Chin. Ann. Math. Ser. B
46(5), 2025, 647–662
DOI: 10.1007/s11401-025-0033-0

Chinese Annals of Mathematics, Series B

The Parabolic Quaternionic Monge-Ampère Type Equation on HyperKähler Manifolds*

Jixiang FU[1] Xin XU[2] Dekai ZHANG[3]

Abstract This paper proves the long-time existence and uniqueness of solutions to a parabolic quaternionic Monge-Ampère type equation on compact hyperKähler manifolds. Moreover, it is shown that after normalization, the solution converges smoothly to the unique solution of the Monge-Ampère equation for $(n-1)$-quaternionic psh functions.

Keywords Quaternionic Monge-Ampère type equation, Parabolic equation, HyperKähler manifold
2020 MR Subject Classification 53C26, 58J35, 32W20

1 Introduction

A hypercomplex manifold is a smooth manifold M together with a triple (I,J,K) of complex structures satisfying the quaternionic relation $IJ=-JI=K$. A hyperhermitian metric on a hypercomplex manifold (M,I,J,K) is a Riemannian metric g which is hermitian with respect to I, J and K.

On a hyperhermitian manifold (M,I,J,K,g), let $\Omega=\omega_J-\mathrm{i}\omega_K$ where ω_J and ω_K are the fundamental forms corresponding to J and K, respectively. Then g is called hyperKähler (HK for short) if $\mathrm{d}\Omega=0$, and called hyperKähler with torsion (HKT for short) if $\partial\Omega=0$. Throughout this paper we use ∂ and $\overline{\partial}$ to denote the complex partial differential operator with respect to the complex structure I.

Analogous to the complex Calabi-Yau equation on Kähler manifolds which was solved by Yau [26], Alesker and Verbitsky introduced a quaternionic Calabi-Yau equation on hyperhermitian manifolds in [4],

$$\begin{aligned}(\Omega+\partial\partial_J u)^n&=\mathrm{e}^f\Omega^n,\\ \Omega+\partial\partial_J u&>0,\end{aligned}\tag{1.1}$$

where f is a given smooth function on M and $\partial_J:=J^{-1}\circ\overline{\partial}\circ J$. They conjectured that the equation is solvable on HKT manifolds with holomorphically trivial canonical bundle with

Manuscript received October 28, 2023. Revised March 15, 2024.
[1]Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China.
E-mail: majxfu@fudan.edu.cn
[2]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: 20110180015@fudan.edu.cn
[3]Corresponding author. School of Mathematical Sciences, Key Laboratory of Mathematics and Engineering Applications (Ministry of Education), Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China. E-mail: dkzhang@math.ecnu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 12141104, 11901102).

respect to I and further obtained the C^0 estimate in this setting (cf. [4]). Alesker [1] solved the equation on a flat hyperKähler manifold and the parabolic case was solved by Bedulli-Gentili-Vezzoni [5] and Zhang [27]. In [2], Alesker and Shelukhin proved the C^0 estimate without any extra assumptions and the proof was later simplified by Sroka [22]. Recently Dinew and Sroka [11] solved the equation on a compact HK manifold. Bedulli, Gentili and Vezzoni [6] considered the parabolic method. More partial results can be found in [3–5, 16–17, 23, 27] and the conjecture remains open.

By adopting the techniques of Dinew and Sroka [11], we solved the quaternionic form-type Calabi-Yau equation in [15] on compact HK manifolds, which is parallel to the complex case where the form-type Calabi-Yau equation was proposed by Fu, Wang and Wu [13–14] and solved by Tosatti and Weinkove [25] on Kähler manifolds.

Specifically, let (M, I, J, K, g, Ω) be a hyperhermitian manifold of quaternionic dimension n, and g_0 be another hyperhermitian metric on M with induced $(2,0)$-form Ω_0. Given a smooth function f on M, the quaternionic form-type Calabi-Yau equation is

$$\Omega_u^n = \mathrm{e}^{f+b}\Omega^n \tag{1.2}$$

in which b is a uniquely determined constant, and Ω_u is determined by

$$\Omega_u^{n-1} = \Omega_0^{n-1} + \partial\partial_J(u\Omega^{n-2}), \tag{1.3}$$

where $\Omega_0^{n-1} + \partial\partial_J(u\Omega^{n-2})$ is strictly positive. When Ω is HKT, i.e., $\partial\Omega = 0$, (1.2) is equivalent to the following Monge-Ampère equation for $(n-1)$-quaternionic psh functions

$$\begin{aligned} \Big(\Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}u\Big)\Omega - \partial\partial_J u\Big)\Big)^n &= \mathrm{e}^{f+b}\Omega^n, \\ \Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}u\Big)\Omega - \partial\partial_J u\Big) &> 0, \end{aligned} \tag{1.4}$$

where Ω_h is related to Ω_0 by $(n-1)! * \Omega_h = \Omega_0^{n-1}$ with $*$ being a Hodge star-type operator. This is explained in [15, Section 2].

On locally flat compact HK manifolds which admits quaternionic coordinates, Gentili and Zhang solved a class of fully non-linear elliptic equations including (1.4) in [19] and the parabolic case in [18]. In [15], using the approach by Dinew and Sroka [11], we solved (1.4) on compact HK manifolds without the flatness assumption in [19].

In this article, we consider the parabolic version of (1.4) on a compact hyperKähler manifold

$$\frac{\partial}{\partial t}u = \log \frac{\Big(\Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}u\Big)\Omega - \partial\partial_J u\Big)\Big)^n}{\Omega^n} - f \tag{1.5}$$

with $u(\cdot, 0) = u_0 \in C^\infty(M, \mathbb{R})$ satisfying

$$\Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}u_0\Big)\Omega - \partial\partial_J u_0\Big) > 0. \tag{1.6}$$

Our main result is as follows.

Theorem 1.1 *Let (M, I, J, K, g, Ω) be a compact hyperKähler manifold of quaternionic dimension n, and Ω_h be a strictly positive $(2,0)$-form with respect to I. Let f be a smooth function on M. Then there exists a unique solution u to (1.5) on $M \times [0, \infty)$ with $u(\cdot, 0) = u_0$ satisfying (1.6). And if we normalize u by*

$$\widetilde{u} := u - \frac{\int_M u\,\Omega^n \wedge \overline{\Omega}^n}{\int_M \Omega^n \wedge \overline{\Omega}^n}, \tag{1.7}$$

then $\widetilde{u}$ converges smoothly to a function $\widetilde{u}_\infty$ as $t \to \infty$, and $\widetilde{u}_\infty$ is the unique solution to (1.4) up to a constant $\widetilde{b} \in \mathbb{R}$.

This gives a parabolic solution to the original equation (1.4). There are plenty of results on parabolic flows on compact complex manifolds, for example, [8, 10, 12, 20–21, 28].

The article is organized as follows. In Section 2, we introduce some basic notations and useful lemmas. In Section 3, we prove the u_t and the C^0 estimate. We derive the C^1 estimate in Section 4 and the complex Hessian estimate in Section 5. The Theorem 1.1 is proved in Section 6.

2 Preliminaries

On a hyperhermitian manifold (M, I, J, K, g) of quaternionic dimension n, we denote the (p, q)-forms with respect to I by $\Lambda_I^{p,q}(M)$. A form $\alpha \in \Lambda_I^{2k,0}(M)$ is called J-real if $J\alpha = \overline{\alpha}$, and denoted by $\alpha \in \Lambda_{I,\mathbb{R}}^{2k,0}(M)$. In particular, we have $\Omega = \omega_J - \mathrm{i}\omega_K$ is a J-real $(2,0)$-form.

Definition 2.1 (cf. [15, Definition 2.2]) *A J-real $(2,0)$-form α is said to be positive (resp. strictly positive) if $\alpha(X, \overline{X}J) \geq 0$ (resp. $\alpha(X, \overline{X}J) > 0$) for any non-zero $(1,0)$-vector X. We denote all strictly positive J-real $(2,0)$-forms by $\Lambda_{I,\mathbb{R}}^{2,0}(M)_{>0}$.*

Note that Ω is determined by g and is strictly positive. Conversely any $\Omega \in \Lambda_{I,\mathbb{R}}^{2k,0}(M)_{>0}$ induces a hyperhermitian metric by $g = \mathrm{Re}(\Omega(\cdot, \cdot J))$. Thus there is a bijection between strictly positive J-real $(2,0)$-forms and hyperhermitian metrics.

Definition 2.2 *For $\chi \in \Lambda_{I,\mathbb{R}}^{2,0}(M)$, define*

$$S_m(\chi) = \frac{C_n^m \chi^m \wedge \Omega^{n-m}}{\Omega^n} \quad \text{for } 0 \leq m \leq n. \tag{2.1}$$

In particular for $u \in C^\infty(M, \mathbb{R})$ we have

$$S_1(\partial\partial_J u) = \frac{1}{2}\Delta_{I,g} u. \tag{2.2}$$

For convenience we denote

$$\widetilde{\Omega} = \Omega_h + \frac{1}{n-1}(S_1(\partial\partial_J u)\Omega - \partial\partial_J u). \tag{2.3}$$

It is easily checked that $\widetilde{\Omega}$ is a J-real $(2,0)$-form, thus one can define the corresponding hyperhermitian metric and the induced fundamental form by

$$g_u = \mathrm{Re}(\widetilde{\Omega}(\cdot, \cdot J)), \quad \omega_u = g_u(\cdot I, \cdot). \tag{2.4}$$

Lemma 2.1

$$\omega_u = \omega_h + \frac{1}{n-1}\Big(S_1(\partial\partial_J u)\omega - \frac{1}{2}(\mathrm{i}\partial\overline{\partial}u - \mathrm{i}J\partial\overline{\partial}u)\Big). \tag{2.5}$$

Proof It is shown in [23, Proposition 3.2] that

$$\mathrm{Re}(\partial\partial_J u(\cdot I, \cdot J)) = \frac{1}{2}(\mathrm{i}\partial\overline{\partial}u - \mathrm{i}J\partial\overline{\partial}u).$$

Hence by definition

$$\begin{aligned}\omega_u &= g_u(\cdot I, \cdot) = \mathrm{Re}(\widetilde{\Omega}(\cdot I, \cdot J))\\ &= \mathrm{Re}(\Omega_h(\cdot I, \cdot J)) + \frac{1}{n-1}(S_1(\partial\partial_J u)\,\mathrm{Re}(\Omega(\cdot I, \cdot J)) - \mathrm{Re}(\partial\partial_J u(\cdot I, \cdot J)))\\ &= \omega_h + \frac{1}{n-1}\Big(S_1(\partial\partial_J u)\omega - \frac{1}{2}(\mathrm{i}\partial\overline{\partial}u - \mathrm{i}J\partial\overline{\partial}u)\Big).\end{aligned}$$

We also need the following lemma.

Lemma 2.2 (cf. [15, Lemma 3.2])

$$S_1(\partial\partial_J u) = S_1(\widetilde{\Omega}) - S_1(\Omega_h), \tag{2.6}$$

$$\partial\partial_J u = (n-1)\Omega_h - S_1(\Omega_h)\Omega + S_1(\widetilde{\Omega})\Omega - (n-1)\widetilde{\Omega}. \tag{2.7}$$

Remark 2.1 On a hyperhermitian manifold (M, I, J, K, g, Ω) of quaternionic dimension n, we can find local I-holomorphic geodesic coordinates such that Ω and another J-real $(2,0)$-form $\widetilde{\Omega}$ are simultaneously diagonalizable at a point $x \in M$, i.e.,

$$\Omega = \sum_{i=0}^{n-1} \mathrm{d}z^{2i} \wedge \mathrm{d}z^{2i+1}, \quad \widetilde{\Omega} = \sum_{i=0}^{n-1} \widetilde{\Omega}_{2i2i+1}\mathrm{d}z^{2i} \wedge \mathrm{d}z^{2i+1},$$

and the Christoffel symbol of ∇^O and first derivatives of J vanish at x, i.e.,

$$J^{l}_{\overline{k},i} = J^{\overline{l}}_{k,i} = J^{\overline{l}}_{k,\overline{i}} = J^{l}_{\overline{k},\overline{i}} = 0.$$

Such local coordinates which were introduced in [11] , are called the normal coordinates around the point x.

The linearized operator $\mathcal{P}$ of the flow (1.5) is derived in the following lemma.

Lemma 2.3 *The linearized operator $\mathcal{P}$ has the form:*

$$\mathcal{P}(v) = v_t - \frac{A \wedge \partial\partial_J(v)}{\widetilde{\Omega}^n}, \tag{2.8}$$

where $A = \frac{n}{n-1}(S_{n-1}(\widetilde{\Omega})\Omega^{n-1} - \widetilde{\Omega}^{n-1})$ and $v \in C^{2,1}(M \times [0,T))$.

Proof Let $w(s)$ be the variation of u and $v = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} w(s)$. It is sufficient to compute the variation of $\widetilde{\Omega}^n = \big(\Omega_h + \frac{1}{n-1}(S_1(\partial\partial_J u)\Omega - \partial\partial_J u)\big)^n$. We have

$$\delta(\widetilde{\Omega}^n) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(\Omega_h + \frac{1}{n-1}(S_1(\partial\partial_J w(s))\Omega - \partial\partial_J w(s))\Big)^n$$

$$
\begin{aligned}
&= \frac{n}{n-1}\widetilde{\Omega}^{n-1} \wedge (S_1(\partial\partial_J v)\Omega - \partial\partial_J v) \\
&= \frac{n}{n-1}\widetilde{\Omega}^{n-1} \wedge \Omega \cdot \frac{n\Omega^{n-1} \wedge \partial\partial_J v}{\Omega^n} - \frac{n}{n-1}\widetilde{\Omega}^{n-1} \wedge \partial\partial_J v \\
&= \frac{n}{n-1} S_{n-1}(\widetilde{\Omega})\Omega^{n-1} \wedge \partial\partial_J v - \frac{n}{n-1}\widetilde{\Omega}^{n-1} \wedge \partial\partial_J v \\
&= A \wedge \partial\partial_J v.
\end{aligned}
$$

Then

$$
\mathcal{P}(v) = v_t - \delta\Big(\log \frac{\widetilde{\Omega}^n}{\Omega^n}\Big) = v_t - \frac{A \wedge \partial\partial_J(v)}{\widetilde{\Omega}^n}
$$

as claimed.

3 The u_t Estimate and C^0 Estimate

We first prove the uniform estimate of u_t.

Lemma 3.1 *Let u be a solution to* (1.5) *on $M \times [0,T)$. Then there exists a constant C depending only on the fixed data $(I, J, K, g, \Omega, \Omega_h)$ and f such that*

$$
\sup_{M\times[0,T)} \big|u_t\big| \le C. \tag{3.1}
$$

Proof One can see that u_t satisfies

$$
\mathcal{P}(u_t) = \frac{\partial}{\partial t}(u_t) - \frac{A \wedge \partial\partial_J(u_t)}{\widetilde{\Omega}^n} = 0. \tag{3.2}
$$

For any $T_0 \in (0,T)$, by maximum principle,

$$
\begin{aligned}
\max_{M\times[0,T_0]} |u_t| &\le \max_M |u_t(x,0)| \\
&\le \max_M \Big| \log \frac{\Big(\Omega_h + \frac{1}{n-1}(S_1(\partial\partial_J u_0)\Omega - \partial\partial_J u_0)\Big)^n}{\Omega^n} \Big| + \max_M |f|.
\end{aligned}
$$

Since T_0 is arbitrary, we have the desired estimate.

Using the C^0 estimate for the elliptic equation, which has been proved by Sroka [23] and Fu, Xu and Zhang [15], we have the following Lemma.

Lemma 3.2 *Let u be a solution to* (1.5) *on $M\times[0,T)$. Then there exists a uniform constant C depending only on the fixed data $(I, J, K, g, \Omega, \Omega_h)$ and f such that*

$$
\sup_{M\times[0,T)} |\widetilde{u}| \le \sup_{t\in[0,T)} \Big(\sup_{x\in M} u(x,t) - \inf_{x\in M} u(x,t) \Big) \le C. \tag{3.3}
$$

Proof The flow is equivalent to the following

$$
\widetilde{\Omega}^n = \mathrm{e}^{u_t+f}\Omega^n. \tag{3.4}
$$

Since u_t is uniformly bounded, we can apply the C^0-estimate for the elliptic equation such that for any $t \in (0, T)$,

$$|u(x,t) - \sup_M u(\cdot,t)| \le C, \quad \forall x \in M. \tag{3.5}$$

Since $\int_M \widetilde{u}(\cdot,t)\,\Omega^n \wedge \overline{\Omega}^n = 0$, there exists $x_0 \in M$ such that $\widetilde{u}(x_0,t) = 0$. Then we have

$$\begin{aligned}|\widetilde{u}(x,t)| &= |\widetilde{u}(x,t) - \widetilde{u}(x_0,t)| = |u(x,t) - u(x_0,t)| \\ &\le |u(x,t) - \sup_M u(\cdot,t)| + |u(x_0,t) - \sup_M u(\cdot,t)| \\ &\le 2C, \quad \forall x \in M.\end{aligned}$$

Hence the C^0 estimate follows.

4 The C^1 Estimate

Although the gradient estimate is unnecessary for the proof of the main result, we provide it as the gradient estimate for fully nonlinear equations has independent interest.

Theorem 4.1 *Let u be a solution to (1.5) on $M \times [0,T)$. Then there exists a constant C depending only on the fixed data $(I, J, K, g, \Omega, \Omega_h)$ and f such that*

$$\sup_{M\times[0,T)} |\mathrm{d}u|_g \le C. \tag{4.1}$$

Proof A simple computation in local coordinates shows that

$$n\partial u \wedge \partial_J u \wedge \Omega^{n-1} = \frac{1}{4}|\mathrm{d}u|_g^2 \Omega^n.$$

Define

$$\beta := \frac{1}{4}|\mathrm{d}u|_g^2.$$

Following [7], we consider

$$G = \log\beta - \varphi(\widetilde{u}),$$

where φ is a function to be determined and $\widetilde{u}$ is the normalization of u. For any $T_0 \in (0,T)$, suppose $\max_{M\times[0,T_0]} G = G(p_0,t_0)$ with $(p_0,t_0) \in M\times[0,T_0]$. We want to show $\beta(p_0,t_0)$ is uniformly bounded. If $t_0 = 0$, we have the estimate. In the following, we assume $t_0 > 0$.

We choose the normal coordinates around p_0 (see Remark 2.1) and all the calculation is at (p_0,t_0),

$$\begin{aligned} 0 \le \partial_t G &= \frac{\beta_t}{\beta} - \varphi'\widetilde{u}_t; \\ \partial G &= \frac{\partial\beta}{\beta} - \varphi'\partial u = 0; \\ \partial_J G &= \frac{\partial_J\beta}{\beta} - \varphi'\partial_J u = 0; \end{aligned}$$

$$\begin{aligned}\partial\partial_J G &= \frac{\partial\partial_J\beta}{\beta} - \frac{\partial\beta\wedge\partial_J\beta}{\beta^2} - \varphi''\partial u\wedge\partial_J u - \varphi'\partial\partial_J u\\ &= \frac{\partial\partial_J\beta}{\beta} - ((\varphi')^2+\varphi'')\partial u\wedge\partial_J u - \varphi'\partial\partial_J u.\end{aligned}$$

Then we have

$$\begin{aligned}0\le \mathcal{P}(G) &= G_t - \frac{\partial\partial_J G\wedge A\wedge\overline{\Omega}^n}{\widetilde{\Omega}^n\wedge\overline{\Omega}^n}\\ &= \frac{\beta_t}{\beta} - \varphi'\widetilde{u}_t - \frac{\partial\partial_J\beta\wedge A\wedge\overline{\Omega}^n}{\beta\widetilde{\Omega}^n\wedge\overline{\Omega}^n} + ((\varphi')^2+\varphi'')\frac{\partial u\wedge\partial_J u\wedge A\wedge\overline{\Omega}^n}{\widetilde{\Omega}^n\wedge\overline{\Omega}^n}\\ &\quad + \varphi'\frac{\partial\partial_J u\wedge A\wedge\overline{\Omega}^n}{\widetilde{\Omega}^n\wedge\overline{\Omega}^n}.\end{aligned} \tag{4.2}$$

We first deal with $\partial_t\beta$. By taking ∂_t on both sides of $\beta\Omega^n = n\partial u\wedge\partial_J u\wedge\Omega^{n-1}$, we get

$$\beta_t = \sum_{j=0}^{2n-1}(u_{t,j}u_{\overline{j}} + u_j u_{t,\overline{j}}). \tag{4.3}$$

We next compute $\partial\partial_J\beta$. Taking ∂_J on both sides of $\beta\overline{\Omega}^n = n\overline{\partial}u\wedge\overline{\partial_J}u\wedge\overline{\Omega}^{n-1}$ and noticing $\partial_J\overline{\Omega}=0$ (since Ω is hyperKähler), we have

$$\partial_J\beta\wedge\overline{\Omega}^n = n\partial_J\overline{\partial}u\wedge\overline{\partial_J}u\wedge\overline{\Omega}^{n-1} - n\overline{\partial}u\wedge\partial_J\overline{\partial_J}u\wedge\overline{\Omega}^{n-1}.$$

Then taking ∂ on both sides, we obtain

$$\begin{aligned}\partial\partial_J\beta\wedge\overline{\Omega}^n &= n\partial\partial_J\overline{\partial}u\wedge\overline{\partial_J}u\wedge\overline{\Omega}^{n-1} + n\partial_J\overline{\partial}u\wedge\partial\overline{\partial_J}u\wedge\overline{\Omega}^{n-1}\\ &\quad - n\partial\overline{\partial}u\wedge\partial_J\overline{\partial_J}u\wedge\overline{\Omega}^{n-1} + n\overline{\partial}u\wedge\partial\partial_J\overline{\partial_J}u\wedge\overline{\Omega}^{n-1}.\end{aligned}$$

From the equation

$$\widetilde{\Omega}^n = \mathrm{e}^{u_t+f}\Omega^n, \tag{4.4}$$

by taking $\overline{\partial}$ on both sides we get

$$n(\overline{\partial}S_1(\partial\partial_J u)\wedge\Omega - \overline{\partial}\partial\partial_J u)\wedge\widetilde{\Omega}^{n-1} = (n-1)(\overline{\partial}\mathrm{e}^{u_t+f}\wedge\Omega^n - n\overline{\partial}\Omega_h\wedge\widetilde{\Omega}^{n-1}).$$

The left hand side can be calculated as the following:

$$\begin{aligned}&n(\overline{\partial}S_1(\partial\partial_J u)\wedge\Omega - \overline{\partial}\partial\partial_J u)\wedge\widetilde{\Omega}^{n-1}\\ &= n\Big(\overline{\partial}S_1(\partial\partial_J u)\wedge\Omega^n\cdot\frac{\Omega\wedge\widetilde{\Omega}^{n-1}}{\Omega^n} - \overline{\partial}\partial\partial_J u\wedge\widetilde{\Omega}^{n-1}\Big)\\ &= n\Big(\overline{\partial}\Big(\frac{\partial\partial_J u\wedge\Omega^{n-1}}{\Omega^n}\cdot\Omega^n\Big)\cdot S_{n-1}(\widetilde{\Omega}) - \overline{\partial}\partial\partial_J u\wedge\widetilde{\Omega}^{n-1}\Big)\\ &= (S_{n-1}(\widetilde{\Omega})\Omega^{n-1} - \widetilde{\Omega}^{n-1})\wedge n\overline{\partial}\partial\partial_J u\\ &= (n-1)A\wedge\overline{\partial}\partial\partial_J u.\end{aligned}$$

Hence we obtain

$$A\wedge n\overline{\partial}\partial\partial_J u = -n^2\widetilde{\Omega}^{n-1}\wedge\overline{\partial}\Omega_h + n\overline{\partial}\mathrm{e}^{u_t+f}\wedge\Omega^n.$$

By taking $\overline{\partial_J}$ on both sides of (4.4), we obtain

$$A \wedge n\overline{\partial_J}\partial\partial_J u = -n^2\widetilde{\Omega}^{n-1} \wedge \overline{\partial_J}\Omega_h + n\overline{\partial_J}\mathrm{e}^{u_t+f} \wedge \Omega^n.$$

Thus for the third term of (4.2), we have

$$\partial\partial_J\beta \wedge A \wedge \overline{\Omega}^n = \mathrm{I}_1 + \mathrm{I}_2 + n\partial_J\overline{\partial}u \wedge \partial\overline{\partial_J}u \wedge \overline{\Omega}^{n-1} \wedge A - n\partial\overline{\partial}u \wedge \partial_J\overline{\partial_J}u \wedge \overline{\Omega}^{n-1} \wedge A, \tag{4.5}$$

where

$$\begin{aligned}\mathrm{I}_1 &= (-n^2\widetilde{\Omega}^{n-1} \wedge \overline{\partial}\Omega_h + n\overline{\partial}\mathrm{e}^{u_t+f} \wedge \Omega^n) \wedge \overline{\partial_J}u \wedge \overline{\Omega}^{n-1},\\ \mathrm{I}_2 &= (n^2\widetilde{\Omega}^{n-1} \wedge \overline{\partial_J}\Omega_h - n\overline{\partial_J}\mathrm{e}^{u_t+f} \wedge \Omega^n) \wedge \overline{\partial}u \wedge \overline{\Omega}^{n-1}.\end{aligned}$$

By direct computation,

$$\begin{aligned}\partial_J\overline{\partial}u &= \sum u_{\overline{j}\overline{i}}J^{-1}\mathrm{d}\overline{z^i} \wedge \mathrm{d}\overline{z^j},\\ \partial\overline{\partial_J}u &= \sum u_{ij}\mathrm{d}z^j \wedge J^{-1}\mathrm{d}z^i,\\ \partial\overline{\partial}u &= \sum u_{i\overline{j}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z^j},\\ \partial_J\overline{\partial_J}u &= \sum u_{i\overline{j}}J^{-1}\mathrm{d}\overline{z^j} \wedge J^{-1}\mathrm{d}z^i,\end{aligned}$$

the third term of (4.5) becomes

$$\begin{aligned}&n\partial_J\overline{\partial}u \wedge \partial\overline{\partial_J}u \wedge \overline{\Omega}^{n-1} \wedge A\\ &= \frac{1}{n-1}\sum_{k=0}^{n-1}\sum_{j=0}^{2n-1}\Big(\sum_{i\neq k}\frac{1}{\widetilde{\Omega}_{2i2i+1}}\Big)(|u_{2kj}|^2 + |u_{2k+1j}|^2)\widetilde{\Omega}^n \wedge \overline{\Omega}^n,\end{aligned} \tag{4.6}$$

and the forth term

$$\begin{aligned}&-n\partial\overline{\partial}u \wedge \partial_J\overline{\partial_J}u \wedge \overline{\Omega}^{n-1} \wedge A\\ &= \frac{1}{n-1}\sum_{k=0}^{n-1}\sum_{j=0}^{2n-1}\Big(\sum_{i\neq k}\frac{1}{\widetilde{\Omega}_{2i2i+1}}\Big)(|u_{2k\overline{j}}|^2 + |u_{2k+1\overline{j}}|^2)\widetilde{\Omega}^n \wedge \overline{\Omega}^n.\end{aligned} \tag{4.7}$$

For I_1 and I_2 we have

$$\begin{aligned}\mathrm{I}_1 &= -n^2\widetilde{\Omega}^{n-1} \wedge \overline{\partial}\Omega_h \wedge \overline{\partial_J}u \wedge \overline{\Omega}^{n-1} - n\overline{\partial_J}u \wedge \overline{\partial}\mathrm{e}^{u_t+f} \wedge \Omega^n \wedge \overline{\Omega}^{n-1}\\ &= -\sum_{i=0}^{n-1}\sum_{j=0}^{2n-1}\frac{(\Omega_h)_{2i2i+1,\overline{j}}u_j}{\widetilde{\Omega}_{2i2i+1}}\widetilde{\Omega}^n \wedge \overline{\Omega}^n + \sum_{j=0}^{2n-1}u_j(u_t+f)_{\overline{j}}\widetilde{\Omega}^n \wedge \overline{\Omega}^n\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\mathrm{I}_2 &= n\widetilde{\Omega}^{n-1} \wedge \overline{\partial_J}\Omega_h \wedge \overline{\partial}u \wedge \overline{\Omega}^{n-1} + \overline{\partial}u \wedge \overline{\partial_J}\mathrm{e}^{u_t+f} \wedge \Omega^n \wedge \overline{\Omega}^{n-1}\\ &= -\sum_{i=0}^{n-1}\sum_{j=0}^{2n-1}\frac{(\overline{\Omega}_h)_{2i2i+1,j}u_{\overline{j}}}{\widetilde{\Omega}_{2i2i+1}}\widetilde{\Omega}^n \wedge \overline{\Omega}^n + \sum_{j=0}^{2n-1}u_{\overline{j}}(u_t+f)_j\widetilde{\Omega}^n \wedge \overline{\Omega}^n.\end{aligned} \tag{4.9}$$

Combining (4.6)–(4.9), we obtain estimate of (4.5),

$$\frac{\partial\partial_J\beta \wedge A \wedge \overline{\Omega}^n}{\beta\widetilde{\Omega}^n \wedge \overline{\Omega}^n}$$

$$
\begin{aligned}
&= -\frac{1}{\beta}\sum_{i=0}^{n-1}\sum_{j=0}^{2n-1}\frac{(\Omega_h)_{2i2i+1,\bar{j}}u_j + (\overline{\Omega}_h)_{2i2i+1,j}u_{\bar{j}}}{\widetilde{\Omega}_{2i2i+1}} \\
&\quad + \frac{1}{\beta}\sum_{j=0}^{2n-1}(u_j(u_t+f)_{\bar{j}} + u_{\bar{j}}(u_t+f)_j) \\
&\quad + \frac{1}{(n-1)\beta}\sum_{k=0}^{n-1}\sum_{j=0}^{2n-1}\sum_{i\neq k}\frac{|u_{2kj}|^2 + |u_{2k+1j}|^2 + |u_{2k\bar{j}}|^2 + |u_{2k+1\bar{j}}|^2}{\widetilde{\Omega}_{2i2i+1}}. \qquad (4.10)
\end{aligned}
$$

Again by direct computation, the forth term of (4.2) is

$$
\partial u \wedge \partial_J u \wedge A \wedge \overline{\Omega}^n = \frac{1}{n-1}\sum_{i=0}^{n-1}\Big(\sum_{k\neq i}\frac{1}{\widetilde{\Omega}_{2k2k+1}}\Big)(|u_{2i}|^2 + |u_{2i+1}|^2)\widetilde{\Omega}^n \wedge \overline{\Omega}^n. \qquad (4.11)
$$

For the fifth term of (4.2), we compute

$$
\begin{aligned}
\partial\partial_J u \wedge A &= \frac{n}{n-1}\partial\partial_J u \wedge \Big(\frac{n\widetilde{\Omega}^{n-1}\wedge\Omega}{\Omega^n}\Omega^{n-1} - \widetilde{\Omega}^{n-1}\Big) \\
&= \frac{n}{n-1}(S_1(\partial\partial_J u)\Omega - \partial\partial_J u)\wedge\widetilde{\Omega}^{n-1} \\
&= n(\widetilde{\Omega}^n - \Omega_h \wedge \widetilde{\Omega}^{n-1}).
\end{aligned}
$$

By compactness of M, there exists $\varepsilon > 0$ such that $\Omega_h \geq \varepsilon\Omega$. Hence we obtain

$$
\begin{aligned}
\varphi'\frac{\partial\partial_J u \wedge A \wedge \overline{\Omega}^n}{\widetilde{\Omega}^n \wedge \overline{\Omega}^n} &= n\varphi' - n\varphi'\frac{\Omega_h \wedge \widetilde{\Omega}^{n-1}\wedge\overline{\Omega}^n}{\widetilde{\Omega}^n\wedge\overline{\Omega}^n} \\
&\leq n\varphi' - \varepsilon\varphi'\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}}. \qquad (4.12)
\end{aligned}
$$

We assume $\beta \gg 1$ otherwise we are finished. By (4.3) and (4.10)–(4.12), the inequality (4.2) becomes

$$
\begin{aligned}
0 &\leq -\frac{1}{\beta}\sum_{i=0}^{2n-1}(u_i(f)_{\bar{i}} + u_{\bar{i}}(f)_i) \\
&\quad + \frac{(\varphi')^2 + \varphi''}{n-1}\sum_{i=0}^{n-1}\Big(\sum_{k\neq i}\frac{1}{\widetilde{\Omega}_{2k2k+1}}\Big)(|u_{2i}|^2 + |u_{2i+1}|^2) \\
&\quad + n\varphi' - \Big(\varepsilon\varphi' - C_1\frac{\Sigma|u_j|}{\beta} - C_2\frac{\Sigma|u_{\bar{j}}|}{\beta}\Big)\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}} - \varphi'\widetilde{u}_t. \qquad (4.13)
\end{aligned}
$$

The first term is bounded from above. Now we take

$$
\varphi(s) = \frac{\log(2s + C_0)}{2},
$$

where C_0 is determined by C^0 estimate. Then (4.13) becomes

$$
C_3 \geq C_4\sum_{i=0}^{n-1}\Big(\sum_{k\neq i}\frac{1}{\widetilde{\Omega}_{2k2k+1}}\Big)(|u_{2i}|^2 + |u_{2i+1}|^2) + C_5\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}}. \qquad (4.14)
$$

Thus for any fixed i,

$$\widetilde{\Omega}_{2i2i+1} \geq \frac{C_5}{C_3} \geq C.$$

By (4.4) we also have

$$\frac{1}{\widetilde{\Omega}_{2i2i+1}} = \mathrm{e}^{-u_t - f} \prod_{j \neq i} \widetilde{\Omega}_{2j2j+1} \geq \frac{C^{n-1}}{\sup\limits_M \mathrm{e}^{u_t + f}}, \quad 0 \leq i \leq n-1.$$

Then by (4.14) we obtain β is uniformly bounded.

5 Bound on $\partial\partial_J u$

Theorem 5.1 *Let u be a solution to (1.5) on $M \times [0, T)$. Then there exists a constant C depending only on the fixed data $(I, J, K, g, \Omega, \Omega_h)$ and f such that*

$$\sup_{M \times [0,T)} |\partial\partial_J u|_g \leq C. \tag{5.1}$$

Proof For simplicity denote

$$\eta = S_1(\partial\partial_J u).$$

Consider the function

$$G = \log \eta - \varphi(\widetilde{u}),$$

where φ is the same as before. For any $T_0 \in (0, T)$, suppose $\max\limits_{M \times [0, T_0]} G = G(p_0, t_0)$ with $(p_0, t_0) \in M \times [0, T_0)$. We want to show $\eta(p_0, t_0)$ is uniformly bounded. We choose the normal coordinates around p_0. All the calculations are carried at (p_0, t_0). We have

$$\begin{aligned} 0 \leq \partial_t G &= \frac{\eta_t}{\eta} - \varphi' \widetilde{u}_t, \\ \partial G &= \frac{\partial \eta}{\eta} - \varphi' \partial u = 0, \\ \partial_J G &= \frac{\partial_J \eta}{\eta} - \varphi' \partial_J u = 0, \\ \partial\partial_J G &= \frac{\partial\partial_J \eta}{\eta} - ((\varphi')^2 + \varphi'')\partial u \wedge \partial_J u - \varphi' \partial\partial_J u. \end{aligned}$$

We further have

$$\begin{aligned} 0 \leq \mathcal{P}(G) &= G_t - \frac{\partial\partial_J G \wedge A \wedge \overline{\Omega}^n}{\widetilde{\Omega}^n \wedge \overline{\Omega}^n} \\ &= \frac{\eta_t}{\eta} - \varphi' \widetilde{u}_t - \frac{\partial\partial_J \eta \wedge A \wedge \overline{\Omega}^n}{\eta \widetilde{\Omega}^n \wedge \overline{\Omega}^n} + ((\varphi')^2 + \varphi'') \frac{\partial u \wedge \partial_J u \wedge A \wedge \overline{\Omega}^n}{\widetilde{\Omega}^n \wedge \overline{\Omega}^n} \\ &\quad + \varphi' \frac{\partial\partial_J u \wedge A \wedge \overline{\Omega}^n}{\widetilde{\Omega}^n \wedge \overline{\Omega}^n}. \end{aligned} \tag{5.2}$$

The last two terms were dealt with in the previous section. Since

$$\eta \Omega^n = n \partial\partial_J u \wedge \Omega^{n-1},$$

by taking ∂_t on both sides we have for η_t in the first term

$$\eta_t = u_{t,p\overline{p}}. \tag{5.3}$$

We now focus on $\partial\partial_J\eta$ in the third term of (5.2). By definition η is real, and

$$\eta\overline{\Omega}^n = n\overline{\partial}\,\overline{\partial}_J u \wedge \overline{\Omega}^{n-1}.$$

Under the hyperKähler condition $\mathrm{d}\Omega = 0$, differentiating twice the above equation gives

$$\partial\partial_J\eta \wedge \overline{\Omega}^n = n\partial\partial_J\overline{\partial}\,\overline{\partial}_J u \wedge \overline{\Omega}^{n-1} = n\overline{\partial}\,\overline{\partial}_J\partial\partial_J u \wedge \overline{\Omega}^{n-1}.$$

We know that (see (2.7))

$$\partial\partial_J u = (n-1)\Omega_h - S_1(\Omega_h)\Omega + S_1(\widetilde{\Omega})\Omega - (n-1)\widetilde{\Omega}.$$

Thus

$$\overline{\partial}\,\overline{\partial}_J\partial\partial_J u = (n-1)\overline{\partial}\,\overline{\partial}_J\Omega_h - \overline{\partial}\,\overline{\partial}_J S_1(\Omega_h) \wedge \Omega + \overline{\partial}\,\overline{\partial}_J S_1(\widetilde{\Omega}) \wedge \Omega - (n-1)\overline{\partial}\,\overline{\partial}_J\widetilde{\Omega}, \tag{5.4}$$

where we used the hyperKähler condition on Ω. Now we have

$$\begin{aligned}\partial\partial_J\eta \wedge A \wedge \overline{\Omega}^n &= nA \wedge \overline{\partial}\,\overline{\partial}_J\partial\partial_J u \wedge \overline{\Omega}^{n-1}\\ &= n(n-1)A \wedge \overline{\partial}\,\overline{\partial}_J\Omega_h \wedge \overline{\Omega}^{n-1} - n\overline{\partial}\,\overline{\partial}_J S_1(\Omega_h) \wedge A \wedge \Omega \wedge \overline{\Omega}^{n-1}\\ &\quad + n\overline{\partial}\,\overline{\partial}_J S_1(\widetilde{\Omega}) \wedge A \wedge \Omega \wedge \overline{\Omega}^{n-1} - n(n-1)A \wedge \overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \overline{\Omega}^{n-1}.\end{aligned} \tag{5.5}$$

Note that

$$A \wedge \Omega = \frac{n}{n-1}\big(S_{n-1}(\widetilde{\Omega})\Omega^{n-1} - \widetilde{\Omega}^{n-1}\big) \wedge \Omega = S_{n-1}(\widetilde{\Omega})\Omega^n$$

and

$$\overline{\partial}\,\overline{\partial}_J S_1(\widetilde{\Omega}) \wedge \Omega^n = n\overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \Omega^{n-1}.$$

The third term of (5.5) becomes

$$\begin{aligned}n\overline{\partial}\,\overline{\partial}_J S_1(\widetilde{\Omega}) \wedge A \wedge \Omega \wedge \overline{\Omega}^{n-1} &= n\overline{\partial}\,\overline{\partial}_J S_1(\widetilde{\Omega}) \wedge (\Omega^n \cdot S_{n-1}(\widetilde{\Omega})) \wedge \overline{\Omega}^{n-1}\\ &= n^2 S_{n-1}(\widetilde{\Omega})\overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1}.\end{aligned}$$

The forth term is

$$n(n-1)A \wedge \overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \overline{\Omega}^{n-1} = n^2 S_{n-1}(\widetilde{\Omega})\overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - n^2\widetilde{\Omega}^{n-1} \wedge \overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \overline{\Omega}^{n-1}.$$

The first two terms of (5.5) are similar and we get

$$\partial\partial_J\eta \wedge A \wedge \overline{\Omega}^n = n^2\overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \widetilde{\Omega}^{n-1} \wedge \overline{\Omega}^{n-1} - n^2\overline{\partial}\,\overline{\partial}_J\Omega_h \wedge \widetilde{\Omega}^{n-1} \wedge \overline{\Omega}^{n-1}$$

and

$$\frac{\partial\partial_J\eta \wedge A \wedge \overline{\Omega}^n}{\eta\widetilde{\Omega}^n \wedge \overline{\Omega}^n} = n^2\frac{\overline{\partial}\,\overline{\partial}_J\widetilde{\Omega} \wedge \widetilde{\Omega}^{n-1} \wedge \overline{\Omega}^{n-1}}{\eta\widetilde{\Omega}^n \wedge \overline{\Omega}^n} - n^2\frac{\overline{\partial}\,\overline{\partial}_J\Omega_h \wedge \widetilde{\Omega}^{n-1} \wedge \overline{\Omega}^{n-1}}{\eta\widetilde{\Omega}^n \wedge \overline{\Omega}^n}$$

$$
\begin{aligned}
&= \frac{1}{\eta}\sum_{i=0}^{n-1}\sum_{p=0}^{2n-1}\frac{\widetilde{\Omega}_{2i2i+1,p\overline{p}}}{\widetilde{\Omega}_{2i2i+1}} - \frac{1}{\eta}\sum_{i=0}^{n-1}\sum_{p=0}^{2n-1}\frac{(\Omega_h)_{2i2i+1,p\overline{p}}}{\widetilde{\Omega}_{2i2i+1}} \\
&\geq \frac{1}{\eta}\sum_{i=0}^{n-1}\sum_{p=0}^{2n-1}\frac{\widetilde{\Omega}_{2i2i+1,p\overline{p}}}{\widetilde{\Omega}_{2i2i+1}} - \frac{C_1}{\eta}\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}}. \qquad (5.6)
\end{aligned}
$$

We now rewrite the right hand side of (5.6) using the equation

$$
\mathrm{Pf}(\widetilde{\Omega}_{ij}) = \mathrm{e}^{u_t+f}\mathrm{Pf}(\Omega_{ij}), \tag{5.7}
$$

where $\Omega^n = n!\mathrm{Pf}(\Omega_{ij})\mathrm{d}z^0 \wedge \cdots \wedge \mathrm{d}z^{2n-1}$. Take logarithm of both sides

$$
\log \mathrm{Pf}(\widetilde{\Omega}_{ij}) = u_t + f + \log \mathrm{Pf}(\Omega_{ij}). \tag{5.8}
$$

Since $\overline{\partial}\Omega = 0$, we have $\overline{\partial}\mathrm{Pf}(\Omega) = 0$. By taking $\overline{\partial}$ of (5.8) and using $\mathrm{Pf}(\widetilde{\Omega}_{ij})^2 = \det(\widetilde{\Omega}_{ij})$, we get

$$
\frac{1}{2}\sum \widetilde{\Omega}^{ij}\widetilde{\Omega}_{ji,\overline{p}} = u_{t,\overline{p}} + f_{\overline{p}}. \tag{5.9}
$$

By taking ∂ of both sides we obtain

$$
\frac{1}{2}\sum \widetilde{\Omega}^{ij}\widetilde{\Omega}_{ji,\overline{p}p} = \frac{1}{2}\sum \widetilde{\Omega}^{ik}\widetilde{\Omega}_{kl,p}\widetilde{\Omega}^{lj}\widetilde{\Omega}_{ji,\overline{p}} + f_{p\overline{p}} + u_{t,p\overline{p}}. \tag{5.10}
$$

In local coordinates, the left hand side of (5.10) is

$$
\frac{1}{2}\sum \widetilde{\Omega}^{2i2i+1}\widetilde{\Omega}_{2i+12i,p\overline{p}} + \frac{1}{2}\sum \widetilde{\Omega}^{2i+12i}\widetilde{\Omega}_{2i2i+1,p\overline{p}} = \sum \frac{\widetilde{\Omega}_{2i2i+1,p\overline{p}}}{\widetilde{\Omega}_{2i2i+1}}. \tag{5.11}
$$

It was proved in [15] that the first term of the right hand side of (5.10) is nonnegative, i.e.,

$$
\sum \widetilde{\Omega}^{ik}\widetilde{\Omega}_{kl,p}\widetilde{\Omega}^{lj}\widetilde{\Omega}_{ji,\overline{p}} \geq 0. \tag{5.12}
$$

Hence we obtain

$$
\frac{\partial\partial_J\eta \wedge A \wedge \overline{\Omega}^n}{\eta\widetilde{\Omega}^n \wedge \overline{\Omega}^n} \geq \frac{1}{2\eta}\Delta_{I,g}f - \frac{C_1}{\eta}\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}} + \frac{1}{\eta}u_{t,p\overline{p}}. \tag{5.13}
$$

Inserting (5.3), (5.13) and (4.11)–(4.12) into (5.2), we have

$$
\begin{aligned}
0 \leq &-\frac{1}{2\eta}\Delta_{I,g}f + \frac{(\varphi')^2 + \varphi''}{n-1}\sum_{i=0}^{n-1}\Big(\sum_{k\neq i}\frac{1}{\widetilde{\Omega}_{2k2k+1}}\Big)(|u_{2i}|^2 + |u_{2i+1}|^2) \\
&+ n\varphi' - \Big(\varepsilon\varphi' - \frac{C_1}{\eta}\Big)\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}} - \varphi'\widetilde{u}_t. \qquad (5.14)
\end{aligned}
$$

Assuming $\eta \gg 1$, we obtain from (5.14),

$$
C_2 \geq C_3\sum_{i=0}^{n-1}\frac{1}{\widetilde{\Omega}_{2i2i+1}}. \tag{5.15}
$$

Hence all $\widetilde{\Omega}_{2i2i+1}$ are uniformly bounded. Since $\eta = S_1(\partial\partial_J u) = S_1(\widetilde{\Omega}) - S_1(\Omega_h)$, we can therefore obtain a uniform bound on η.

6 Proof of the Main Theorem

In [24], Tosatti, Wang, Weinkove and Yang derived $C^{2,\alpha}$ estimates for solutions of some nonlinear elliptic equations based on a bound on the Laplacian of the solution, which was improved and extended to parabolic equations by Chu [9]. Bedulli, Gentili and Vezzoni [6] proved the $C^{2,\alpha}$ for the quaternionic complex Monge-Ampère equation. In this section we apply their techniques to derive the $C^{2,\alpha}$ estimates in our setting. Then the longtime existence and convergence follows.

We first need to rewrite (1.5) in terms of real $(1,1)$-forms, which can be done by using the following relation

$$\frac{\Omega^n \wedge \overline{\Omega}^n}{(n!)^2} = \frac{\omega^{2n}}{(2n)!}.$$

And the equation is reformulated as

$$\omega_u^{2n} = \mathrm{e}^{2(u_t+f)}\omega^{2n}, \tag{6.1}$$

where ω and ω_u are induced by Ω and $\widetilde{\Omega}$, respectively.

Lemma 6.1 *Let u be a solution to (1.5) on $M \times [0,T)$ and $\varepsilon \in (0,T)$, then we have*

$$\|\nabla^2 u\|_{C^\alpha(M\times[\varepsilon,T))} \le C_{\varepsilon,\alpha}, \tag{6.2}$$

where the constant $C_{\varepsilon,\alpha} > 0$ depending only on $(I,J,K,g,\Omega,\Omega_h)$, f, ε and α.

Proof The proof here follows from [9–10, 24]. For any point $p \in M$, choose a local chart around p that corresponds to the unit ball B_1 in $\mathbb{C}^{2n}$ with I-holomorphic coordinates $(z^0,\cdots,z^{2n-1})$. We have $\omega = \sqrt{-1}g_{i\overline{j}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$ where $(g_{i\overline{j}}(x))$ is a positive definite $2n\times 2n$ hermitian matrix given by the metric at any point $x \in B_1$. We introduce the real coordinates by $z^i = x^i + \sqrt{-1}x^{2n+i}$ for $i = 0,\cdots,2n-1$.

The complex structure I corresponds to an endomorphism of the real tangent space which we still denote by I, written in matrix form

$$I = \begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix},$$

where I_{2n} denotes the identity matrix.

For any $2n\times 2n$ hermitian matrix $H = A + \sqrt{-1}B$, the standard way to identify H with a real symmetric matrix $\iota(H) \in \mathrm{Sym}(4n)$ is defined as

$$\iota(H) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Let $Q_{(x,t)}(r)$ denote the domain $B_x(r)\times(t-r^2,t]$. We want to check (6.1) is of the following form as in [9, p. 14],

$$u_t(x,t) - F(S(x,t) + T(D^2_{\mathbb{R}}u,x,t),x,t) = h(x,t), \tag{6.3}$$

where u is defined in $Q_{(0,0)}(1)$ up to scaling and translation, $D^2_{\mathbb{R}}u$ is the real Hessian and the functions F, S and T are defined as the following:

$$F : \mathrm{Sym}(4n) \times Q_{(0,0)}(1) \to \mathbb{R}, \quad F(N, x, t) := \frac{1}{2} \log\det(N),$$
$$S : Q_{(0,0)}(1) \to \mathrm{Sym}(4n), \quad S(x, t) := \iota(g_{i\bar{j}}(x))$$

and

$$T : \mathrm{Sym}(4n) \times Q_{(0,0)}(1) \to \mathrm{Sym}(4n),$$
$$T(N, x, t) := \frac{1}{n-1}\Big(\frac{1}{8}\operatorname{tr}(\iota(g_{i\bar{j}}(x))^{-1}p(N))\iota(g_{i\bar{j}}(x)) - G(N, x)\Big),$$

where

$$p(N) := \frac{1}{2}(N + {}^tINI),$$
$$G(N, x) := \frac{1}{4}(p(N) + \iota({}^tJ(x))p(N)\iota(J(x))).$$

Here we are using $J(x)$ as the matrix representation of the complex structure J. Observe that $p(D^2_{\mathbb{R}}u) = 2\iota(D^2_{\mathbb{C}}u)$, we have

$$G(D^2_{\mathbb{R}}u, x) = \frac{1}{2}(\iota(u_{i\bar{j}}) + \iota(J)_i^{\bar{k}}\iota(D^2_{\mathbb{C}}u)_{l\bar{k}}\iota(J)^{l}_{\bar{j}})(x) = \frac{1}{2}\iota(\mathrm{Re}(\partial\partial_J u(\cdot I, \cdot J))_{i\bar{j}})(x).$$

Moreover, one can verify that

$$\operatorname{tr}(\iota(g_{i\bar{j}}(x))^{-1}p(D^2_{\mathbb{R}}u)) = 4\operatorname{tr}(g^{-1}_{i\bar{j}}(x)D^2_{\mathbb{C}}u) = 4\Delta_{I,g}u.$$

Notice that for a hermitian matrix H, $\det(\iota(H)) = \det(H)^2$, hence we get

$$\begin{aligned}
&u_t(x, t) - F(S(x, t) + T(D^2_{\mathbb{R}}u, x, t), x, t)\\
&= \frac{1}{2}\log\det\Big(\iota(g_{i\bar{j}}(x)) + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}u\Big)\iota(g_{i\bar{j}}(x)) - \frac{1}{2}\iota(\mathrm{Re}(\partial\partial_J u(\cdot I, \cdot J))_{i\bar{j}})(x)\Big)\Big)\\
&= \log\det\Big(g_{i\bar{j}}(x) + \frac{1}{n-1}\Big(S_1(\partial\partial_J u)g_{i\bar{j}}(x) - \frac{1}{2}\iota(\mathrm{Re}(\partial\partial_J u(\cdot I, \cdot J))_{i\bar{j}})(x)\Big)\Big)\\
&= -2f(x) - \log\det(g_{i\bar{j}}(x)).
\end{aligned}$$

Thus (6.1) is indeed of form (6.3).

It remains to verify that the functions F, S and T defined above satisfies all the assumptions in [9, H1–H3, p. 14]. From Theorem 5.1 we have $\mathrm{tr}_g g_u \le C$, thus we get

$$C_0^{-1}I_{4n} \le S(x, t) + T(D^2_{\mathbb{R}}u, x, t) \le C_0 I_{4n}.$$

Take the convex set $\mathcal{E}$ to be the set of matrices $N \in \mathrm{Sym}(4n)$ with

$$C_0^{-1}I_{4n} \le N \le C_0 I_{4n}.$$

It is straightforward that H1, H3 and H2(1), H2(2) hold (cf. [9]). For H2(3), we choose local coordinates such that $g(x) = Id$ and J is block diagonal with only J^{2i}_{2i+1} and J^{2i+1}_{2i} non-zero, while $p(P)$ is diagonal with eigenvalues $\lambda_1, \lambda_1, \cdots, \lambda_{2n}, \lambda_{2n} \ge 0$. Then one computes the eigenvalues of $T(P,x,t)$ are $\frac{1}{2}\sum_{i\neq j}\lambda_i \geq 0$. Thus for $P \geq 0$ we have $T(P,x,t) \geq 0$ and let $K = 2(n-1)$, then

$$K^{-1}\|P\| \leq \|T(P,x,t)\| \leq K\|P\|.$$

Finally, to apply [9, Theorem 5.1], we need overcome the lack of C^0 bound of u using the same argument as in [10, Lemma 6.1]. Specifically, we split into two cases $T < 1$ and $T \geq 1$. If $T < 1$ then we have a C^0 bound on u since by Lemma 3.1 $\sup_{M\times[0,T)} |u_t| \leq C$. Hence, [9, Theorem 5.1] applies directly in this case.

If $T \geq 1$, for any $b \in (0, T-1)$, we consider

$$u_b(x,t) = u(x,t+b) - \inf_{M\times[b,b+1)} u(x,t)$$

for all $t \in [0,1)$. By Lemma 3.2, we have $\sup_{M\times[0,1)} |u_b(x,t)| \leq C$. Moreover, it is obvious that u_b also satisfies the equation, thus we have a Laplacian bound on u_b. By applying Theorem 5.1 in [9] to u_b, for any $\varepsilon \in \left(0, \frac{1}{2}\right)$, we have

$$\|\nabla^2 u\|_{C^\alpha(M\times[b+\varepsilon,b+1))} = \|\nabla^2 u_b\|_{C^\alpha(M\times[\varepsilon,1))} \leq C_{\varepsilon,\alpha},$$

where $C_{\varepsilon,\alpha}$ is a uniform constant depending only on the fixed data $(I,J,K,g,\Omega,\Omega_h)$, f, ε and α. Since $b \in (0, T-1)$ is arbitrary, we obtain the estimate.

Proof of Theorem 1.1 Once we have the $C^{2,\alpha}$ estimates, we obtain the longtime existence and the exponential convergence of $\widetilde{u}$ similar as the argument in [20]. Let $\widetilde{u}_\infty = \lim_{t\to\infty} \widetilde{u}(\cdot,t)$, then $\widetilde{u}_\infty$ satisfies

$$\Big(\Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}\widetilde{u}_\infty\Big)\Omega - \partial\partial_J\widetilde{u}_\infty\Big)\Big)^n = \mathrm{e}^{f+\widetilde{b}}\Omega^n$$
$$\Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}\widetilde{u}_\infty\Big)\Omega - \partial\partial_J\widetilde{u}_\infty\Big) > 0,$$

where

$$\widetilde{b} = \Big(\int_M \Omega^n \wedge \overline{\Omega}^n\Big)^{-1} \int_M \Bigg(\log \frac{\Big(\Omega_h + \frac{1}{n-1}\Big(\Big(\frac{1}{2}\Delta_{I,g}\widetilde{u}_\infty\Big)\Omega - \partial\partial_J\widetilde{u}_\infty\Big)\Big)^n}{\Omega^n} - f\Bigg)\Omega^n \wedge \overline{\Omega}^n.$$

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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