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Number of Singular Points on Projective Surfaces*

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Abstract The number of singular points on a klt Fano surface X is less than or equal to $2\rho(X)+2$.

Keywords Singular points, Fano variety, Canonical bundle formula
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1 Introduction

We work over the field $\mathbb{C}$ of complex numbers. For any normal projective variety X, let $\rho(X)$ be the Picard number of X.

Let X be a klt Fano surface, i.e., a klt projective surface such that $-K_X$ is ample. It is interesting to ask when the number of singular points of X is bounded from above, and to give an estimate of the maximal number of singular points on X.

For simplicity, for any surface X, let $n(X)$ be the number of singular points on X. When X is klt Fano, Keel and McKernan showed that $n(X)\leq 5$ when $\rho(X)=1$ (see [20, p. 72]). This is strengthened by Belousov who showed that $n(X)\leq 4$.

Theorem 1.1 (see [1, Theorem 1.2, 2, Theorem 1.1]) *Let X be a klt Fano surface such that $\rho(X)=1$. Then $n(X)\leq 4$.*

This bound is optimal even for Fano surfaces with canonical singularities by [23] (see also [12, 27–29] and Example 4.2(1)). In this note, we show that $n(X)$ is bounded from above by a number depending only on $\rho(X)$.

Theorem 1.2 *Let X be a klt Fano surface. Then $n(X)\leq 2\rho(X)+2$.*

It is easy to see that Theorems 1.1–1.2 are equivalent when $\rho(X)=1$.

In fact, we can relax the assumption "klt Fano" to "(X,B) is klt log Calabi-Yau for some boundary $B\neq 0$" without changing the bound $2\rho(X)+2$. Moreover, we can relax the assump-

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tion "klt Fano" to "(X, B) is lc and $-(K_X + B)$ is nef for some boundary B" if we allow a small increase on the bound $2\rho(X) + 2$. We have the following result.

Theorem 1.3 *Let (X, B) be an lc surface pair such that $-(K_X + B)$ is nef. Then*

(1) $n(X) \leq \max\{2\rho(X) + 10, 16\}$.

(2) *If X is of Fano type, then $n(X) \leq 2\rho(X) + 2$.*

(3) *If X is klt and $K_X \not\equiv 0$, then $n(X) \leq 2\rho(X) + 4$.*

(4) *If X is klt but not canonical and $K_X \equiv 0$, then $n(X) \leq 2\rho(X) + 7$.*

(5) *If X is canonical and $K_X \equiv 0$, then $n(X) \leq 16$.*

(6) *If X is not klt, then $n(X) \leq 2\rho(X) + 10$.*

(7) *If X is not klt and $-K_X$ is big and nef, then $n(X) \leq 2\rho(X) + 7$.*

Remark 1.1 (1) The assumption of Theorem 1.3(2) includes the case when X is klt Fano, hence immediately implies Theorem 1.2.

(2) Theorem 1.2 may be well-known to experts, but we cannot find any references except [1–2, 20], and we cannot find any similar results in papers citing (see [1–2, 20]), so we believe that Theorem 1.2 is new.

(3) The assumption "$-(K_X+B)$ is nef" in Theorem 1.3 cannot be further relaxed to "$-K_X$ is pseudo-effective" even when X is canonical and $-K_X$ is effective (see Example-Proposition 4.1(1)).

(4) The assumption "(X, B) is lc" in Theorem 1.3 cannot be further relaxed even when $\rho(X) = 1$ and X is Fano, otherwise $n(X)$ may be unbounded (see Example 4.2(3)).

(5) The bounds for Theorem 1.3(2)–(3) are optimal at least for low Picard numbers and the bounds for Theorem 1.3(5) are optimal. We do not know if the bounds for Theorem 1.3(4) and (6) are optimal even for small values of $\rho(X)$ (see [6, Theorem D, 30, Theorem 4.1]), however $2\rho(X) + 2$ is not satisfied even when $\rho(X) = 1$ and X is Fano (see Example 4.2(2)).

(6) We expect some boundedness results on singular points to hold in high dimensions (see Section 5). We prove the boundedness on the number of torus invariant singular points for toric varieties with bounded Picard numbers (see Theorem 5.1), but one needs to be careful for non-toric varieties due to Example-Proposition 5.1.

2 Preliminaries

We adopt the standard notation and definitions in [4, 21].

2.1 Pairs and singularities

Definition 2.1 *A pair (X, B) consists of a normal quasi-projective variety X and an $\mathbb{R}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{R}$-Cartier. If $B \in [0, 1]$, then B is called a boundary.*

Let E be a prime divisor on X and D be an $\mathbb{R}$-divisor on X. We define $\mathrm{mult}_E D$ *to be the multiplicity of E along D. Let $\phi : W \to X$ be any log resolution of (X, B) and let*

$$K_W + B_W := \phi^*(K_X + B).$$

The log discrepancy of a prime divisor D on W with respect to (X, B) is $1 - \mathrm{mult}_D B_W$ and it is denoted by $a(D, X, B)$. We say that (X, B) is lc (resp. klt) if $a(D, X, B) \geq 0$ (resp. > 0) for every log resolution $\phi : W \to X$ as above and every prime divisor D on W.

A germ $X \ni x$ consists of a normal quasi-projective variety X and a closed point $x \in X$.

Definition 2.2 *Let $f : X \dashrightarrow Y$ be a birational map which does not extract any divisor, $p : W \to X$ and $q : W \to Y$ be a common resolution, and D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D_Y := f_*D$ is $\mathbb{R}$-Cartier. We say that f is D-negative if*

$$p^*D = q^*D_Y + E$$

for some $E \geq 0$, and $\mathrm{Supp}(p_*E)$ *equals the set of f-exceptional divisors.*

Definition 2.3 *Let X be a normal projective variety. We say that X is Fano if $-K_X$ is ample. We say that X is of Fano type if (X, B) is klt and $-(K_X + B)$ is ample for some boundary B on X. We say that (X, B) is log Calabi-Yau if $K_X + B \equiv 0$.*

2.2 Surfaces

Definition 2.4 *A surface is a normal quasi-projective variety of dimension 2. For any non-negative integer m, the Hirzebruch surface $\mathbb{F}_m$ is given by $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$.*

In some references, a klt Fano surface is also called a log del Pezzo surface.

Definition 2.5 (Dual graph) *Let n be a non-negative integer, and $C = \bigcup_{i=1}^{n} C_i$ be a collection of irreducible curves on a smooth surface U. We define the dual graph $\mathcal{D}(C)$ of C as follows.*

(1) *The vertices $v_i = v_i(C_i)$ of $\mathcal{D}(C)$ correspond to the curves C_i.*

(2) *For $i \neq j$, the vertices v_i and v_j are connected by $C_i \cdot C_j$ edges.*

For any birational morphism $f : Y \to X$ between surfaces, let $E = \bigcup_{i=1}^{n} E_i$ be the reduced exceptional divisor for some non-negative integer n. We define $\mathcal{D}(f) := \mathcal{D}(E)$.

A dual graph is called a tree if the graph contains no cycles.

Lemma 2.1 (1) *Let $f' : Y' \to X \ni x$ be a resolution of a klt surface germ $X \ni x$. Then $\mathcal{D}(f')$ is a tree whose vertices are all smooth rational curves.*

(2) *Let $f' : Y' \to X$ be a projective morphism between smooth surfaces. Then $\mathcal{D}(f')$ is a tree whose vertices are all smooth rational curves.*

Proof (1) Follows from [17, Lemma 3.10] and the classification of klt surface singularities by taking $f : Y \to X$ to be the minimal resolution of $X \ni x$. (2) Follows from (1) because Y' is a resolution of X.

Lemma 2.2 *Let (X, B) be an lc surface pair. Then K_X is $\mathbb{Q}$-Cartier.*

Proof Pick any closed point $x \in X$. If $(X, 0)$ is numerically dlt near x, then K_X is $\mathbb{Q}$-Cartier near x by [21, Proposition 4.11]. If $(X, 0)$ is not numerically dlt near x, since (X, B) is lc, (X, B) is numerically lc near x. By [21, Corollary 4.2], $x \notin B$, hence K_X is $\mathbb{Q}$-Cartier near x. Thus K_X is $\mathbb{Q}$-Cartier.

2.3 g-Pairs

We need the following definitions on generalized pairs (g-pairs for short). See [5] for more details.

Definition 2.6 (**b**-divisors) *Let X be a normal quasi-projective variety. We call Y a birational model over X if there exists a projective birational morphism $Y \to X$.*

*Let $X \dashrightarrow X'$ be a birational map. For any valuation ν over X, we define $\nu_{X'}$ to be the center of ν on X'. A **b**-divisor $\mathbf{M}$ over X is a formal sum $\mathbf{M} = \sum_{\nu} r_\nu \nu$ where ν are valuations over X, such that ν_X is not a divisor except for finitely many ν. If in addition, $r_\nu \in \mathbb{Q}$ for every ν, then $\mathbf{M}$ is called a $\mathbb{Q}$-**b**-divisor. The trace of $\mathbf{M}$ on X' is the $\mathbb{R}$-divisor*

$$\mathbf{M}_{X'} := \sum_{\nu_{i,X'} \text{ is a divisor}} r_i \nu_{i,X'}.$$

*If $\mathbf{M}_{X'}$ is $\mathbb{R}$-Cartier and $\mathbf{M}_Y$ is the pullback of $\mathbf{M}_{X'}$ on Y for any birational model Y of X', we say that $\mathbf{M}$ descends to X', and write $\mathbf{M} = \overline{\mathbf{M}_{X'}}$. If X is projective and $\mathbf{M}$ is a **b**-divisor over X, such that $\mathbf{M}$ descends to some birational model Y over X and $\mathbf{M}_Y$ is nef, then we say that $\mathbf{M}$ is nef.*

Definition 2.7 (g-Pairs) *A projective* g-*pair $(X, B, \mathbf{M})$ consists of a normal projective variety X, an $\mathbb{R}$-divisor $B \geq 0$ on X, and a nef **b**-divisor $\mathbf{M}$ over X, such that $K_X + B + \mathbf{M}_X$ is $\mathbb{R}$-Cartier. If B is a $\mathbb{Q}$-divisor and $\mathbf{M}$ is a $\mathbb{Q}$-**b**-divisor, then we say that $(X, B, \mathbf{M})$ is a* $\mathbb{Q}$-g-*pair.*

Let $(X, B, \mathbf{M})$ be a projective g-*pair, $\phi : W \to X$ any log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W, and*

$$K_W + B_W + \mathbf{M}_W := \phi^*(K_X + B + \mathbf{M}_X).$$

We say that $(X, B, \mathbf{M})$ is glc if the coefficients of B_W are ≤ 1.

For any projective glc g-*pair $(X, B, \mathbf{M})$ and $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D \geq 0$ on X, we define*

$$\operatorname{glct}(X, B, \mathbf{M}; D) := \sup\{t \mid (X, B + tD; \mathbf{M}) \textit{ is glc}\}$$

to be the glc threshold of D with respect to $(X, B, \mathbf{M})$.

3 Proofs of the Main Theorems

Lemma 3.1 *Let (X, B) be an lc pair such that (X, B) is lc and $-(K_X + B)$ is nef (resp. $K_X + B \equiv 0$). Then there exists a $\mathbb{Q}$-divisor B' on X such that (X, B') is lc and $-(K_X + B')$ is nef (resp. $K_X + B \equiv 0$).*

Proof See [15, Proposition 2.6], [16, Corollary 3.5] and [14, Lemma 5.4, Theorem 5.6].

Lemma 3.2 *Let X be a klt surface, $f : X \to Y$ be a K_X-negative divisorial contraction of a curve C. Then C contains at most 2 singular points of X.*

Proof We may assume that C contains n singular points of X for some integer $n \geq 3$. Let $g : W \to X$ be the minimal resolution of X near C with exceptional divisors $E_1, \cdots, E_m$ for some integer $m \geq n$. Let $C_W := g_*^{-1}C$. Possibly reordering indices, we may assume that C_W intersects E_1, E_2 and E_3.

Since $f \circ g$ is a resolution of $Y \ni y := f(C)$, by Lemma 2.1(1), $C_W \cong \mathbb{P}^1$. If $C_W^2 \leq -2$ then $f \circ g$ is actually the minimal resolution of $Y \ni y$. But $a(C, Y, 0) > 1$ since f is K_X-negative, thus C is not contained in the minimal resolution of $Y \ni y$. Hence $C_W^2 = -1$ and we may let $p : W \to T$ be the contraction of C_W. Then there exists an induced morphism $h : T \to Y$ which is a resolution of $Y \ni y$. Let $E_{i,T} := p_*E_i$ for each i, then $E_{i,T} \cdot E_{j,T} \geq 1$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$. Thus $\mathcal{D}\big(\bigcup\limits_{i=1}^{m} E_{i,T}\big) = \mathcal{D}(h)$ is not a tree, which contradicts Lemma 2.1(1).

Lemma 3.3 *Let X be a klt surface, $f : X \to Z$ be a K_X-Mori fiber space such that* $\dim Z = 1$, *and $z \in Z$ be a closed point. If f^*z is reduced, then X is smooth near $f^{-1}z$.*

Proof Since $f : X \to Z$ is a K_X-Mori fiber space, $f^{-1}z$ is an irreducible curve and $R^i f_* \mathcal{O}_X = 0$ for any $i > 0$. Since Z is regular and X is Cohen-Macaulay, f is flat (see [22, Theorem 23.1]). If f^*z is reduced, then by Cohomology and Base change (see [18, III 12.11]), $H^1(X_z, \mathcal{O}_{X_z}) = 0$ so $X_z \cong \mathbb{P}^1$. Combining with the fact that f is flat, we deduce that X is regular along $f^{-1}z$ because both $X_z = f^{-1}z$ and Z are regular (see [22, Theorem 23.7]).

Lemma 3.4 *Let (X, B) be an lc projective surface pair such that $-(K_X + B)$ is nef, and $f : X \to Z$ be a K_X-Mori fiber space such that* $\dim Z = 1$. *Then*

(1) *any fiber of f contains at most 2 singular points of X,*

(2) (a) *at most four fibers of f contain singular point (s) of X, and*

(b) *if X is of Fano type, then at most three fibers of f contain singular point (s) of X.*

Proof By Lemma 3.1, we may assume that B is a $\mathbb{Q}$-divisor. There exists a non-negative integer n, closed points $z_1, \cdots, z_n \in Z$ and fibers $F_i := f^{-1}z_i$ for each i, such that $F_1, \cdots, F_n$ are the only closed fibers of f which contain singular points of X. If $n = 0$, there is nothing left to prove, so in the rest of the proof, we may assume that $n \geq 1$.

First we prove (1). Suppose that there exists a fiber F of f, such that F contains at least 3 singular points of X and $F = f^{-1}z$ for some closed point $z \in Z$. We let $g : W \to X$ be the minimal resolution of X, $E_1, \cdots, E_m$ be the g-exceptional divisors for some integer $m \geq 3$ such that $\operatorname{center}_X E_i \in F$ for each i, and $F_W := g_*^{-1}F$. Then $E_i^2 \leq -2$ for each i. Possibly reordering indices, we may assume that F_W intersects E_1, E_2, E_3.

We may run a K_W-MMP over Z, which induces a birational contraction $h : W \to Y$ between smooth projective varieties and a K_Y-Mori fiber space $f' : Y \to Z$, such that Y is a geometrically ruled surface. In particular, h contracts m elements of $\{F_W, E_1, \cdots, E_m\}$. Since Y is smooth and X is not smooth, F_W is contracted by h. Since W is smooth, h is a K_W-MMP over Z and $E_i^2 \leq -2$, we have that $F_W \cong \mathbb{P}^1$ and $F_W^2 = -1$. Thus we may let $p : W \to T$ be the contraction of F_W, and there is an induced morphism $q : T \to Y$. Let $E_{i,T} := p_*E_i$ for each i, then $E_{i,T} \cdot E_{j,T} \geq 1$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$. Thus $\mathcal{D}\big(\bigcup_{i=1}^{m} E_{i,T}\big)$ is not a tree, hence $\mathcal{D}(q)$ is not a tree, which contradicts Lemma 2.1(2).

Now we prove (2)(a). Let $\mathbf{M}_X := -(K_X + B)$ and $\mathbf{M} := \overline{\mathbf{M}_X}$. Then $(X, B, \mathbf{M})$ is a projective glc $\mathbb{Q}$-g-pair. By the generalized canonical bundle formula (see [10, Theorem 1.4, 16, Theorem 1.2]), we have

$$0 = K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

such that M_Z is pseudo-effective and

$$\operatorname{mult}_z B_Z = 1 - \operatorname{glct}(X, B, \mathbf{M}; f^*z)$$

for any point $z \in Z$. By Lemma 3.3, each f^*z_i is not reduced, hence $\operatorname{mult}_{z_i} B_Z \geq \frac{1}{2}$ for each i. Thus

$$0 = \deg(K_Z + B_Z + M_Z) \geq -2 + n \cdot \frac{1}{2} + 0 = -2 + \frac{n}{2},$$

which implies that $n \leq 4$. Since $n \geq 1$, we have $\deg(K_Z) < 0$, so $Z \cong \mathbb{P}^1$. Moreover, $n = 4$ if and only if $M_Z \sim_{\mathbb{Q}} 0$ and $B_Z = \frac{1}{2}\sum_{i=1}^{4} z_i$.

Under the assumptions of (2)(b), we can find a boundary $\widetilde{B} \sim_{\mathbb{Q}} -K_X$ such that $(X, \widetilde{B})$ is klt. We can further assume that $\operatorname{Supp}\widetilde{B}$ contains a general smooth fiber $X_{z'}$, which is away from the singular points on X. Now let

$$K_X + \widetilde{B} \sim_{\mathbb{Q}} f^*(K_Z + \widetilde{B}_Z + \widetilde{M}_Z)$$

be the canonical bundle formula for $K_X + \widetilde{B}$. Assume that $n = 4$, then as the above shows, $\widetilde{M}_Z \sim_{\mathbb{Q}} 0$ and $\widetilde{B}_Z = \frac{1}{2}\sum_{i=1}^{4} z_i$. However, by the definition of the canonical bundle formula, we have

$$\operatorname{mult}_{z'} \widetilde{B}_Z = 1 - \operatorname{lct}(X, \widetilde{B}; f^*z') > 0$$

since $X_{z'} = f^*z'$ is contained in the support of B as our assumption. Therefore z' (as a divisor on Z) should be contained in the support of $\widetilde{B}_Z$. It is impossible since $z' \neq z_i$ by our assumption on $X_{z'}$.

Proof of Theorem 1.3(2)–(3) Since K_X is not pseudo-effective, we may run a K_X-MMP which terminates with a Mori fiber space $f : Y \to Z$. Let $g : X \to Y$ be the induced morphism and $B_Y := g_*B$, then $-(K_Y + B_Y)$ is nef. Moreover, if (X, B) is of Fano type, then (Y, B_Y) is of Fano type.

Case 1 $\dim Z = 0$. In this case, $\rho(Y) = 1$ and Y is klt Fano, so g is a composition of $\rho(X) - 1$ divisorial contractions between klt surfaces. By Lemma 3.2 and Theorem 1.1,

$$n(X) \le n(Y) + 2(\rho - 1) \le 4 + 2(\rho - 1) = 2\rho + 2.$$

Case 2 $\dim Z = 1$. In this case, $\rho(Y) = 2$, so f is a composition of $\rho(X) - 2$ divisorial contractions between klt surfaces. By Lemma 3.2, $n(X) \le n(Y) + 2(\rho - 2)$. By Lemma 3.4, $n(Y) \le 8$ and $n(Y) \le 6$ when (X, B) is of Fano type. Thus $n(X) \le 2\rho(X) + 4$ and $n(X) \le 2\rho(X) + 2$ when (X, B) is of Fano type.

Proof of Theorem 1.3(4) Since X is klt but not canonical and $K_X \equiv 0$, there exists an extraction $f : Y \to X$ of a prime divisor E such that Y is klt and $K_Y + aE = f^*K_X \equiv 0$ for some positive real number a. By Theorem 1.3(3),

$$n(Y) \le 2\rho(Y) + 4 = 2\rho(X) + 6,$$

thus $n(X) \le n(Y) + 1 \le 2\rho(X) + 7$.

Proof of Theorem 1.3(5) By abundance, $K_X \sim_{\mathbb{Q}} 0$, hence there exists the smallest positive integer m such that $mK_X \sim 0$. Since K_X is Cartier, there exists an étale cyclic cover $Y \to X$ of degree m such that $K_Y \sim 0$. In particular, Y is canonical and $n(X) \le n(Y)$ (see [21, Lemma 2.51]).

Let $f : W \to Y$ be the minimal resolution of Y. Then $K_W = f^*K_Y \sim 0$, hence W is either an abelian surface or a smooth K3 surface. If W is an abelian surface, then W does not contain any rational curves, so $W = Y$ and hence $n(Y) = 0$. If W is a smooth K3 surface, then Y is a K3 surface with at most canonical singularities. By [24, Corollary 4.6], $n(Y) \le 16$. Thus $n(X) \le n(Y) \le 16$.

Lemma 3.5 *Let $X \ni x$ be a surface germ that is lc but not klt. Then there exists a birational morphism $f : Y \to X$ which extracts a prime divisor E over $X \ni x$, such that $a(E, X, 0) = 0$ and Y is klt.*

Proof Let E be any lc place in the dual graph of the minimal resolution of $X \ni x$ and let $f : Y \to X$ be the extraction of E. Then (Y, E) is lc and all lc centers of (Y, E) are contained in E. Thus Y is klt.

Proof of Theorem 1.3(6)–(7) By Lemma 2.2, K_X is $\mathbb{Q}$-Cartier. By assumption X is not klt, hence there exists at least 1 point on X where X is not klt. By applying the connectedness theorem (see [25, Proposition 3.3.2, 13, Theorem 1.2, 3, Theorem 1.2(1)]) to (X, B) (or apply

[11, Theorem 1.1] to the g-pair $(X, B, \mathbf{M} := \overline{-(K_X + B)})$; see also [26, Lemma 6.9]), we know that there exist at most 2 points on X where X is not klt. If $-K_X$ is big and nef, then by the Shokurov-Kollár connectedness principle, there exists exactly 1 point on X where X is not klt.

By Lemma 3.5, there exists an extraction $f : Y \to X$ and a divisor $E \geq 0$ on X, such that Y is klt, $1 \leq \rho(Y) - \rho(X) \leq 2$, $K_Y + E = f^*(K_X + B)$, (Y, E) is lc and $-(K_Y + E)$ is nef. Moreover, $\rho(Y) - \rho(X) = 1$ when $-K_X$ is big and nef. In particular, $K_Y \not\equiv 0$. By Theorem 1.3(3), $n(Y) \leq 2\rho(Y)+4$, hence $n(X) \leq n(Y)+2 \leq 2\rho(X)+10$ and $n(X) \leq n(Y)+1 \leq 2\rho(X)+7$ when $-K_X$ is big and nef.

Proof of Theorem 1.3 We are only left to prove (1), which follows from (3)–(6).

Proof of Theorem 1.2 It follows from Theorem 1.3(2).

4 Examples on Surfaces

In this section, we discuss how far our bounds in Theorem 1.3 are away from being optimal. The following Example-Proposition shows that even when $\rho(X) = 2$,

(1) the assumption "$-(K_X + B)$ is nef" is necessary in Theorem 1.3,

(2) Theorem 1.3(2) is optimal even when X is klt Fano, and

(3) Theorem 1.3(3) is optimal.

Example-Proposition 4.1 Let n be a positive integer, $Z := \mathbb{P}^1 \times \mathbb{P}^1$, and $z_i := (u_i, v_i) \in Z$ closed points in Z for any $i \in \{1, 2, \cdots, n\}$ such that $u_i \neq u_j$ for any $i \neq j$. Let $p_1 : Z \to \mathbb{P}^1$ and $p_2 : Z \to \mathbb{P}^1$ be the first and second projection of Z to $\mathbb{P}^1$, and $L_i := p_1^* u_i$ and $R_i := p_2^* v_i$ for each i.

Let $f : Y \to Z$ be the blow-up of $z_1, \cdots, z_n$. For each i, let E_i be the exceptional curve of f over z_i, $L_{i,Y} := f_*^{-1} L_i$, $R_{i,Y} := f_*^{-1} R_i$ and $y_i := L_{i,Y} \cap E_i$.

Let $g : X \to Y$ be the blow-up of $y_1, \cdots, y_n$. For each i, let F_i be the exceptional curve of g over y_i, $L_{i,X} := g_*^{-1} L_{i,Y}$, $R_{i,X} := g_*^{-1} R_{i,Y}$ and $E_{i,X} := g_*^{-1} E_i$.

Let $h : X \to S$ be the contraction of $E_{1,X}, \cdots, E_{n,X}$ and $L_{1,X}, \cdots, L_{n,X}$. For each i, let $F_{i,S} := h_* F_i$, $R_{i,S} := h_* R_{i,X}$, $s_i := h(E_{i,X})$ and $t_i := h(L_{i,X})$. Then $s_1, \cdots, s_n$ and $t_1, \cdots, t_n$ are the only singular points on S and are $\frac{1}{2}(1, 1)$ singularities.

(1) When $v_1 = v_2 = \cdots = v_n$, $-K_S$ is effective.

(2) When $n = 4$ and $v_1 = v_3 \neq v_2 = v_4$, (S, B) is lc log Calabi-Yau for some B.

(3) When $n = 3$ and $v_i \neq v_j$ for any $i \neq j$, S is klt Fano.

Proof Most of the proofs are elementary computations on pullbacks and pushforwards of divisors which we omit. In (1), $-K_S \sim 4F_{1,S} + 2R_{1,S} \geq 0$. In (2), we may pick $B = R_{1,S} + R_{2,S}$. In (3), $-K_S \sim 2R_{i,S}$, $R_{i,S}^2 = \frac{1}{2}$, $s_i \in R_{i,S}$ and $t_i \in R_{j,S}$ for any $i \neq j$. Thus $-K_S$ is nef and big and we may let $\phi : S \to T$ be the ample model of $-K_S$.

If $S \neq T$, then $-K_S$ is not ample, and ϕ contracts an irreducible curve $C \subset S$ such that

$-K_S \cdot C = 0$. Since $\rho(S) = 2$, T is a klt Fano variety and $\rho(T) = 1$. Since $-K_S \sim 2R_{i,S}$ for any i and $R_{i,S}^2 > 0$, C does not intersect $R_{i,S}$ for any i, so C is contained in the smooth locus of S. Thus $n(T) \geq n(S) = 6$, which contradicts Theorem 1.1.

Thus $S = T$, hence $-K_S$ is ample, and we are done.

The following example shows that even when $\rho(X) = 1$ and X is Fano,

(1) Theorem 1.1 is optimal,

(2) the bound "$2\rho(X) + 2$" is not enough if X is not klt, and

(3) the assumption "(X, B) is lc" is necessary for Theorem 1.3.

Example 4.2 Assumptions and notations as in Example-Proposition 4.1 and assume that $t_1 = t_2 \cdots = t_n$. Let $R' := p_2^* v$ for some $v \neq v_1$ and $R'_S := h_*((f \circ g)_*^{-1} R')$. Since the intersection matrix of $R_{1,X} \bigcup_{i=1}^{n} (E_{i,X} \cup L_{i,X})$ is negative definite, there exists a contraction $\phi : S \to T$ of $R_{1,S}$. In particular, $\rho(T) = 1$. Since

$$D := -\Big(K_S + \frac{2(n-2)}{n} R_{1,S}\Big) \sim 4F_{1,S} + \frac{4}{n} R_{1,S}$$

is big and nef and ϕ-trivial, and since $nD \sim 4nF_{1,S} + 4R_{1,S} \sim 4nF_{2,S} + 4R_{1,S} \sim 4R'_S$, $|nD|$ is base-point-free and defines ϕ. Thus $nD \sim \phi^*\phi_*(nD)$, and in particular, $-K_T = \phi_* D$ is ample. Since $a(R_{1,S}, T, 0) = \frac{4-n}{n}$, we have

(1) when $n = 3$, T is a klt Fano surface, $\rho(T) = 1$ and $n(T) = 4$.

(2) When $n = 4$, T is an lc Fano surface, $\rho(T) = 1$ and $n(T) = 5 > 2\rho(T) + 2$.

(3) When $n \geq 5$, T is a non-lc Fano surface, $\rho(T) = 1$ and $n(T) = n + 1$. When $n \to +\infty$, $n(T) \to +\infty$.

The following well-known example shows that Theorem 1.3(5) is optimal.

Example 4.3 Some Kummer surfaces are canonical K3 surfaces with 16 singular points.

We do not know if Theorem 1.3(4) and (6) are optimal or not even when $\rho(X) = 1$, and we do not know if Theorem 1.3(2)–(3) are optimal when $\rho(X)$ is large. We guess that under the assumption of Theorem 1.3, $n(X) \leq \rho(X) + C$ for some constant number C, but we do not know how to prove this yet. The next example shows that the linear term $\rho(X)$ is necessary in an expression of an upper bound of $n(X)$ even when X is klt Fano.

Example 4.4 Fix a positive integer $n \geq 2$, let $e_1 = (1, 0),\ e_2 = (0, 1) \in \mathbb{R}^2$ and $u_{-1} = -e_1,\ u_i = ie_1 + (i^2 - 1)e_2$ $(0 \leq i \leq n)$. Then each u_i is primitive. Now let Σ be the complete fan in $N_{\mathbb{R}} = \mathbb{R}^2$ generated by rays $u_{-1}, u_0, \cdots, u_n$. Then the projective toric surface X_Σ is klt Fano with $\rho(X_\Sigma) = n + 2 - 2 = n$. The number of singular points corresponds to the number of non-smooth maximal cones in $\Sigma(2) = \{\mathrm{Cone}(u_{i-1}, u_i), \mathrm{Cone}(u_n, u_{-1}) \mid 0 \leq i \leq n\}$. Notice that $\mathrm{Cone}(u_n, u_{-1}), \mathrm{Cone}(u_{i-1}, u_i)$ $(2 \leq i \leq n)$ are not smooth because none of $\{u_n, u_{-1}\}, \{u_{i-1}, u_i\}$ $(2 \leq i \leq n)$ generates $N = \mathbb{Z}^2$. Thus X_Σ has exactly n singular points.

5 Discussions

For toric varieties, the singular locus is torus invariant and thus can be nicely described as a disjoint union of torus orbits.

Theorem 5.1 *If X is a proper $\mathbb{Q}$-factorial toric variety of dimension d, then for any $2 \leq k \leq d$, there exists a polynomial h_k of degree $\leq \min\{k, d-1\}$ such that the number of torus invariant singular points of codimension k on X is $\leq h_k(\rho(X))$.*

Proof Let Σ be the complete fan in $N_{\mathbb{R}} \cong \mathbb{R}^d$ which defines X, then the cones in Σ are all simplicial and naturally give a triangulation of $S^{d-1} \cong \{\mathbb{R}^d - 0\}/(x \sim \lambda x)$, where each cone of dimension $k \geq 1$ corresponds to a $(k-1)$-simplex.

Recall $\Sigma(k)$ is the set of k dimensional cones in Σ, then we have $\rho(X) + d = |\Sigma(1)|$ and $|\Sigma(k)| \leq \binom{|\Sigma(1)|}{k}$. Thus any $|\Sigma(k)|$ $(1 \leq k \leq d-1)$ is bounded by a polynomial of $\rho(X)$ with degree $\leq k$. Also, we have $1 - (-1)^d = \chi(S^{d-1}) = \sum\limits_{k=1}^{d} (-1)^{k-1}|\Sigma(k)|$. Hence $|\Sigma(d)|$ is bounded by a polynomial of $\rho(X)$ with degree $\leq d-1$. Since the torus invariant singular points correspond to torus orbits in $\mathrm{Sing}(X)$, the statements follows directly by the orbit-cone correspondence theorem.

It is natural to ask whether we can have a bound on the number of singular points in high dimensions for non-toric klt Fano varieties with bounded Picard number as well. However, the first question is: Since the singular locus may be of dimension > 0, how can we effectively define the "number of singular points" for a non-toric variety?

The most straightforward idea is to consider the number of isolated singular points. Unfortunately, we have the following counterexample for klt Fano varieties with only isolated singularities of Picard number 1 even in dimension 3. This example is given by Chen Jiang.

Example-Proposition 5.1 Fix a positive integer k. Let $X = X_{6k+3} \subset \mathbb{P}(1, 3, 3, 3k+1, 3k+2)$ be a general hypersurface of degree $6k+3$. Then

(1) X is quasismooth klt Fano of Picard number 1, and

(2) X contains exactly the following singularities:

(a) A cyclic quotient singularity of type $\frac{1}{3k+1}(1, 3, 3)$,

(b) a cyclic quotient singularity of type $\frac{1}{3k+2}(1, 3, 3)$, and

(c) $(2k+1)$ cyclic quotient singularities of type $\frac{1}{3}(1, 1, 2)$.

Proof (1) Follows from [19, Theorem 8.1] (see also [8, Theorem 2.7, 9, Theorem 3.2.4(i)]. (2) Follows from (1) and [19, Section 9–10] (see also [8, Theorem 2.8]).

Nevertheless, we may still ask the following questions. These questions arise in personal communications of the first author with Paolo Cascini, Christopher D. Hacon, Jingjun Han and Chen Jiang during the summer of 2020.

Question 5.1 Let d, ρ be two positive integers. Does there exist a positive integer $N_1 =$

$N_1(d,\rho)$, such that for any klt Fano variety X of dimension d with $\rho(X) \leq \rho$, the number of isolated non-terminal singularities of X is $\leq N_1$?

Question 5.2 Let d, ρ be two positive integers. Does there exist a positive integer $N_2 = N_2(d,\rho)$, such that for any klt Fano variety X of dimension d with $\rho(X) \leq \rho$, the number of codimension 2 singularities of X is $\leq N_2$?

Theorem 1.2 answers these two questions when $d = 2$, but both questions seem to be widely open in dimension ≥ 3 even when $\rho = 1$. We remark that if we have satisfactory answers for these questions in the Picard number 1 case, then the methods used in our paper are expected to be applied to prove the bounded Picard number cases.

For similar questions and results, we also refer the readers to [7].

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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