Chin. Ann. Math. Ser. B
46(5), 2025, 663–694
DOI: 10.1007/s11401-025-0041-0

Chinese Annals of Mathematics, Series B

Non-topological Condensates for the Self-dual Maxwell-Chern-Simons Model*

Weiwei AO[1] Chao LIU[2]

Abstract In this paper, the authors study the elliptic system arising from the study of Maxwell-Chern-Simons model. They show that there exists a family of non-topological solutions with magnetic field concentrated at some of the vortex points as the two physical parameters satisfying almost optimal conditions where the limiting profile is the singular Liouville equation.

Keywords Self-dual Maxwell-Chern-Simons model, Non-topological condensates, Vortex concentration
2020 MR Subject Classification 35B40, 35J47

1 Introduction

Since the pioneering work by Ginzburg and Landau, there have been many studies on the classical Abelian-Higgs (AH for short) model (see [5, 7, 29, 33, 41, 52, 55]). However, in the situation with both electronic and magnetic charge, one should consider both the Maxwell term and Chern-Simons (CS for short) term in the model. But a naive inclusion of both AH term and CS term in the Lagrangian will lose the self-dual structure, which is a reduced first-order equation from the original complicated second order equation of motion (see [6, 38]). The pure CS model was proposed by Hong-Kim-Pac in [27] and Jackiw-Weinberg in [28] independently, and there have been extensive literatures for the various types of solutions for CS model (see [8–9, 12–16, 18–19, 21–25, 30–32, 35–37, 39–40, 44–46, 48–49, 54, 56]). As a unified self-dual system of AH and CS, Lee-Lee-Min in [34] proposed the Maxwell-Chern-Simons (MCS for short) model by introducing a neutral scalar field, and showed formally that the limiting problems for MCS model could be AH model and CS model depending on the behavior of Chern-Simons mass scale and electric charge (see [20]). The mathematically rigorous proofs for the formal statements in [20, 34] have been obtained in [2–3, 10–11, 42–43] according to the classes of solutions and domains.

If one is restricted to the energy minimizers of the Euler-Lagrangian equation for MCS model, and apply Jaffe-Taubes argument in [29, 52], then the following reduced elliptic system

Manuscript received February 10, 2023. Revised December 21, 2023.
[1]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail: wwao@whu.edu.cn
[2]Corresponding author. School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China.
E-mail: chaoliuwh@whu.edu.cn
*This work was supported by the National Key Research and Development Program of China (No. 2022YFA1006800), the National Natural Science Foundation of China (Nos. 12471111, 12131017, 12221001) and the Fundamental Research Funds for the Central Universities (No. 2682025CX024).

can be obtained (see [10, 20, 26, 43, 49–51, 56] for the detail):

$$\begin{cases} \Delta u = \lambda\mu e^u - \mu\mathcal{N} + 4\pi \sum_{i=1}^{N} \delta_{p_i} & \text{in } \Omega, \\ \Delta\mathcal{N} = \mu(\mu + \lambda e^u)\mathcal{N} - \lambda\mu(\mu+\lambda)e^u & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, Ω is flat torus in $\mathbb{R}^2$, δ_{p_i} stands for the Dirac measure concentrated at p_i, and each p_i is called a vortex point (repeated according to their multiplicity).

The periodic patterns of vortex configurations have been observed in the experiment for the superconductivity (see [1, 4, 47]). Due to the theory suggested by 't Hooft in [53], we consider the above equation in a flat 2-dimensional torus Ω.

On a flat 2-dimensional torus Ω, (1.1) has two different kinds of periodic solutions (see [2, Remark 3]):

(i) topological solution: $\big(u_{\lambda,\mu}, \frac{\mathcal{N}_{\lambda,\mu}}{\lambda}\big) \to (0,1)$ a.e. on Ω as $\mu \gg \lambda \gg 1$;

(ii) nontopological solution: $\big(u_{\lambda,\mu}, \frac{\mathcal{N}_{\lambda,\mu}}{\lambda}\big) \to (-\infty,0)$ a.e. on Ω as $\mu \gg \lambda \gg 1$.

We recall the following asymptotic behavior of solutions in [2–3, 10–11, 42–43].

Theorem A (see [2, Theorem 1.1]) *We assume that* $\{(u_{\lambda,\mu},\mathcal{N}_{\lambda,\mu})\}$ *is a sequence of solutions of* (1.1). *Then*

$$\lim_{\lambda,\mu\to\infty,\ \frac{\lambda}{\mu}\to 0} \Big\| e^{u_{\lambda,\mu}} - \frac{\mathcal{N}_{\lambda,\mu}}{\lambda} \Big\|_{L^\infty(\Omega)} = 0. \tag{1.2}$$

Note that this system is equivalent to

$$\begin{cases} \Delta\Big(u + \dfrac{\mathcal{N}}{\mu}\Big) = -\lambda^2 e^u\Big(1 - \dfrac{\mathcal{N}}{\lambda}\Big) + 4\pi \sum_{i=1}^{N} \delta_{p_i} & \text{in } \Omega, \\ \Delta\mathcal{N} = \mu^2\Big(1 + \dfrac{\lambda}{\mu}e^u\Big)\mathcal{N} - \lambda\mu^2\Big(1+\dfrac{\lambda}{\mu}\Big)e^u & \text{in } \Omega. \end{cases} \tag{1.3}$$

Due to the estimation (1.2), (1.3) will be regarded as a perturbation of the following Chern-Simons equation:

$$\Delta u = -\lambda^2 e^u(1 - e^u) + 4\pi \sum_{i=1}^{N} \delta_{p_i} \quad \text{in } \Omega. \tag{1.4}$$

In view of this observation, we have proved the existence of non-topological solutions with the concentrating property at a single point (at some vortex point or away from all the vortex points) in [2]. We note that the maximum of the first component for solutions in [2] has a finite lower bound since the profile of approximate solutions comes from the entire solution of CS model.

In [3] we also constructed non-topological solutions of (1.1) whose first component concentrates at points far away from the vortex points and which tends to $-\infty$ uniformly. In this case, the profile of approximate solutions comes from the entire solution of regular Liouville equation, so one need that N be an even number.

In this paper, we continue with this construction, our aim is to prove the existence of non-topological solution concentrated at some vortex points of (1.1) such that the first component for solutions tends to $-\infty$ uniformly. Our construction is inspired by [17], where the authors proved the existence of non-topological solutions to (1.4) with magnetic field concentrated at some of the vortex points. It is natural to use the "singular" Liouville profiles as the approximating solutions. As we will see, the concentration mass at p_l will be $8\pi(n_l+1)$ where n_l is the multiplicity of p_l in the set $\{p_1,\cdots,p_N\}$, then there must hold the relation $2\pi N = 4\pi\sum\limits_{l=1}^{m}(n_l+1)$. In order to describe our main result, let $G(x,y)$ be the Green's function satisfying

$$-\Delta_x G(x,y) = \delta_y - \frac{1}{|\Omega|}, \quad \int_\Omega G(x,y)\mathrm{d}y = 0, \tag{1.5}$$

where $|\Omega|$ is the measure of Ω. And we denote the regular part of $G(x,y)$ by

$$\gamma(x,y) = G(x,y) + \frac{1}{2\pi}\ln|x-y|.$$

Let

$$u_0(x) = -4\pi\sum_{i=1}^{N} G(x,p_i).$$

We replace u by $u+u_0$, and assume $|\Omega|=1$. Then (1.3) is equivalent to

$$\begin{cases} \Delta\Big(u+\dfrac{\mathcal{N}}{\mu}\Big) = -\lambda^2\mathrm{e}^{u+u_0}\Big(1-\dfrac{\mathcal{N}}{\lambda}\Big) + 4\pi N & \text{in } \Omega, \\ \Delta\mathcal{N} = \mu(\mu+\lambda\mathrm{e}^{u+u_0})\mathcal{N} - \lambda\mu(\lambda+\mu)\mathrm{e}^{u+u_0} & \text{in } \Omega. \end{cases} \tag{1.6}$$

For $\{p_1,\cdots,p_m\}\subset\{p_1,\cdots,p_N\}$, we define

$$D_0 = \frac{1}{\pi}\Big[\int_{\Omega\setminus\sigma_0^{-1}(B_\rho(0))} \mathrm{e}^{u_0(z)+8\pi\sum\limits_{l=1}^{m}(n_l+1)G(z,p_l)}\mathrm{d}z - \sum_{l=1}^{m}(n_l+1)\int_{\mathbb{R}^2\setminus B_\rho(0)}\frac{\mathrm{d}y}{|y|^4}\Big] \tag{1.7}$$

for small $\rho>0$.

For simplicity, we write $\int_\Omega f(y)\mathrm{d}y$ as $\int_\Omega f$ in the following text. Our main result can be stated as follows.

Theorem 1.1 *Let $\{p_1,\cdots,p_m\}$ be a subset of the vortex set $\{p_1,\cdots,p_N\}\notin\partial\Omega$, $\{p_j\}_j$ be remaining points and n_l, n_j be the corresponding multiplicities so that*

$$2\pi N = 4\pi\sum_{l=1}^{m}(n_l+1). \tag{1.8}$$

Letting $\mathcal{H}_0$ be a meromorphic function in Ω so that $|\mathcal{H}_0|^2 = \mathrm{e}^{u_0+8\pi\sum\limits_{l=1}^{m}(n_l+1)G(z,p_l)}$ (which exists and is unique up to rotations), assume that $\mathcal{H}_0$ has a zero residue at each $p_1,\cdots,p_m$. Letting $\sigma_0 := -\big(\int^z\mathcal{H}_0(w)\mathrm{d}w\big)^{-1}$ (a well-defined meromorphic function, the notation $\int^z g(\omega)\mathrm{d}\omega$ denotes the anti-derivative of $g(z)$), assume that $D_0<0$ in (1.7) and the "non-degeneracy condition" $\det\mathcal{A}\neq 0$, where $\mathcal{A}$ is given by (6.34). Assume that λ and μ are large enough and $\frac{\lambda}{\mu}$ is small enough. Then (1.6) has a solution $(u_{\lambda,\mu},\mathcal{N}_{\lambda,\mu})$ satisfying

- $\lambda^2\mathrm{e}^{u_{\lambda,\mu}+u_0}\big(1-\frac{\mathcal{N}_{\lambda,\mu}}{\lambda}\big) \rightharpoonup 8\pi\sum\limits_{l=1}^{m}(n_l+1)\delta_{p_l}$ *and* $\frac{\mathrm{e}^{u_{\lambda,\mu}+u_0}}{\int_\Omega \mathrm{e}^{u_{\lambda,\mu}+u_0}} \rightharpoonup \frac{2\sum\limits_{l=1}^{m}(n_l+1)\delta_{p_l}}{N}$ *weakly in the sense of measure as $\lambda,\mu\to\infty$, $\frac{\lambda}{\mu}\to 0$;*
- $\lim\limits_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}(\max\limits_\Omega u_{\lambda,\mu}) = -\infty$ *and* $\lim\limits_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}\frac{\|\mathcal{N}_{\lambda,\mu}\|_{L^\infty(\Omega)}}{\lambda} = 0.$

Remark 1.1 In [17], the authors gave some explicit examples which show that the assumption $D_0 < 0$ and "non-degeneracy condition" in Theorem 1.1 can be valid (see [17, Sections 5–6]).

The paper is organized as follows. In Section 2, we introduce the main problem and review some properties of the approximate solutions. In Section 3, we solve the projected problem. In Section 4, we prove Theorem 1.1 for the case of one point condensate, i.e., $m = 1$. In Section 5, we prove Theorem 1.1 when $m \geq 2$. The approximate solutions which were given by [17] are introduced in Section 6 (Appendix).

2 Preliminaries

In this section, we will recall some results in [17]. First let us introduce the following transform for any solution $\big(u, \frac{\mathcal{N}}{\lambda}\big)$ of (1.6)

$$\widetilde{u} = u + \frac{\mathcal{N}}{\mu} \quad \text{and} \quad \widetilde{v} = \frac{\mathcal{N}}{\lambda}, \tag{2.1}$$

then (1.6) is equivalent to

$$\begin{cases} \Delta\widetilde{u} = -\lambda^2 e^{u_0+\widetilde{u}-\frac{\lambda}{\mu}\widetilde{v}}(1-\widetilde{v}) + 4\pi N & \text{in } \Omega, \\ \Delta\widetilde{v} = \mu^2(\widetilde{v} - e^{u_0+\widetilde{u}-\frac{\lambda}{\mu}\widetilde{v}}) - \mu\lambda e^{u_0+\widetilde{u}-\frac{\lambda}{\mu}\widetilde{v}}(1-\widetilde{v}) & \text{in } \Omega. \end{cases} \tag{2.2}$$

Furthermore, (2.2) can be written as follows:

$$\begin{cases} \Delta\widetilde{u} = -\lambda^2 e^{u_0+\widetilde{u}}(1-e^{u_0+\widetilde{u}}) + 4\pi N + h_1(\widetilde{u},\widetilde{v}) & \text{in } \Omega, \\ \Delta\widetilde{v} = \mu^2(\widetilde{v} - e^{u_0+\widetilde{u}-\frac{\lambda}{\mu}\widetilde{v}}) - \mu\lambda e^{u_0+\widetilde{u}-\frac{\lambda}{\mu}\widetilde{v}}(1-\widetilde{v}) & \text{in } \Omega, \end{cases} \tag{2.3}$$

where $h_1(\widetilde{u},\widetilde{v}) = \lambda^2 e^{u_0+\widetilde{u}}(1-e^{u_0+\widetilde{u}}) - \lambda^2 e^{u_0+\widetilde{u}-\frac{\lambda}{\mu}\widetilde{v}}(1-\widetilde{v})$.

We will find a solution $(\widetilde{u},\widetilde{v})$ of (2.3) such that

$$\lambda^2\int_\Omega e^{u_0+\widetilde{u}}(1-e^{u_0+\widetilde{u}}) = 4\pi N \quad \text{and} \quad \int_\Omega h_1(\widetilde{u},\widetilde{v}) = 0. \tag{2.4}$$

Let us decompose $\widetilde{u}$ as $\widetilde{u} = \widetilde{w} + c$, where $c = \frac{1}{|\Omega|}\int_\Omega \widetilde{u}$. Using (2.4), we have

$$e^{2c}\int_\Omega e^{2u_0+2\widetilde{w}} - e^c\int_\Omega e^{u_0+\widetilde{w}} + \frac{4N\pi}{\lambda^2} = 0.$$

Hence, we obtain that

$$e^{c_\pm(\widetilde{w})} = \frac{8N\pi}{\lambda^2\int_\Omega e^{u_0+\widetilde{w}} \mp \lambda\sqrt{\Big(\lambda\int_\Omega e^{u_0+\widetilde{w}}\Big)^2 - 16N\pi\int_\Omega e^{2u_0+2\widetilde{w}}}}. \tag{2.5}$$

Since we are interested in non-topological solutions, it is natural to restrict our attention to the case $c = c_-$. Thus, using (2.5), we obtain that

$$c(\widetilde{w}) = c_-(\widetilde{w}) = \log\Bigg(\frac{8N\pi}{\lambda^2\int_\Omega e^{u_0+\widetilde{w}} + \lambda\sqrt{\Big(\lambda\int_\Omega e^{u_0+\widetilde{w}}\Big)^2 - 16N\pi\int_\Omega e^{2u_0+2\widetilde{w}}}}\Bigg). \tag{2.6}$$

(2.3) can be reduced to the following equation in Ω:

$$
\begin{cases}
-\Delta\widetilde{w} = 4N\pi\Big(\dfrac{\mathrm{e}^{u_0+\widetilde{w}}}{\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}} - 1\Big) - h_1(\widetilde{w}+c(\widetilde{w}),\widetilde{v}) \\
\qquad + \dfrac{64\pi^2N^2\Big(\mathrm{e}^{u_0+\widetilde{w}}\displaystyle\int_\Omega \mathrm{e}^{2u_0+2\widetilde{w}}\Big(\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}\Big)^{-1} - \mathrm{e}^{2u_0+2\widetilde{w}}\Big)}{\lambda^2\Big[\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}} + \sqrt{\Big(\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}\Big)^2 - \frac{16N\pi}{\lambda^2}\int_\Omega \mathrm{e}^{2u_0+2\widetilde{w}}}\Big]^2} & \text{in } \Omega, \\
\Delta\widetilde{v} = \mu^2(\widetilde{v} - \mathrm{e}^{\widetilde{w}+u_0+c(\widetilde{w})-\frac{\lambda}{\mu}\widetilde{v}}) - \mu\lambda\mathrm{e}^{\widetilde{w}+u_0+c(\widetilde{w})-\frac{\lambda}{\mu}\widetilde{v}}(1-\widetilde{v}) & \text{in } \Omega, \\
\displaystyle\int_\Omega \widetilde{w} = 0.
\end{cases}
\tag{2.7}
$$

In the next sections, we will first prove Theorem 1.1 for one point condensate case. We denote this point by p. Assume that p is present n-times in $\{p_1,\cdots,p_N\}$, and denote by $p_j's$ the remaining points in the set $\{p_1,\cdots,p_N\}$ with corresponding multiplicities $n_j's$. For simplicity in the notations let us assume $p=0$.

Since the first equation of (2.7):

$$
\begin{aligned}
-\Delta\widetilde{w} = {} & 4N\pi\Big(\frac{\mathrm{e}^{u_0+\widetilde{w}}}{\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}} - 1\Big) - h_1(\widetilde{w}+c(\widetilde{w}),\widetilde{v}) \\
& + \frac{64\pi^2N^2\Big(\mathrm{e}^{u_0+\widetilde{w}}\displaystyle\int_\Omega \mathrm{e}^{2u_0+2\widetilde{w}}\Big(\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}\Big)^{-1} - \mathrm{e}^{2u_0+2\widetilde{w}}\Big)}{\lambda^2\Big[\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}} + \sqrt{\Big(\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}\Big)^2 - \frac{16N\pi}{\lambda^2}\int_\Omega \mathrm{e}^{2u_0+2\widetilde{w}}}\Big]^2}
\end{aligned}
\tag{2.8}
$$

can be seen as a perturbed mean-field equation with potential e^{u_0} and unperturbed part

$$
-\Delta\widetilde{w} = 4N\pi\Big(\frac{\mathrm{e}^{u_0+\widetilde{w}}}{\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}} - 1\Big),
\tag{2.9}
$$

and e^{u_0} vanishes like $|z|^{2n}$ near 0, the local profile of $\widetilde{w}$ near 0 will be given in terms of the "singular" Liouville equation:

$$
-\Delta U = |z|^{2n}\mathrm{e}^U.
\tag{2.10}
$$

Following the idea in [17], we will choose the approximate solution as $PU_{\delta,a,\sigma}$ which is given in Section 2 of [17], we postpone the definition and estimates of $PU_{\delta,a,\sigma}$ to Section 6.

In the following, we will list some estimates of the approximate solution $W = PU_{\delta,a,\sigma}$ defined in (6.3). In order to simplify the notations, we set $U_{\delta,a} = U_{\delta,a,\sigma}$ and $c_a = c_{a,\sigma_a}$ which are defined in (6.2) and (6.15), and omit the subscript a in σ_a is a function defined in (6.13).

Lemma 2.1 (see [17, (2.27)]) *There holds*

$$
\begin{aligned}
W + u_0 = {} & U_{\delta,a} - \log(8\delta^2) + \log|\sigma'(z)|^2 + 2\pi\sum_{k=0}^{n}|a_k|^2 \\
& + 2\mathrm{Re}[c_a z^{n+1}] + \theta_{\delta,a,\sigma} + 2\delta^2 f_{a,\sigma} + O(\delta^4)
\end{aligned}
\tag{2.11}
$$

in $C^1(\overline{\Omega})$*, as* $\delta \to 0$*, uniformly for* $|a| < \rho$ *and* $\sigma \in \mathcal{B}_r$ *defined by* (6.12)*, where*

$$\theta_{\delta,a,\sigma} = -\frac{1}{|\Omega|}\int_\Omega \log \frac{|\sigma(z)-a|^4}{(\delta^2+|\sigma(z)-a|^2)^2} \tag{2.12}$$

and

$$f_{a,\sigma}(z) = \int_{\partial\Omega} \Big[\partial_\nu \frac{1}{|\sigma(w)-a|^2} G(w,z) - \frac{1}{|\sigma(w)-a|^2}\partial_\nu G(w,z)\Big] \mathrm{d}s(w), \tag{2.13}$$

ν *is the unit outward normal of* $\partial\Omega$ *and* G *is given by* (1.5).

Lemma 2.2 *The following expansions hold:*

$$\frac{\mathrm{e}^{u_0+W}}{\displaystyle\int_\Omega \mathrm{e}^{u_0+W}} = \frac{|\sigma'(z)|^2 \mathrm{e}^{U_{\delta,a}}}{4N\pi}[1+O(|c_a||z|^{n+1}+|c_a||a|+\delta^2|\log\delta|)], \tag{2.14}$$

$$\frac{64(n+1)^3}{\delta^{\frac{2}{n+1}}}|\alpha_a|^{-\frac{2}{n+1}}\frac{\mathrm{e}^{2u_0+2W}}{\displaystyle\int_\Omega \mathrm{e}^{2u_0+2W}}E_{a,\delta} = |\sigma'(z)|^4\mathrm{e}^{2U_{\delta,a}}[1+O(|c_a||z|^{n+1})+o(1)] \tag{2.15}$$

and

$$\begin{aligned}&\frac{\delta^2}{\pi(n+1)\mathrm{e}^{\frac{2\pi}{|\Omega|}\sum\limits_{k=0}^{n}|a_k|^2+\theta_{\delta,a,\sigma}+2\delta^2 f_{a,\sigma}(0)}}\int_\Omega \mathrm{e}^{u_0+W}\\ &= 1+2\mathrm{Re}[c_aF_a(a)]+|c_a|^2\mathrm{Re}G_a(a)+\frac{1}{2}|c_a|^2\Delta\mathrm{Re}G_a(a)\delta^2\log\frac{1}{\delta}+\frac{\delta^2}{n+1}D_a\\ &\quad+O(\delta^2|a|^{\frac{1}{n+1}}+\delta^2|c_a|+\delta^{\frac{2n+3}{n+1}}),\end{aligned} \tag{2.16}$$

where

$$\alpha_a = \lim_{z\to 0}\frac{z^{n+1}}{\sigma(z)} \neq 0 \ (\textit{since } \sigma\in\mathcal{B}_r), \tag{2.17}$$

$$E_{a,\sigma} := \begin{cases} \displaystyle\int_{\mathbb{R}^2}\frac{|y+\frac{a}{\delta}|}{(1+|y|^2)^4}\mathrm{d}y, & \textit{if } |a| = O(\delta),\\ \displaystyle\frac{\pi}{3}\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}}, & \textit{if } |a| \gg \delta,\end{cases}$$

and

$$F_a(y) = \sum_{k=0}^{+\infty}\alpha_a^{k(n+1)}y^{k+1},$$

$$G_a(y) = |y|^2\Big[2\sum_{k=0}^{+\infty}\sum_{m=1}^{+\infty}\overline{\alpha}_a^k\alpha_a^{k+m(n+1)}|y|^{\frac{2k}{k+1}}y^m+\sum_{k=0}^{+\infty}|\alpha_a^k|^2|y|^{\frac{2k}{n+1}}\Big],$$

where $\{\alpha_a^k\}_{k=0}^{+\infty}$ *are the coefficients of the Taylor expansion:* $\mathrm{e}^{c_a(q^{-1}(y))} = 1+c_ay^{n+1}\sum\limits_{k=0}^{+\infty}\alpha_a^ky^k$*,* $q(z) = zQ^{\frac{1}{n+1}}(z)$*,* $Q(z) = \frac{\sigma(z)}{z^{n+1}}$*(see* [17*, Appendix A*] *for details). Here* D_a *is defined by*

$$\pi D_a = \int_{\Omega\setminus\sigma_0^{-1}(B_\rho(0))}\mathrm{e}^{u_0(z)+8\pi\sum\limits_{k=0}^{n}G(z,a_k)-\frac{2\pi}{|\Omega|}\sum\limits_{k=0}^{n}|a_k|^2}\mathrm{d}z - \int_{\mathbb{R}^2\setminus B_\rho(0)}\frac{n+1}{|y|^4}\mathrm{d}y. \tag{2.18}$$

Proof Please refer to [17, (2.42), (2.51)–(2.52)].

Lemma 2.3 (see [17, Theorem 2.3]) *Let* $|a| < \frac{\rho}{2}$ *and set*

$$\eta = \lambda^{-2}\delta^{-\frac{2}{n+1}} \max\Big\{1, \frac{|a|}{\delta}\Big\}. \tag{2.19}$$

The following expansions hold:

$$\begin{aligned}
&\Delta W + 4\pi N\Big(\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}} - 1\Big)\\
&= |\sigma'(z)|^2 \mathrm{e}^{U_{\delta,a}}\Big[\frac{\mathrm{e}^{2\mathrm{Re}[c_a z^{n+1}]}}{1 + 2\mathrm{Re}[c_a F_a(a)] + |c_a|^2 \mathrm{Re} G_a(a) + \frac{1}{2}|c_a|^2 \Delta \mathrm{Re} G_a(a)\delta^2 \log\frac{1}{\delta} + \frac{\delta^2}{n+1} D_a} - 1\Big]\\
&\quad + |\sigma'(z)|^2 \mathrm{e}^{U_{\delta,a}} O(\delta^2|z| + \delta^2|a|^{\frac{1}{n+1}} + \delta^2|c_a| + \delta^{\frac{2n+3}{n+1}}) + O(\delta^2)
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&\frac{64\pi^2 N^2\Big(\mathrm{e}^{u_0+W}\int_\Omega \mathrm{e}^{2u_0+2W}\Big(\int_\Omega \mathrm{e}^{u_0+W}\Big)^{-1} - \mathrm{e}^{2u_0+2W}\Big)}{\lambda^2\Big[\int_\Omega \mathrm{e}^{u_0+W} + \sqrt{\Big(\int_\Omega \mathrm{e}^{u_0+W}\Big)^2 - \frac{16N\pi}{\lambda^2}\int_\Omega \mathrm{e}^{2u_0+2W}}\Big]^2}\\
&= |\sigma'(z)|^2 \mathrm{e}^{U_{\delta,a}}\Big[\frac{8(n+1)^2}{\lambda^2\pi|\alpha_a|^{\frac{2}{n+1}}\delta^{\frac{2}{n+1}}} E_{a,\delta} - \frac{1}{\lambda^2}|\sigma'(z)|^2 \mathrm{e}^{U_{\delta,a}}\Big][1 + O(|c_a||z|^{n+1} + \eta) + o(1)],
\end{aligned} \tag{2.21}$$

where α_a, F_a, G_a, D_a, $E_{a,\delta}$ *are given in Lemma* 2.2.

The following lemma shows some properties of the approximate solution $W := PU_{\delta,a,\sigma}$, which will be used in the next sections.

Lemma 2.4 *Assume* $|a| = O(\delta)$, *then the following estimates hold*:

- $\frac{\int_\Omega \mathrm{e}^{2(u_0+W)}}{\big(\int_\Omega \mathrm{e}^{u_0+W}\big)^2} = O(\delta^{-\frac{2}{n+1}})$;
- $\|\mathrm{e}^{W+u_0}\|_\infty = O(\delta^{-2-\frac{2}{n+1}})$;
- $\mathrm{e}^{c(W)} = O(\frac{\delta^2}{\lambda^2})$;
- $\|\Delta W\|_\infty = O(\delta^{-\frac{2}{n+1}})$, $\|W\|_\infty = O(|\log\delta|)$.

Proof Since $|a| = O(\delta)$, using Lemma 2.2, we have that

$$\frac{\int_\Omega \mathrm{e}^{2(u_0+W)}}{\Big(\int_\Omega \mathrm{e}^{u_0+W}\Big)^2} = \frac{n+1}{\pi^2\delta^{\frac{2}{n+1}}|\alpha_a|^{\frac{2}{n+1}}}(1+o(1))\int_{\mathbb{R}^2}\frac{|1+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^4}\mathrm{d}y,$$

where $\alpha_a = \lim\limits_{z\to 0}\frac{z^{n+1}}{\sigma(z)} \neq 0$. Since $\lim\limits_{a\to 0}\alpha_a = \alpha_0 := \frac{\mathcal{H}(0)}{n+1} \neq 0$, and $\mathcal{H}$ is defined in (6.4), we derive that $\frac{\int_\Omega \mathrm{e}^{2(u_0+W)}}{\big(\int_\Omega \mathrm{e}^{u_0+W}\big)^2} = O(\delta^{-\frac{2}{n+1}})$.

According to Lemma 2.1, following the argument as for [17, (2.42), (2.45)], for any $p > 1$, we have

$$
\begin{aligned}
&\frac{8^p\delta^{2p+\frac{2(p-1)}{n+1}}}{e^{2p\pi\sum\limits_{k=0}^{n}|a_k|^2+p\theta_{\delta,a}}}\int_\Omega e^{p(W+u_0)}\\
&=\delta^{\frac{2(p-1)}{n+1}}\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^{2p}e^{pU_{\delta,a}+O(p|c_a||z|^{n+1}+p\delta^2)}dz+O(\delta^{2p+\frac{2p}{n+1}})\quad(\text{by } y=\sigma(z))\\
&=\frac{8^p\delta^{2p+\frac{2(p-1)}{n+1}}(n+1)^{2p-1}}{|\alpha_a|^{\frac{2}{n+1}(p-1)}}\int_{B_\rho(0)}\frac{|y|^{\frac{2n(p-1)}{n+1}}(1+O(|y|^{\frac{1}{n+1}}))^{p-1}}{(\delta^2+|y-a|^2)^{2p}}\\
&\quad\times[1+O(|c_ay|+|y|^{\frac{1}{n+1}}+\delta^2)]^pdy+O(\delta^{2p+\frac{2p}{n+1}})\quad(\text{by } \delta\overline{y}=y-a)\\
&=\frac{8^p(n+1)^{2p-1}}{|\alpha_a|^{\frac{2}{n+1}(p-1)}}\int_{B_{\frac{\rho}{\delta}}(-a)}\frac{|\overline{y}+\frac{a}{\delta}|^{\frac{2n(p-1)}{n+1}}}{(1+|\overline{y}|^2)^{2p}}\\
&\quad\times[1+O(|a|^{\frac{1}{n+1}}+\delta^{\frac{1}{n+1}}|y|^{\frac{1}{n+1}}+\delta^2)^{2p-1}]dy+O(\delta^{2p+\frac{2p}{n+1}}),
\end{aligned}\tag{2.22}
$$

in view of

$$
\begin{aligned}
|\sigma'(z)|^2&=(n+1)^2|\alpha_a|^{-2}|z|^{2n}(1+O(|z|))\\
&=(n+1)^2|\alpha_a|^{-\frac{2}{n+1}}|\sigma(z)|^{\frac{2n}{n+1}}(1+O(|\sigma(z)|^{\frac{1}{n+1}})),
\end{aligned}
$$

where $\alpha_a=\lim\limits_{z\to0}\frac{z^{n+1}}{\sigma(z)}\neq0$. Since $|a|=O(\delta)$ and $\lim\limits_{a\to0}\alpha_a=\alpha_0=\frac{\mathcal{H}(0)}{n+1}\neq0$, and $\mathcal{H}$ is defined in (6.4), we have

$$
\|e^{W+u_0}\|_\infty=\lim_{p\to\infty}\Big(\int_\Omega e^{p(u_0+W)}\Big)^{\frac{1}{p}}=O(\delta^{-2-\frac{2}{n+1}}).
$$

Using the definition of c in (2.6), we have

$$
\begin{aligned}
e^{c(W)}&=\frac{8N\pi}{\lambda^2}\frac{1}{\displaystyle\int_\Omega e^{u_0+W}+\sqrt{\Big(\int_\Omega e^{u_0+W}\Big)^2-16N\pi\lambda^{-2}\int_\Omega e^{2u_0+2W}}}\\
&=\frac{8N\pi}{\lambda^2\displaystyle\int_\Omega e^{u_0+W}}\frac{1}{1+\sqrt{1-\dfrac{16N\pi}{\lambda^2}\dfrac{\int_\Omega e^{2(u_0+W)}}{\Big(\int_\Omega e^{u_0+W}\Big)^2}}}\quad(\text{using Lemma 2.2})\\
&=O\Big(\frac{\delta^2}{\lambda^2}\Big).
\end{aligned}
$$

Since e^{u_0} is a smooth function in $\overline{\Omega}$ and $\|e^{W+u_0}\|_\infty=O(\delta^{-2-\frac{2}{n+1}})$, we have $\|e^W\|_\infty=O(\delta^{-2-\frac{2}{n+1}})$. Moreover, $\|W\|_\infty=O(|\log\delta|)$.

Using the expansion of ΔW in Lemma 2.3, we obtain that

$$
\|\Delta W\|_\infty=O(\||\sigma'(z)|^2e^{U_{\delta,a}}\|_\infty).\tag{2.23}
$$

So we need to calculate $\||\sigma'(z)|^2e^{U_{\delta,a}}\|_\infty$. For any $p>1$, by a similar argument as in (2.22), we obtain that

$$\delta^{\frac{2p}{n+1}}\int_\Omega |\sigma'(z)|^{2p}\mathrm{e}^{pU_{\delta,a}} = \frac{8^p(n+1)^{2p-1}}{|\alpha_a|^{\frac{2}{n+1}(p-1)}}\int_{B_{\frac{\varrho}{\delta}}(-a)} \frac{|\overline{y}+\frac{a}{\delta}|^{\frac{2n(p-1)}{n+1}}}{(1+|\overline{y}|^2)^{2p}} \times [1+O(|a|^{\frac{1}{n+1}}+\delta^{\frac{1}{n+1}}|y|^{\frac{1}{n+1}}+\delta^2)^{2p-1}]\mathrm{d}y. \tag{2.24}$$

Since $|a| = O(\delta)$ and $\lim\limits_{a\to 0}\alpha_a = \alpha_0 = \frac{\mathcal{H}(0)}{n+1} \neq 0$, and $\mathcal{H}$ is defined in (6.4), we know that $\||\sigma'(z)|^2\mathrm{e}^{U_{\delta,a}}\|_\infty = \delta^{-\frac{2}{n+1}}$. Thus, using (2.23), we get that $\|\Delta W\|_\infty = O(\delta^{-\frac{2}{n+1}})$.

3 Resolution of Projected Problem

In the previous section, we have introduced an approximate solution of the form $W := PU_{\delta,a,\sigma}$ given by (6.3). We are now looking for solutions $(\widetilde{w},\widetilde{v})$ of (2.7) of the form

$$(\widetilde{w},\widetilde{v}) = (W+\phi,\ \mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)+S),$$

where (ϕ, S) is a small correcting term, and using (2.6), we have

$$c(W+\phi) = \log\Big(\frac{8N\pi}{\lambda^2\int_\Omega \mathrm{e}^{u_0+W+\phi}+\lambda\sqrt{\Big(\lambda\int_\Omega \mathrm{e}^{u_0+W+\phi}\Big)^2-16N\pi\int_\Omega \mathrm{e}^{2u_0+2W+2\phi}}}\Big). \tag{3.1}$$

For convenience, we denote

$$f(\phi,S) := \mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)+S. \tag{3.2}$$

Let $L_2(S) := \Delta S-\mu^2 S$ and the linear operator

$$L_1(\phi) := \Delta\phi+\mathcal{K}[\phi+\gamma(\phi)], \tag{3.3}$$

where

$$\mathcal{K} = 4\pi N\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}+\frac{4\pi NB(W)}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)^2}\Big(\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}-2\frac{\mathrm{e}^{2(u_0+W)}}{\int_\Omega \mathrm{e}^{2(u_0+W)}}\Big) \tag{3.4}$$

and

$$\gamma(\phi) = -\frac{\int_\Omega \mathrm{e}^{u_0+W}\phi}{\int_\Omega \mathrm{e}^{u_0+W}}+\frac{B(W)}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)\sqrt{1-\frac{B(W)}{\lambda^2}}}\Big(\frac{\int_\Omega \mathrm{e}^{2(u_0+W)}\phi}{\int_\Omega \mathrm{e}^{2(u_0+W)}}-\frac{\int_\Omega \mathrm{e}^{u_0+W}\phi}{\int_\Omega \mathrm{e}^{u_0+W}}\Big). \tag{3.5}$$

Then problem (2.7) is equivalent to find a solution (ϕ,S) of

$$\begin{cases} L_1(\phi) = -[R+N(\phi)]-h_1(W+\phi+c(W+\phi),f(\phi,S)) & \text{in } \Omega,\\ L_2(S) = h_2(\phi,S) & \text{in } \Omega,\\ \phi \text{ is a doubly periodic function with } \int_\Omega \phi = 0, \end{cases} \tag{3.6}$$

where h_1 is given by (2.3) and

$$\begin{aligned} h_2(\phi,S) = &-\Delta[\mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)] \\ &+\mu^2[\mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)-\mathrm{e}^{u_0+W+\phi+c(W+\phi)-\frac{\lambda}{\mu}f(\phi,S)}] \\ &-\lambda\mu\mathrm{e}^{u_0+W+\phi+c(W+\phi)-\frac{\lambda}{\mu}f(\phi,S)}(1-S-\mathrm{e}^{u_0+W+c(W+\phi)}). \end{aligned} \tag{3.7}$$

The error term R is defined by

$$\begin{aligned} R := &\Delta W + 4N\pi\Big(\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}-1\Big) \\ &+\frac{64\pi^2N^2\Big(\mathrm{e}^{u_0+W}\int_\Omega \mathrm{e}^{2u_0+2W}\Big(\int_\Omega \mathrm{e}^{u_0+W}\Big)^{-1}-\mathrm{e}^{2u_0+2W}\Big)}{\lambda^2\Big[\int_\Omega \mathrm{e}^{u_0+W}+\sqrt{\Big(\int_\Omega \mathrm{e}^{u_0+W}\Big)^2-\frac{16N\pi}{\lambda^2}\int_\Omega \mathrm{e}^{2u_0+2W}}\Big]^2}. \end{aligned} \tag{3.8}$$

We define $B(W) := 16\pi N\big(\int_\Omega \mathrm{e}^{2u_0+2W}\big)\big(\int_\Omega \mathrm{e}^{u_0+W}\big)^{-2}$, the nonlinear term $N(\phi)$, which is quadratic in ϕ, is given by

$$\begin{aligned} N(\phi) = &4\pi N\Big[\frac{\mathrm{e}^{u_0+W+\phi}}{\int_\Omega \mathrm{e}^{u_0+W+\phi}}-\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}-\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}\Big(\phi-\frac{\int_\Omega \mathrm{e}^{u_0+W}\phi}{\int_\Omega \mathrm{e}^{u_0+W}}\Big)\Big] \\ &+\Big[\frac{4\pi NB(W+\phi)}{\lambda^2\Big(1+\sqrt{1-\frac{B(W+\phi)}{\lambda^2}}\Big)^2}-\frac{4\pi NB(W)}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)^2} \\ &-\frac{4N\pi DB(W)[\phi]}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)^2\sqrt{1-\frac{B(W)}{\lambda^2}}}\Big]\Big(\frac{\mathrm{e}^{u_0+W+\phi}}{\int_\Omega \mathrm{e}^{u_0+W+\phi}}-\frac{\mathrm{e}^{2(u_0+W+\phi)}}{\int_\Omega \mathrm{e}^{2(u_0+W+\phi)}}\Big) \\ &+\frac{4N\pi B(W)}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)^2}\Big[\frac{\mathrm{e}^{u_0+W+\phi}}{\int_\Omega \mathrm{e}^{u_0+W+\phi}}-\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}-\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}\Big(\phi-\frac{\int_\Omega \mathrm{e}^{u_0+W}\phi}{\int_\Omega \mathrm{e}^{u_0+W}}\Big)\Big] \\ &-\frac{4N\pi B(W)}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)^2}\Big[\frac{\mathrm{e}^{2(u_0+W+\phi)}}{\int_\Omega \mathrm{e}^{2(u_0+W+\phi)}}-\frac{\mathrm{e}^{2(u_0+W)}}{\int_\Omega \mathrm{e}^{2(u_0+W)}} \\ &-2\frac{\mathrm{e}^{2(u_0+W)}}{\int_\Omega \mathrm{e}^{2(u_0+W)}}\Big(\phi-\frac{\int_\Omega \mathrm{e}^{2(u_0+W)}\phi}{\int_\Omega \mathrm{e}^{2(u_0+W)}}\Big)\Big] \\ &+\frac{4N\pi DB(W)[\phi]}{\lambda^2\Big(1+\sqrt{1-\frac{B(W)}{\lambda^2}}\Big)^2\sqrt{1-\frac{B(W)}{\lambda^2}}}\Big(\frac{\mathrm{e}^{u_0+W+\phi}}{\int_\Omega \mathrm{e}^{u_0+W+\phi}}-\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}} \\ &-\frac{\mathrm{e}^{2(u_0+W+\phi)}}{\int_\Omega \mathrm{e}^{2(u_0+W+\phi)}}+\frac{\mathrm{e}^{2(u_0+W)}}{\int_\Omega \mathrm{e}^{2(u_0+W)}}\Big) \end{aligned} \tag{3.9}$$

with

$$DB(W)[\phi] = 2B(W)\Big(\frac{\int_\Omega e^{2(u_0+W)}\phi}{\int_\Omega e^{2(u_0+W)}} - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\Big).$$

We notice that

$$\int_\Omega R = \int_\Omega L_1(\phi) = \int_\Omega N(\phi) = 0.$$

According to [17], we know that L_1 is not invertible and has a kernel which is almost generated by PZ_0, PZ and $\overline{PZ}$, where PZ_0 and PZ are the unique solutions with zero average of $\Delta PZ_0 = \Delta Z_0 - \frac{1}{|\Omega|}\int_\Omega \Delta Z_0$ and $\Delta PZ = \Delta Z - \frac{1}{|\Omega|}\int_\Omega \Delta Z$ in Ω. Here, the functions Z_0 and Z are defined as follows:

$$Z_0(z) = \frac{\delta^2 - |\sigma(z)-a|^2}{\delta^2 + |\sigma(z)-a|^2} \quad \text{and} \quad Z(z) = \frac{\delta(\sigma(z)-a)}{\delta^2 + |\sigma(z)-a|^2},$$

which are (not doubly-periodic) solutions of $-\Delta\phi = |\sigma'(z)|^2 e^{U_{\delta,a,\sigma}}\phi$ in Ω.

We list the expansions of $\int_\Omega RPZ_0$ and $\int_\Omega RPZ$ from [17], which is crucial to solve problem (3.6).

Proposition 3.1 *Assume $|a| \le C_0\delta$ for some $C_0 > 0$. The following expansions hold as $\delta, \tau \to 0$:*

$$\begin{aligned}\int_\Omega RPZ_0 &= -16(n+1)|\alpha_a|^2|c_a|^2\delta^2\log\frac{1}{\delta} - 8\pi\delta^2 D_a \\ &\quad + 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\tau\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}dy \\ &\quad + o(\delta^2+\tau) + O(\delta^2|c_a| + |a|^{\frac{1}{n+1}}\delta^2|\log\delta| + \tau^2)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\int_\Omega RPZ &= 4\pi\delta(n+1)\overline{\alpha_a c_a} - 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\tau\int_{\mathbb{R}^2}\frac{y|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}dy \\ &\quad + o(\delta|c_a| + \delta|a| + \tau + \delta^2) + O(\tau^2),\end{aligned} \tag{3.11}$$

where $\tau = \lambda^{-2}\delta^{-\frac{2}{n+1}}$ and $\alpha_a, D_a, c_a = c_{a,\sigma_a}$, are given by (2.17)–(2.18) *and* (6.15), *respectively.*

Remark 3.1 According to [17, Remarks 3.2–3.3], the range $|a| \gg \delta$ is not compatible while solving simultaneously $\int_\Omega RPZ = 0$ and $\int_\Omega RPZ_0 = 0$.

Thus, $L_1(\phi) = -[R + N(\phi)] - h_1$ is not generally solvable. To solve (3.6), we first consider the following projected problem:

$$\begin{cases} L_1(\phi) = -[R+N(\phi)] - h_1(W+\phi+c(W+\phi), f(\phi,S)) \\ \qquad\quad +d_0\Delta PZ_0 + \mathrm{Re}[d\Delta PZ] & \text{in } \Omega, \\ L_2(S) := \Delta S - \mu^2 S = h_2(\phi,S) & \text{in } \Omega, \\ \int_\Omega \phi = \int_\Omega \phi\Delta PZ_0 = \int_\Omega \phi\Delta PZ = 0. \end{cases} \tag{3.12}$$

Remark 3.2 Let us recall that $c_a = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\big[\frac{g^2_{a,\sigma}(z)}{g^2_{a,\sigma}}\frac{g^2_{0,\sigma}(0)}{g^2_{0,\sigma}(z)}\frac{\mathcal{H}_{a,\sigma}(z)}{\mathcal{H}_{a,\sigma}(0)}\big](0)$, then $\lim\limits_{a\to 0} c_a = c_0 = \frac{1}{\mathcal{H}(0)(n+1)!}\frac{\mathrm{d}^{n+1}\mathcal{H}}{\mathrm{d}z^{n+1}}(0)$. We will solve problem (3.12) under the assumption that $c_0 = 0$ and $\lambda^{-2}\delta^{-\frac{2}{n+1}} \sim \delta^2$, which is reasonable according to Remark 4.1.

The resolution of problem (3.12) is based on the following linear theory. We introduce the weighted norm:

$$\|\widetilde{h}\|_* = \sup_{z\in\Omega}\frac{(\delta^2+|\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}}{\delta^\gamma(|\sigma'(z)|^2+\delta^{\frac{2n}{n+1}})}|\widetilde{h}(z)| \tag{3.13}$$

for any $\widetilde{h}\in L^\infty(\Omega)$, where $0<\gamma<1$ is small fixed constant.

Proposition 3.2 *Let $M_0 > 0$. There exists λ_0 large such that for all $0<\delta\le\frac{1}{\lambda_0}$, $\delta^{-\frac{2}{n+1}}\lambda^{-2}\sim\delta^2$, $|a|\le M_0\delta$ and $g_1, g_2\in L^\infty(\Omega)$ with $\int_\Omega g_1 = 0$ there is a unique solution $\phi := T_1(g_1), S := T_2(g_2), d_0\in\mathbb{R}$ and $d\in\mathbb{C}$ of the following problem*

$$\begin{cases} L_1(\phi) = g_1 + d_0\Delta PZ_0 + \mathrm{Re}[d\Delta PZ] & \text{in } \Omega,\\ \displaystyle\int_\Omega\phi = \int_\Omega\phi\Delta PZ_0 = \int_\Omega\phi\Delta PZ = 0, & \\ L_2(S) := \Delta S - \mu^2 S = g_2 & \text{in } \Omega. \end{cases} \tag{3.14}$$

Moreover, there is a constant $C>0$ such that

$$\delta^{\frac{2}{n+1}}\|\Delta\phi\|_\infty + \|\phi\|_\infty \le C\Big(\log\frac{1}{\delta}\Big)\|g_1\|_*, \quad |d_0|+|d|\le C\|g_1\|_*,$$
$$\mu^2\|S\|_{L^\infty(\Omega)}\le C\|g_2\|_{L^\infty(\Omega)}.$$

Proof According [2, Theorem 4.2] and [17, Proposition 4.1], problem (3.14) has a unique solution (ϕ, S). Moreover, there exists $C>0$ such that

$$\|\phi\|_\infty \le C\Big(\log\frac{1}{\delta}\Big)\|g_1\|_*, \quad |d_0|+|d|\le C\|g_1\|_*, \quad \mu^2\|S\|_{L^\infty(\Omega)}\le C\|g_2\|_{L^\infty(\Omega)}. \tag{3.15}$$

By the definition of L_1 in (3.3), we get that

$$\begin{aligned}\|\Delta\phi\|_\infty &= \|\mathcal{K}[\phi+\gamma(\phi)] + g_1 + d_0\Delta PZ_0 + \mathrm{Re}[d\Delta PZ]\|_\infty\\ &\le \|\mathcal{K}\|_\infty[\|\phi\|_\infty + \|\overline{\gamma}(\phi)\|_\infty] + \|g_1\|_\infty + C_2(|d_0|+|d|).\end{aligned} \tag{3.16}$$

Since $\Delta PZ = O(1)$ and $\Delta PZ_0 = O(1)$, where $\mathcal{K}$ and $\gamma(\phi)$ are given by (3.4)–(3.5), respectively. Using (2.14)–(2.16), $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2$ and Lemma 2.4, it is straightforward but tedious computation to show that

$$\|\mathcal{K}\|_\infty = O(\delta^{-\frac{2}{n+1}}) \quad\text{and}\quad \|\overline{\gamma}(\phi)\|_\infty = \|\phi\|_\infty O(\delta^{-\frac{2}{n+1}}). \tag{3.17}$$

For $\|g_1\|_\infty$, using the definition of $\|\cdot\|_*$ in (3.13), we have

$$\begin{aligned}|g_1(z)| &\le \frac{\delta^\gamma(|\sigma'(z)|^2+\delta^{\frac{2n}{n+1}})}{(\delta^2+|\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}}\|g_1\|_*\\ &\le \delta^{-\frac{2}{n+1}}\|g_1\|_* + \frac{\delta^\gamma|\sigma'(z)|^2}{(\delta^2+|\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}}\|g_1\|_*.\end{aligned}$$

By the same argument as in (2.22), we obtain that

$$\Big\|\frac{\delta^\gamma|\sigma'(z)|^2}{(\delta^2+|\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}}\Big\|_\infty = \lim_{p\to\infty}\Big(\int_\Omega\Big|\frac{\delta^\gamma|\sigma'(z)|^2}{(\delta^2+|\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}}\Big|^p\Big)^{\frac{1}{p}} = O(\delta^{-\frac{2}{n+1}}).$$

Furthermore, combining (3.15)–(3.17), we have that $\delta^{\frac{2}{n+1}}\|\Delta\phi\|_\infty \le C\|g_1\|_*$ for some $C>0$. The proof is completed.

Hereafter, we denote $\|\cdot\|_{L^\infty(\Omega)}$ by $\|\cdot\|_\infty$. Define the space

$$\begin{aligned} M_\infty := \{&(\phi,S) \mid S\in W^{2,2}(\Omega)\cap L^\infty(\Omega);\ \phi\in L^\infty(\Omega),\\ &\Delta\phi\in L^\infty(\Omega) \text{ and } \phi \text{ is doubly-periodic in } \Omega\} \end{aligned} \tag{3.18}$$

and a subset of M_∞,

$$\mathcal{F}=\{(\phi,S)\in M_\infty \mid \delta^{\frac{2}{n+1}}\|\Delta\phi\|_\infty+\|\phi\|_\infty+\frac{|\log\delta|^2}{\delta^\gamma}\|S\|_\infty \le \delta^{2-\gamma}|\log\delta|^2\}, \tag{3.19}$$

where γ is a small constant given by (3.13). Denote the operator

$$\mathcal{A}(\phi,S) := (T_1[-R-N(\phi)-h_1(W+\phi+c(W+\phi),f(\phi,S))],T_2[h_2(\phi,S)]), \tag{3.20}$$

where $R, N(\phi), h_1$, h_2 and f have been given by (3.8), (3.9), (2.3), (3.7) and (3.2), respectively. Here T_1 and T_2 have been defined in Proposition 3.2.

Next we will show that the operator $\mathcal{A}$ is a contraction operator, which then has a fixed point (ϕ,S) in $\mathcal{F}$. According to Proposition 3.2, we have that the fixed point (ϕ,S) is a solution of (3.12). First, we give some estimates, which will be used to show that $\mathcal{A}$ is a contraction mapping.

According to [17, Corollary 2.4, (C.3), (2.51)–(2.52)], we easily obtain the following results.

Lemma 3.1 *Recall* $c_0=\frac{1}{\mathcal{H}(0)(n+1)!}\frac{\mathrm{d}^{n+1}\mathcal{H}}{\mathrm{d}z^{n+1}}(0)$. *Assume that* $c_0=0$, $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2$ *and* $|a|\le M_0\delta$ *for some constant* $M_0>0$, *then we have the following estimates hold*

- $\|R\|_*=O(\delta^{2-\gamma})$, *where* R *is given by* (3.8);
- *There exists a constant* C_1 *such that* $\|N(\phi_1)-N(\phi_2)\|_*\le C_1(\|\phi_1\|_\infty+\|\phi_2\|_\infty)\|\phi_1-\phi_2\|_\infty$ *for any* $\|\phi_i\|_\infty\le\delta^{2-\gamma}$, $i=1,2$, *where* $N(\phi)$ *is given by* (3.9);
- $\big\|\frac{\mathrm{e}^{2u_0+2W}}{\int_\Omega \mathrm{e}^{2u_0+2W}}\big\|_*=\big\|\frac{\mathrm{e}^{u_0+W}}{\int_\Omega \mathrm{e}^{u_0+W}}\big\|_*=O(1)$.

The next lemma shows that $\mathcal{A}$ is contraction mapping from $\mathcal{F}$ to itself.

Lemma 3.2 *There exists* ε_0 *small and* $\lambda_0,\mu_0>0$ *large such that for all* $\lambda>\lambda_0$, $\mu>\mu_0$, $\frac{\lambda}{\mu}<\varepsilon_0$, *and* δ *satisfying* $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2$, *for* $|a|\le\frac{1}{\lambda_0}\delta$, *then operator* $\mathcal{A}$ *admits a unique fixed point* $(\phi,S)\in\mathcal{F}$.

Proof **Step 1** We first estimate the L^∞ norm of $h_2(\varphi,S)$. Recall that

$$\begin{aligned} h_2(\phi,S) &= -\Delta[\mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)]\\ &\quad+\mu^2[\mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)-\mathrm{e}^{u_0+W+\phi+c(W+\phi)-\frac{\lambda}{\mu}f(\phi,S)}]\\ &\quad-\lambda\mu\mathrm{e}^{W+\phi+c(W+\phi)+u_0-\frac{\lambda}{\mu}f(\phi,S)}(1-S-\mathrm{e}^{u_0+W+c(W+\phi)})\\ &:= \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3. \end{aligned}$$

Next we will estimate $\|\mathrm{I}_1\|_\infty$, $\|\mathrm{I}_2\|_\infty$ and $\|\mathrm{I}_3\|_\infty$, respectively.

Since e^{u_0} is $C^2(\overline{\Omega})$, for any $p>1$, we have

$$\begin{aligned}
\|\mathrm{I}_1\|_p &= \|\Delta[\mathrm{e}^{W+u_0+c(W+\phi)}(1+\phi)]\|_p \\
&\le \|\mathrm{e}^{c(W+\phi)}\|_\infty \|\Delta[\mathrm{e}^{W+u_0}(1+\phi)]\|_p \\
&\le \frac{\delta^2}{\lambda^2} O\{\|(1+\phi)\Delta \mathrm{e}^{W+u_0}\|_p + 2\|\nabla\phi\nabla \mathrm{e}^{W+u_0}\|_p + \|\mathrm{e}^{W+u_0}\Delta\phi\|_p\} \\
&\le \frac{\delta^2}{\lambda^2} O\{(1+\|\phi\|_\infty)[\|\mathrm{e}^{W+u_0}(\Delta W + |\nabla W|^2)\|_p + \|\mathrm{e}^W\|_p + \|\mathrm{e}^W|\nabla W|\|_p] \\
&\quad + \|\mathrm{e}^{W+u_0}\nabla\phi\nabla W\|_p + \|\mathrm{e}^W\nabla\phi\|_p + \|\mathrm{e}^{W+u_0}\Delta\phi\|_p\} \\
&\le \frac{\delta^2}{\lambda^2} O\Big\{(1+\|\phi\|_\infty)\frac{1}{\delta^{2+\frac{6}{n+1}}} + \frac{1}{\delta^{2+\frac{2}{n+1}}}\Big(\frac{1}{\delta^{\frac{2}{n+1}}} + |\log\delta|\Big)(\|\Delta\phi\|_\infty \\
&\quad + \|\phi\|_\infty) + \frac{\|\Delta\phi\|_\infty + \|\phi\|_\infty}{\delta^2} + \frac{\|\Delta\phi\|_\infty}{\delta^{2+\frac{2}{n+1}}}\Big\} \\
&\le \frac{1}{\lambda^2\delta^{\frac{6}{n+1}}} O\{1+\|\Delta\phi\|_\infty + \|\phi\|_\infty\} = \delta^{2-\frac{4}{n+1}} O\{1+\|\Delta\phi\|_\infty + \|\phi\|_\infty\},
\end{aligned} \tag{3.21}$$

where we have used $W^{2,p}$ estimate and Lemma 2.4. Thus, we derive that $\|I_1\|_\infty = \lim_{p\to\infty}\|I_1\|_p \le O(\delta^{2-\frac{4}{n+1}})$.

Recall that $f(\phi,S) = \mathrm{e}^{W+u_0+c(W+\phi)}(1+\phi) + S$, using Lemma 2.4, we have

$$\Big\|\frac{\lambda}{\mu} f(\phi,S)\Big\|_\infty = \frac{1}{\mu\lambda\delta^{\frac{2}{n+1}}} O(1+\|\phi\|_\infty) + \frac{\lambda}{\mu}\|S\|_\infty = o(\delta^2). \tag{3.22}$$

Together with the above estimate and the mean value theorem, we obtain that

$$\begin{aligned}
\|\mathrm{I}_2\|_\infty &= \mu^2 \|\mathrm{e}^{W+u_0+c(W+\phi)}[1+\phi - \mathrm{e}^{\phi - \frac{\lambda}{\mu} f(\phi,S)}]\|_\infty \\
&\le \frac{\mu^2}{\lambda^2\delta^{\frac{2}{n+1}}} O\Big(\Big\|\frac{\lambda}{\mu} f(\phi,S)\Big\|_\infty + \|\phi\|_\infty^2\Big) \\
&\le O\Big(\frac{\mu}{\lambda^3\delta^{\frac{4}{n+1}}} + \frac{\mu\|S\|_\infty}{\lambda\delta^{\frac{2}{n+1}}} + \frac{\mu^2\|\phi\|_\infty^2}{\lambda^2\delta^{\frac{2}{n+1}}}\Big)
\end{aligned} \tag{3.23}$$

and

$$\|\mathrm{I}_3\|_\infty \le \frac{\mu}{\lambda\delta^{\frac{2}{n+1}}}\Big(1 + \|S\|_\infty + \frac{\|\phi\|_\infty}{\lambda^2\delta^{\frac{2}{n+1}}}\Big). \tag{3.24}$$

Since $\lambda^{-2}\delta^{-\frac{2}{n+1}} \sim \delta^2$, combining the above estimates, we get that

$$\frac{1}{\mu^2}\|h_2(\phi,S)\|_\infty = o(\delta^2). \tag{3.25}$$

Next we calculate $\|h_1(W+\phi+c(W+\phi), f(\phi,S))\|_*$, where $\|\cdot\|_*$ is given by (3.13). Using the mean value theorem, (3.22), Lemmas 2.4 and 3.1, we have

$$\begin{aligned}
&\|h_1(W+\phi+c(W+\phi), f(\phi,S))\|_* \\
&\le \lambda^2 O\Big\{\|\mathrm{e}^{W+u_0+\phi+c(W+\phi)}\|_* \Big\|\frac{\lambda}{\mu} f(\phi,S)\Big\|_\infty + \|\mathrm{e}^{2W+2u_0+2\phi+2c(W+\phi)}\|_* \\
&\quad \times \Big[\Big\|\frac{\lambda}{\mu} f(\phi,S)\Big\|_\infty + \|S\|_\infty + \|\phi\|_\infty^2\Big]\Big\} \\
&\le \Big(\frac{\lambda}{\mu} + \frac{1}{\lambda^2\delta^{\frac{2}{n+1}}}\Big)\|S\|_\infty + \frac{1}{\mu\lambda\delta^{\frac{2}{n+1}}}(1+\|\phi\|_\infty).
\end{aligned} \tag{3.26}$$

Combining the above result and Lemma 3.1, we have

$$\|R+N(\phi)+h_1(W+\phi+c(W+\phi),f(\phi,S))\|_*\le O(\delta^{2-\gamma}). \tag{3.27}$$

Combining Proposition 3.2, (3.25) and (3.27), we derive that $\mathcal{A}(\mathcal{F})\subset\mathcal{F}$.

Step 2 Next for any (ϕ_1,S_1) and $(\phi_2,S_2)\in\mathcal{F}$, by the same arguments as in (3.21), (3.23)–(3.24) and (3.26), using $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2$ and $\lim\limits_{\lambda,\mu\to\infty}\frac{\lambda}{\mu}=0$, we get that

$$\begin{aligned}
&\frac{1}{\mu^2}\|h_2(\phi_1,S_1)-h_2(\phi_2,S_2)\|_\infty\\
&\le\Big(\delta^2+\frac{\lambda}{\mu}\delta^{2-\frac{2}{n+1}}+\frac{\lambda^2}{\mu^2}\delta^{4-\frac{4}{n+1}}\Big)\|\phi_1-\phi_2\|_\infty+\frac{\lambda^2}{\mu^2}\delta^{4-\frac{4}{n+1}}\|\Delta\phi_1-\Delta\phi_2\|_\infty\\
&\quad+\delta^2\|S_1-S_2\|_\infty\\
&\le o\Big(\|\phi_1-\phi_2\|_\infty+\delta^{\frac{2}{n+1}}\|\Delta\phi_1-\Delta\phi_2\|_\infty+\frac{|\log\delta|^2}{\delta^\gamma}\|S_1-S_2\|_\infty\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\|h_1(W+\phi_1+c(W+\phi_1),f(\phi_1,S_1))-h_1(W+\phi_2+c(W+\phi_2),f(\phi_2,S_2))\|_*\\
&\le O\Big(\frac{\lambda}{\mu}\|S_1-S_2\|_\infty+\delta^2\|\phi_1-\phi_2\|_\infty\Big)\\
&\le o\Big(\|\phi_1-\phi_2\|_\infty+\frac{|\log\delta|^2}{\delta^\gamma}\|S_1-S_2\|_\infty\Big).
\end{aligned}$$

According to Proposition 3.2 and the above estimates, since $\|N(\phi_1)-N(\phi_2)\|_*\le C_1(\|\phi_1\|_\infty+\|\phi_2\|_\infty)\|\phi_1-\phi_2\|_\infty$ in Lemma 3.1 for some $C_1>0$ independent of ϕ_1,ϕ_2, we have $\mathcal{A}$ is a contraction mapping of $\mathcal{F}$ into itself when λ_0 and μ_0 are sufficiently large. Furthermore, the operator has a fixed point (ϕ,S) in $\mathcal{F}$.

According to the above Lemma 3.2, we can obtain the following result.

Proposition 3.3 *There exists ε_0 small and $\lambda_0,\mu_0>0$ large such that for all $\lambda>\lambda_0$, $\mu>\mu_0$, $\frac{\lambda}{\mu}<\varepsilon_0$, and δ satisfying $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2$, for $|a|\le\frac{1}{\lambda_0}\delta$, then problem* (3.12) *has a unique solution $\phi=\phi(\delta,a)$, $S=S(\delta,a)$, $d_0=d_0(\delta,a)\in R$, $d=d(\delta,a)\in\mathbb{C}$. Moreover, the map $(\delta,a)\mapsto(\phi(\delta,a),S(\delta,a))$ is C^1 with*

$$\|\phi\|_\infty+\frac{|\log\delta|^2}{\delta^\gamma}\|S\|_\infty\le\delta^{2-\gamma}|\log\delta|^2.$$

4 Proof of the Main Result

By Proposition 3.3, the function $(W+\phi,\mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)+S)$ will be a true solution of (2.7) once we adjust δ and a to have $d_0(\delta,a)=d(\delta,a)=0$. The crucial point is given by the following Lemma. Its proof is the same as in [17, Lemma 4.3].

Lemma 4.1 *Let $\phi=\phi(\delta,a)$, $S=S(\delta,a)$, $d_0=d_0(\delta,a)\in\mathbb{R}$, $d=d(\delta,a)\in\mathbb{C}$ be the solution of* (3.12) *given by Proposition* 3.3. *There exists $\eta_0>0$ such that if $0<\delta\le\eta_0$, $|a|\le\eta_0$,*

$$\begin{cases}\displaystyle\int_\Omega[L_1(\phi)+N(\phi)+R+h_1(W+\phi+c(W+\phi),f(\phi,S))]PZ_0=0,\\ \displaystyle\int_\Omega[L_1(\phi)+N(\phi)+R+h_1(W+\phi+c(W+\phi),f(\phi,S))]PZ=0\end{cases} \tag{4.1}$$

hold, then $(W+\phi,\ e^{u_0+W+c(W+\phi)}(1+\phi)+S)$ *is a solution of* (2.7)*, i.e.,* $d_0(\delta,a)=d(\delta,a)=0$.

Remark 4.1 According to the previous estimates, we know that $h_1(W+\phi+c(W+\phi), f(\phi,S))$ is small since ϕ is sufficiently small. Thus, system (4.1) could be viewed as a perturbation of the reduced equations $\int_\Omega RPZ_0=0$ and $\int_\Omega RPZ=0$. The integral coefficient in (3.10) is negative for all $\frac{a}{\delta}$, which is given in [17, Appendix D]. Since $\alpha_a\to\alpha_0=\frac{\mathcal{H}(0)}{n+1}\neq 0$ and $c_a\to c_0$ as $a\to 0$, we can always exclude the case $c_0\neq 0$. Indeed, in such a case the equation $\int_\Omega RPZ_0=0$ will yield to $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2|\log\delta|$ as $\delta\to 0$ by mean of (3.10) (we are implicity assuming $\lambda^{-2}\delta^{-\frac{2}{n+1}}\to 0$, which is a natural range for solving the reducing equation through (3.10)–(3.11)). This is not compatible with $\int_\Omega RPZ=0$, which allows at most $\delta^2=O(\lambda^{-2}\delta^{-\frac{2}{n+1}})$ by (3.11).

Proposition 4.1 *Assume* $c_0=0$ *and* $|a|\le M_0\delta$ *for some* $M_0>0$. *We have the following results hold as* $\delta\to 0$, $\lambda,\mu\to\infty$ *and* $\frac{\lambda}{\mu}\to 0$:

$$\begin{aligned}&\int_\Omega[L_1(\phi)+N(\phi)+R+h_1(W+\phi+c(W+\phi),f(\phi,S))]PZ_0\\&=-8\pi\delta^2D_0+64(n+1)^{\frac{3n+5}{n+1}}|\mathcal{H}(0)|^{-\frac{2}{n+1}}\lambda^{-2}\delta^{-\frac{2}{n+1}}\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\frac{a}{\delta}|}{(1+|y|^2)^5}\mathrm{d}y\\&\quad+o(\delta^2+\lambda^{-2}\delta^{-\frac{2}{n+1}})+O(\lambda^{-4}\delta^{-\frac{2}{n+1}}|\log\delta|^2+\lambda^{-8}\delta^{-\frac{4}{n+1}}|\log\delta|^2)\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}&\int_\Omega[L_1(\phi)+N(\phi)+R+h_1(W+\phi+c(W+\phi),f(\phi,S))]PZ\\&=4\pi\delta(\overline{\Upsilon}a+\overline{\Gamma}\overline{a})-64(n+1)^{\frac{3n+5}{n+1}}|\mathcal{H}(0)|^{-\frac{2}{n+1}}\lambda^{-2}\delta^{-\frac{2}{n+1}}\int_{\mathbb{R}^2}\frac{y|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\mathrm{d}y\\&\quad+o(\delta^2+\lambda^{-2}\delta^{-\frac{2}{n+1}})+O(\lambda^{-4}\delta^{-\frac{2}{n+1}}|\log\delta|^2+\lambda^{-8}\delta^{-\frac{4}{n+1}}|\log\delta|^2),\end{aligned}\tag{4.3}$$

where D_0 *and* Γ, Υ *are defined by* (1.7) *and* (6.17) *respectively.*

Proof According to [17, Proposition 4.5], we have that the above results do hold when $h_1(W+\phi+c(W+\phi),f(\phi,S))=0$. Therefore, we just consider the term of h_1 in the above expansions. Using (3.26), we have

$$\begin{aligned}&\int_\Omega h_1(W+\phi+c(W+\phi),f(\phi,S))PZ_0\\&=O(\|h_1(W+\phi+c(W+\phi),f(\phi,S))\|_*\|PZ_0\|_\infty)\\&=O\Big(\frac{\lambda}{\mu}\delta^2\Big)=o(\delta^2),\end{aligned}$$

since $PZ_0=O(1)$ and $\|\phi\|_\infty+\frac{|\log\delta|^2}{\delta^\gamma}\|S\|_\infty\le\delta^{2-\gamma}|\log\delta|^2$. By the same argument, we also have

$$\int_\Omega h_1(W+\phi+c(W+\phi),f(\phi,S))PZ=O\Big(\frac{\lambda}{\mu}\delta^2\Big)=o(\delta^2).$$

Thus combining the above estimates, we have that the proposition does hold.

Owing to (4.2) and (4.3), our aim is to find $(\delta(\lambda,\mu), a(\lambda,\mu))$ so that (4.1) does hold. To simplify the notations, we denote

$$\varphi_0(\delta,a,\lambda,\mu)=\int_\Omega [L_1(\phi)+N(\phi)+R+h_1(W+\phi+c(W+\phi),f(\phi,S))]PZ_0$$

and

$$\varphi(\delta,a,\lambda,\mu)=\overline{\int_\Omega [L_1(\phi)+N(\phi)+R+h_1(W+\phi+c(W+\phi),f(\phi,S))]PZ}.$$

Thus, (4.1) reduces to find a solution of

$$\varphi_0(\delta(\lambda,\mu),a(\lambda,\mu),\lambda,\mu)=\varphi(\delta(\lambda,\mu),a(\lambda,\mu),\lambda,\mu)=0 \tag{4.4}$$

for λ large. Next we prove our main result, which clearly implies the validity of Theorem 1.1 with $m=1$.

Theorem 4.1 *Let $\mathcal{H}_0=\frac{\mathcal{H}}{z^{n+2}}$, where $\mathcal{H}$ is given by* (6.4)*, be meromorphic function in Ω with $|\mathcal{H}|^2=\mathrm{e}^{u_0+8\pi(n+1)G(z,0)}$(which exists in view of* (6.1) *and is unique up to rotations), and $\sigma_0=-\big(\int^z H_0(w)\mathrm{d}w\big)^{-1}$. Assume that*

$$\frac{\mathrm{d}^{n+1}\mathcal{H}}{\mathrm{d}z^{n+1}}(0)=0 \tag{4.5}$$

and for some small $\rho>0$

$$D_0:=\frac{1}{\pi}\Big[\int_{\Omega\setminus\sigma_0^{-1}(B_\rho(0))}\mathrm{e}^{u_0+8\pi(n+1)G(z,0)}-\int_{\mathbb{R}^2\setminus B_\rho(0)}\frac{n+1}{|y|^4}\mathrm{d}y\Big]<0. \tag{4.6}$$

If the "non-degeneracy condition"

$$|\Gamma|\neq\Big|\Upsilon+\frac{n(2n+3)}{n+1}D_0\Big| \tag{4.7}$$

does hold where Γ and Υ are given in (6.17)*, for λ,μ are large enough with $\lambda\ll\mu$, then there exist $\delta(\lambda,\mu)$ and $a(\lambda,\mu)$ small so that $(\widetilde{w},\widetilde{v})=(W_{\lambda,\mu}+\phi_{\lambda,\ \mu},\ \mathrm{e}^{u_0+W_{\lambda,\mu}+c(W_{\lambda,\mu}+\phi_{\lambda,\mu})}(1+\phi_{\lambda,\mu})+S_{\lambda,\mu})$ is a solution of* (2.7)*, where $W_{\lambda,\mu}=PU_{\delta(\lambda,\mu),\ a(\lambda,\mu),\ \sigma_{a(\lambda,\mu)}}$. Furthermore, $(u_{\lambda,\mu},N_{\lambda,\mu})=\big(\widetilde{w}+c(\widetilde{w})-\frac{\lambda}{\mu}\widetilde{v},\lambda\widetilde{v}\big)$ does solve* (1.6) *and satisfies*

- $\lambda^2\mathrm{e}^{u_{\lambda,\mu}+u_0}\big(1-\frac{\mathcal{N}_{\lambda,\mu}}{\lambda}\big)\rightharpoonup 8\pi(n+1)\delta_0$ *and* $\frac{\mathrm{e}^{u_{\lambda,\mu}+u_0}}{\int_\Omega \mathrm{e}^{u_{\lambda,\mu}+u_0}}\rightharpoonup\delta_0$ *in the sense of measure as* $\lambda,\mu\to\infty,\ \frac{\lambda}{\mu}\to 0$;
- $\lim\limits_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}(\max\limits_\Omega u_{\lambda,\mu})=-\infty$ *and* $\lim\limits_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}\frac{\|\mathcal{N}_{\lambda,\mu}\|_{L^\infty(\Omega)}}{\lambda}=0.$

Proof First of all, we need to solve (4.4). Since $\varphi_0(\delta(\lambda,\mu),a(\lambda,\mu),\lambda,\mu)=0$ naturally requires $\lambda^{-2}\delta^{-\frac{2}{n+1}}\sim\delta^2$ in view of (4.2), we make the following change of variables: $\delta=\big[\frac{(n+1)\lambda^{-(n+1)}}{|\mathcal{H}(0)|}\big]^{\frac{1}{n+2}}\eta$ and $\zeta=\frac{a}{\delta}$. Then system (4.4) is equivalent to find zeros of

$$\begin{aligned}\Gamma_{\lambda,\mu}(\eta,\zeta):=&\Big[\frac{(n+1)\lambda^{-(n+1)}}{|\mathcal{H}(0)|}\Big]^{-\frac{2}{n+2}}\\&\times\Big(-\frac{1}{8}\varphi_0,\frac{1}{4\pi\eta^2}\varphi\Big)\Big(\Big[\frac{(n+1)\lambda^{-(n+1)}}{|\mathcal{H}(0)|}\Big]^{\frac{1}{n+2}}\eta,\Big[\frac{(n+1)\lambda^{-(n+1)}}{|\mathcal{H}(0)|}\Big]^{\frac{1}{n+2}}\eta\zeta,\lambda,\mu\Big),\end{aligned}$$

which has the expansion $\Gamma_{\lambda,\mu} = \Gamma_0(\eta,\zeta) + o(1)$ as $\lambda,\mu \to \infty$ and $\frac{\lambda}{\mu} \to 0^+$, uniformly for η in compact subset of $(0,+\infty)$, in view of (4.2)–(4.3), where the map $\Gamma_0 : \mathbb{R}\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}$ is defined as

$$\Gamma_0(\eta,\zeta) = \Big(\pi D_0\eta^2 - \frac{8(n+1)^3}{\eta^{\frac{2}{n+1}}}\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\mathrm{d}y, \Gamma\zeta + \Upsilon\overline{\zeta} - \frac{16(n+1)^3}{\pi\eta^{\frac{2(n+2)}{n+1}}}\int_{\mathbb{R}^2}\frac{\overline{y}|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\mathrm{d}y\Big).$$

We need to exhibit "stable" zeros of Γ_0 in $(0,+\infty)\times\mathbb{C}$, which will persist under L^∞-small perturbations yielding to zeros of $\Gamma_{\lambda,\mu}$ as required. The easiest case is given by the point $(\eta_0, 0)$, that solves $\Gamma_0 = 0$ for $\eta_0 = \big(\frac{8(n+1)^3 I_0}{D_0\pi}\big)^{\frac{n+1}{2(n+2)}} > 0$ in view of $D_0 < 0$ and

$$I_0 := \int_{\mathbb{R}^2}\frac{(|y|^2-1)|y|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\mathrm{d}y = \frac{\pi}{2}\int_1^{+\infty}\frac{(1-r)(r^{\frac{n+2}{n+1}} - r^{\frac{n}{n+1}})}{(1+r)^5}\mathrm{d}r < 0.$$

Regarding Γ_0 as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ and setting $\Gamma = \Gamma_1 + \mathrm{i}\Gamma_2$, $\Upsilon = \Upsilon_1 + \mathrm{i}\Upsilon_2$, we have that

$$D\Gamma_0(\eta_0,0) = \begin{pmatrix} \frac{2(n+2)}{n+1}\pi D_0\eta_0 & 0 & 0 \\ 0 & \Gamma_1 + \Upsilon_1 + \frac{2(n+2)}{n+1}D_0 & \Upsilon_2 - \Gamma_2 \\ 0 & \Gamma_2 + \Upsilon_2 & \Gamma_1 - \Upsilon_1 - \frac{2(n+2)}{n+1}D_0 \end{pmatrix}$$

in view of [17, (D.7)] and

$$\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\mathrm{d}y = \pi\int_0^\infty\frac{\rho^{\frac{n}{n+1}}}{(1+\rho)^5}\mathrm{d}\rho = \pi\mathrm{I}_5^{\frac{n}{n+1}},$$

where $\mathrm{I}_p^q := \int_0^\infty \frac{\rho^p}{(1+\rho)^q}\mathrm{d}\rho$.

Using the assumption (4.7), we have

$$\det D\Gamma_0(\eta_0,0) = \frac{2(n+2)}{n+1}\pi D_0\eta_0\Big(|\Gamma|^2 - \Big|\Upsilon + \frac{n(2n+3)}{n+1}D_0\Big|^2\Big) \neq 0.$$

Hence the point $(\eta_0, 0)$ is an isolated zero of Γ_0 with nontrivial local index. By a Taylor expansion of Γ_0, using $\Gamma_0(\eta_0,0) = 0$, we can find $r_0 > 0$ small so that

$$\begin{aligned}|\Gamma_{\lambda,\mu}(\eta,\zeta)| &= |\Gamma_0(\eta,\zeta)| + o(1) \\ &= |D\Gamma_0(\eta_0,0))[\eta-\eta_0,\zeta] + O(|\eta-\eta_0|^2 + |\zeta|^2)| + o(1) \\ &\geq \frac{\nu}{2}|\eta-\eta_0,\zeta|\end{aligned}$$

for all $(\eta,\zeta) \in \partial B_r(\mu_0, 0)$ and all $r \leq r_0$, for $\frac{\lambda}{\mu} \to 0$, λ and μ sufficiently large depending on r. Then, the map $\Gamma_{\lambda,\mu}$ has in $B_{r_0}(\mu_0, 0)$ well-defined degree for all λ,μ large with $\frac{\lambda}{\mu} \to 0$, and it then coincides with the local index of Γ_0 at $(\mu_0, 0)$. In this way, the map $\Gamma_{\lambda,\mu}$ has a zero of the form $(\eta_{\lambda,\mu}, \zeta_{\lambda,\mu})$ with $(\eta_{\lambda,\mu}, \zeta_{\lambda,\mu}) \to (\mu_0, 0)$ as $\lambda,\mu \to \infty$ and $\frac{\lambda}{\mu} \to 0$. Therefore, we have solved (4.4) for $\delta(\lambda,\mu) = \big[\frac{(n+1)\lambda^{-(n+1)}}{\mathcal{H}(0)}\big]^{\frac{1}{n+2}}\eta_{\lambda,\mu}$ and $a(\lambda,\mu) = \delta(\lambda,\mu)\zeta_{\lambda,\mu}$, and the corresponding

$(\widetilde{w},\widetilde{v})=(W+\phi_{\lambda,\ \mu},\ e^{u_0+W}(1+\phi_{\lambda,\mu})+S_{\lambda,\mu})$ does solve (2.7), where $W=PU_{\delta(\lambda,\mu),\ a(\lambda,\mu),\ \sigma_{a(\lambda,\mu)}}$. Furthermore, $(u_{\lambda,\mu},N_{\lambda,\mu})=\big(\widetilde{w}+c(\widetilde{w})-\frac{\lambda}{\mu}\widetilde{v},\lambda\widetilde{v}\big)$ is a solution of (1.6).

We next show that $(u_{\lambda,\mu},N_{\lambda,\mu})$ satisfies the concentration properties stated in Theorem 4.1.

By the construction of $(\widetilde{w},\widetilde{v})$, (2.21) and (3.26), we have

$$\begin{aligned}&4N\pi\frac{e^{u_0+\widetilde{w}}}{\displaystyle\int_\Omega e^{u_0+\widetilde{w}}}+\frac{64\pi^2N^2\Big(e^{u_0+\widetilde{w}}\displaystyle\int_\Omega e^{2u_0+2\widetilde{w}}\Big(\int_\Omega e^{u_0+\widetilde{w}}\Big)^{-1}-e^{2u_0+2\widetilde{w}}\Big)}{\lambda^2\Big[\displaystyle\int_\Omega e^{u_0+\widetilde{w}}+\sqrt{\Big(\int_\Omega e^{u_0+\widetilde{w}}\Big)^2-\frac{16N\pi}{\lambda^2}\int_\Omega e^{2u_0+2\widetilde{w}}}\Big]^2}\\&-h_1(\widetilde{w}+c_-(\widetilde{w}),\widetilde{v})\rightharpoonup 8\pi(n+1)\delta_0,\end{aligned}$$

in sense of measures as $\lambda,\ \mu\to\infty$ and $\frac{\lambda}{\mu}\to 0$. Notice that the second term and the last term in the above equality are all go to zero. Using the balance condition (6.1), we get that $\lambda^2e^{u_{\lambda,\mu}+u_0}\big(1-\frac{\mathcal{N}_{\lambda,\mu}}{\lambda}\big)\rightharpoonup 8\pi(n+1)\delta_0$ and $\frac{e^{u_{\lambda,\mu}+u_0}}{\int_\Omega e^{u_{\lambda,\mu}+u_0}}\rightharpoonup\delta_0$ in the sense of measure as $\lambda,\mu\to\infty$, $\frac{\lambda}{\mu}\to 0$. Moreover, using the Lemma 2.4,

$$\lim_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}(\max_\Omega u_{\lambda,\mu})=-\infty\quad\text{and}\quad\lim_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}\frac{\|\mathcal{N}_{\lambda,\mu}\|_{L^\infty(\Omega)}}{\lambda}=0.$$

Remark 4.2 By the same argument as [17, Remark 4.8], we can show that (4.2)–(4.3) do hold in a C^1 sense.

Remark 4.3 The assumptions (4.5)–(4.7) are valid supported by some examples (see [17, Section 5].

5 A More General Result

In this section, we study the general case $m\geq 2$ in Theorem 1.1. For more clearness, let us denote the concentration point as ξ_l, $l=1,\cdots,m$, the remaining points in the vortex set as p_j and by n_l, n_j the corresponding multiplicities. Inspired by [17], we will choose W as an approximate solution of (2.8), where W is defined by (6.24), to find a solution of (2.7) in the form $(W+\phi,\ e^{u_0+W+c(W+\phi)}(1+\phi)+S)$. The approximating solution W will be introduced in detail in the next section.

By the same argument as in Lemma 2.4, using (6.35)–(6.37), we can obtain the following results.

Lemma 5.1 *Assume $|a|=O(\delta)$, then the following estimates hold:*

- $\dfrac{\int_\Omega e^{2(u_0+W)}}{\big(\int_\Omega e^{u_0+W}\big)^2}=O(\delta^{-\frac{2}{n+1}})$;
- $\|e^{W+u_0}\|_\infty=O(\delta^{-2-\frac{2}{n+1}})$;
- $e^{c(W)}=O\big(\frac{\delta^2}{\lambda^2}\big)$;
- $\|\Delta W\|_\infty=O(\delta^{-\frac{2}{n+1}})$, $\|W\|_\infty=O(|\log\delta|)$.

We restrict our attention to the case $c_0^l=0$ given by (6.21) for all $l=1,\cdots,m$, which is necessary in our context and is simply a re-formulation of the assumption that $\mathcal{H}_0$ defined by (6.19) has zero residues at $p_1,\cdots,p_m$. As in Theorem 4.1, we will work in the parameter's range:

$$a_l=o(\delta),\quad\delta\sim\lambda^{-\frac{n+1}{n+2}}$$

as $\lambda \to \infty$, where a_l given by Lemma 6.2. Since then

$$K^{-1} \le \frac{\delta^2 + |z - \xi_l|^{2n_l+2}}{\delta^2 + |\sigma_l(z) - a_l|^2} \le K, \quad K^{-1}|z - \xi_l|^{2n_l} \le |\sigma'(z)|^2 \le K|z - \xi_l|^{2n_l}$$

in $B_{2\eta}(\xi_l)$ for all $\sigma_l \in \mathcal{B}_r^l$ given by (6.25) and $l = 1, \cdots, m$, where $K > 1$, the norm (3.9) can be now simply defined as

$$\|h(z)\|_* = \sup_{z\in\Omega} \Big[\sum_{l=1}^{m} \frac{\delta^\gamma(|z-\xi_l|^{2n_l} + \delta^{\frac{2n_l}{n_l+1}})}{(\delta^2 + |z-\xi_l|^{2n_l+2})^{1+\frac{\gamma}{2}}}\Big]^{-1} |h(z)|$$

for any $h \in L^\infty(\Omega)$, where $0 < \gamma < 1$ is small fixed constant. Recall that the error term R given by (3.8), one has the following result from [17, Lemma 6.2].

Lemma 5.2 *There exists a constant $C > 0$ independent of δ such that*

$$\|R\|_* \le C\delta^{2-\gamma}. \tag{5.1}$$

As mentioned in Section 3, we can look for a solution of (2.7) in the form $(W+\phi,\ e^{u_0+W+c}(1+\phi) + S)$ by solving (3.6). In order to state the invertibility of the linear operator L_1 given by (3.3) in a suitable functional setting, for $l = 1, \cdots, m$, let us introduce the functions:

$$Z_{0l}(z) = \frac{\delta^2 - |\sigma_l(z) - a_l|^2}{\delta^2 + |\sigma_l(z) - a_l|^2}, \quad Z_l(z) = \frac{\delta(\sigma_l(z) - a_l)}{\delta^2 + |\sigma_l(z) - a_l|^2}, \quad z \in B_{2\eta}(\xi_l).$$

Also, let PZ_{0l} and PZ_l be the unique solutions with zero average of

$$\Delta PZ_{0l} = \chi_l \Delta Z_{0l} - \frac{1}{|\Omega|}\int_\Omega \chi_l \Delta Z_{0l}, \quad \Delta PZ_l = \chi_l \Delta Z_l - \frac{1}{|\Omega|}\int_\Omega \chi_l \Delta Z_l,$$

where $\chi_l(z) := \chi(|z - \xi_l|)$ defined in (6.23), and set $PZ_0 = \sum\limits_{l=1}^{m} PZ_{0l}$. According to Lemmas 5.1–5.2, as in Propositions 3.2–3.3, it is possible to prove the following result.

Proposition 5.1 *Let $M_0 > 0$. There exist $\lambda_0, \mu_0 > 0$ large such that for all $\lambda > \lambda_0$, $\mu > \mu_0$, $\frac{\lambda}{\mu} \to 0$, and δ satisfying $\lambda^{-2}\delta^{-\frac{2}{n+1}} \sim \delta^2$, for $|a| \le M_0\delta$, there exists a unique solution $\phi = \phi(\delta, a)$, $S = S(\delta, a)$, $d_0 = d_0(\delta, a) \in R$, $d_l = d_l(\delta, a) \in \mathbb{C}$, $l = 1, \cdots, m$, to*

$$\begin{cases} L_1(\phi) = -[R + N(\phi)] - h_1(W + \phi + c(W+\phi), f(\phi, S)) \\ \qquad\qquad + d_0\Delta PZ_0 + \sum\limits_{l=1}^{m} \mathrm{Re}[d_l \Delta PZ_l], & \text{in } \Omega; \\ L_2(S) := \Delta S - \mu^2 S = h_2(\phi, S), & \text{in } \Omega; \\ \displaystyle\int_\Omega \phi = \int_\Omega \phi \Delta PZ_l = 0, & l = 0, \cdots, m. \end{cases}$$

Moreover, the map $(\delta, a) \mapsto (\phi(\delta, a), S(\delta, a))$ is C^1 with

$$\|\phi\|_\infty + \frac{|\log\delta|^2}{\delta^\gamma}\|S\|_\infty \le \delta^{2-\gamma}|\log\delta|^2.$$

The function $(W + \phi,\ e^{u_0+W+c}(1+\phi) + S)$ will be a true solution of (2.7) once we adjust δ and a to have $d_l(\delta, a) = 0$ for all $l = 0, \cdots, m$. Similar to Lemma 4.1, we have the following result.

Lemma 5.3 *Let $\phi = \phi(\delta, a)$, $S = S(\delta, a)$, $d_0 = d_0(\delta, a) \in R$, $d_l = d_l(\delta, a) \in \mathbb{C}$ $(1 \le l \le m)$ be the solution of* (3.12) *given by Proposition* 5.1. *There is $\eta_0 > 0$ such that if $0 < \delta \le \eta_0$, $|a| \le \eta_0$,*

$$\int_\Omega [L_1(\phi) + N(\phi) + R + h_1(W + \phi + c(W + \phi), f(\phi, S))]PZ_l = 0 \tag{5.2}$$

holds for $l = 0, \cdots, m$, then $(W + \phi, \mathrm{e}^{u_0 + W + c(W+\phi)}(1 + \phi) + S)$ is a solution of (2.7), *i.e., $d_i(\delta, a) = 0$ for all $l = 0, \cdots, m$.*

According to [17, Lemma 6.5] and the same argument as in Proposition 4.1, we can deduce the following expansion for (5.2).

Lemma 5.4

$$\begin{aligned}
&\int_\Omega [L_1(\phi) + N(\phi) + R + h_1(W + \phi + c(W + \phi), f(\phi, S))]PZ_0 \\
&= -8\pi D_0 \delta^2 + 64(n+1)^{\frac{3n+5}{n+1}} \lambda^{-2} \delta^{-\frac{2}{n+1}} \sum_{l=1}^{m'} |\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}} \int_{\mathbb{R}^2} \frac{(|y|^2 - 1)|y + \frac{a_l}{\delta}|^{\frac{2n}{n+1}}}{(1 + |y|^2)^5} \mathrm{d}y \\
&\quad + o(\delta^2 + \lambda^{-2}\delta^{-\frac{1}{n+1}}) + O(\lambda^{-4}\delta^{-\frac{2}{n+1}} |\log \delta|^2 + \lambda^{-8}\delta^{-\frac{4}{n+1}} |\log \delta|^2)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega [L_1(\phi) + N(\phi) + R + h_1(W + \phi + c(W + \phi), f(\phi, S))]PZ_l \\
&= 4\pi\delta \sum_{l'=1}^{m} (\overline{\Upsilon^{ll'}} a_{l'} + \overline{\Gamma^{ll'}} \overline{a}_{l'}) - 64(n+1)^{\frac{3n+5}{n+1}} \lambda^{-2} \delta^{-\frac{2}{n+1}} |\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}} \chi_{M'}(l) \\
&\quad \times \int_{\mathbb{R}^2} \frac{|y + \frac{a_l}{\delta}|^{\frac{2n}{n+1}}}{(1 + |y|^2)^5} \mathrm{d}y + o(\delta^2 + \lambda^{-2}\delta^{-\frac{2}{n+1}}) + O(\lambda^{-4}\delta^{-\frac{2}{n+1}} |\log \delta|^2 \\
&\quad + \lambda^{-8}\delta^{-\frac{4}{n+1}} |\log \delta|^2),
\end{aligned}$$

where D_0 is defined in (1.7) *and $\chi_{M'}$ is the characteristic function of the set $M' = \{1, \cdots, m'\}$.*

Finally, arguing as in the proof of Theorem 4.1, we can establish Theorem 1.1 thanks to $D_0 < 0$ and the invertibility of the matrix $\mathcal{A}$.

6 Appendix

In this section, we will introduce the approximate solutions for $m = 1$ and $m \ge 2$ respectively, which have been considered in [17]. Since the choice of the approximate solution is quite complicated, and not so direct, we write it here in detail for completeness.

First we introduce an approximate solution $PU_{\delta,a,\sigma}$ for $m = 1$. Follows from Section 2, assume that $p = 0$ is present n-times in $\{p_1, \cdots, p_N\}$, and denote by $p_j's$ the remaining points in the set $\{p_1, \cdots, p_N\}$ with corresponding multiplicities $n_j's$. Recall that by Liouville formula the function

$$\log \frac{8|F'|^2}{(1 + |F|^2)^2}$$

does solve $-\Delta U = \mathrm{e}^U$ in the set $\{F' \neq 0\}$, for any holomorphic map F. It is well known that all the entire finite-solutions of (2.10) are classified as

$$U_{\delta,a}(z) = \log \frac{8(n+1)^2\delta^2}{(\delta^2 + |z^{n+1} - a|^2)^2}, \quad \delta > 0,\ a \in \mathbb{C}.$$

That is taking $F = \frac{z^{n+1}-a}{\delta}$. Moreover, we have that $\int_{\mathbb{R}^2} |z|^{2n}\mathrm{e}^{U_{\delta,a}}\mathrm{d}z = 8\pi(n+1)$. Since by construction the corresponding $\widetilde{u} = \widetilde{w} + c_-(\widetilde{w})$ will satisfy

$$\lambda^2 \mathrm{e}^{u_0+\widetilde{u}}(1 - \mathrm{e}^{u_0+\widetilde{u}}) \rightharpoonup 8\pi(n+1)\delta_0$$

in the sense of measures, the balance condition

$$2\pi N = 4\pi(n+1) \tag{6.1}$$

is necessary in view of (2.4).

To correct $U_{\delta,a}$ into a doubly-periodic function, we consider the projection $PU_{\delta,a}$ of $U_{\delta,a}$ as the solution of

$$\begin{cases} -\Delta PU_{\delta,a} = -\Delta U_{\delta,a} + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \Delta U_{\delta,a} \quad \text{in } \Omega, \\ \displaystyle\int_\Omega PU_{\delta,a} = 0. \end{cases}$$

Thus, we obtain the constant term

$$\int_\Omega \Delta U_{\delta,a} = -\int_\Omega |z|^{2n}\mathrm{e}^{U_{\delta,a}} \to -\frac{4\pi N}{|\Omega|} \quad \text{as } \delta \to 0$$

in view of (6.1), and we need to check that the difference between $-\Delta U_{\delta,a} = |z|^{2n}\mathrm{e}^{U_{\delta,a}}$ and $4\pi N \frac{|z|^{2n}\mathrm{e}^{PU_{\delta,a}}}{\int_\Omega |z|^{2n}\mathrm{e}^{PU_{\delta,a}}}$ is asymptotically small, which unfortunately is in general still not enough, we refer to [17] for detials. In [17], they took advantage of the Liouville formula to use the inner parameter σ, present in the Liouville formula, to get improved file.

Next we will introduce the approximate solution $PU_{\delta,a,\sigma}$ in [17]. Hereafter, let us fix an open simply-connected domain $\widetilde{\Omega}$ so that $\overline{\Omega} \subset \widetilde{\Omega}$ and $\widetilde{\Omega} \cap (\mathbf{a}_1\mathbb{Z} + \mathbf{a}_2\mathbb{Z}) = \{0\}$ and set $\mathcal{M}(\overline{\Omega}) = \{\sigma|_{\overline{\Omega}} : \sigma \text{ meromorphic in } \widetilde{\Omega}\}$. Let $\delta \in (0, +\infty)$, $a \in \mathbb{C}$ and $\sigma \in \mathcal{M}(\overline{\Omega})$ be a function which vanishes only at 0 with multiplicity $n+1$. Since $\log|\sigma'(z)|^2$ is harmonic in $\{\sigma' \neq 0\}$, the choice $F(z) = \frac{\sigma(z)-a}{\delta}$ yields to solutions

$$U_{\delta,a,\sigma}(z) = \log \frac{8\delta^2}{(\delta^2 + |\sigma(z) - a|^2)^2} \tag{6.2}$$

of $-\Delta U = |\sigma'(z)|^2\mathrm{e}^U$ in $\Omega \setminus \{\text{poles of } \sigma\}$, for $U_{\delta,a,\sigma}$ is a smooth function up to $\{\sigma' = 0\}$.

The guess is so to find a better local approximating function $PU_{\delta,a,\sigma}$ for a suitable choice of σ, where $PU_{\delta,a,\sigma}$ does solve

$$\begin{cases} -\Delta PU_{\delta,a,\sigma} = |\sigma'(z)|^2\mathrm{e}^{U_{\delta,a,\sigma}} - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |\sigma'(z)|^2\mathrm{e}^{U_{\delta,a,\sigma}} \quad \text{in } \Omega, \\ \displaystyle\int_\Omega PU_{\delta,a,\sigma} = 0. \end{cases} \tag{6.3}$$

Notice that $PU_{\delta,a,\sigma}$ is well-defined and smooth as long as $\sigma \in \mathcal{M}(\overline{\Omega})$, no matter σ has poles or not.

Recall that $G(z,0)$ can be thought as a doubly-periodic function in $\mathbb{C}$ with singularities on the lattice $\mathbf{a}_1\mathbb{Z}+\mathbf{a}_2\mathbb{Z}$, and $H(z)=G(z,0)+\frac{1}{2\pi}\log|z|$ is then a smooth function in 2Ω with $\Delta H=1$. Since 2Ω is simply-connected, we can find a holomorphic function H^* in 2Ω having the harmonic function $H-\frac{|z|^2}{4|\Omega|}$ as real part. Since $p_j\in\Omega$, take $\widetilde{\Omega}$ close to Ω so that $\widetilde{\Omega}-p_j\subset 2\Omega$ for all $j=1,\cdots,N$. The function

$$\mathcal{H}(z)=\prod_j(z-p_j)^{n_j}\exp\Big(4\pi(n+1)H^*(z)-2\pi\sum_{j=1}^N H^*(z-p_j)
-\frac{\pi}{2|\Omega|}\sum_{j=1}^N|p_j|^2+\frac{\pi}{|\Omega|}z\overline{\sum_{j=1}^N p_j}\Big) \tag{6.4}$$

is the holomorphic in $\widetilde{\Omega}$ with

$$|\mathcal{H}(z)|^2=\frac{1}{|z|^{2n}}\mathrm{e}^{u_0+8\pi(n+1)H(z)}=\mathrm{e}^{4\pi(n+2)H(z)-4\pi\sum_j n_jG(z,p_j)}\quad\text{in }\widetilde{\Omega} \tag{6.5}$$

in view of (6.1). The meromorphic function $\mathcal{H}_0(z)=\frac{\mathcal{H}(z)}{z^{n+2}}$ does satisfy $|\mathcal{H}_0(z)|^2=\mathrm{e}^{u_0+8\pi(n+1)G(z,0)}$ in $\widetilde{\Omega}$.

Hereafter, for a meromorphic function g in $\widetilde{\Omega}$ the notation $\int^z g(w)\mathrm{d}w$ stands for the anti-derivative of $g(z)$ which is a well-defined meromorphic function in the simply-connected domain Ω as soon as g has zero residues at each of its poles. Since $\mathcal{H}(0)\neq 0$ by (6.5), we define

$$\sigma_0(z)=-\Big(\int^z\mathcal{H}_0(w)\mathrm{e}^{-c_0w^{n+1}}\mathrm{d}w\Big)^{-1}=-\Big(\int^z\frac{\mathcal{H}(w)\mathrm{e}^{-c_0w^{n+1}}}{w^{n+2}}\mathrm{d}w\Big)^{-1}, \tag{6.6}$$

where

$$c_0=\frac{1}{\mathcal{H}(0)(n+1)!}\frac{\mathrm{d}^{n+1}\mathcal{H}}{\mathrm{d}z^{n+1}}(0) \tag{6.7}$$

guarantees that the residue of $\mathcal{H}_0(z)\mathrm{e}^{-c_0z^{n+1}}$ at 0 vanishes. By construction $\sigma_0\in\mathcal{M}(\overline{\Omega})$ vanishes only at 0 with multiplicity $n+1$, as needed, with

$$\lim_{z\to 0}\frac{z^{n+1}}{\sigma_0(z)}=\frac{\mathcal{H}(0)}{n+1}, \tag{6.8}$$

and does solve

$$|\sigma_0'|^2=|\sigma_0(z)|^4\mathrm{e}^{u_0+8\pi(n+1)G(z,0)}\mathrm{e}^{-2\mathrm{Re}[c_0z^{n+1}]} \tag{6.9}$$

in view of (6.5). Let $\sigma\in\mathcal{M}(\overline{\Omega})$ be a function which only at zero with multiplicity $n+1$. For $a\in\mathbb{C}$ small there exist $a_0,\cdots,a_n$ so that $\{z\in\widetilde{\Omega}:\sigma(z)=a\}=\{a_0,\cdots,a_n\}$ (distinct points when $a\neq 0$). For a small the function

$$\mathcal{H}_{a,\sigma}(z)=\prod_{j=1}^N(z-p_j)^{n_j}\exp\Big(4\pi\sum_{k=0}^n H^*(z-a_k)-\frac{2\pi}{|\Omega|}z\overline{\sum_{k=0}^n a_k}
-2\pi\sum_{j=1}^N H^*(z-p_j)-\frac{\pi}{2|\Omega|}\sum_{j=1}^N|p_j|^2+\frac{\pi}{|\Omega|}z\overline{\sum_{j=1}^N p_j}\Big) \tag{6.10}$$

is holomorphic in $\widetilde{\Omega}$ with

$$|\mathcal{H}_{a,\sigma}(z)|^2 = \frac{1}{|z|^{2n}} \mathrm{e}^{u_0 + 8\pi \sum\limits_{k=0}^{n} H(z-a_k) - \frac{2\pi}{|\Omega|} \sum\limits_{k=0}^{n} |a_k|^2} \tag{6.11}$$

in view of (6.1).

Endowed with the norm $\|\sigma\| := \left\|\frac{\sigma}{\sigma_0}\right\|_{\infty,\widetilde{\Omega}}$, the set $\mathcal{M}'(\overline{\Omega}) = \{\sigma \in \mathcal{M}(\overline{\Omega}) : \ \|\sigma\| < \infty\}$ is a Banach space, and let $\mathcal{B}_r$ be the closed ball centered at σ_0 and radius $r > 0$, i.e.,

$$\mathcal{B}_r = \Big\{\sigma \in \mathcal{M}(\overline{\Omega}) : \ \Big\|\frac{\sigma}{\sigma_0} - 1\Big\|_{\infty,\widetilde{\Omega}} \le r\Big\}. \tag{6.12}$$

For $a \neq 0$ and r small, the aim is to find a solution $\sigma_a \in \mathcal{B}_r$ of

$$\sigma(z) = -\Big[\int^z \Big(\frac{\sigma(w) - a}{\prod\limits_{k=0}^{n}(w - a_k)} \frac{w^{n+1}}{\sigma(w)}\Big)^2 \frac{\mathcal{H}_{a,\sigma}(w)}{w^{n+2}} \mathrm{e}^{-c_{a,\sigma} w^{n+1}} \mathrm{d}w\Big]^{-1} \tag{6.13}$$

for a suitable coefficient $c_{a,\sigma}$. To be more precise, letting

$$g_{a,\sigma}(z) = \frac{\sigma(z) - a}{\prod\limits_{k=0}^{n}(z - a_k)}$$

for $|a| < \rho$ and $\sigma \in \mathcal{B}_r$, by [17, Lemma A.1], we have that $g_{a,\sigma} \in \mathcal{M}(\overline{\Omega})$ never vanishes, and the problem above gets re-written as

$$\sigma(z) = -\Big[\int^z \frac{g^2_{a,\sigma}(w)}{g^2_{0,\sigma}(w)} \frac{\mathcal{H}_{a,\sigma}(w)}{w^{n+2}} \mathrm{e}^{-c_{a,\sigma} w^{n+1}} \mathrm{d}w\Big]^{-1}. \tag{6.14}$$

The choice

$$c_{a,\sigma} = \frac{1}{(n+1)!} \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \Big[\frac{g^2_{a,\sigma}(z)}{g^2_{a,\sigma}} \frac{g^2_{0,\sigma}(0)}{g^2_{0,\sigma}(z)} \frac{\mathcal{H}_{a,\sigma}(z)}{\mathcal{H}_{a,\sigma}(0)}\Big](0) \tag{6.15}$$

lets vanish the residue of the integrand function in (6.14) making the R.H.S. well-defined. Since $\sigma_a \in \mathcal{B}_r$, the function σ_a vanishes only at zero with multiplicity $n+1$, and satisfies

$$\begin{aligned} |\sigma'_a(z)|^2 = |\sigma_a(z) - a|^4 \exp\Big[u_0 + 8\pi \sum_{k=0}^{n} G(z, a_k) \\ - 2\pi \sum_{k=0}^{n} |a_k|^2 - 2\mathrm{Re}[c_{a,\sigma_a} z^{n+1}]\Big] \end{aligned} \tag{6.16}$$

in view of (6.1). The resolution of problem (6.15)–(6.16) has been given in [17, Appendix A]. Next we list their main result of the resolution of problem (6.15)–(6.16).

Lemma 6.1 (see [17, Lemma A.2]) *Up to take ρ smaller, there exists a C^1-map $a \in B_\rho(0) \to \sigma_a \in \mathcal{B}_r$ so that σ_a solves problem* (6.15)–(6.16). *Moreover, the map $a \in B_\rho(0) \to c_a = c_{a,\sigma_a}$ is C^1 with*

$$\begin{aligned} \Gamma &:= \mathcal{H}(0)\partial_a c_a \,|_{a=0} = \frac{1}{n!} \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} [\mathcal{H}(z) f_{n+1}(z)](0), \\ \Upsilon &:= \mathcal{H}(0)\partial_{\overline{a}} c_a \,|_{a=0} = \frac{2\pi(n+1)}{|\Omega| n!} \overline{b_{n+1} \frac{\mathrm{d}^n \mathcal{H}}{\mathrm{d}z^n}(0)}, \end{aligned} \tag{6.17}$$

where

$$f_{n+1}(z) = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}w^{n+1}}\Big[2\log\frac{w-q_0(z)}{q_0^{-1}(w)-z} + 4\pi H^*(z-q_0^{-1}(z))\Big](0),$$
$$b_{n+1} = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}q_0^{-1}}{\mathrm{d}w^{n+1}}(0).$$

And $q_0 = zQ_0^{\frac{1}{n+1}}(z)$, $Q_0(z) = \frac{\sigma_0(z)}{z^{n+1}}$ *(see* [17, *Appendix A*] *for details),* σ_0 *given by* (6.6).

Next we introduce the approximate solution $PU_{\delta_l,a_l,\sigma_{a,l}}$, for $m \geq 2$ given in [17, Section 6].

Follow the notations in Section 4, we denote the concentration point as ξ_l, $l = 1, \cdots, m$, the remaining points in the vortex set as p_j and by n_l, n_j the corresponding multiplicities. From the previous argument recall that $H(z,0) = G(z,0) + \frac{1}{2\pi}\log|z|$ is a smooth function in 2Ω with $\Delta H = \frac{1}{|\Omega|}$ and H^* is a holomorphic function in 2Ω with $\mathrm{Re}H^* = H - \frac{|z|^2}{|\Omega|}$. Up to a translation, we are assuming that $p_j \in \Omega$ for all $j = 1, \cdots, N$, and taking $\widetilde{\Omega}$ close to Ω so that $\widetilde{\Omega} - p_j \subset 2\Omega$ for all $j = 1, \cdots, N$. Arguing as for (6.4), the function

$$\mathcal{H}(z) = \prod_j (z-p_j)^{n_j} \exp\Big(4\pi\sum_{l=1}^m (n_l+1)H^*(z-\xi_l) - 2\pi\sum_{j=1}^N H^*(z-p_j)$$
$$+ \frac{\pi}{|\Omega|}\sum_{l=1}^m (n_l+1)(\xi_l - 2z)\overline{\xi_l} - \frac{\pi}{2|\Omega|}\sum_{j=1}^N |p_j|^2 + \frac{\pi}{|\Omega|}z\overline{\sum_{j=1}^N p_j}\Big)$$

is holomorphic in $\widetilde{\Omega}$ and satisfies

$$|\mathcal{H}(z)|^2 = \Big(\prod_{l=1}^m |z-\xi_l|^{-2n_l}\Big)\exp\Big(u_0 + 8\pi\sum_{l=1}^m (n_l+1)H(z-\xi_l)\Big)$$

in view of (1.8). For $l = 1, \cdots, m$, the function

$$\mathcal{H}^l(z) = \mathcal{H}(z)\prod_{l'\neq l}(z-\xi_{l'})^{-(n_l+2)}$$

is holomorphic near ξ_l and satisfies

$$|\mathcal{H}^l(z)|^2 = \exp\Big(4\pi(n_l+2)H(z-\xi_l) + 4\pi\sum_{l'\neq l}(n_{l'}+2)G(z,\xi_{l'})$$
$$- 4\pi\sum_j n_j G(z,p_j)\Big). \tag{6.18}$$

Setting

$$\mathcal{H}_0 = \frac{\mathcal{H}}{(z-\xi_1)^{n_1+2}\dots(z-\xi_m)^{n_m+2}}, \tag{6.19}$$

we now define σ_0 as

$$\sigma_0(z) = -\Big(\int^z \mathcal{H}_0(w)\exp\Big[-\sum_{l=1}^m c_0^l(w-\xi_l)^{n_l+1}\prod_{l'\neq l}(w-\xi_l)^{n_{l'}+2}\Big]\mathrm{d}w\Big)^{-1}, \tag{6.20}$$

where

$$c_0^l = \frac{1}{\mathcal{H}_0(\xi_l)(n_l+1)!} \frac{\mathrm{d}^{n_l+1}\mathcal{H}^l}{\mathrm{d}z^{n_l+1}}(\xi_l), \quad l = 1, \cdots, m \tag{6.21}$$

guarantees that all the residues of the integrand function in the definition of σ_0 vanish. The presence of the term $\prod\limits_{l'\neq l}(w-\xi_l)^{n_{l'}+2}$ is crucial to compute explicitly the c_0^l's for

$$c_0^l(w-\xi_l)^{n_l+1}\prod_{l'\neq l}(w-\xi_l)^{n_{l'}+2} = O((w-\xi_{l'})^{n_{l'}+2})$$

has a high-order effect near any other $\xi_{l'}$, $l' \neq l$. By construction $\sigma_0 \in \mathcal{M}(\overline{\Omega})$ vanishes only at the ξ_l's with multiplicity n_l+1 and

$$\lim_{z\to\xi_l} \frac{(z-\xi_l)^{n_l+1}}{\sigma_0(z)} = \frac{\mathcal{H}^l(\xi_l)}{n_l+1},$$

and satisfies

$$\begin{aligned}|\sigma_0'(z)|^2 = |\sigma_0(z)|^4 \exp\Big(&u_0 + 8\pi\sum_{l=1}^m (n_l+1)G(z,\xi_l) \\ &- 2\sum_{l=1}^m \mathrm{Re}\Big[c_0^l(z-\xi_l)^{n_l+1}\prod_{l'\neq l}(z-\xi_{l'})^{n_{l'}+2}\Big]\Big).\end{aligned}$$

Under the assumptions of Theorem 1.1, notice that $c_0^l = 0$ for all $l = 1, \cdots, m$ and

$$\Big|\Big(\frac{1}{\sigma_0}\Big)'(z)\Big|^2 = |\mathcal{H}_0(z)|^2 = \mathrm{e}^{u_0 + 8\pi\sum\limits_{l=1}^m (n_l+1)G(z,\xi_l)}.$$

Since each ξ_l gives a contribution to the dimension of the kernel for the linearized operator (3.3), the parameters δ and a are no longer enough to recover all the degeneracies induced by the ansatz $PU_{\delta,a,\sigma}$ for $\sigma \in \mathcal{M}(\overline{\Omega})$ a function which vanishes only at the points ξ_l, $l = 1, \cdots, m$, with multiplicity n_l+1. In our construction, the correct number of parameters to use is $2m+1$, given by m small complex numbers $a_1, \cdots, a_m$ and $\delta > 0$ small, where the latter gives rise to the concentration parameter δ_l at ξ_l, $l = 1, \cdots, m$, by means of (6.38). The request that all the δ_l's tend to zero with the same rate is necessary as we will discuss later.

We need to construct an ansatz that looks as $PU_{\delta_l,a_l,\sigma_{a,l}}$ near each ξ_l, for a suitable $\sigma_{a,l}$ which makes the approximation near ξ_l good enough. In order to localize our previous construction, let us define $PU_{\delta_l,a_l,\sigma}$ as the solution of

$$\begin{cases} -\Delta PU_{\delta_l,a_l,\sigma} = \chi(|z-\xi_l|)|\sigma'(z)|^2\mathrm{e}^{U_{\delta_l,a_l,\sigma}} - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \chi(|z-\xi_l|)|\sigma'(z)|^2\mathrm{e}^{U_{\delta_l,a_l,\sigma}} & \text{in } \Omega, \\ \displaystyle\int_\Omega PU_{\delta_l,a_l,\sigma} = 0, \end{cases} \tag{6.22}$$

where χ is a smooth radial cut-off function so that

$$\chi = \begin{cases} 1 & \text{in } [-\eta,\eta], \\ 0 & \text{in } (-\infty,-2\eta] \cup [2\eta,+\infty), \end{cases} \tag{6.23}$$

and $0 < \eta < \frac{1}{2}\min\{|\xi_l - \xi_l'|, \mathrm{dist}(\xi_l, \partial\Omega) : l, l' = 1, \cdots, m, l \neq l'\}$. The approximating function is then built as

$$W = \sum_{l=1}^{m} PU_l, \tag{6.24}$$

where $U_{\delta_l, a_l, \sigma_{a,l}}$ and $PU_{\delta_l, a_l, \sigma_{a,l}}$ will be simply denoted by U_l and PU_l.

Let us now explain how to find the solution $\sigma_{a,l}$, $l = 1, \cdots, m$. Setting

$$\mathcal{B}_r^l = \Big\{\sigma \text{ holomorphic in } B_{2\eta}(\xi_l) : \ \Big\|\frac{\sigma}{\sigma_0} - 1\Big\|_{\infty, B_{2\eta}(\xi_l)} \leq r\Big\} \tag{6.25}$$

for $l = 1, \cdots, m$, [17, Lemma A.1] still holds in this context for all $\sigma \in \mathcal{B}_r^l$, by simply replacing 0, n with ξ_l, n_l and $\widetilde{\Omega}$ with $B_{2\eta}(\xi_l)$. Then, for all $\sigma = (\sigma_1, \cdots, \sigma_m) \in \mathcal{B}_r := \mathcal{B}_r^1 \times \cdots \times \mathcal{B}_r^m$ and $a = (a_1, \cdots, a_m) \in \mathbb{C}^m$ with $\|a\|_\infty < \rho$, there exist points a_i^l, $l = 1, \cdots, m$ and $i = 0, \cdots, n_l$, so that $\{z \in B_{2\eta}(\xi_l) : \sigma_l(z) = a_l\} = \{\xi_l + a_0^l, \cdots, \xi_l + a_{n_l}^l\}$ for all $l = 1, \cdots, m$. Arguing as for (6.10) and $l = 1, \cdots, m$, the function

$$\begin{aligned} H_{a,\sigma}^l(z) = {} & \prod_j (z - p_j)^{n_j} \prod_{l' \neq l} (z - \xi_{l'})^{n_{l'}} \prod_{l' \neq l} \prod_{i=0}^{n_{l'}} (z - \xi_{l'} - a_i^{l'})^{-2} \\ & \times \exp\Big(4\pi \sum_{l'=1}^{m} \sum_{i=0}^{n_{l'}} H^*(z - \xi_{l'} - a_i^{l'}) - 2\pi \sum_{j=1}^{N} H^*(z - p_j) + \frac{\pi}{|\Omega|} \sum_{l'=1}^{m} (\xi_{l'} - 2z)\overline{\xi_{l'}} \\ & - \frac{\pi}{2|\Omega|} \sum_{j=1}^{N} |p_j|^2 - \frac{2\pi}{|\Omega|} \sum_{l'=1}^{m} (z - \xi_{l'}) \overline{\sum_{i=0}^{n_{l'}} a_i^{l'}} + \frac{\pi}{|\Omega|} z \overline{\sum_{j=1}^{N} p_j}\Big) \end{aligned}$$

is holomorphic near ξ_l and satisfies

$$\begin{aligned} |\mathcal{H}_{a,\sigma}^l(z)|^2 = {} & |z - \xi_l|^{-2n_l} \exp\Big[u_0 + 8\pi \sum_{i=0}^{n_l} H(z - \xi_l - a_i^l)\Big] \\ & + \sum_{l' \neq l} \sum_{i=0}^{n_{l'}} G(z, \xi_{l'} + a_i^{l'}) - \frac{2\pi}{|\Omega|} \sum_{l'=1}^{m} \sum_{i=0}^{n_{l'}} |a_i^{l'}|^2 \end{aligned} \tag{6.26}$$

in view of (1.8). Setting

$$g_{a_l, \sigma_l(z)}^l(z) = \frac{\sigma_l(z) - a_l}{\prod_{i=0}^{n_l} (z - \xi_l - a_i^l)}, \quad z \in B_{2\eta}(\xi_l)$$

and

$$c_{a,\sigma}^l = \frac{\prod_{l' \neq l} (\xi_l - \xi_{l'})^{-(n_{l'}+2)}}{(n_l + 1)!} \frac{\mathrm{d}^{n_l+1}}{\mathrm{d}z^{n_l+1}} \Big[\Big(\frac{g_{a_l,\sigma_l}^l(z) g_{0,\sigma_l}^l(\xi_l)}{g_{a_l,\sigma_l}^l(\xi_l) g_{0,\sigma_l}^l(z)}\Big)^2 \frac{\mathcal{H}_{a,\sigma}^l(z)}{\mathcal{H}_{a,\sigma}^l(\xi_l)}\Big](\xi_l), \tag{6.27}$$

the aim is to find a solution $\sigma_a = (\sigma_{a,1}, \cdots, \sigma_{a,m}) \in \mathcal{B}_r$ of the system ($l = 1, \cdots, m$):

$$\begin{aligned} \sigma_l(z) = {} & -\Big(\int^z \Big(\frac{g_{a_l,\sigma_l}^l(w)}{g_{0,\sigma_l}^l(w)}\Big)^2 \frac{\mathcal{H}_{a,\sigma}^l(w)}{(w - \xi_l)^{n_l+2}} \\ & \times \exp\Big[-\sum_{l'=1}^{m} c_{a,\sigma}^{l'} (w - \xi_{l'})^{n_{l'}+1} \prod_{l'' \neq l'} (w - \xi_{l''})^{n_{l''}+2}\Big] \mathrm{d}w\Big)^{-1}, \end{aligned} \tag{6.28}$$

where the definition of $c^l_{a,\sigma}$ makes null the residue at ξ_l of the integrand function in (6.27). The function $\sigma_{a,l}$ will vanish only at ξ_l with multiplicity n_l+1 and satisfy

$$\begin{aligned}|\sigma'_{a,l}(z)|^2 = |\sigma_{a,l}(z)-a_l|^4 \exp\Big(u_0 + 8\pi\sum_{l'=1}^{m}\sum_{i=0}^{n_{l'}} G(z,\xi_{l''}+a_i^{l'})\\ - \frac{2\pi}{|\Omega|}\sum_{l'=1}^{m}\sum_{i=0}^{n'_l}|a_i^{l'}|^2 - 2\sum_{l'=1}^{m}\mathrm{Re}\Big[c^{l'}_{a,\sigma_a}(z-\xi_{l'})^{n_{l'}+1}\prod_{l''\neq l'}(z-\xi_{l''})^{n_{l''}+2}\Big]\Big)\end{aligned} \tag{6.29}$$

in view of (6.26).

Since $\mathcal{H}^l_{0,\sigma}=\mathcal{H}_l$ and $c^l_{0,\sigma}=c^l_0$ for all $l=1,\cdots,m$, when $a=0$ the system (6.27) is reduced to m-copies of (6.20) in each $B_{2\eta}(\xi_l)$ $(l=1,\cdots,m)$ and it is natural to find σ_a branching off $(\sigma_0,\cdots,\sigma_0)$ for a small by IFT. Let us emphasize that each $\sigma_{a,l}(l=1,\cdots,m)$ is close to $\sigma_0|_{B_{2\eta}(\xi_l)}$, a crucial property to have D_0 defined in terms of a unique σ_0 (see (1.7)). Letting $q_{0,l}$ be the function so that $\sigma_0=q_{0,l}^{n_l+1}$ near ξ_l, we have the following result.

Lemma 6.2 (see [17, Lemma 6.1]) *Up to take ρ smaller, there exists a C^1-map $a\in B_\rho(0)\to\sigma_a\in\mathcal{B}_r$ so that σ_a solves the system* (6.27)–(6.28). *Moreover, the map $a\in B_\rho(0)\to c^l_a := c^l_{a,\sigma_a}$ is C^1 with*

$$\Gamma^{ll} := \mathcal{H}(\xi_l)\partial_{a_l}c^l_a\,|_{a=0} = \frac{1}{n_l!}\frac{\mathrm{d}^{n_l+1}}{\mathrm{d}z^{n_l+1}}[\mathcal{H}^l(z)f^l_{n_l+1}(z)](\xi_l), \tag{6.30}$$

$$\Upsilon^{ll} := \mathcal{H}(\xi_l)\partial_{\overline{a}_l}c^l_a\,|_{a=0} = \frac{2\pi(n_l+1)}{|\Omega|n_l!}\overline{b^l_{n_l+1}\frac{\mathrm{d}^{n_l}\mathcal{H}^l}{\mathrm{d}z^{n_l}}}(\xi_l), \tag{6.31}$$

and for $j\neq l$,

$$\Gamma^{lj} := \mathcal{H}(\xi_l)\partial_{a_j}c^l_a\,|_{a=0} = \frac{1}{n_l!}\frac{\mathrm{d}^{n_j+1}}{\mathrm{d}z^{n_l+1}}[\mathcal{H}^l(z)\widetilde{f}^j_{n_j+1}(z)](\xi_l), \tag{6.32}$$

$$\Upsilon^{lj} := \mathcal{H}(\xi_l)\partial_{\overline{a}_j}c^l_a\,|_{a=0} = \frac{2\pi(n_j+1)}{|\Omega|n_l!}\overline{b^j_{n_j+1}\frac{\mathrm{d}^{n_l}\mathcal{H}^l}{\mathrm{d}z^{n_l}}}(\xi_l), \tag{6.33}$$

where

$$f^l_{n+1}(z) = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}w^{n+1}}\Big[2\log\frac{w-q_{0,l}(z)}{q_{0,l}^{-1}(w)-z} + 4\pi H^*(z-q_{0,l}^{-1}(w))\Big](0),$$

$$b^l_{n+1} = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}q_{0,l}^{-1}}{\mathrm{d}w^{n+1}}(0),$$

and for $j\neq l$

$$\widetilde{f}^j_{n+1} = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}w^{n+1}}[-2\log(z-q_{0,j}^{-1}(w) + 4\pi H^*(z-q_{0,j}^{-1}(w)))](0).$$

Letting $n=\min\{n_l : l=1,\cdots,m\}$, up to re-ordering, assume that $n=n_1=\cdots=n_{m'}<n_l$ for all $l=m'+1,\cdots,m$, where $1\le m'\le m$. The matrix $\mathcal{A}$ in Theorem 1.1 is the $2m\times 2m$-matrix in the form

$$\mathcal{A} := \begin{pmatrix} A^{1,2}_{1,2} & \cdots & A^{2m-1,2m}_{1,2}\\ \vdots & \vdots & \vdots \\ A^{1,2}_{2m-1,2m} & \cdots & A^{2m-1,2m}_{2m-1,2m}\end{pmatrix}, \tag{6.34}$$

where the 2×2-blocks $A^{2l'-1,2l'}_{2l-1,2l}$ are given by

$$\begin{pmatrix} \mathrm{Re}\Big[\Gamma^{ll'}+\Upsilon^{ll'}+\frac{n(2n+3)D_0\delta_{ll'}|\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}}}{(n+1)\sum\limits_{j=1}^{m'}|\mathcal{H}^j(\xi_j)|^{-\frac{2}{n+1}}}\Big] & \mathrm{Im}[\Upsilon^{ll'}-\Gamma^{ll'}] \\ \mathrm{Im}[\Upsilon^{ll'}+\Gamma^{ll'}] & \mathrm{Re}\Big[\Gamma^{ll'}-\Upsilon^{ll'}-\frac{n(2n+3)D_0\delta_{ll'}|\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}}}{(n+1)\sum\limits_{j=1}^{m'}|\mathcal{H}^j(\xi_j)|^{-\frac{2}{n+1}}}\Big] \end{pmatrix},$$

when $l=m'+1,\cdots,m$, with $\Gamma^{ll'}$ and $\Upsilon^{ll'}$ given by (6.30), (6.32) and (6.31), (6.33), respectively, and $\delta_{ll'}$ the Kronecker's symbol.

Next we list some expansions of the approximate solution $W=\sum\limits_{l=1}^{m}PU_{\delta_l,a_l,\sigma_l}$ from [17, Section 6].

Lemma 6.3 *For $l=1,\cdots,m$, the following expansions hold:*

$$\begin{aligned} PU_{\delta_l,a_l,\sigma_l} &= \chi(|z-\xi_l|)[U_{\delta_l,a_l,\sigma_l}-\log(8\delta^2)+4\log|g^l_{a_l,\sigma_l}|] \\ &\quad + 8\pi\sum_{i=0}^{n_l}\Big[\frac{1}{2\pi}(\chi(|z-\xi_l|)-1)\log|z-\xi_l-a^l_i|+H(z-\xi_l-a^l_i)\Big] \\ &\quad + \theta_{\delta_l,a_l,\sigma_l}+2\delta_l^2 f_{a_l,\sigma_l}+O(\delta_l^4) \end{aligned}$$

and

$$\begin{aligned} PU_{\delta_l,a_l,\sigma_l} &= 8\pi\sum_{i=0}^{n_l}G(z,\xi_l+a^l_i)+\theta_{\delta_l,a_l,\sigma_l} \\ &\quad + 2\delta_l^2\Big(f_{a_l,\sigma_l}-\frac{\chi(|z-\xi_l|)}{|\sigma_l(z)-a_l|^2}\Big)+O(\delta_l^4) \end{aligned}$$

in $C(\overline{\Omega})$ and $C_{loc}(\overline{\Omega}\setminus\{\xi_l\})$, respectively, uniformly for $|a|<\rho$ and $\sigma_l\in\mathcal{B}^l_r$, where

$$\theta_{\delta_l,a_l,\sigma_l}=-\frac{1}{|\Omega|}\int_\Omega \chi(|z-\xi_l|)\log\frac{|\sigma_l(z)-a_l|^4}{(\delta_l^2+|\sigma_l(z)-a_l|^2)^2}$$

and f_{a_l,σ_l} is a smooth function in z (with a uniform control in a_l and σ_l of it and its derivatives in z).

Choosing $\sigma_l=\sigma_{a,l}$ and summing up over $l=1,\cdots,m$, by (6.29) for our approximate function there hold

$$\begin{aligned} W &= U_{\delta_l,a_l,\sigma_l}-\log(8\delta_l^2)+\log|\sigma_l'|^2-u_0+\frac{2\pi}{|\Omega|}\sum_{i=0}^{n_{l'}}|a^{l'}_i|^2+\theta^l(a,\delta) \\ &\quad + 2\mathrm{Re}\Big[c^l_{a,\sigma_l}(z-\xi_l)^{n_l+1}\prod_{l'\neq l}(z-\xi_{l'})^{n_{l'}+2}\Big] \\ &\quad + O(|z-\xi_l|^{n_l+2}\sum_{l'\neq l}|c^{l'}_{a,\sigma_{l'}}|)+\sum_{l'=1}^{m}O(\delta_{l'}^2|z-\xi_l|+\delta_{l'}^4) \end{aligned} \tag{6.35}$$

and

$$W=8\pi\sum_{l=1}^{m}\sum_{i=0}^{n_l}G(z,\xi_l+a^l_i)+O\Big(\sum_{l'=1}^{m}\delta_{l'}^2\log|\delta_{l'}|\Big)$$

uniformly in $B_\eta(\xi_l)$ and in $\Omega\setminus\cup_{l=1}^m B_\eta(\xi_l)$, respectively, where

$$\Theta^l(a,\delta):=\sum_{l'=1}^m[\Theta_{\delta_{l'},a_{l'},\sigma_{l'}}+\delta_{l'}^2 f_{a_{l'},\sigma_{l'}}(\xi_l)].$$

As a consequence, we have that

$$\begin{aligned}\int_\Omega \mathrm{e}^{u_0+W}&=\sum_{l'=1}^m\Big[\int_{B_\rho(0)}\frac{n_{l'}+1}{(\delta_{l'}^2+|y-a_{l'}|^2)^2}+o\Big(\frac{1}{\delta_{l'}^2}\Big)\Big]\\&=\pi\sum_{l'=1}^m\frac{n_{l'}+1}{\delta^2}[1+o(1)]\end{aligned}\tag{6.36}$$

and then near ξ_l there holds

$$4\pi N\frac{\mathrm{e}^{u_0+W}}{\displaystyle\int_\Omega \mathrm{e}^{u_0+W}}=4\pi N\frac{|\sigma_l'|^2\mathrm{e}^{U_{\delta_l,a_l,\sigma_l}+O(|z-\xi_l|^{n_l+1})+o(1)}}{8\pi\sum\limits_{l'=1}^m(n_{l'}+1)\delta_l^2\delta_{l'}^{-2}(1+o(1))}.\tag{6.37}$$

In order to construct an N-condensate $(u_{\lambda,\mu},\mathcal{N}_{\lambda,\mu})$ which satisfies (1.6) as $\lambda,\mu\to+\infty$ and $\frac{\lambda}{\mu}\to 0$, we look for a solution of (2.7) in the form $(\widetilde{w},\widetilde{v})=(W+\phi,\ \mathrm{e}^{u_0+W+c(W+\phi)}(1+\phi)+S)$, where ϕ,S are the small remained terms, $W=\sum\limits_{l=1}^m PU_{\delta_l,a_l,\sigma_l}$ and $\delta_l(\lambda,\mu)$, $a_l=a_l(\lambda,\mu)$ are suitable small parameters, so that

$$\begin{aligned}&4N\pi\frac{\mathrm{e}^{u_0+\widetilde{w}}}{\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}}+\frac{64\pi^2N^2\Big(\mathrm{e}^{u_0+\widetilde{w}}\displaystyle\int_\Omega \mathrm{e}^{2u_0+2\widetilde{w}}\Big(\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}\Big)^{-1}-\mathrm{e}^{2u_0+2\widetilde{w}}\Big)}{\lambda^2\Big[\displaystyle\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}+\sqrt{\Big(\int_\Omega \mathrm{e}^{u_0+\widetilde{w}}\Big)^2-\frac{16N\pi}{\lambda^2}\int_\Omega \mathrm{e}^{2u_0+2\widetilde{w}}}\Big]^2}\\&\rightharpoonup 8\pi\sum_{l=1}^m(n_l+1)\delta_{\xi_l}\end{aligned}$$

in the sense of measures as $\lambda,\mu\to\infty$ and $\frac{\lambda}{\mu}\to 0$. Since $|\sigma'(z)|^2\mathrm{e}^{U_{\delta_l,a_l,\sigma_l}}\rightharpoonup 8\pi(n_l+1)\delta_{\xi_l}$ as $\delta_l,a_l\to 0$, to have the correct concentration property we need that

$$8\pi\sum_{l'=1}^m(n_{l'}+1)\delta_l^2\delta_{l'}^{-2}\to 4\pi N$$

for all $l=1,\cdots,m$, and then $\frac{\delta_l}{\delta_{l'}}\to 1$ for all $l,l'=1,\cdots,m$ in view of (1.8). It is natural to introduce just one parameter δ and to choose the δ_l's as

$$\delta_l=\delta,\quad l=1,\cdots,m.\tag{6.38}$$

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Abrikosov, A. A., On the magnetic properties of superconductors of the second group, *Sov. Phys. JETP*, **5**, 1957, 1174–1182.

[2] Ao, W. W., Kwon, O. and Lee, Y., Periodic Maxwell-Chern-Simons vortices with concentration property, *Math. Ann.*, **381**(3–4), 2021, 1885–1942.

[3] Ao, W. W., Kwon, O. and Lee, Y., Multi-bubbling condenstes for the Maxwell-Chern-Simons model, *Calc. Var. Partial Differential Equations*, **61**(1), 2022, 34, 19 pp.

[4] Aref, H. and Stremler, M. A, On the motion of three point vortices in a periodic strip, *J. Fluid Mech.*, **314**, 1996, 1–25.

[5] Bethuel, F., Brezis, H. and Hélein, F., Ginzburg-Landau Vortices, Birkhäuser Boston, Inc. Boston. MA, 1994.

[6] Bogomol'nyĭ, E., The stability of classical solutions, *Sov. J. Nucl. Phys.*, **24**, 1976, 449–454.

[7] Boutet de Monvel-Berthier, A., Georgescu, V. and Purice, R., A boundary value problem related to the Ginzburg-Landau model, *Comm. Math. Phys.*, **142**, 1991, 1–23.

[8] Caffarelli, L. A. and Yang, Y. S., Vortex condensation in Chern-Simons-Higgs model: An existence theorem, *Comm. Math. Phys.*, **168**, 1995, 321–336.

[9] Chae, D. and Imanuvilov, Y., The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory, *Comm. Math. Phys.*, **215**, 2000, 119–142.

[10] Chae, D. and Kim, N., Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs system, *J. Differential Equations*, **134**, 1997, 154–182.

[11] Chae, D. and Kim, N., Vortex condensates in the relativistic self-dual Maxwell-Chern-Simons-Higgs system, 1997, arXiv: cond-mat/9711211.

[12] Chan, H., Fu, C. C. and Lin, C. S., Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation, *Comm. Math. Phys.*, **231**, 2002, 189–221.

[13] Choe, K., Uniqueness of the topological multivortex solution in the selfdual Chern-Simons theory, *J. Math. Phys.*, **46**, 2005, 012305, 22 pp.

[14] Choe, K., Asymptotic behavior of condensate solutions in the Chern-Simons-Higgs theory, *J. Math. Phys.*, **48**, 2007, 103501.

[15] Choe, K. and Kim, N., Blow-up solutions of the self-dual Chern-Simons-Higgs vortex equation, *Ann. Inst. H. Poincaré Anal. Non Linaire*, **25**, 2008, 313–338.

[16] Choe, K., Kim, N. and Lin, C. S., Existence of self-dual non-topological solutions in the Chren-Simons-Higgs model, *Ann. Inst. H. Poincaré Anal. Non Linaire*, **28**, 2011, 837–852.

[17] Del Pino, M. A., Esposito, P., Figueroa, P. and Musso, M., Nontopological condensates for the self-dual Chern-Simons-Higgs model, *Comm. Pure Appl. Math.*, **68**, 2015, 1191–1283.

[18] Ding, W. Y., Jost, J., Li, J. Y., et al., Self-duality equations for Ginzburg-Landau and Seiberg-Witten type functionals with 6th order potentials, *Comm. Math. Phys.*, **217**, 2001, 383–407.

[19] Ding, W. Y., Jost, J., Li, J. Y. and Wang, G. F., An analysis of the two-vortex case in the Chern-Simons Higgs model, *Calc. Var. Partial Differential Equations*, **7**, 1989, 87–97.

[20] Dunne, G., Self-dual Chern-Simons Theories, Lecture Notes in Physics, Monographs, Springer-Verlag, New York, 1995.

[21] Fan, Y. W., Lee, Y. and Lin, C. S., Mixed type solutions of the $SU(3)$ models on a torus, *Comm. Math. Phys.*, **343**(1), 2016, 233–271.

[22] Han, J., Topological solutions in the self-dual Chern-Simons-Higgs theory in a background metric, *Lett. Math. Phys.*, **65**, 2003, 37–47.

[23] Han, J., Asymptotic limit for condensate solutions in the Abelian Chern-Simons Higgs model, *Proc. Amer. Math. Soc.*, **131**(6), 2003, 1839–1845 .

[24] Han, J., Asymptotic limit for condensate solutions in the Abelian Chern-Simons Higgs model II, *Proc. Amer. Math. Soc.*, **131**(12), 2003, 3827–3832.

[25] Han, J. and Jang, J., Self-dual Chern-Simons vortices on bounded domains, *Lett. Math. Phys.*, **64**, 2003, 45–56.

[26] Han, J. and Kim, N., Nonself-dual Chern-Simons and Maxwell-Chern-Simons vortices on bounded domains, *J. Funct. Anal.*, **221**(1), 2005, 167–204.

[27] Hong, J., Kim, Y. and Pac, P. Y., Multivortex Solutions of the Abelian Chern-Simons-Higgs Theory, *Phys. Rev. Lett.*, **64**, 1990, 2230–2233.

[28] Jackiw, R. and Weinberg, E. J., Self-dual Chen-Simons vortices, *Phys. Rev. Lett.*, **64**, 1990, 2234–2237.

[29] Jaffe, A. and Taubes, C. H., Vortices and Monopoles, Birkhäuser, Boston, MA, 1980.

[30] Kim, S., Solitons of the self-dual Chern-Simons theory on a cylinder, *Lett. Math. Phys.*, **61**, 2002, 113–122.

[31] Kim, S. and Kim, Y., Self-dual Chern-Simons vortices on Riemann surfaces, *J. Math. Phys.*, **43**, 2002, 2355–2362.

[32] Kurata, K., Existence of nontopological solutions for a nonlinear elliptic equation from Chern-Simons-Higgs theory in a general background metric, *Differential Integral Equations*, **14**, 2001, 925–935.

[33] Landau, L. and Lifschitz, E., The Classical Theory of Fields, Addison-Wesley, Cambridge MA, 1951.

[34] Lee, C., Lee, K. and Min, H., Self-dual Maxwell-Chern-Simons solitons, *Phys. Lett. B*, **252**, 1990, 79–83.

[35] Lee, Y., Bubbling mixed type solutions of the $SU(3)$ models on a torus, *J. Math. Phys.*, **58**, 2017, 111508, 16 pp.

[36] Lin, C. S. and Yan, S. S., Bubbling solutions for the $SU(3)$ Chern-Simons Model on a torus, *Comm. Pure Appl. Math.*, **66**, 2013, 991–1027.

[37] Lin, C. S. and Yan, S. S., Existence of bubbling solutions for Chern-Simons model on a torus, *Arch. Ration. Mech. Anal.*, **207**, 2013, 353–392.

[38] Nielsen, H. and Olesen, P., Vortex-Line models for dual strings, *Nucl. Phys. B*, **61**, 1973, 45–61.

[39] Nolasco, M. and Tarantello, G., On a sharp type inequality on two dimensional compact manifolds, *Arch. Rational Mech. Anal.*, **145**, 1998, 161–195.

[40] Nolasco, M. and Tarantello, G., Double vortex condensates in the Chern-Simons-Higgs theory, *Calc. Var. Partial Differential Equations*, **9**, 1999, 31–94.

[41] Pacard, F. and Riviere, T., Linear and nonlinear aspects of vortices, The Ginzburg-Landau model, Progress in Nonlinear Differential Equations and their Applications, **39**, Birkhauser Boston, Inc., Boston, MA, 2000.

[42] Ricciardi, T., Asymptotics for Maxwell-Chern-Simons multivortices, *Nonlinear Anal.*, **50**, 2002, 1093–1106.

[43] Ricciardi, T. and Tarantello, G., Vortices in the Maxwell-Chern-Simons theory, *Comm. Pure Appl. Math.*, **53**, 2000, 811–851.

[44] Schiff, J., Integrability of Chern-Simons-Higgs and Abelian Higgs vortex equations in a background metric, *J. Math. Phys.*, **32**, 1991, 753–761.

[45] Spruck, J. and Yang, Y. S., The existence of nontopological solitons in the self-dual Chern-Simons theory, *Comm. Math. Phys.*, **149**, 1992, 361–376.

[46] Spruck, J. and Yang, Y. S., Topological solutions in the self-dual Chern-Simons theory, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, **12**, 1995, 75–97.

[47] Stremler, M., Relative equilibria of singly periodic point vortex arrays, *Phys. Fluids.*, **12**, 2003, 3767–3775.

[48] Struwe, M. and Tarantello, G., On multivortex solutions in Chern-Simons gauge theory, *Boll. Uni. Mat. Ital. Sez. B Artic. Ric. Mat.*, **8**(1), 1998, 109–121.

[49] Tarantello, G., Multiple condensate solutions for the Chern-Simons-Higgs theory, *J. Math. Phys.*, **37**, 1996, 3769–3796.

[50] Tarantello, G., Selfdual Maxwell-Chern-Simons vortices, *Milan J. Math.*, **72**, 2004, 29–80.

[51] Tarantello, G., Selfdual Gauge field vortices, An analytical approach, Progress in Nonlinear Differential Equations and their Applications, Birkhauser Boston, Inc., Boston, 2008.

[52] Taubes, C. H., Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations, *Comm. Math. Phys.*, **72**(3), 1980, 277–292.

[53] 't Hooft, G., A property of electric and magnetic flux in non-Abelian gauge theories, *Nucl. Phys. B*, **153**, 1979, 141–160.

[54] Wang, R. G., The existence of Chern-Simons vortices, *Comm. Math. Phys.*, **137**, 1991, 587–597.

[55] Wang, S. and Yang, Y. S., Abrikosov's vortices in the critical coupling, *SIAM J. Math. Anal.*, **23**, 1992, 1125–1140.

[56] Yang, Y. S., Solitons in Field Theory and Nonlinear Analysis, Springer Monograph in Mathematics, Springer-Verlag, New York, 2001.