundedness and compactness of composition operators on various holomorphic functions spaces have been studied intensively in the past few decades. Interested readers may refer to books [1, 7]. If $g \in H(\mathbb{D})$, the Volterra operator J_g is defined by

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta)\mathrm{d}\zeta$$

and another integral operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta)\mathrm{d}\zeta,$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The operators J_g and I_g are close companions because of their relations to the multiplication operator $M_g f(z) = g(z)f(z)$. To see this, integration by parts gives

$$M_g f = f(0)g(0) + J_g f + I_g f.$$

The discussion of Volterra type operators J_g and I_g first arose in connection with semigroup of composition operators, and readers can refer to [8] for the background.

If X and Y are two Banach spaces, the symbol $\mathscr{B}(X,Y)$ denotes the collection of all bounded linear operators from X to Y. Let $\mathcal{K}(X,Y)$ be the collection of all compact elements of $\mathscr{B}(X,Y)$, and $\mathscr{Q}(X,Y)$ be the quotient set $\mathscr{B}(X,Y)/\mathcal{K}(X,Y)$.

For $A\in\mathscr{B}(X,X), B\in\mathscr{B}(Y,Y)$ and $T\in\mathscr{B}(X,Y)$, the phrase "T intertwines A and B in $\mathscr{Q}(X,Y)$" (or "T intertwines A and B compactly") means that

$$TA = BT \mod \mathcal{K}(X,Y) \quad \text{with} \quad T\neq 0. \tag{1.2}$$

To be more intuitive, the compact intertwining relation means the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{A} & X \\ \Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T} \\ Y & \xrightarrow{B} & Y \end{array} \quad \mod \mathcal{K}(X,Y).$$

We use notation $A\propto_K B\ (T)$ to represent (1.2). In the series papers [9–11], Yuan, Zhou and the second author firstly investigate the intertwining relations $C_\varphi\propto_K C_\varphi\ (V_g)$ on the Bergman spaces, bounded analytic function spaces and Bloch spaces in the unit disk. The motivation of this paper is to continue this topic and solve the "compact intertwining problem" (CIP for short) for composition operators and Volterra type operators between the mixed norm spaces and Zygmund spaces. Our aim in this paper is to answer the following question:

CIP What properties should $g\in H(\mathbb{D})$ have, if

$$C_\varphi|_{H_{p,q,\gamma}} \propto_K C_\varphi|_{\mathcal{Z}^\alpha} \quad (V_g : H_{p,q,\gamma}\to\mathcal{Z}^\alpha)$$

holds for every bounded C_φ?

To state our results compactly and clearly, we use the symbol $\Omega_{co}(V)$ to represent the class of g in CIP. The lower symbol "co" stands for "composition operator", and "V" in the bracket for Volterra operators. Our main result reads as follows.

Theorem 1.1 *To answer CIP for Volterra operators J_g and I_g, we have*
(i) $\Omega_{co}(J_g)=\mathcal{B}_0^{\alpha-s-1}\cap\mathcal{Z}_0^{\alpha-s}$, *and*
(ii) $\Omega_{co}(I_g)=\mathcal{B}_0^{\alpha-s-1}\cap H_0^{\infty,\alpha-s-2}$,
where $s=\frac{1}{p}+\frac{\gamma+1}{q}$.

In the following discussion, we write $A\lesssim B$ if there exists an absolute constant $C>0$ such that $A\leq C\cdot B$, and $A\approx B$ represents $A\lesssim B$ and $B\lesssim A$.

2 Preliminaries

Before the discussion of our main results, we state a couple of lemmas which will be used in the proofs of the main results. The following result is well known.

Lemma 2.1 (see [5, Lemma 1]) *Assume that $0<p,q<\infty$, $-1<\gamma<\infty$ and $f\in H_{p,q,\gamma}$. Then for every nonnegative integer n, there exists a positive constant C, independent of f, such that*

$$|f^{(n)}(z)|\leq C\frac{\|f\|_{H_{p,q,\gamma}}}{(1-|z|^2)^{s+n}},$$

where $s = \frac{1}{p} + \frac{\gamma+1}{q}$ *and does not change in the sequel unless specifically stated.*

Boundedness of J_g and I_g are characterized respectively in [4] and [6], and we summarize them as the following lemma.

Lemma 2.2 *Assume that* $0 < p, q < \infty$, $\alpha > 0$, $-1 < \gamma < \infty$ *and* $g \in H(\mathbb{D})$. *Then the operator* $J_g : H_{p,q,\gamma} \to \mathcal{Z}^\alpha$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-s-1}|g'(z)| < \infty$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-s}|g''(z)| < \infty. \tag{2.1}$$

The operator $I_g : H(p,q,\gamma) \to \mathcal{Z}^\alpha$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-s-2}|g(z)| < \infty$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-s-1}|g'(z)| < \infty.$$

Next lemma is the main result of [3].

Lemma 2.3 *Assume that* $\varphi \in S(\mathbb{D})$, *then*

(1) *for* $0 < \alpha < 1$, C_φ *is bounded on* $\mathcal{Z}^\alpha$ *if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|^2 < \infty$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha|\varphi''(z)| < \infty.$$

(2) *For* $\alpha = 1$, C_φ *is bounded on* $\mathcal{Z}^\alpha$ *if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}|\varphi'(z)|^2 < \infty$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|\varphi''(z)| \log \frac{1}{1-|\varphi(z)|^2} < \infty. \tag{2.2}$$

(3) *For* $\alpha > 1$, C_φ *is bounded on* $\mathcal{Z}^\alpha$ *if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|^2 < \infty$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}}|\varphi''(z)| < \infty. \tag{2.3}$$

The following inequality is also well known (see [2, Theorem 1.1]).

Lemma 2.4 *Let φ be an analytic function in $\mathbb{D}$ and $|\varphi(z)| < 1$ for each $z \in \mathbb{D}$. Then*

$$|\varphi^{(n)}(z)| \le \frac{n!(1-|\varphi(z)|^2)}{(1-|z|^2)^n}(1+|z|)^{n-1}.$$

Let $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ and $S_{\varphi,g} = C_\varphi I_g - I_g C_\varphi$, both of which are from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$. The following lemma is a classic criterion for compactness, whose proof is an easy modification of [1, Proposition 3.11].

Lemma 2.5 *Suppose that $0 < p, q < \infty$, $\alpha > 0$ and $-1 < \gamma < \infty$. Suppose further that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g}$ (resp. $S_{\varphi,g}$) is a compact operator from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$ if and only if $T_{\varphi,g}$ (resp. $S_{\varphi,g}$) is bounded, and for any bounded sequence $\{f_k\}_{k\in N}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|T_{\varphi,g} f_k\|_{\mathcal{Z}^\alpha} \to 0$ (resp. $\|S_{\varphi,g} f_k\|_{\mathcal{Z}^\alpha} \to 0$) as $k \to \infty$.*

3 Proof of Theorem 1.1(i)

First we consider the compactness of $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$.

Theorem 3.1 *Let $0 < p, q < \infty$, $\alpha > 0$ and $-1 < \gamma < \infty$. Assume that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ is a bounded operator from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}} |\varphi'(z)||(g\circ\varphi)'(z) - g'(z)| < \infty \tag{3.1}$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s}} |(g\circ\varphi)''(z) - g''(z)| < \infty. \tag{3.2}$$

Proof Suppose that (3.1)–(3.2) hold. For any $f \in H_{p,q,\gamma}$, we have

$$\begin{aligned} T_{\varphi,g} f(z) &= C_\varphi\Big(\int_0^z f(\zeta)g'(\zeta)\mathrm{d}\zeta\Big) - J_g(f\circ\varphi)(z) \\ &= \int_0^{\varphi(z)} f(\zeta)g'(\zeta)\mathrm{d}\zeta - \int_0^z f(\varphi(\zeta))g'(\zeta)\mathrm{d}\zeta. \end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned} \|T_{\varphi,g} f\|_{\mathcal{Z}^\alpha} \approx& \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |(T_{\varphi,g} f)''(z)| \\ \le& \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\varphi'(z)||f'(\varphi(z))||(g\circ\varphi)'(z) - g'(z)| \\ &+ \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f(\varphi(z))||(g\circ\varphi)''(z) - g''(z)| \\ \lesssim& \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}} |\varphi'(z)||(g\circ\varphi)'(z) - g'(z)| \|f\|_{H_{p,q,\gamma}} \\ &+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s}} |(g\circ\varphi)''(z) - g''(z)| \|f\|_{H_{p,q,\gamma}} \\ \lesssim& \|f\|_{H_{p,q,\gamma}}, \end{aligned}$$

where the last inequality follows from (3.1)–(3.2). Hence $T_{\varphi,g}$ is bounded from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$.

Conversely, suppose $T_{\varphi,g}$ is bounded. There exists a constant $C>0$ such that $\|T_{\varphi,g}f\|_{\mathcal{Z}^\alpha} \le C\|f\|_{H_{p,q,\gamma}}$ for any $f \in H_{p,q,\gamma}$.

Taking $f(z)\equiv 1$, we deduce that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(g\circ\varphi)''(z)-g''(z)|<\infty. \tag{3.3}$$

Taking $f(z)=z$, we get

$$\begin{aligned}&(1-|z|^2)^\alpha|\varphi'(z)||(g\circ\varphi)'(z)-g'(z)|\\ \le&(1-|z|^2)^\alpha|(T_{\varphi,g}f)''(z)|+(1-|z|^2)^\alpha|\varphi(z)||(g\circ\varphi)''(z)-g''(z)|\\ \le&(1-|z|^2)^\alpha|(T_{\varphi,g}f)''(z)|+(1-|z|^2)^\alpha|(g\circ\varphi)''(z)-g''(z)|.\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi'(z)||(g\circ\varphi)'(z)-g'(z)|<\infty. \tag{3.5}$$

Now, for any fixed $w\in\mathbb{D}$, we use the test function

$$f_w(z)=\Big(\frac{1-|w|^2}{(1-\bar{w}z)^2}\Big)^s,\quad \forall z\in\mathbb{D}. \tag{3.6}$$

It is from straightforward computations that $f_w\in H_{p,q,\gamma}$ with $\sup\limits_{w\in\mathbb{D}}\|f_w\|_{H_{p,q,\gamma}}<C$, and

$$|f'_{\varphi(\lambda)}(\varphi(\lambda))|=2\Big(\frac{1}{p}+\frac{\gamma+1}{q}\Big)\frac{|\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{s+1}}.$$

Therefore,

$$\begin{aligned}\|T_{\varphi,g}\|_{H_{p,q,\gamma}\to\mathcal{Z}^\alpha}\gtrsim&\|T_{\varphi,g}f_{\varphi(\lambda)}\|_{\mathcal{Z}^\alpha}\\ \ge&(1-|\lambda|^2)^\alpha|f_{\varphi(\lambda)}(\varphi(\lambda))||(g\circ\varphi)''(\lambda)-g''(\lambda)|\\ &-(1-|\lambda|^2)^\alpha|f'_{\varphi(\lambda)}(\varphi(\lambda))||\varphi'(\lambda)||(g\circ\varphi)'(\lambda)-g'(\lambda)|\\ \approx&\frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^s}|(g\circ\varphi)''(\lambda)-g''(\lambda)|\\ &-\frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+1}}|\varphi(\lambda)||\varphi'(\lambda)||(g\circ\varphi)'(\lambda)-g'(\lambda)|.\end{aligned}$$

Rearranging the inequality above, we have

$$\begin{aligned}&\frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^s}|(g\circ\varphi)''(\lambda)-g''(\lambda)|\\ \lesssim&\frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+1}}|\varphi(\lambda)||\varphi'(\lambda)||(g\circ\varphi)'(\lambda)-g'(\lambda)|+\|T_{\varphi,g}\|_{H_{p,q,\gamma}\to\mathcal{Z}^\alpha}.\end{aligned} \tag{3.7}$$

For any fixed $\lambda\in\mathbb{D}$ we set

$$h_\lambda(z):=\Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^s-\frac{1}{(1-|\varphi(\lambda)|^2)^s},\quad\forall z\in\mathbb{D}.$$

Then $h_\lambda(z) \in H_{p,q,\gamma}$ with $\sup\limits_{\lambda\in\mathbb{D}} \|h_\lambda(z)\|_{H_{p,q,\gamma}} < C$ and $h_\lambda(\varphi(\lambda)) = 0$, $|h'_\lambda(\varphi(\lambda))| = |f'_{\varphi(\lambda)}(\varphi(\lambda))|$. Hence

$$\begin{aligned}\|T_{\varphi,g}\|_{H_{p,q,\gamma}\to\mathcal{Z}^\alpha} &\gtrsim \|T_{\varphi,g}h_\lambda\|_{\mathcal{Z}^\alpha}\\ &\gtrsim \frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+1}}|\varphi(\lambda)||\varphi'(\lambda)||(g\circ\varphi)'(\lambda)-g'(\lambda)|. \end{aligned}\tag{3.8}$$

It follows from (3.5) and (3.8) that

$$\begin{aligned}&\sup_{|\varphi(\lambda)|\le\frac12}\frac{(1-|\lambda|^2)^\alpha|\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{s+1}}|(g\circ\varphi)'(\lambda)-g'(\lambda)|\\ \le&\sup_{|\varphi(\lambda)|\le\frac12}\Big(\frac43\Big)^{s+1}(1-|\lambda|^2)^\alpha|\varphi'(\lambda)||(g\circ\varphi)'(\lambda)-g'(\lambda)|<\infty\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}&\sup_{|\varphi(\lambda)|>\frac12}\frac{(1-|\lambda|^2)^\alpha|\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{s+1}}|(g\circ\varphi)'(\lambda)-g'(\lambda)|\\ \le&\sup_{|\varphi(\lambda)|>\frac12}2\frac{(1-|\lambda|^2)^\alpha|\varphi(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{s+1}}|(g\circ\varphi)'(\lambda)-g'(\lambda)|<\infty.\end{aligned}\tag{3.10}$$

The desired inequality (3.1) can be deduced from (3.9)–(3.10). Similarly (3.2) can be deduced from (3.3) and (3.7).

Theorem 3.2 *Let $0 < p, q < \infty$, $\alpha > 0$ and $-1 < \gamma < \infty$. Assume that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g}$ is compact from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$ if and only if $T_{\varphi,g}$ is bounded and the following two conditions are satisfied*

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}}|\varphi'(z)||(g\circ\varphi)'(z)-g'(z)|=0\tag{3.11}$$

and

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s}}|(g\circ\varphi)''(z)-g''(z)|=0.\tag{3.12}$$

Proof We first suppose that $T_{\varphi,g} : H_{p,q,\gamma} \to \mathcal{Z}^\alpha$ is bounded and (3.11)–(3.12) hold. Let $\{f_k\}$ be an arbitrary bounded sequence with $\|f_k\|_{H_{p,q,\gamma}} \le M$, and which converges to zero uniformly on compact subsets of $\mathbb{D}$. From (3.11)–(3.12), for any small $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}}|\varphi'(z)||(g\circ\varphi)'(z)-g'(z)|<\frac{\varepsilon}{M}$$

and

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s}}|(g\circ\varphi)''(z)-g''(z)|<\frac{\varepsilon}{M},$$

whenever $\varphi(z) \in \mathbb{D}\backslash(1-\delta)\mathbb{D}$. Hence we have

$$\|T_{\varphi,g}f_k\|_{\mathcal{Z}^\alpha}\approx\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(T_{\varphi,g}f_k)''(z)|$$

$$
\begin{aligned}
&\le \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}}|\varphi'(z)||(g\circ\varphi)'(z)-g'(z)|(1-|\varphi(z)|^2)^{s+1}|f_k'(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s}}|(g\circ\varphi)''(z)-g''(z)|(1-|\varphi(z)|^2)^{s}|f_k(\varphi(z))| \\
&= \max\Big\{ \sup_{\varphi(z)\in(1-\delta)\mathbb{D}} Q_1, \sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} Q_1 \Big\} \\
&\quad + \max\Big\{ \sup_{\varphi(z)\in(1-\delta)\mathbb{D}} Q_2, \sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} Q_2 \Big\},
\end{aligned}
$$

where

$$Q_1 = \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}}|\varphi'(z)||(g\circ\varphi)'(z)-g'(z)|(1-|\varphi(z)|^2)^{s+1}|f_k'(\varphi(z))|$$

and

$$Q_2 = \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s}}|(g\circ\varphi)''(z)-g''(z)|(1-|\varphi(z)|^2)^{s}|f_k(\varphi(z))|.$$

According to Cauchy's estimate, the first supremum of Q_1 will be smaller than ε for sufficient large k since f_k converges to zero on compact subsets of $\mathbb{D}$. Note that

$$\sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} Q_1 < \frac{\varepsilon}{M}\cdot\sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{s+1}|f_k'(\varphi(z))| = \frac{\varepsilon}{M}C\|f_k\|_{H_{p,q,\gamma}} < \varepsilon C.$$

It is clear that we can also do the same estimates of the supremums of Q_2.

Conversely, suppose $T_{\varphi,g}$ is compact, hence bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. We consider the test functions defined by

$$f_k(z) := \Big(\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^2}\Big)^s.$$

A direct computation shows that $f_k\in H_{p,q,\gamma}$ with $\sup_k\|f_k\|_{H_{p,q,\gamma}} < C$ and $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. And one can show that

$$|f_{\varphi(z_k)}'(\varphi(z_k))| = 2\Big(\frac{1}{p}+\frac{\gamma+1}{q}\Big)\frac{|\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{s+1}}. \tag{3.13}$$

It follows from Lemma 2.5 that

$$\lim_{k\to\infty}\|T_{\varphi,g}f_k\|_{\mathcal{Z}^\alpha} = 0.$$

Therefore,

$$
\begin{aligned}
\|T_{\varphi,g}f_k\|_{\mathcal{Z}^\alpha} \ge{}& \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s}}|(g\circ\varphi)''(z_k)-g''(z_k)| \\
&- \frac{2s(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi(z_k)||\varphi'(z_k)||(g\circ\varphi)'(z_k)-g'(z_k)|.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s}}|(g\circ\varphi)''(z_k)-g''(z_k)| \\
&= \lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi(z_k)||\varphi'(z_k)||(g\circ\varphi)'(z_k)-g'(z_k)| \qquad (3.14)
\end{aligned}
$$

as long as one of these two limits exists.

Next, we set

$$p_k(z) := \Big(\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^2}\Big)^s - \frac{1}{(1-|\varphi(z_k)|^2)^s}.$$

Then $\{p_k\} \subset H_{p,q,\gamma}$ with $\sup_k \|p_k\|_{H_{p,q,\gamma}} < C$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $T_{\varphi,g}$ is compact, we have $\|T_{\varphi,g}p_k\|_{\mathcal{Z}^\alpha} \to 0$. Note that $p_k(\varphi(z_k)) = 0$ and $p_k'(\varphi(z_k)) = f_k'(\varphi(z_k))$. We have

$$\|T_{\varphi,g}p_k\|_{\mathcal{Z}^\alpha} \gtrsim \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi(z_k)||\varphi'(z_k)||(g\circ\varphi)'(z_k) - g'(z_k)|.$$

Therefore,

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi(z_k)||\varphi'(z_k)||(g\circ\varphi)'(z_k) - g'(z_k)| \\ &= \lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi'(z_k)||(g\circ\varphi)'(z_k) - g'(z_k)|. \end{aligned}$$

This together with (3.14) implies

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s}}|(g\circ\varphi)''(z_k) - g''(z_k)| = 0.$$

Now we are ready to prove the first assertion of Theorem 1.1.

Proof of Theorem 1.1(i) First, we prove $\mathcal{B}_0^{\alpha-s-1} \cap \mathcal{Z}_0^{\alpha-s} \subset \Omega_{co}(J_g)$. For every $g \in \mathcal{B}_0^{\alpha-s-1} \cap \mathcal{Z}_0^{\alpha-s}$, we have

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-s-1}|g'(z)| = 0$$

and

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-s}|g''(z)| = 0.$$

By [1, Corollary 2.40], we can see that $\frac{1-|z|}{1-|\varphi(z)|} \le \frac{1+|z||\varphi(0)|}{1-|\varphi(0)|}$ for every $\varphi \in S(\mathbb{D})$ and $z \in \mathbb{D}$. Together with Lemma 2.4, we have

$$\begin{aligned} &\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{1+s}}|\varphi'(z)||(g\circ\varphi)'(z) - g'(z)| \\ &\le \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{1+s}}|\varphi'(z)|(|g'(\varphi(z))||\varphi'(z)| + |g'(z)|) \\ &\le \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)|^2(1-|\varphi(z)|^2)^{\alpha-s-1}|g'(\varphi(z))| \\ &\quad + 2^s\Big(\frac{1+|z||\varphi(0)|}{1-|\varphi(0)|}\Big)^s(1-|z|^2)^{\alpha-s-1}|g'(z)|, \end{aligned}$$

which converges to 0 as $|\varphi(z)| \to 1$ by the boundedness of C_φ on $\mathcal{Z}^\alpha$. Therefore we obtain (3.11). Next we see that

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^s}|(g\circ\varphi)''(z) - g''(z)|$$

$$
\begin{aligned}
&\le \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^s}(|g''(\varphi(z))||\varphi'(z)|^2+|g'(\varphi(z))||\varphi''(z)|+|g''(z)|)\\
&\le \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|^2(1-|\varphi(z)|^2)^{\alpha-s}|g''(\varphi(z))|\\
&\quad+\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}}|\varphi''(z)|(1-|\varphi(z)|^2)^{\alpha-s-1}|g'(\varphi(z))|\\
&\quad+\frac{(1-|z|^2)^s}{(1-|\varphi(z)|^2)^s}(1-|z|^2)^{\alpha-s}|g''(z)|:=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}
$$

It is obvious that both I_1 and I_3 converge to 0 as $|\varphi(z)|\to 1$. When $\alpha>1$, we can see I_2 converges to 0 as $|\varphi(z)|\to 1$ by (2.3). When $\alpha=1$, by (2.2), (1.1) and $g\in\mathcal{Z}^1$, we obtain

$$
\begin{aligned}
\mathrm{I}_2&=(1-|z|^2)|\varphi''(z)|(1-|\varphi(z)|^2)^{-s}|g'(\varphi(z))|\\
&=(1-|z|^2)|\varphi''(z)|\log\frac{1}{1-|\varphi(z)|^2}\frac{1}{\log\frac{1}{1-|\varphi(z)|^2}}\frac{|g'(\varphi(z))|}{(1-|\varphi(z)|^2)^s}\\
&\le C\frac{1}{\log\frac{1}{1-|\varphi(z)|^2}}\frac{|g'(\varphi(z))|}{(1-|\varphi(z)|^2)^s}\\
&\lesssim \frac{\|g\|_{\mathcal{Z}}}{|g'(\varphi(z))|}\frac{|g'(\varphi(z))|}{(1-|\varphi(z)|^2)^s}\\
&=\frac{(1-|z|^2)^s}{(1-|\varphi(z)|^2)^s}(1-|z|^2)^{1-s}|g''(z)|.
\end{aligned}
$$

Therefore I_2 converges to 0. And hence (3.12) holds true for every $\varphi\in S(\mathbb{D})$. That is $g\in\Omega_{co}(J_g)$.

To prove $\Omega_{co}(J_g)\subset\mathcal{B}_0^{\alpha-s-1}\cap\mathcal{Z}_0^{\alpha-s}$, suppose $g\in\Omega_{co}(J_g)$, then Lemmas 2.2–2.3 and Theorem 3.2 hold for every $\varphi\in S(\mathbb{D})$. Putting $\varphi(z)=\mathrm{e}^{\mathrm{i}\theta}z$ in (3.11)–(3.12) where θ ranges over $[0,2\pi]$, we have

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-s}|\mathrm{e}^{\mathrm{i}2\theta}g''(\mathrm{e}^{\mathrm{i}\theta}z)-g''(z)|=0 \tag{3.15}$$

and

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-s-1}|\mathrm{e}^{\mathrm{i}\theta}g'(\mathrm{e}^{\mathrm{i}\theta}z)-g'(z)|=0. \tag{3.16}$$

It is necessary to estimate the upper bound of the left hand side in (3.15),

$$
\begin{aligned}
&(1-|z|^2)^{\alpha-s}|\mathrm{e}^{\mathrm{i}2\theta}g''(\mathrm{e}^{\mathrm{i}\theta}z)-g''(z)|\\
&\le (1-|z|^2)^{\alpha-s}|\mathrm{e}^{\mathrm{i}2\theta}g''(\mathrm{e}^{\mathrm{i}\theta}z)|+(1-|z|^2)^{\alpha-s}|g''(z)|\\
&=(1-|\mathrm{e}^{\mathrm{i}\theta}z|^2)^{\alpha-s}|g''(\mathrm{e}^{\mathrm{i}\theta}z)|+(1-|z|^2)^{\alpha-s}|g''(z)|:=\mathrm{I}_4+\mathrm{I}_5.
\end{aligned}
$$

(2.1) implies I_4 and I_5 are finite, thus the left-hand-side in (3.15) is bounded independent of θ. Similarly, the left-hand-side in (3.16) is also bounded independent of θ.

We write $g(z)=\sum\limits_{n=0}^{\infty}a_nz^n$, and integrate the left-hand-side of (3.15) with respect to θ from 0 to 2π,

$$0=\int_0^{2\pi}\lim_{|z|\to 1}(1-|z|^2)^{\alpha-s}|\mathrm{e}^{\mathrm{i}2\theta}g'(\mathrm{e}^{\mathrm{i}\theta}z)-g'(z)|\mathrm{d}\theta$$

$$
\begin{aligned}
&= \lim_{|z|\to 1}\int_0^{2\pi}(1-|z|^2)^{\alpha-s}|\mathrm{e}^{\mathrm{i}2\theta}g'(\mathrm{e}^{\mathrm{i}\theta}z)-g'(z)|\mathrm{d}\theta\\
&= \lim_{|z|\to 1}\int_0^{2\pi}(1-|z|^2)^{\alpha-s}\Big|\sum_{n=1}^{\infty}n(n-1)a_n z^{n-2}(\mathrm{e}^{\mathrm{i}n\theta}-1)\Big|\mathrm{d}\theta\\
&\geq \lim_{|z|\to 1}(1-|z|^2)^{\alpha-s}\Big|\sum_{n=1}^{\infty}n(n-1)a_n z^{n-2}\int_0^{2\pi}(\mathrm{e}^{\mathrm{i}n\theta}-1)\mathrm{d}\theta\Big|\\
&= 2\pi\lim_{|z|\to 1}(1-|z|^2)^{\alpha-s}|g''(z)|,
\end{aligned}
$$

where dominant convergent theorem is applied in second line. Thus $g\in\mathcal{Z}_0^{\alpha-s}$. Similarly, integrate the left-hand-side of (3.16) with respect to θ from 0 to 2π, we can get $g\in\mathcal{B}_0^{\alpha-s-1}$, hence $g\in\mathcal{B}_0^{\alpha-s-1}\cap\mathcal{Z}_0^{\alpha-s}$.

4 Proof of Theorem 1.1(ii)

We consider compact intertwining relations $C_\varphi\propto_K C_\varphi$ $(I_g: H_{p,q,\gamma}\to\mathcal{Z}^\alpha)$ in this section. The first step is to investigate $S_{\varphi,g}: H_{p,q,\gamma}\to\mathcal{Z}^\alpha$.

Theorem 4.1 *Let $0<p,q<\infty$, $\alpha>0$ and $-1<\gamma<\infty$. Assume that $\varphi\in S(\mathbb{D})$ and $g\in H(\mathbb{D})$. Then $S_{\varphi,g}: H_{p,q,\gamma}\to\mathcal{Z}^\alpha$ is bounded if and only if the following two supremums*

$$
\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}}|\varphi''(z)[g(\varphi(z))-g(z)]+\varphi'(z)[(g\circ\varphi)'(z)-g'(z)]| \tag{4.1}
$$

and

$$
\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+2}}|\varphi'(z)|^2|g(\varphi(z))-g(z)| \tag{4.2}
$$

are finite.

Proof Suppose $S_{\varphi,g}$ is bounded from $H_{p,q,\gamma}$ to $\mathcal{Z}^\alpha$. Note that

$$
S_{\varphi,g}f(z)=\int_0^{\varphi(z)}f'(\zeta)g(\zeta)\mathrm{d}\zeta-\int_0^z(f\circ\varphi)'(\zeta)g(\zeta)\mathrm{d}\zeta
$$

and

$$
\begin{aligned}
(S_{\varphi,g}f)''(z)=&\,\varphi''(z)f'(\varphi(z))[g(\varphi(z))-g(z)]+\varphi'(z)f'(\varphi(z))[(g\circ\varphi)'(z)-g'(z)]\\
&+f''(\varphi(z))(\varphi'(z))^2[g(\varphi(z))-g(z)].
\end{aligned}
$$

Taking $f(z)=z$, we deduce that

$$
\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi''(z)[g(\varphi(z))-g(z)]+\varphi'(z)[(g\circ\varphi)'(z)-g'(z)]|<\infty. \tag{4.3}
$$

Taking $f(z)=z^2$, note that $|\varphi(z)|\leq 1$, and we get

$$
\begin{aligned}
&2(1-|z|^2)^\alpha|\varphi'(z)|^2|g(\varphi(z))-g(z)|\\
&\leq(1-|z|^2)^\alpha|(S_{\varphi,g}f)''(z)|\\
&\quad+2(1-|z|^2)^\alpha|\varphi(z)||\varphi''(z)[g(\varphi(z))-g(z)]+\varphi'(z)[(g\circ\varphi)'(z)-g'(z)]|
\end{aligned}
$$

$$\begin{aligned}&\leq (1-|z|^2)^\alpha|(S_{\varphi,g}f)''(z)| + 2(1-|z|^2)^\alpha|\varphi''(z)[g(\varphi(z)) - g(z)]\\&\quad + \varphi'(z)[(g\circ\varphi)'(z) - g'(z)]|. \end{aligned}\tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi'(z)|^2|g(\varphi(z)) - g(z)| < \infty. \tag{4.5}$$

Now choosing the test function (3.6) for $\lambda\in\mathbb{D}$, one can see that

$$|f''_{\varphi(\lambda)}(\varphi(\lambda))| \approx \frac{|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{s+2}}.$$

Therefore,

$$\begin{aligned}&\|S_{\varphi,g}\|_{H_{p,q,\gamma}\to\mathcal{Z}^\alpha} \gtrsim \|S_{\varphi,g}f_{\varphi(\lambda)}\|_{\mathcal{Z}^\alpha}\\ &\geq (1-|\lambda|^2)^\alpha|\varphi''(\lambda)f'_{\varphi(\lambda)}(\varphi(\lambda))[g(\varphi(\lambda)) - g(\lambda)] + \varphi'(\lambda)f'_{\varphi(\lambda)}(\varphi(\lambda))[(g\circ\varphi)'(\lambda) - g'(\lambda)]|\\ &\quad - (1-|\lambda|^2)^\alpha|f''_{\varphi(\lambda)}(\varphi(\lambda))||\varphi'(\lambda)|^2|g(\varphi(\lambda)) - g(\lambda)|\\ &\approx \frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+1}}|\varphi(\lambda)||\varphi''(\lambda)[g(\varphi(\lambda)) - g(\lambda)] + \varphi'(\lambda)[(g\circ\varphi)'(\lambda) - g'(\lambda)]|\\ &\quad - \frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+2}}|\varphi(\lambda)|^2|\varphi'(\lambda)|^2|g(\varphi(\lambda)) - g(\lambda)|.\end{aligned}$$

Hence, we get

$$\begin{aligned}&\frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+1}}|\varphi(\lambda)||\varphi''(\lambda)[g(\varphi(\lambda)) - g(\lambda)] + \varphi'(\lambda)[(g\circ\varphi)'(\lambda) - g'(\lambda)]|\\ &\lesssim \frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+2}}|\varphi(\lambda)|^2|\varphi'(\lambda)|^2|g(\varphi(\lambda)) - g(\lambda)| + \|S_{\varphi,g}\|_{H_{p,q,\gamma}\to\mathcal{Z}^\alpha}.\end{aligned}\tag{4.6}$$

Then we set

$$q_\lambda(z) = \frac{2s(1-|\varphi(\lambda)|^2)^{s+1}}{(1-\overline{\varphi(\lambda)}z)^{2s+1}} - \frac{(2s+1)(1-|\varphi(\lambda)|^2)^s}{(1-\overline{\varphi(\lambda)}z)^{2s}}.$$

A direct computation shows that $q_\lambda(z)\in H_{p,q,\gamma}$ with $\sup_k\|q_\lambda(z)\|_{H_{p,q,\gamma}} < C$ and $q'_\lambda(\varphi(\lambda)) = 0$, $|q''_\lambda(\varphi(\lambda))| \approx |f''_{\varphi(\lambda)}(\varphi(\lambda))|$. Hence

$$\begin{aligned}\|S_{\varphi,g}\|_{H_{p,q,\gamma}\to\mathcal{Z}^\alpha} &\gtrsim \|S_{\varphi,g}q_\lambda\|_{\mathcal{Z}^\alpha}\\ &\gtrsim \frac{(1-|\lambda|^2)^\alpha}{(1-|\varphi(\lambda)|^2)^{s+2}}|\varphi(\lambda)|^2|\varphi'(\lambda)|^2|g(\varphi(\lambda)) - g(\lambda)|.\end{aligned}\tag{4.7}$$

It follows from (4.5) and (4.7) that

$$\begin{aligned}&\sup_{|\varphi(\lambda)|\leq\frac{1}{2}}\frac{(1-|\lambda|^2)^\alpha|\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{s+2}}|g(\varphi(\lambda)) - g(\lambda)|\\ &\leq \sup_{|\varphi(\lambda)|\leq\frac{1}{2}}\left(\frac{4}{3}\right)^{s+2}(1-|\lambda|^2)^\alpha|\varphi'(\lambda)|^2|g(\varphi(\lambda)) - g(\lambda)| < \infty\end{aligned}\tag{4.8}$$

and

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}}\frac{(1-|\lambda|^2)^\alpha|\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{s+2}}|g(\varphi(\lambda)) - g(\lambda)|$$

$$\leq \sup_{|\varphi(\lambda)|>\frac{1}{2}} 4\frac{(1-|\lambda|^2)^\alpha|\varphi(\lambda)|^2|\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{s+2}}|g(\varphi(\lambda))-g(\lambda)|<\infty. \tag{4.9}$$

(4.8)–(4.9) give the desired result (4.2). Similarly, (4.3) and (4.6) give (4.1).

And by Lemma 2.1, we can get sufficient part. The proof is paralleled to those of Theorem 3.1 and hence omitted.

Theorem 4.2 *Let $0<p,q<\infty$, $\alpha>0$ and $-1<\gamma<\infty$. Assume that $\varphi\in S(\mathbb{D})$ and $g\in H(\mathbb{D})$. Then $S_{\varphi,g}: H_{p,q,\gamma}\to\mathcal{Z}^\alpha$ is compact if and only if $S_{\varphi,g}$ is bounded and the following conditions are satisfied:*

$$\begin{aligned}\lim_{|\varphi(z)|\to 1}&\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+1}}|\varphi''(z)[g(\varphi(z))-g(z)]\\&+\varphi'(z)[(g\circ\varphi)'(z)-g'(z)]|=0\end{aligned} \tag{4.10}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{s+2}}|\varphi'(z)|^2|g(\varphi(z))-g(z)|=0. \tag{4.11}$$

Proof Suppose $S_{\varphi,g}$ is compact, hence bounded. Let $\{z_k\}_{k\in N}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. We choose the test function (3.6). And one can show that

$$|f''_{\varphi(z_k)}(\varphi(z_k))|\approx\frac{|\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{s+2}}. \tag{4.12}$$

Since $S_{\varphi,g}=C_\varphi J_g-J_gC_\varphi$ is compact, it follows from Lemma 2.5 that

$$\lim_{k\to\infty}\|S_{\varphi,g}f_k\|_{\mathcal{Z}^\alpha}=0.$$

Therefore by (3.13) and (4.12), we get

$$\begin{aligned}\|S_{\varphi,g}f_k\|_{\mathcal{Z}^\alpha}\gtrsim&\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi(z_k)||\varphi''(z_k)[g(\varphi(z_k))-g(z_k)]\\&+\varphi'(z_k)[(g\circ\varphi)'(z_k)-g'(z_k)]|\\&-\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+2}}|\varphi(z_k)|^2|\varphi'(z_k)|^2|g(\varphi(z_k))-g(z_k)|.\end{aligned}$$

Consequently,

$$\begin{aligned}&\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+1}}|\varphi(z_k)||\varphi''(z_k)[(g\circ\varphi)(z_k)-g(z_k)]\\&+\varphi'(z_k)[(g\circ\varphi)'(z_k)-g'(z_k)]|\\=&\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{s+2}}|\varphi(z_k)|^2|\varphi'(z_k)|^2|(g\circ\varphi)(z_k)-g(z_k)|\end{aligned}$$

if one of these limits exists. The rest of proof is similar as Theorem 3.2.

For the sufficient part of the theorem, suppose that (4.10)–(4.11) holds. For any bounded sequence $\{f_k\}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, and by Cauchy's estimate, f'_k and f''_k also converge to zero on compact subsets of $\mathbb{D}$. Then we can prove the sufficiency by the same arguments as the proof of Theorem 3.2.

According to the compactness of $S_{\varphi,g} : H_{p,q,\gamma} \to \mathcal{Z}^\alpha$ in the above theorem, we can use the same arguments as the proof of Theorem 1.1 (i) to obtain

$$\Omega_{co}(I_g) = \mathcal{B}_0^{\alpha-s-1} \cap H_0^{\infty,\alpha-s-2}.$$

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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