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On Higher Moments of the Error Term in the Rankin-Selberg Problem*

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Abstract Let $\Delta_1(x;\varphi)$ denote the error term in the classical Rankin-Selberg problem. In this paper, the authors consider the higher power moments of $\Delta_1(x;\varphi)$ and derive the asymptotic formulas for 3-rd, 4-th and 5-th power moments, which improve the previous results.

Keywords The Rankin-Selberg problem, Power moment, Voronoï formula
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1 Introduction

Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$, that is,

$$\varphi\Big(\frac{az+b}{cz+d}\Big)=(cz+d)^{\kappa}\varphi(z),\quad a,b,c,d\in\mathbb{Z},\ ad-bc=1. \tag{1.1}$$

We denote by $a(n)$ the n-th Fourier coefficient of $\varphi(z)$ and suppose that $\varphi(z)$ is normalized such that $a(1)=1$ and $T(n)\varphi=a(n)\varphi$ for every $n\in\mathbb{N}$, where $T(n)$ is the Hecke operator of order n. Rankin [11] and Selberg [13] independently introduced the function

$$Z(s)=\zeta(2s)\sum_{n=1}^{\infty}|a(n)|^2n^{1-\kappa-s},$$

where $\zeta(s)$ is the Riemann zeta-function. In the half plane $\sigma=\Re s>1$ the function $Z(s)$ has the absolutely convergent Dirichlet series expansion

$$Z(s)=\sum_{n=1}^{\infty}c_nn^{-s},$$

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where c_n is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} \Big| a\Big(\frac{n}{m^2}\Big)\Big|^2.$$

In 1974, Deligne [1] proved that $|a(n)| \le n^{\frac{\kappa-1}{2}} d(n)$, where $d(n)$ is the Dirichlet divisor function. Then we know that $c_n \ll n^{\varepsilon}$. Here and in what follows ε denotes an arbitrarily small positive number which is not necessarily the same at each occurrence.

Rankin [11] considered the analytic behaviour of $Z(s)$, and consequently he obtained

$$\sum_{n\le x} c_n = Cx + \Delta(x;\varphi),$$

where

$$\Delta(x;\varphi) = O(x^{\frac{3}{5}}) \tag{1.2}$$

and

$$C = \frac{1}{6}\pi^2 \kappa R_0, \quad R_0 = \frac{12(4\pi)^{\kappa-1}}{\Gamma(\kappa+1)} \iint_{\mathfrak{F}} y^{\kappa-2} |\varphi(z)|^2 \mathrm{d}x\mathrm{d}y,$$

the integral being taken over a fundamental domain $\mathfrak{F}$ of $SL(2,\mathbb{Z})$. Selberg [13] also briefly sketched how to get the above results. The classical Rankin-Selberg problem is to study properties of $\Delta(x;\varphi)$. Recently, The exponent $\frac{3}{5}$ in (1.2) has been improved to $\frac{3}{5}-\delta, \delta > 0$ by Huang [2], which is a breakthrough in the Rankin-Selberg problem.

Ivić wrote a series of papers about the Rankin-Selberg problem [3–8]. In [3] he proved that $\Delta(x;\varphi) = \Omega_{\pm}(x^{\frac{3}{8}})$ and conjectured that $\Delta(x;\varphi) = O(x^{\frac{3}{8}+\varepsilon})$. In [4–5] Ivić studied the mean square of $\Delta(x;\varphi)$ and proved that

$$\int_1^T \Delta^2(x;\varphi)\mathrm{d}x \ll T^{1+2\beta+\varepsilon}, \tag{1.3}$$

where

$$\beta = \frac{2}{5-2\mu\Big(\frac{1}{2}\Big)}, \quad \mu(\sigma) := \limsup_{t\to+\infty} \frac{\log|\zeta(\sigma+\mathrm{i}t)|}{\log t}.$$

In [6] he studied the fourth moment of $\Delta(x;\varphi)$ and proved that

$$\int_1^T \Delta^4(x;\varphi)\mathrm{d}x \ll T^{3+\varepsilon}. \tag{1.4}$$

In 1999, Ivić, Matsumoto and Tanigawa [9] considered the Riesz mean of the type

$$D_\rho(x;\varphi) := \frac{1}{\Gamma(\rho+1)} \sum_{n\le x} (x-n)^\rho c_n$$

for any fixed $\rho \ge 0$ and defined the error term $\Delta_\rho(x;\varphi)$ by

$$D_\rho(x;\varphi) = \frac{\pi^2\kappa R_0}{6\Gamma(\rho+2)} x^{\rho+1} + \frac{Z(0)}{\Gamma(\rho+1)} x^\rho + \Delta_\rho(x;\varphi).$$

Ivić, Matsumoto and Tanigawa considered the relation between $\Delta(x;\varphi)$ and $\Delta_1(x;\varphi)$. For some $\alpha \geq 0$, if $\Delta_1(x;\varphi) = O(x^{\alpha})$, they obtained $\Delta(x;\varphi) = O(x^{\frac{\alpha}{2}})$. They also obtained

$$\Delta_1(x;\varphi) = O(x^{\frac{6}{5}})$$

and

$$\int_1^T \Delta_1^2(x;\varphi)\mathrm{d}x = \frac{2}{13}(2\pi)^{-4}\Big(\sum_{n=1}^{\infty} c_n^2 n^{-\frac{7}{4}}\Big)T^{\frac{13}{4}} + O(T^{3+\varepsilon}). \tag{1.5}$$

In 2014, Matsumoto [10] proved that the error term $O(T^{3+\varepsilon})$ in (1.5) can be replaced by $O(T^3(\log T)^{3+\varepsilon})$. In [6], Ivić also studied the fourth moment of $\Delta_1(x;\varphi)$ and proved that

$$\int_1^T \Delta_1^4(x;\varphi)\mathrm{d}x \ll T^{\frac{11}{2}+\varepsilon}. \tag{1.6}$$

Tanigawa, Zhai and Zhang [14] studied the third, fourth and fifth power moments of $\Delta_1(x;\varphi)$ and proved that

$$\begin{aligned}
\int_T^{2T} \Delta_1^3(x;\varphi)\mathrm{d}x &= \frac{B_3(c)}{1120\pi^6}T^{\frac{35}{8}} + O(T^{\frac{35}{8}-\frac{1}{36}+\varepsilon}),\\
\int_T^{2T} \Delta_1^4(x;\varphi)\mathrm{d}x &= \frac{B_4(c)}{11264\pi^8}T^{\frac{11}{2}} + O(T^{\frac{11}{2}-\frac{1}{221}+\varepsilon}),\\
\int_T^{2T} \Delta_1^5(x;\varphi)\mathrm{d}x &= \frac{B_5(c)}{108544\pi^{10}}T^{\frac{53}{8}} + O(T^{\frac{53}{8}-\frac{1}{1731}+\varepsilon}),
\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}
B_k(f) &:= \sum_{l=1}^{k-1}\binom{k-1}{l} s_{k;l}(f)\cos\frac{\pi(k-2l)}{4},\\
s_{k;l} &:= \sum_{\sqrt[4]{n_1}+\cdots+\sqrt[4]{n_l}=\sqrt[4]{n_{l+1}}+\cdots+\sqrt[4]{n_k}} \frac{f(n_1)\cdots f(n_k)}{(n_1\cdots n_k)^{\frac{7}{8}}}, \quad 1\leq l\leq k.
\end{aligned}$$

In this paper, we shall prove the following theorem, which improves (1.7).

Theorem 1.1 *Let $k\in\{3,4,5\}$. We have the asymptotic formula*

$$\int_T^{2T} \Delta_1^k(x;\varphi)\mathrm{d}x = \frac{B_k(c)}{(8+9k)2^{3k-4}\pi^{2k}}T^{1+\frac{9k}{8}} + O(T^{1+\frac{9k}{8}-\delta_k+\varepsilon}), \tag{1.8}$$

where

$$\delta_3 = \frac{3}{62}, \quad \delta_4 = \frac{3}{256}, \quad \delta_5 = \frac{1}{680}.$$

2 Some Preliminary Lemmas

Lemma 2.1 (see [14, Lemma 2.1]) *Let $x > 1$ be a real number. For $1 \ll N \ll x^2$ a parameter we have*

$$\Delta_1(x;\varphi) = \frac{1}{(2\pi)^2}\mathcal{R}(x;N) + O(x^{1+\varepsilon} + x^{\frac{3}{2}+\varepsilon}N^{-\frac{1}{2}}), \tag{2.1}$$

where

$$\mathcal{R} := \mathcal{R}(x;N) = x^{\frac{9}{8}} \sum_{n\le N} \frac{c_n}{n^{\frac{7}{8}}} \cos\Big(8\pi \sqrt[4]{nx} - \frac{\pi}{4}\Big).$$

Lemma 2.2 (see [14, Lemma 2.3]) *Let* $k \ge 3$, $(i_1, \cdots, i_{k-1}) \in \{0,1\}^{k-1}$ *such that*

$$\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k} \ne 0,$$

then we have

$$|\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k}| \gg \max(n_1, \cdots, n_k)^{-(4^{k-2}-4^{-1})}.$$

Lemma 2.3 *Suppose* $T \ge 3$ *is a large parameter,* α *and* β *are real numbers such that* $\beta \ne 0$. *Then we have*

$$\int_T^{2T} x^\alpha g(\beta x^{\frac{1}{4}})\mathrm{d}x \ll \frac{T^{\alpha+\frac{3}{4}}}{|\beta|},$$

where $g(x) = \cos(2\pi x)$, *or* $\sin(2\pi x)$, *or* $\mathrm{e}(x) := \mathrm{e}^{2\pi \mathrm{i} x}$.

Proof This follows from the first derivative test.

Lemma 2.4 *Suppose* $k \ge 3$, $(i_1, \cdots, i_{k-1}) \in \{0,1\}^{k-1}$, $(i_1, \cdots, i_{k-1}) \ne (0, \cdots, 0)$, $N_1, \cdots, N_k > 1$, $0 < \Delta \ll H^{\frac{1}{4}}$, $H = \max(N_1, \cdots, N_k)$. *Let* $\mathcal{A}$ *denote the number of solutions of the inequality*

$$|\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k}| < \Delta \tag{2.2}$$

with $N_j < n_j \le 2N_j$, $1 \le j \le k$, *where*

$$\mathcal{A} = \mathcal{A}(N_1, \cdots, N_k; i_1, \cdots, i_{k-1}; \Delta).$$

Then we have

$$\mathcal{A} \ll \Delta H^{-\frac{1}{4}} N_1 \cdots N_k + H^{-1} N_1 \cdots N_k.$$

Proof The proof of this lemma is similar to the proof of [15, Lemma 2.4]. Suppose $H = N_k$. If $(n_1, \cdots, n_k)$ satisfies (2.2), then for some $|\theta| < 1$, we can obtain

$$\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt[4]{n_{k-1}} = (-1)^{i_{k-1}+1}\sqrt[4]{n_k} + \theta\Delta,$$

thus we have

$$(\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt[4]{n_{k-1}})^4 = n_k + O(\Delta N_k^{\frac{3}{4}}).$$

Therefore, for fixed $(n_1, \cdots, n_{k-1})$, the number of n_k is $\ll 1 + \Delta N_k^{\frac{3}{4}}$ and so

$$\mathcal{A} \ll \Delta N_k^{\frac{3}{4}} N_1 \cdots N_{k-1} + N_1 \cdots N_{k-1}.$$

Lemma 2.5 (see [12, Theorem 2]) *Let $c > 0$ be a non-integer real number, $M \geq 2$ be a large parameter, $\delta > 0$ be any real number. Let $\mathcal{A}(M, \delta; c)$ denote the number of solutions of the inequality*

$$|m_1^c + m_2^c - m_3^c - m_4^c| \leq \delta M^c, \quad M < m_1, m_2, m_3, m_4 \leq 2M.$$

Then we have

$$\mathcal{A}(M, \delta) \ll M^{\varepsilon}(M^2 + \delta M^4).$$

Let $T \geq 10$ be a large parameter and y is a real number such that $T^{\varepsilon} \ll y \ll T$. For any $T \leq x \leq 2T$ define

$$\mathcal{R}_1(x; y) := \frac{x^{\frac{9}{8}}}{4\pi^2} \sum_{n \leq y} \frac{c_n}{n^{\frac{7}{8}}} \cos\Big(8\pi(nx)^{\frac{1}{4}} - \frac{\pi}{4}\Big),$$
$$\mathcal{R}_2(x; y) := \Delta_1(x; \varphi) - \mathcal{R}_1(x; y).$$

Lemma 2.6 *If $y \ll T^{\frac{1}{2}}$, then we have the estimates*

$$\int_T^{2T} |\mathcal{R}_1(x; y)|^{2\ell} \mathrm{d}x \ll T^{1 + \frac{9\ell}{4} + \varepsilon}, \quad \ell = 1, 2, 3.$$

Proof We only need to consider the case $\ell = 3$. Using the large value technique of [14] to $\mathcal{R}_1(x; y)$ directly we get that the estimate

$$\int_T^{2T} |\mathcal{R}_1(x; y)|^6 \mathrm{d}x \ll T^{\frac{31}{4} + \varepsilon}$$

holds for $y \ll T^{\frac{1}{2}}$. We omit the details.

Lemma 2.7 *If $y \ll T^{\frac{1}{12}}$, then we have the estimates*

$$\int_T^{2T} |\mathcal{R}_2(x; y)|^3 \mathrm{d}x \ll T^{\frac{35}{8} + \varepsilon} y^{-\frac{9}{8}}, \tag{2.3}$$
$$\int_T^{2T} |\mathcal{R}_2(x; y)|^4 \mathrm{d}x \ll T^{\frac{11}{2} + \varepsilon} y^{-\frac{3}{2}}, \tag{2.4}$$
$$\int_T^{2T} |\mathcal{R}_2(x; y)|^5 \mathrm{d}x \ll T^{\frac{53}{8} + \varepsilon} y^{-\frac{3}{8}}. \tag{2.5}$$

Proof We estimate (2.4) first. For any $T^{\varepsilon} \ll y \ll T^{\frac{1}{3}}$, we have the estimate

$$\int_T^{2T} |\mathcal{R}_2(x; y)|^2 \mathrm{d}x \ll \frac{T^{\frac{13}{4} + \varepsilon}}{y^{\frac{3}{4}}}, \tag{2.6}$$

which is (4.8) in [14].

By Lemma 2.1 we can write

$$\mathcal{R}_2(x; y) = \frac{x^{\frac{9}{8}}}{4\pi^2} \sum_{y < n \leq \sqrt{T}} \frac{c_n}{n^{\frac{7}{8}}} \cos\Big(8\pi(nx)^{\frac{1}{4}} - \frac{\pi}{4}\Big) + \mathcal{R}_2(x; \sqrt{T}).$$

So we have

$$\int_T^{2T} |\mathcal{R}_2(x;y)|^4 \mathrm{d}x \ll T^{\frac{9}{2}} \int_T^{2T} \Big| \sum_{y<n\le\sqrt{T}} \frac{c_n}{n^{\frac{7}{8}}} \mathrm{e}(4(nx)^{\frac{1}{4}}) \Big|^4 \mathrm{d}x + \int_T^{2T} |\mathcal{R}_2(x;\sqrt{T})|^4 \mathrm{d}x. \tag{2.7}$$

By a splitting argument we get for some $y \ll N \ll \sqrt{T}$ that

$$\begin{aligned} &T^{\frac{9}{2}} \int_T^{2T} \Big| \sum_{y<n\le\sqrt{T}} \frac{c_n}{n^{\frac{7}{8}}} e(4(nx)^{\frac{1}{4}}) \Big|^4 \mathrm{d}x \\ &\ll T^{\frac{9}{2}} \log^4 T \times \sum_{n_1,n_2,n_3,n_4\sim N} \frac{c_{n_1}c_{n_2}c_{n_3}c_{n_4}}{(n_1n_2n_3n_4)^{\frac{7}{8}}} \int_T^{2T} \mathrm{e}(4\rho\sqrt[4]{x})\mathrm{d}x \\ &\ll T^{\frac{21}{4}} \log^4 T \times \sum_{n_1,n_2,n_3,n_4\sim N} \frac{c_{n_1}c_{n_2}c_{n_3}c_{n_4}}{(n_1n_2n_3n_4)^{\frac{7}{8}}} \min\Big(\sqrt[4]{T}, \frac{1}{|\rho|}\Big), \end{aligned}$$

where $\rho = n_1^{\frac{1}{4}} + n_2^{\frac{1}{4}} - n_3^{\frac{1}{4}} - n_4^{\frac{1}{4}}$.

By Lemma 2.5, the contribution of $\sqrt[4]{T}$ (in this case $|\rho| \le T^{-\frac{1}{4}}$) is

$$\begin{aligned} &\ll \frac{T^{\frac{11}{2}+\varepsilon}}{N^{\frac{7}{2}}}(N^2 + T^{-\frac{1}{4}}N^{-\frac{1}{4}}N^4) \ll T^{\frac{11}{2}+\varepsilon}N^{-\frac{3}{2}} + T^{\frac{21}{4}+\varepsilon}N^{\frac{1}{4}} \\ &\ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{2}} + T^{\frac{21}{4}+\varepsilon}N^{\frac{1}{4}} \ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{2}} + T^{\frac{43}{8}+\varepsilon} \\ &\ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{2}}, \end{aligned}$$

if $y \ll T^{\frac{1}{12}}$.

Now we consider the contribution of $\frac{1}{|\rho|}$ for which $T^{-\frac{1}{4}} \ll |\rho| \ll y^{\frac{1}{4}}$. We divide the range of ρ into $O(\log T)$ subcases of the form $\xi < |\rho| \le 2\xi$. By Lemma 2.5 again we get that the contribution of $\frac{1}{|\rho|}$ is

$$\begin{aligned} &\ll \frac{T^{\frac{21}{4}+\varepsilon}}{N^{\frac{7}{2}}} \max_{T^{-\frac{1}{4}}\ll\xi\ll y^{\frac{1}{4}}} \sum_{\xi<|\rho|\le 2\xi} \frac{1}{|\rho|} \\ &\ll \frac{T^{\frac{21}{4}+\varepsilon}}{N^{\frac{7}{2}}} \max_{T^{-\frac{1}{4}}\ll\xi\ll y^{\frac{1}{4}}} \frac{1}{\xi}(N^2 + \xi N^{-\frac{1}{4}}N^4) \\ &\ll T^{\frac{11}{2}+\varepsilon}N^{-\frac{3}{2}} + T^{\frac{21}{4}+\varepsilon}N^{\frac{1}{4}} \ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{2}}. \end{aligned}$$

From the above three estimates we get

$$T^{\frac{9}{2}} \int_T^{2T} \Big| \sum_{y<n\le\sqrt{T}} \frac{c_n}{n^{\frac{7}{8}}} \mathrm{e}(4(nx)^{\frac{1}{4}}) \Big|^4 \mathrm{d}x \ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{2}} \tag{2.8}$$

for $y \ll T^{\frac{1}{12}}$.

By Lemma 2.1 we have $\mathcal{R}_2(x;\sqrt{T}) \ll T^{\frac{5}{4}+\varepsilon}$, which combining (2.6) gives

$$\int_T^{2T} |\mathcal{R}_2(x;\sqrt{T})|^4 \mathrm{d}x \ll T^{\frac{5}{2}+\varepsilon} \int_T^{2T} |\mathcal{R}_2(x;\sqrt{T})|^2 \mathrm{d}x \ll T^{\frac{43}{8}+\varepsilon} \ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{2}} \tag{2.9}$$

by noting that $y \ll T^{\frac{1}{12}}$.

Now the estimate (2.4) follows from (2.7)–(2.9). The estimate (2.3) follows from (2.6), (2.4) and Cauchy's inequality.

Now we estimate (2.5). From Lemma 2.6 with $\ell = 3$ we get easily that the estimate

$$\int_T^{2T} |\mathcal{R}_1(x;y)|^{\frac{16}{3}} \mathrm{d}x \ll T^{7+\varepsilon} \tag{2.10}$$

holds for $y \ll T^{\frac{1}{2}}$. From [14] we have

$$\int_T^{2T} |\Delta_1(x;\varphi)|^{\frac{16}{3}} \mathrm{d}x \ll T^{7+\varepsilon}. \tag{2.11}$$

From (2.10)–(2.11) we see that the estimate

$$\int_T^{2T} |\mathcal{R}_2(x;y)|^{\frac{16}{3}} \mathrm{d}x \ll \int_T^{2T} |\Delta_1(x;\varphi) - \mathcal{R}_1(x;y)|^{\frac{16}{3}} \mathrm{d}x \ll T^{7+\varepsilon} \tag{2.12}$$

holds for $y \ll T^{\frac{1}{2}}$. Now by (2.4), (2.12) and Hölder's inequality we get that

$$\int_T^{2T} |\mathcal{R}_2(x;y)|^5 \mathrm{d}x \ll \Big(\int_T^{2T} |\mathcal{R}_2(x;y)|^4 \mathrm{d}x\Big)^{\frac{1}{4}} \Big(\int_T^{2T} |\mathcal{R}_2(x;y)|^{\frac{16}{3}} \mathrm{d}x\Big)^{\frac{3}{4}} \ll T^{\frac{53}{8}+\varepsilon} y^{-\frac{3}{8}}.$$

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. Suppose that $T \geq 10$ is a real number. It suffices for us to evaluate the integral $\int_T^{2T} \Delta_1^k(x;\varphi)\mathrm{d}x$ for any $k \in \{3,4,5\}$. Suppose that y is a parameter such that $T^\varepsilon < y \leq T^{\frac{1}{12}}$.

3.1 The evaluation of the integral $\int_T^{2T} \mathcal{R}_1^k(x;y)\mathrm{d}x$

Let $\mathbb{I} = \{0,1\}$, $\mathbf{i} = (i_1, \cdots, i_{k-1}) \in \mathbb{I}^{k-1}$, $\mathbf{n} = (n_1, \cdots, n_k) \in \mathbb{N}^k$. Define

$$\begin{aligned} \alpha(\mathbf{n};\mathbf{i}) &:= \sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt[4]{n_k}, \\ \beta(\mathbf{i}) &:= 1 + (-1)^{i_1} + (-1)^{i_2} + \cdots + (-1)^{i_{k-1}}. \end{aligned}$$

The formula (4.1) of [14] reads

$$\mathcal{R}_1^k(x;y) = \frac{1}{(2\pi)^{2k} 2^{k-1}} (S_1(x) + S_2(x)), \tag{3.1}$$

where

$$\begin{aligned} S_1(x) &:= x^{\frac{9k}{8}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos\Big(-\frac{\pi\beta(\mathbf{i})}{4}\Big) \sum_{\substack{n_j\leq y, 1\leq j\leq k \\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}}, \\ S_2(x) &:= x^{\frac{9k}{8}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\leq y, 1\leq j\leq k \\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}} \cos\Big(8\pi\alpha(\mathbf{n};\mathbf{i})\sqrt[4]{x} - \frac{\pi\beta(\mathbf{i})}{4}\Big). \end{aligned}$$

From (4.3)–(4.4) of [14] we have

$$\int_T^{2T} S_1(x)\mathrm{d}x = B_k(c) \int_T^{2T} x^{\frac{9k}{8}} \mathrm{d}x + O(T^{1+\frac{9k}{8}+\varepsilon} y^{-\frac{3}{4}}). \tag{3.2}$$

We now estimate the contribution of $S_2(x)$. By Lemma 2.3 we have

$$\int_T^{2T} S_2(x)\mathrm{d}x \ll T^{\frac{9k}{8}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\le y, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}} \times \frac{T^{\frac{3}{4}}}{|\alpha(\mathbf{n};\mathbf{i})|}. \tag{3.3}$$

The sum in the right-hand side of (3.3) can be divided into $O(\log^k T)$ sums of the form

$$\begin{aligned} S(T;N_1,\cdots,N_k) :&= T^{\frac{9k}{8}+\frac{3}{4}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\sim N_j, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}} \times \frac{1}{|\alpha(\mathbf{n};\mathbf{i})|} \\ &\ll \frac{T^{\frac{9k}{8}+\frac{3}{4}+\varepsilon}}{(N_1\cdots N_k)^{\frac{7}{8}}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{n_j\sim N_j, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{1}{|\alpha(\mathbf{n};\mathbf{i})|} \end{aligned} \tag{3.4}$$

by noting that $c_n \ll n^{\varepsilon}$, where $1 \ll N_j \ll y,\ 1 \le j \le k$. We only need to bound the sum

$$W_{\mathbf{i}}(T;N_1,\cdots,N_k) = \sum_{\substack{n_j\sim N_j, 1\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{1}{|\alpha(\mathbf{n};\mathbf{i})|}$$

for each $\mathbf{i} \in \mathbb{I}^{k-1}$.

Let $H = \max(N_1,\cdots,N_k)$. If $\mathbf{i} = (0,\cdots,0)$, then we have

$$W_{\mathbf{i}}(T;N_1,\cdots,N_k) \ll \sum_{n_j\sim N_j, 1\le j\le k} \frac{1}{n_1^{\frac{1}{4}}+\cdots+n_k^{\frac{1}{4}}} \ll \frac{N_1\cdots N_k}{H^{\frac{1}{4}}}. \tag{3.5}$$

Now we suppose $\mathbf{i} \neq (0,\cdots,0)$. The sum $W_{\mathbf{i}}(T;N_1,\cdots,N_k)$ can be divided into $O(\log T)$ sums for which $0 < \Delta < |\alpha(\mathbf{n};\mathbf{i})| \le 2\Delta$. So by Lemma 2.4, we get

$$\begin{aligned} W_{\mathbf{i}}(T;N_1,\cdots,N_k) &\ll \frac{1}{\Delta}\Big(\frac{\Delta N_1\cdots N_k}{H^{\frac{1}{4}}} + \frac{N_1\cdots N_k}{H}\Big) \\ &\ll \frac{N_1\cdots N_k}{H^{\frac{1}{4}}} + \frac{N_1\cdots N_k}{\Delta H} \\ &\ll \frac{N_1\cdots N_k}{H^{\frac{1}{4}}} + N_1\cdots N_k H^{4^{k-2}-\frac{5}{4}} \\ &\ll N_1\cdots N_k H^{4^{k-2}-\frac{5}{4}}, \end{aligned} \tag{3.6}$$

where in the third step we used Lemma 2.2.

From (3.4)–(3.6) we get that

$$S(T;N_1,\cdots,N_k) \ll T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon} H^{4^{k-2}-\frac{5}{4}+\frac{k}{8}} \ll T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon} y^{4^{k-2}-\frac{5}{4}+\frac{k}{8}}. \tag{3.7}$$

From (3.3) and (3.7) we get

$$\int_T^{2T} S_2(x)\mathrm{d}x \ll T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon} y^{4^{k-2}-\frac{5}{4}+\frac{k}{8}}. \tag{3.8}$$

From (3.1)–(3.2) and (3.8) we get

$$\begin{aligned} \int_T^{2T} \mathcal{R}_1^k(x;y)\mathrm{d}x = \frac{B_k(c)}{(2\pi)^{2k}2^{k-1}} \int_T^{2T} x^{\frac{9k}{8}}\mathrm{d}x + O(T^{1+\frac{9k}{8}+\varepsilon} y^{-\frac{3}{4}}) \\ + O(T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon} y^{4^{k-2}-\frac{5}{4}+\frac{k}{8}}). \end{aligned} \tag{3.9}$$

3.2 Estimate of $\int_T^{2T} \mathcal{R}_1^{k-1}(x;y)\mathcal{R}_2(x;y)\mathrm{d}x$

We begin with the formula (4.13) of [14], which reads

$$\int_T^{2T} \mathcal{R}_1^{k-1}(x;y)\mathcal{R}_2(x;y)\mathrm{d}x = \int_T^{2T} \mathcal{R}_1^{k-1}(x;y)\mathcal{R}_2^*(x;y)\mathrm{d}x + O(T^{1+\frac{9k}{8}-\frac{1}{8}+\varepsilon}), \tag{3.10}$$

where

$$\mathcal{R}_2^*(x;y) = (2\pi)^{-2}x^{\frac{9}{8}} \sum_{y<n\le T} \frac{c_n}{n^{\frac{7}{8}}}\cos\Big(8\pi\sqrt[4]{nx}-\frac{\pi}{4}\Big).$$

Write

$$\mathcal{R}_2^*(x;y) = \mathcal{R}_{21}^*(x;y) + \mathcal{R}_{22}^*(x;y), \tag{3.11}$$

where

$$\mathcal{R}_{21}^*(x;y) = (2\pi)^{-2}x^{\frac{9}{8}} \sum_{y<n\le 2ky} \frac{c_n}{n^{\frac{7}{8}}}\cos\Big(8\pi\sqrt[4]{nx}-\frac{\pi}{4}\Big),$$
$$\mathcal{R}_{22}^*(x;y) = (2\pi)^{-2}x^{\frac{9}{8}} \sum_{2ky<n\le T} \frac{c_n}{n^{\frac{7}{8}}}\cos\Big(8\pi\sqrt[4]{nx}-\frac{\pi}{4}\Big).$$

Similar to (4.1) of [14], we can write

$$\mathcal{R}_1^{k-1}(x;y)\mathcal{R}_{21}^*(x;y) = \frac{1}{(2\pi)^{2k}2^{k-1}}(S_3(x)+S_4(x)), \tag{3.12}$$

where

$$S_3(x) := x^{\frac{9k}{8}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \cos\Big(-\frac{\pi\beta(\mathbf{i})}{4}\Big) \sum_{\substack{y<n_1<2ky,n_j\le y,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}},$$
$$S_4(x) := x^{\frac{9k}{8}} \sum_{\mathbf{i}\in\mathbb{I}^{k-1}} \sum_{\substack{y<n_1<2ky,n_j\le y,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}}\cos\Big(8\pi\alpha(\mathbf{n};\mathbf{i})\sqrt[4]{x}-\frac{\pi\beta(\mathbf{i})}{4}\Big).$$

Similar to (4.14) of [14], we have

$$\int_T^{2T} S_3(x)\mathrm{d}x \ll T^{1+\frac{9k}{8}+\varepsilon}y^{-\frac{3}{4}}. \tag{3.13}$$

Similar to (3.8), we have

$$\int_T^{2T} S_4(x)\mathrm{d}x \ll T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon}y^{4^{k-2}-\frac{5}{4}+\frac{k}{8}}. \tag{3.14}$$

Similar to (3.12), we have

$$\mathcal{R}_1^{k-1}(x;y)\mathcal{R}_{22}^*(x;y) = \frac{1}{(2\pi)^{2k}2^{k-1}}S_5(x), \tag{3.15}$$

where

$$S_5(x) := x^{\frac{9k}{8}} \sum_{\boldsymbol{i}\in\mathbb{I}^{k-1}} \sum_{\substack{2ky<n_1\le T,n_j\le y,2\le j\le k\\ \alpha(\mathbf{n};\mathbf{i})\neq 0}} \frac{c_{n_1}\cdots c_{n_k}}{(n_1\cdots n_k)^{\frac{7}{8}}}\cos\Big(8\pi\alpha(\mathbf{n};\mathbf{i})\sqrt[4]{x}-\frac{\pi\beta(\mathbf{i})}{4}\Big).$$

Note that in this case we always have $|\alpha(\mathbf{n};\mathbf{i})| \gg n_1^{\frac{1}{4}}$. By Lemma 2.3 we have

$$\begin{aligned}\int_T^{2T} S_5(x)\mathrm{d}x &\ll T^{\frac{3}{4}+\frac{9k}{8}} \sum_{2ky<n_1\le T} \sum_{n_2\le y,\cdots,n_k\le y} \frac{c_{n_1}c_{n_2}\cdots c_{n_k}}{n_1^{\frac{9}{8}}(n_2\cdots n_k)^{\frac{7}{8}}} \\ &\ll T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon} y^{\frac{k-2}{8}}.\end{aligned}$$

Thus we have

$$\int_T^{2T} \mathcal{R}_1^{k-1}(x;y)\mathcal{R}_{22}^*(x;y)\mathrm{d}x \ll T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon} y^{\frac{k-2}{8}}. \tag{3.16}$$

Collecting (3.10)–(3.16), we get

$$\begin{aligned}\int_T^{2T} \mathcal{R}_1^{k-1}(x;y)\mathcal{R}_2(x;y)\mathrm{d}x &\ll T^{1+\frac{9k}{8}-\frac{1}{8}+\varepsilon} + T^{1+\frac{9k}{8}+\varepsilon}y^{-\frac{3}{4}} + T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon}y^{4^{k-2}-\frac{5}{4}+\frac{k}{8}} \\ &\ll T^{1+\frac{9k}{8}+\varepsilon}y^{-\frac{3}{4}} + T^{\frac{3}{4}+\frac{9k}{8}+\varepsilon}y^{4^{k-2}-\frac{5}{4}+\frac{k}{8}}.\end{aligned} \tag{3.17}$$

3.3 Proof of theorem for the case $k = 3$

We have

$$\begin{aligned}\Delta_1^3(x;\varphi) &= (\mathcal{R}_1(x;y) + \mathcal{R}_2(x;y))^3 \\ &= \mathcal{R}_1^3(x;y) + 3\mathcal{R}_1^2(x;y)\mathcal{R}_2(x;y) + 3\mathcal{R}_1(x;y)\mathcal{R}_2^2(x;y) + \mathcal{R}_2^3(x;y).\end{aligned} \tag{3.18}$$

Taking $k = 3$ in (3.9) we have

$$\int_T^{2T} \mathcal{R}_1^3(x;y)\mathrm{d}x = \frac{B_3(c)}{2^8\pi^6}\int_T^{2T} x^{\frac{27}{8}}\mathrm{d}x + O(T^{\frac{35}{8}+\varepsilon}y^{-\frac{3}{4}}) + O(T^{\frac{33}{8}+\varepsilon}y^{\frac{25}{8}}). \tag{3.19}$$

Taking $k = 3$ in (3.17) we have

$$\int_T^{2T} \mathcal{R}_1^2(x;y)\mathcal{R}_2(x;y)\mathrm{d}x \ll T^{\frac{35}{8}+\varepsilon}y^{-\frac{3}{4}} + T^{\frac{33}{8}+\varepsilon}y^{\frac{25}{8}}. \tag{3.20}$$

From Lemma 2.6 with $\ell = 1$, (2.4) of Lemma 2.7 and Cauchy's inequality we get

$$\int_T^{2T} \mathcal{R}_1(x;y)\mathcal{R}_2^2(x;y)\mathrm{d}x \ll T^{\frac{35}{8}+\varepsilon}y^{-\frac{3}{4}}. \tag{3.21}$$

From (3.18)–(3.21) and (2.3) of Lemma 2.7 we get

$$\begin{aligned}\int_T^{2T} \Delta_1^3(x;\varphi)\mathrm{d}x &= \frac{B_3(c)}{2^8\pi^6}\int_T^{2T} x^{\frac{27}{8}}\mathrm{d}x + O(T^{\frac{35}{8}+\varepsilon}y^{-\frac{3}{4}}) + O(T^{\frac{33}{8}+\varepsilon}y^{\frac{25}{8}}) \\ &= \frac{B_3(c)}{2^8\pi^6}\int_T^{2T} x^{\frac{27}{8}}\mathrm{d}x + O(T^{\frac{35}{8}-\frac{3}{62}+\varepsilon})\end{aligned} \tag{3.22}$$

by choosing $y = T^{\frac{2}{31}}$. Now the case $k = 3$ of theorem follows from (3.22) by a splitting argument.

3.4 Proof of theorem for the case $k = 4$

We have

$$\Delta_1^4(x;\varphi) = \mathcal{R}_1^4(x;y) + 4\mathcal{R}_1^3(x;y)\mathcal{R}_2(x;y) + O(\mathcal{R}_1^2(x;y)\mathcal{R}_2^2(x;y) + \mathcal{R}_2^4(x;y)). \tag{3.23}$$

Taking $k = 4$ in (3.9) we have

$$\int_T^{2T} \mathcal{R}_1^4(x;y)\mathrm{d}x = \frac{B_4(c)}{2^{11}\pi^8}\int_T^{2T} x^{\frac{9}{2}}\mathrm{d}x + O(T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{4}}) + O(T^{\frac{21}{4}+\varepsilon}y^{\frac{61}{4}}). \tag{3.24}$$

Taking $k = 4$ in (3.17) we have

$$\int_T^{2T} \mathcal{R}_1^3(x;y)\mathcal{R}_2(x;y)\mathrm{d}x \ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{4}} + T^{\frac{21}{4}+\varepsilon}y^{\frac{61}{4}}. \tag{3.25}$$

From Lemma 2.6 with $\ell = 2$, (2.4) of Lemma 2.7 and Cauchy's inequality we get

$$\int_T^{2T} \mathcal{R}_1^2(x;y)\mathcal{R}_2^2(x;y)\mathrm{d}x \ll T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{4}}. \tag{3.26}$$

From (3.23)–(3.26) and (2.4) of Lemma 2.7 we get that

$$\begin{aligned}\int_T^{2T} \Delta_1^4(x;\varphi)\mathrm{d}x &= \frac{B_4(c)}{2^{11}\pi^8}\int_T^{2T} x^{\frac{9}{2}}\mathrm{d}x + O(T^{\frac{11}{2}+\varepsilon}y^{-\frac{3}{4}}) + O(T^{\frac{21}{4}+\varepsilon}y^{\frac{61}{4}}) \\ &= \frac{B_4(c)}{2^{11}\pi^8}\int_T^{2T} x^{\frac{9}{2}}\mathrm{d}x + O(T^{\frac{11}{2}-\frac{3}{256}+\varepsilon})\end{aligned} \tag{3.27}$$

by choosing $y = T^{\frac{1}{64}}$. Now the case $k = 4$ of theorem follows from (3.27) by a splitting argument.

3.5 Proof of theorem for the case $k = 5$

We have

$$\Delta_1^5(x;\varphi) = \mathcal{R}_1^5(x;y) + 4\mathcal{R}_1^4(x;y)\mathcal{R}_2(x;y) + O(|\mathcal{R}_1^3(x;y)|\mathcal{R}_2^2(x;y) + |\mathcal{R}_2^5(x;y)|). \tag{3.28}$$

Taking $k = 5$ in (3.9) we have

$$\int_T^{2T} \mathcal{R}_1^5(x;y)\mathrm{d}x = \frac{B_5(c)}{2^{14}\pi^{10}}\int_T^{2T} x^{\frac{45}{8}}\mathrm{d}x + O(T^{\frac{53}{8}+\varepsilon}y^{-\frac{3}{4}}) + O(T^{\frac{51}{8}+\varepsilon}y^{\frac{507}{8}}). \tag{3.29}$$

Taking $k = 5$ in (3.17) we have

$$\int_T^{2T} \mathcal{R}_1^4(x;y)\mathcal{R}_2(x;y)\mathrm{d}x \ll T^{\frac{53}{8}+\varepsilon}y^{-\frac{3}{4}} + T^{\frac{51}{8}+\varepsilon}y^{\frac{507}{8}}. \tag{3.30}$$

From Lemma 2.6 with $\ell = 3$, (2.4) of Lemma 2.7 and Cauchy's inequality we get

$$\int_T^{2T} |\mathcal{R}_1^3(x;y)|\mathcal{R}_2^2(x;y)\mathrm{d}x \ll T^{\frac{53}{8}+\varepsilon}y^{-\frac{3}{4}}. \tag{3.31}$$

From (3.28)–(3.31) and (2.5) of Lemma 2.7 we get that

$$\int_T^{2T} \Delta_1^5(x;\varphi)\mathrm{d}x = \frac{B_5(c)}{2^{14}\pi^{10}}\int_T^{2T} x^{\frac{45}{8}}\mathrm{d}x + O(T^{\frac{53}{8}+\varepsilon}y^{-\frac{3}{8}}) + O(T^{\frac{51}{8}+\varepsilon}y^{\frac{507}{8}})$$

$$= \frac{B_5(c)}{2^{14}\pi^{10}} \int_T^{2T} x^{\frac{45}{8}} \mathrm{d}x + O(T^{\frac{53}{8} - \frac{1}{680} + \varepsilon}) \tag{3.32}$$

by choosing $y = T^{\frac{1}{255}}$. Now the case $k = 5$ of theorem follows from (3.32) by a splitting argument.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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